17 Advanced Differential Calculus

1 | Product-quotient differentiation 1
2 | Differentiation of composite functions 4
3 | Differentiation of trigonometric functions 6
4 | Differentiation of exponential functions e^x 9
5 | Differentiation of logarithmic functions 11
6 | Differentiation of exponential functions a^x 13
7 | Differentiation of the function x^p 15
8 | High-order derivatives 17
9 | Differentiation of implicit functions 19
10 | Differentiation of functions expressed as parameters 21
11 | Differentiation of logarithms 23
12 | Equation for a tangent 26
13 | Equation for a normal line 28
14 | Function increase/decrease and derivative signs 30
15 | Maximums and minimums of functions 32
16 | Maximum and minimum values of functions 34
17 | Concavity, convexity and inflection points of curves 36
18 | How to draw graphs of functions 39
19 | 2nd-order derivatives and extrema 42
20 | Proving inequalities 45
21 | Number of real roots of equations 47
22 | 1st order approximation of a function $f(x)$ 50

CASIO
Essential Materials

Introduction

These teaching materials were created with the hope of conveying to many teachers and students the appeal of scientific calculators.

(1) Change awareness (emphasizing the thinking process) and boost efficiency in learning mathematics

- By reducing the time spent on manual calculations, we can have learning with a focus on the thinking process that is more efficient.
- This reduces the aversion to mathematics caused by complicated calculations, and allows students to experience the joy of thinking, which is the essence of mathematics.

(2) Diversification of learning materials and problem-solving methods

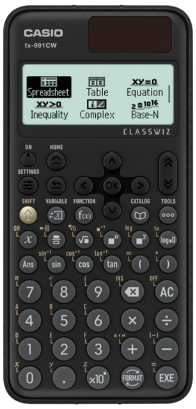
- Making it possible to do difficult calculations manually allows for diversity in learning materials and problem-solving methods.

(3) Promoting understanding of mathematical concepts

- By using the various functions of the scientific calculator in creative ways, students are able to deepen their understanding of mathematical concepts through calculations and discussions from different perspectives than before.
- This allows for exploratory learning through easy trial and error of questions.
- Listing and graphing of numerical values by means of tables allows students to discover laws and to understand visually.

Features of this book

- As well as providing first-time scientific calculator users with opportunities to learn basic scientific calculator functions from the ground up, the book also has material to show people who already use scientific calculators the appeal of scientific calculators described above.
- You can also learn about functions and techniques that are not available on conventional Casio models or other brands of scientific calculators.
- This book covers many units of high school mathematics, allowing students to learn how to use the scientific calculator as they study each topic.
- This book can be used in a variety of situations, from classroom activities to independent study and homework by students.



Better Mathematics Learning
with Scientific Calculator

Structure

Quadratic functions

TARGET To understand the concept of functions.

STUDY GUIDE

Functions

Definition of a function

For 2 variables, x and y, if the value of x is determined and if just 1 corresponding value of y is determined, then we say that **y is a function of x.**

Ex. Given $x=2$ ➡ Functions ($y=x+1$) ➡ We can determine $y=3$

How to describe functions

When y is a function of x, generally y is expressed as $f(x)$ or $g(x)$. Furthermore, in the function $y=f(x)$, then when $x=a$, the value of y is expressed as $f(a)$.

Ex. In the function $f(x)=3x+3$, when $x=1$, then when $f(1)=3\times1+3=6$ and $x=5$, then we get $f(5)=3\times5+3=18$.

Linear functions & Quadratic functions

When y is expressed as a linear equation of x, y is called the **linear function** of x, which is expressed as follows.

$$\boxed{y=ax+b \quad (a\neq0)}$$

When y is expressed as a quadratic equation of x, y is called the **quadratic function** of x, which is expressed as follows.

$$\boxed{y=ax^2+bx+c \quad (a\neq0)}$$

TARGET

Students can identify the objective to learn in each section.

STUDY GUIDE

Mathematical theorems and concepts are explained in detail. A scientific calculator is used to check and derive formulas according to the topic.

EXERCISE

1 For the quadratic function $f(x)=3x^2+2x+6$, find the various following values.

(1) $f(0)$

$f(0)=3\cdot0^2+2\cdot0+6=6$

6

(2) $f(1)$

$f(1)=3\cdot1^2+2\cdot1+6=11$

11

(3) $f(-1)$

$f(-1)=3\cdot(-1)^2+2\cdot(-1)+6=7$

7

(4) $f(109)$

$f(109)=3\cdot109^2+2\cdot109+6=35867$

35867

check

Use the FUNCTION function to calculate the value of $f(x)$.

Press ⓢ, select [Calculate], press OK

Press f(x), select [Define f(x)], press OK 3 x x² + 2 x + 6 EXE

Calculate Statistics Distribution Spreadsheet Table Equation

f(x) g(x) Define f(x) Define g(x)

f(x)=3x²+2x+6

EXERCISE

Students learn basic examples based on the explanation in Study Guide.

check

Explains how to use the scientific calculator to solve problems and check answers.

PRACTICE

1 For the quadratic function $f(x)=-2x^2+4x+5$, find the various following values.

(1) $f(0)$

$f(0)=-2\cdot0^2+4\cdot0+5=5$

5

(2) $f(2)$

$f(2)=-2\cdot2^2+4\cdot2+5=5$

5

(3) $f(-3)$

$f(-3)=-2\cdot(-3)^2+4\cdot(-3)+5=-25$

-25

(4) $f(52)$

$f(52)=-2\cdot52^2+4\cdot52+5=-5195$

-5195

PRACTICE

Students can do practice problems similar to those in EXERCISE. They can also practice using the scientific calculator as they learned to in Check.

ADVANCED

3 We know that when an object is thrown vertically upward from a height of 0 m at a velocity of v m/s, the relation between the height y m of the object and the time x seconds since it was thrown is $y=-4.9x^2+vx$. So, when a ball is thrown vertically from a height of 0 m at a velocity of 9.8 m/s, find the time it takes to reach the highest point and the height of that highest point. Note that the air resistance can be ignored.

$y=-4.9x^2+9.8x=-4.9(x^2-2x)=-4.9\{(x-1)^2-1^2\}=-4.9(x-1)^2+4.9$

Therefore, the graph of this function is convex upward, the axis is $x=1$, and the vertex is (1, 4.9).

Thus, the ball takes 1 second to reach its highest point, and the height of that highest point is 4.9 m.

1 second, 4.9 m

check

Press ⓢ, select [Equation], press OK

Select [Polynomial], press OK, select [ax²+bx+c], press OK

− 4 . 9 EXE 9 . 8 EXE 0 EXE EXE

Calculate Statistics Distribution Spreadsheet Table Equation

Simul Equation Polynomial Solver

ax²+bx+c ax³+bx²+cx+d ax⁴+bx³+cx²+dx+e

EXE EXE EXE **Press ⇧ x, scan the QR code to display a graph.**

Max of y=ax²+bx+c x= 1

Max of y=ax²+bx+c y= 49/10

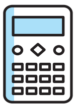
ADVANCED

Practical problems have been included in several topics. Solutions using a scientific calculator are also presented as necessary.

Other marks

Ex.	Simple examples on how to apply equations and theorems
explanation	Formulas and their supplementary explanations
proof	Proofs and checks of mathematical formulas
EXTRA Info.	Knowledge and information on formulas and other supplementary information in other units
OTHER METHODS	Alternative solutions and different verification methods for previously presented problems

Calculator mark



Where to use the scientific calculator

Colors of fonts in the teaching materials

- In STUDY GUIDE, important mathematical terms and formulas are printed in blue.
- In PRACTICE and ADVANCED the answers are printed in red.
  (Separate data is also available without the red parts, so it can be used for exercises.)

Applicable models

The applicable model is fx-991CW.
(Instructions on how to do input are for the fx-991CW, but in many cases similar calculations can be done on other models.)

Related Links

- Information and educational materials relevant to scientific calculators can be viewed on the following site.
  https://edu.casio.com
- The following video can be viewed to learn about the multiple functions of scientific calculators.
  https://www.youtube.com/playlist?list=PLRgxo9AwbIZLurUCZnrbr4cLfZdqY6aZA

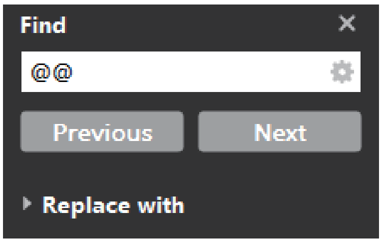
How to use PDF data

About types of data

- Data for all unit editions and data for each unit are available.
- For the above data, the PRACTICE and ADVANCED data without the answers in red is also available.

How to find where the scientific calculator is used

(1) Open a search window in the PDF Viewer.
(2) Type in "@@" as a search term.
(3) You can sequentially check where the calculator marks appear in the data.



How to search for a unit and section

(1) Search for units of data in all unit editions

- The data in all unit editions has a unit table of contents.
- Selecting a unit in the table of contents lets you jump to the first page of that unit.
- There is a bookmark on the first page of each unit, so you can jump from there also.

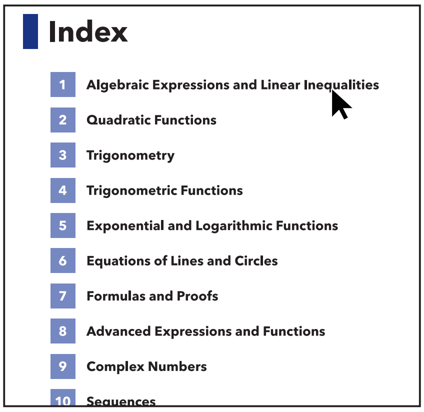
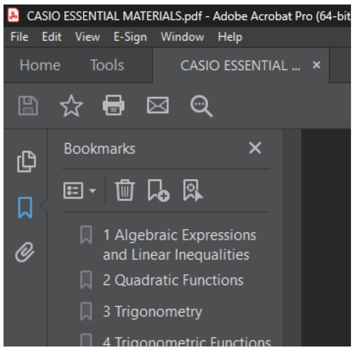


Table of contents of unit



Bookmark of unit

(2) Search for sections

- There are tables of contents for sections on the first page of units.
- Selecting a section in the table of contents takes you to the first page of that section.

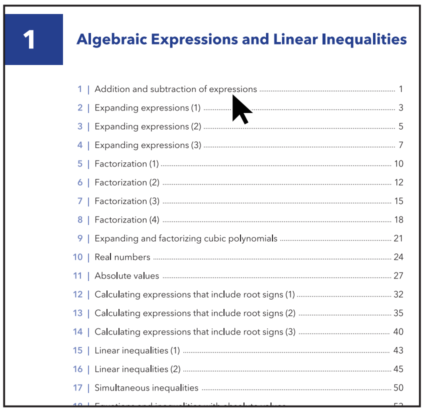


Table of contents of section

Product-quotient differentiation

TARGET To understand how to differentiate a function expressed as a product or a quotient.

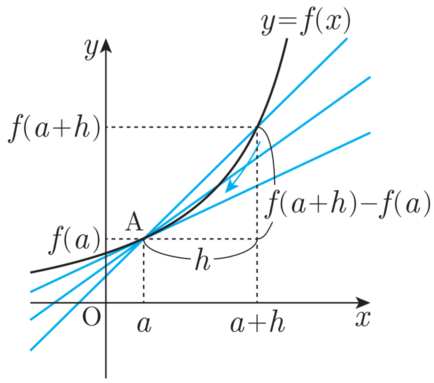
STUDY GUIDE

Product-quotient differentiation

Derivatives

The function $f(x)$ is differentiable with $x=a$, that is to say, it is the **differential coefficient**

When $f'(a) = \lim_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$ exists, then we say that the function derived by the differential coefficient $f'(a)$ corresponds to the value a of x, so it is the **derivative** of $f(x)$, and is expressed as $\dfrac{d}{dx}f(x)$ or $f'(x)$.



$f'(a)$ expresses the gradient of the tangent at point A.

Definition of derivative

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Product-quotient derivatives

Product $\{f(x)g(x)\}' = f'(x)g(x) + f(x)g'(x)$

Quotient $\left\{\dfrac{f(x)}{g(x)}\right\}' = \dfrac{f'(x)g(x) - f(x)g'(x)}{\{g(x)\}^2}$

Specifically, $\left\{\dfrac{1}{g(x)}\right\}' = -\dfrac{g'(x)}{\{g(x)\}^2}$ $\quad (g(x) \neq 0)$

explanation

Proof of derivative of products

$$\begin{aligned}
\{f(x)g(x)\}' &= \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h \to 0} \frac{\{f(x+h) - f(x)\}g(x+h) + f(x)\{g(x+h) - g(x)\}}{h} \\
&= \lim_{h \to 0} \left\{ \frac{f(x+h) - f(x)}{h} \cdot g(x+h) + f(x) \cdot \frac{g(x+h) - g(x)}{h} \right\} \\
&= f'(x)g(x) + f(x)g'(x)
\end{aligned}$$

Proof of derivative of quotients

$$\left\{\frac{f(x)}{g(x)}\right\}' = \lim_{h\to 0}\frac{\dfrac{f(x+h)}{g(x+h)}-\dfrac{f(x)}{g(x)}}{h}$$
$$= \lim_{h\to 0}\left\{\frac{f(x+h)g(x)-f(x)g(x+h)}{g(x+h)g(x)}\cdot\frac{1}{h}\right\}$$
$$= \lim_{h\to 0}\left[\frac{1}{g(x+h)g(x)}\cdot\frac{\{f(x+h)-f(x)\}g(x)-f(x)\{g(x+h)-g(x)\}}{h}\right]$$
$$= \lim_{h\to 0}\left[\frac{1}{g(x+h)g(x)}\cdot\left\{\frac{f(x+h)-f(x)}{h}\cdot g(x)-f(x)\cdot\frac{g(x+h)-g(x)}{h}\right\}\right]$$
$$= \frac{f'(x)g(x)-f(x)g'(x)}{\{g(x)\}^2}$$

EXTRA Info.

$$\{f(x)g(x)h(x)\}' = f'(x)g(x)h(x)+f(x)g'(x)h(x)+f(x)g(x)h'(x)$$

EXERCISE

◆ Differentiate the following functions.

(1) $y=(3x+2)(x-1)$

$$y' = 3(x-1)+(3x+2)\cdot 1$$
$$= 3x-3+3x+2$$
$$= 6x-1$$

$$\boldsymbol{y' = 6x-1}$$

(2) $y=x^3(2x-1)$

$$y' = 3x^2(2x-1)+x^3\cdot 2$$
$$= 6x^3-3x^2+2x^3$$
$$= 8x^3-3x^2$$

$$\boldsymbol{y' = 8x^3-3x^2}$$

(3) $y=(x^2+1)(3x^2+x-3)$

$$y' = 2x(3x^2+x-3)+(x^2+1)(6x+1)$$
$$= 6x^3+2x^2-6x+6x^3+x^2+6x+1$$
$$= 12x^3+3x^2+1$$

$$\boldsymbol{y' = 12x^3+3x^2+1}$$

(4) $y=(x+1)(x+2)(x-3)$

$$y' = 1\cdot(x+2)(x-3)+(x+1)\cdot 1\cdot(x-3)+(x+1)(x+2)\cdot 1$$
$$= (x^2-x-6)+(x^2-2x-3)+(x^2+3x+2)$$
$$= 3x^2-7$$

$$\boldsymbol{y' = 3x^2-7}$$

(5) $y=\dfrac{1}{2x^2+x-5}$

$$y' = -\frac{4x+1}{(2x^2+x-5)^2}$$

$$\boldsymbol{y' = -\frac{4x+1}{(2x^2+x-5)^2}}$$

(6) $y=\dfrac{x-1}{x^2-2x+2}$

$$y' = \frac{1\cdot(x^2-2x+2)-(x-1)(2x-2)}{(x^2-2x+2)^2}$$
$$= \frac{x^2-2x+2-(2x^2-4x+2)}{(x^2-2x+2)^2}$$
$$= \frac{-x^2+2x}{(x^2-2x+2)^2}$$
$$= -\frac{x(x-2)}{(x^2-2x+2)^2}$$

$$\boldsymbol{y' = -\frac{x(x-2)}{(x^2-2x+2)^2}}$$

PRACTICE

◆ Differentiate the following functions.

(1) $y=(4x-1)(x+3)$

$$y' = 4(x+3)+(4x-1)\cdot 1$$
$$= 4x+12+4x-1$$
$$= 8x+11$$

$$y' = 8x+11$$

(2) $y=x^4(x^2+1)$

$$y' = 4x^3(x^2+1)+x^4\cdot 2x$$
$$= 4x^5+4x^3+2x^5$$
$$= 6x^5+4x^3$$

$$y' = 6x^5+4x^3$$

(3) $y=(x^2-x+3)(x^2+x+3)$

$$y' = (2x-1)(x^2+x+3)+(x^2-x+3)(2x+1)$$
$$= 2x^3+x^2+5x-3+(2x^3-x^2+5x+3)$$
$$= 4x^3+10x$$

$$y' = 4x^3+10x$$

(4) $y=(x^3-x+1)(x^2-2)$

$$y' = (3x^2-1)(x^2-2)+(x^3-x+1)\cdot 2x$$
$$= 3x^4-7x^2+2+(2x^4-2x^2+2x)$$
$$= 5x^4-9x^2+2x+2$$

$$y' = 5x^4-9x^2+2x+2$$

(5) $y=(x-2)(x+3)(x-4)$

$$y' = 1\cdot(x+3)(x-4)+(x-2)\cdot 1\cdot(x-4)+(x-2)(x+3)\cdot 1$$
$$= (x^2-x-12)+(x^2-6x+8)+(x^2+x-6)$$
$$= 3x^2-6x-10$$

$$y' = 3x^2-6x-10$$

(6) $y=\dfrac{3}{x^2-2}$

$$y' = -\frac{3\cdot 2x}{(x^2-2)^2}$$
$$= -\frac{6x}{(x^2-2)^2}$$

$$y' = -\frac{6x}{(x^2-2)^2}$$

(7) $y=\dfrac{x^2-2x-3}{x^2-5}$

$$y' = \frac{(2x-2)(x^2-5)-(x^2-2x-3)\cdot 2x}{(x^2-5)^2}$$
$$= \frac{2x^3-2x^2-10x+10-(2x^3-4x^2-6x)}{(x^2-5)^2}$$
$$= \frac{2(x^2-2x+5)}{(x^2-5)^2}$$

$$y' = \frac{2(x^2-2x+5)}{(x^2-5)^2}$$

Differentiation of composite functions

TARGET To understand how to find the derivative of a composite function.

STUDY GUIDE

Differentiation of composite functions

Derivatives of composite functions

When there are 2 functions $y=f(u)$ and $u=g(x)$, the function $y=f\left(g(x)\right)$, formed by eliminating u, is a function of x, which is called a **composite function** of $y=f(u)$ and $u=g(x)$. The derivative of the composite function of 2 differentiable functions $y=f(u)$ and $u=g(x)$ is found as follows.

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx} \quad \text{or} \quad \left\{f\left(g(x)\right)\right\}'=f'\left(g(x)\right)g'(x)$$

explanation

Let the increment Δx of u relative to the increment of x in $u=g(x)$ be Δu, and let the increment Δu of y relative to the increment of u in $y=f(u)$ be Δy.

$\Delta u=g(x+\Delta x)-g(x), \Delta y=f(u+\Delta u)-f(u)$

When $\Delta x\to 0$, it becomes $\Delta u\to 0$, as shown below.

$$\frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}\left(\frac{\Delta y}{\Delta u}\cdot\frac{\Delta u}{\Delta x}\right)=\lim_{\Delta x\to 0}\left(\frac{f(u+\Delta u)-f(u)}{\Delta u}\cdot\frac{g(x+\Delta x)-g(x)}{\Delta x}\right)$$
$$=\lim_{\Delta u\to 0}\frac{f(u+\Delta u)-f(u)}{\Delta u}\cdot\lim_{\Delta x\to 0}\frac{g(x+\Delta x)-g(x)}{\Delta x}=\frac{dy}{du}\cdot\frac{du}{dx}$$

Derivative of x^n (n is a rational number)

For x^n, when n is a rational number, the following formula holds.

$$(x^n)'=nx^{n-1} \quad (n \text{ is a rational number})$$

explanation

Given $n=\frac{p}{q}$ (where p and q are whole numbers, and $q>0$), then from $y=x^n=x^{\frac{p}{q}}$, we get $y^q=x^p \quad \ldots(\mathrm{i})$

By differentiating the left side by x, we get $\frac{d}{dx}y^q=\frac{d}{dy}y^q\cdot\frac{dy}{dx}=qy^{q-1}\cdot\frac{dy}{dx}$, so by differentiating both sides of (i) by x, we get $qy^{q-1}\cdot\frac{dy}{dx}=px^{p-1}$.

Therefore, we get $\frac{dy}{dx}=\frac{px^{p-1}}{qy^{q-1}}=\frac{px^{p-1}}{qx^{p-\frac{p}{q}}}=\frac{p}{q}x^{\frac{p}{q}-1}=nx^{n-1}$.

EXERCISE

◆ Differentiate the following functions.

(1) $y=(3x-4)^4$

Given $f(u)=u^4, g(x)=3x-4$,

we can get $y=f\big(g(x)\big)$.

$$y'=f'\big(g(x)\big)g'(x)$$
$$=4(3x-4)^3\cdot 3$$
$$=12(3x-4)^3$$

$$\boldsymbol{y'=12(3x-4)^3}$$

(2) $y=(2x+1)^3(x-1)$

$$y'=3(2x+1)^2\cdot 2(x-1)+(2x+1)^3\cdot 1$$
$$=(2x+1)^2\{6(x-1)+(2x+1)\}$$
$$=(2x+1)^2(8x-5)$$

$$\boldsymbol{y'=(2x+1)^2(8x-5)}$$

(3) $y=\sqrt{(5-x)^3}$

$$y=(5-x)^{\frac{3}{2}}$$
$$y'=\frac{3}{2}(5-x)^{\frac{1}{2}}\cdot(-1)$$
$$=-\frac{3}{2}\sqrt{5-x}$$

$$\boldsymbol{y'=-\frac{3}{2}\sqrt{5-x}}$$

(4) $y=\sqrt[4]{\frac{x}{x+2}}$

$$y=\left(\frac{x}{x+2}\right)^{\frac{1}{4}}$$
$$y'=\frac{1}{4}\left(\frac{x}{x+2}\right)^{-\frac{3}{4}}\cdot\frac{x+2-x}{(x+2)^2}=\frac{1}{2(x+2)^2}\left(\frac{x+2}{x}\right)^{\frac{3}{4}}$$
$$=\frac{1}{2(x+2)^2}\sqrt[4]{\left(\frac{x+2}{x}\right)^3}$$

$$\boldsymbol{y'=\frac{1}{2(x+2)^2}\sqrt[4]{\left(\frac{x+2}{x}\right)^3}}$$

PRACTICE

◆ Differentiate the following functions.

(1) $y=(x^2-4x+5)^3$

$$y'=3(x^2-4x+5)^2(2x-4)$$
$$=6(x^2-4x+5)^2(x-2)$$

$$\boldsymbol{y'=6(x^2-4x+5)^2(x-2)}$$

(2) $y=\left(3x-\frac{2}{x}\right)^4$

$$y'=4\left(3x-\frac{2}{x}\right)^3\left(3+\frac{2}{x^2}\right)$$

$$\boldsymbol{y'=4\left(3x-\frac{2}{x}\right)^3\left(3+\frac{2}{x^2}\right)}$$

(3) $y=\frac{1}{(2x-3)^3}$

$$y=(2x-3)^{-3}$$
$$y'=-3(2x-3)^{-4}\cdot 2$$
$$=-\frac{6}{(2x-3)^4}$$

$$\boldsymbol{y'=-\frac{6}{(2x-3)^4}}$$

(4) $y=(3x-1)^2(2x+5)^3$

$$y'=2(3x-1)\cdot 3(2x+5)^3+(3x-1)^2\cdot 3(2x+5)^2\cdot 2$$
$$=6(3x-1)(2x+5)^2\{(2x+5)+(3x-1)\}$$
$$=6(3x-1)(2x+5)^2(5x+4)$$

$$\boldsymbol{y'=6(3x-1)(2x+5)^2(5x+4)}$$

Differentiation of trigonometric functions

TARGET To understand how to find the derivatives of functions expressed using trigonometric functions.

STUDY GUIDE

Derivatives of trigonometric functions

The derivatives of trigonometric functions are outlined below. Note that, the units for x are radians.

$$(\sin x)' = \cos x \qquad (\cos x)' = -\sin x \qquad (\tan x)' = \frac{1}{\cos^2 x}$$

explanation

The derivative of $\sin x$ is derived by using the formula for sum → product as shown below.

$$(\sin x)' = \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} = \lim_{h\to 0}\frac{1}{h}\cdot 2\cos\frac{(x+h)+x}{2}\sin\frac{(x+h)-x}{2}$$

$$= \lim_{h\to 0}\frac{2}{h}\cos\left(x+\frac{h}{2}\right)\sin\frac{h}{2} = \lim_{h\to 0}\left\{\cos\left(x+\frac{h}{2}\right)\cdot\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right\} = \cos x$$

The derivative of $\cos x$ and $\sin x$ can be derived by using the formula of the sum → product.

$$(\cos x)' = \lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h} = \lim_{h\to 0}\frac{1}{h}\left\{-2\sin\frac{(x+h)+x}{2}\sin\frac{(x+h)-x}{2}\right\}$$

$$= \lim_{h\to 0}\frac{2}{h}\left\{-\sin\left(x+\frac{h}{2}\right)\sin\frac{h}{2}\right\} = \lim_{h\to 0}\left\{-\sin\left(x+\frac{h}{2}\right)\cdot\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right\} = -\sin x$$

Alternatively, we can also use the $(\sin x)' = \cos x$ formula as shown below.

$$(\cos x)' = \left\{\sin\left(\frac{\pi}{2}-x\right)\right\}' = \left\{\cos\left(\frac{\pi}{2}-x\right)\right\}\cdot(-1) = -\sin x$$

The derivative of $\tan x$ is derived by using the $(\sin x)' = \cos x$ and $(\cos x)' = -\sin x$, and the formula for the derivative of the quotient.

$$(\tan x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{(\sin x)'\cos x - \sin x(\cos x)'}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

EXERCISE

◆ Differentiate the following functions.

(1) $y=\sin(2x-1)$

$$y'=\{\cos(2x-1)\}\cdot 2$$
$$=2\cos(2x-1)$$

$$y'=2\cos(2x-1)$$

(2) $y=\cos^3 x$

$$y'=3\cos^2 x\cdot(-\sin x)$$
$$=-3\cos^2 x\sin x$$

$$y'=-3\cos^2 x\sin x$$

(3) $y=\tan(x^2+1)$

$$y'=\frac{1}{\cos^2(x^2+1)}\cdot 2x$$
$$=\frac{2x}{\cos^2(x^2+1)}$$

$$y'=\frac{2x}{\cos^2(x^2+1)}$$

(4) $y=x^3\sin x$

$$y'=3x^2\sin x+x^3\cos x$$

$$y'=3x^2\sin x+x^3\cos x$$

(5) $y=\dfrac{1+\sin x}{\cos x}$

$$y'=\frac{\cos^2 x-(1+\sin x)\cdot(-\sin x)}{\cos^2 x}$$
$$=\frac{\cos^2 x+\sin x+\sin^2 x}{\cos^2 x}=\frac{1+\sin x}{\cos^2 x}$$
$$=\frac{1+\sin x}{1-\sin^2 x}=\frac{1+\sin x}{(1+\sin x)(1-\sin x)}=\frac{1}{1-\sin x}$$

$$y'=\frac{1}{1-\sin x}$$

(6) $y=2\sin x\cos x$

$$y'=2(\cos^2 x-\sin^2 x)$$
$$=2\cos 2x$$

OTHER METHODS

From $y=2\sin x\cos x=\sin 2x$, we get

$y'=(\cos 2x)\cdot 2=2\cos 2x$

$$y'=2\cos 2x$$

PRACTICE

◆ Differentiate the following functions.

(1) $y = 5\sin^2 x$

$$y' = 5 \cdot 2\sin x\cos x$$
$$= 10\sin x\cos x$$
$$(= 5\sin 2x)$$

$$y' = 10\sin x\cos x$$
$$(y' = 5\sin 2x)$$

(2) $y = \cos^2(3-x)$

$$y' = 2\cos(3-x)\{-\sin(3-x)\}\cdot(-1)$$
$$= 2\cos(3-x)\sin(3-x)$$
$$(= \sin 2(3-x))$$

$$y' = 2\cos(3-x)\sin(3-x)$$
$$(y' = \sin 2(3-x))$$

(3) $y = \cos x + x\sin x$

$$y' = -\sin x + \sin x + x\cos x$$
$$= x\cos x$$

$$y' = x\cos x$$

(4) $y = \sin^2 x\cos^3 x$

$$y' = 2\sin x\cos x\cos^3 x + \sin^2 x\cdot 3\cos^2 x\cdot(-\sin x)$$
$$= 2\sin x\cos^4 x - 3\sin^3 x\cos^2 x$$

$$y' = 2\sin x\cos^4 x - 3\sin^3 x\cos^2 x$$

(5) $y = \dfrac{1}{\tan x}$

$$y' = -\frac{\dfrac{1}{\cos^2 x}}{\tan^2 x}$$
$$= -\frac{1}{\cos^2 x}\left(\frac{\cos x}{\sin x}\right)^2$$
$$= -\frac{1}{\sin^2 x}$$

OTHER METHODS

From $y = \dfrac{1}{\tan x} = \dfrac{\cos x}{\sin x}$**, we get**

$$y' = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = -\frac{1}{\sin^2 x}$$

$$y' = -\frac{1}{\sin^2 x}$$

(6) $y = \dfrac{1-\cos x}{1+\cos x}$

$$y' = \frac{\sin x(1+\cos x) - (1-\cos x)\cdot(-\sin x)}{(1+\cos x)^2}$$
$$= \frac{\sin x + \sin x\cos x + \sin x - \sin x\cos x}{(1+\cos x)^2}$$
$$= \frac{2\sin x}{(1+\cos x)^2}$$

$$y' = \frac{2\sin x}{(1+\cos x)^2}$$

Differentiation of exponential functions e^x

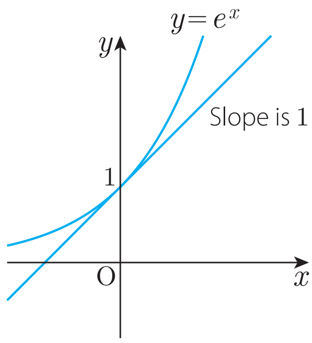
TARGET To understand how to find the derivative of functions expressed using e^x.

STUDY GUIDE

Graphic definition of base e of the natural logarithm

Consider the exponential function $f(x)=a^x\,(a>0, a\neq 1)$.

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{a^{x+h}-a^x}{h}=\lim_{h\to 0}\frac{a^x(a^h-1)}{h}=a^x\lim_{h\to 0}\frac{a^h-1}{h}$$



Here, when $a>1$, there is a point $(0, 1)$ on the curve $y=a^x$ such that the **slope of the tangent is 1**, specifically the base a is $f'(0)=1$, such that from

$f'(0)=a^0\lim_{h\to 0}\frac{a^h-1}{h}=\lim_{h\to 0}\frac{a^h-1}{h}$, we get $\lim_{h\to 0}\frac{a^h-1}{h}=1$.

The value of the base a at this time is e, specifically defined as $\lim_{h\to 0}\frac{e^h-1}{h}=1$. And, we know that $e\simeq 2.72$.

Derivative of the exponential function e^x

The derivative of the exponential function e^x is expressed as shown below.

$$\boxed{(e^x)'=e^x}$$

explanation

The derivative of e^x is derived from the graphic definition of e as shown below.

$$(e^x)'=\lim_{h\to 0}\frac{e^{x+h}-e^x}{h}=\lim_{h\to 0}\frac{e^x(e^h-1)}{h}=e^x\lim_{h\to 0}\frac{e^h-1}{h}=e^x\cdot 1=e^x$$

EXERCISE

◆ Differentiate the following functions.

(1) $y=e^{4x-1}$

$$y'=e^{4x-1}\cdot 4$$
$$=4e^{4x-1}$$

$$y'=4e^{4x-1}$$

(2) $y=e^{-x^2}$

$$y'=e^{-x^2}\cdot(-2x)$$
$$=-2xe^{-x^2}$$

$$y'=-2xe^{-x^2}$$

(3) $y=xe^{-3x}$

$$y'=1\cdot e^{-3x}+xe^{-3x}\cdot(-3)$$
$$=(1-3x)e^{-3x}$$

$$y'=(1-3x)e^{-3x}$$

(4) $y=e^x\sin x$

$$y'=e^x\sin x+e^x\cos x$$
$$=e^x(\sin x+\cos x)$$

$$y'=e^x(\sin x+\cos x)$$

(5) $y = \dfrac{e^x}{2x}$

$$y' = \frac{e^x \cdot 2x - e^x \cdot 2}{(2x)^2} = \frac{2(x-1)e^x}{4x^2} = \frac{(x-1)e^x}{2x^2}$$

$$y' = \frac{(x-1)e^x}{2x^2}$$

(6) $y = \dfrac{e^x - 1}{e^x + 1}$

$$y' = \frac{e^x(e^x+1) - (e^x-1)e^x}{(e^x+1)^2} = \frac{2e^x}{(e^x+1)^2}$$

$$y' = \frac{2e^x}{(e^x+1)^2}$$

PRACTICE

◆ Differentiate the following functions.

(1) $y = e^{-2x+5}$

$$y' = e^{-2x+5} \cdot (-2) = -2e^{-2x+5}$$

$$y' = -2e^{-2x+5}$$

(2) $y = e^{3x^2}$

$$y' = e^{3x^2} \cdot 6x = 6xe^{3x^2}$$

$$y' = 6xe^{3x^2}$$

(3) $y = (x^2+1)e^x$

$$y' = 2xe^x + (x^2+1)e^x = (x^2+2x+1)e^x = (x+1)^2 e^x$$

$$y' = (x+1)^2 e^x$$

(4) $y = e^{5x}\cos 2x$

$$y' = e^{5x} \cdot 5\cos 2x + e^{5x}(-\sin 2x) \cdot 2 = e^{5x}(5\cos 2x - 2\sin 2x)$$

$$y' = e^{5x}(5\cos 2x - 2\sin 2x)$$

(5) $y = \dfrac{e^x}{x^2}$

$$y' = \frac{e^x x^2 - e^x \cdot 2x}{(x^2)^2} = \frac{x(x-2)e^x}{x^4} = \frac{(x-2)e^x}{x^3}$$

$$y' = \frac{(x-2)e^x}{x^3}$$

Differentiation of logarithmic functions

TARGET To understand how to find the derivatives of functions expressed using logarithmic functions.

STUDY GUIDE

Derivatives of logarithmic functions

The derivatives of logarithmic functions are outlined below. Provided that $a>0$ and $a\neq 1$. Furthermore, a logarithm with base e is called the natural logarithm, and the e is usually omitted.

$$(\log x)' = \frac{1}{x} \qquad (\log_a x)' = \frac{1}{x\log a}$$

$$(\log|x|)' = \frac{1}{x} \qquad (\log_a |x|)' = \frac{1}{x\log a}$$

explanation

The derivative of $\log x$ is derived by using the properties of logarithms and $\lim_{h\to 0}(1+h)^{\frac{1}{h}} = e$ as shown below.

$$(\log x)' = \lim_{h\to 0}\frac{\log(x+h)-\log x}{h} = \lim_{h\to 0}\frac{1}{h}\log\frac{x+h}{x} = \lim_{h\to 0}\frac{1}{x}\cdot\frac{x}{h}\log\left(1+\frac{h}{x}\right) = \frac{1}{x}\lim_{h\to 0}\log\left(1+\frac{h}{x}\right)^{\frac{x}{h}}$$

$$= \frac{1}{x}\cdot\log e = \frac{1}{x}\cdot 1 = \frac{1}{x}$$

Also, by using the change-of-base formula, we can derive the derivative of $\log_a x$.

$$(\log_a x)' = \left(\frac{\log x}{\log a}\right)' = \frac{1}{\log a}\cdot\frac{1}{x} = \frac{1}{x\log a}$$

Whereas, when $x<0$, we get $\{\log(-x)\}' = \frac{1}{-x}\cdot(-1) = \frac{1}{x}$ and when $x>0$, then, together with $(\log x)' = \frac{1}{x}$, we get

$$(\log|x|)' = \frac{1}{x}, (\log_a |x|)' = \left(\frac{\log|x|}{\log a}\right)' = \frac{1}{\log a}\cdot\frac{1}{x} = \frac{1}{x\log a}.$$

EXERCISE

◆ Differentiate the following functions.

(1) $y = \log 2x$

$$y' = \frac{1}{2x}\cdot 2$$
$$= \frac{1}{x}$$

$$\boldsymbol{y' = \frac{1}{x}}$$

(2) $y = \log(x^3+4)$

$$y' = \frac{1}{x^3+4}\cdot 3x^2$$
$$= \frac{3x^2}{x^3+4}$$

$$\boldsymbol{y' = \frac{3x^2}{x^3+4}}$$

(3) $y = x^2 \log x$

$$\begin{aligned} y' &= 2x \log x + x^2 \cdot \frac{1}{x} \\ &= 2x \log x + x \\ &= x(2\log x + 1) \end{aligned}$$

$$y' = x(2\log x + 1)$$

(4) $y = (\log x)^3$

$$\begin{aligned} y' &= 3(\log x)^2 \cdot \frac{1}{x} \\ &= \frac{3(\log x)^2}{x} \end{aligned}$$

$$y' = \frac{3(\log x)^2}{x}$$

PRACTICE

◆ Differentiate the following functions.

(1) $y = \log(5 - 3x)$

$$\begin{aligned} y' &= \frac{1}{5 - 3x} \cdot (-3) \\ &= \frac{3}{3x - 5} \end{aligned}$$

$$y' = \frac{3}{3x - 5}$$

(2) $y = \log(x^2 - 2x)$

$$\begin{aligned} y' &= \frac{1}{x^2 - 2x} \cdot (2x - 2) \\ &= \frac{2(x-1)}{x(x-2)} \end{aligned}$$

$$y' = \frac{2(x-1)}{x(x-2)}$$

(3) $y = (x^3 - 2x)\log x$

$$\begin{aligned} y' &= (3x^2 - 2)\log x + (x^3 - 2x) \cdot \frac{1}{x} \\ &= (3x^2 - 2)\log x + x^2 - 2 \end{aligned}$$

$$y' = (3x^2 - 2)\log x + x^2 - 2$$

(4) $y = (\log |x|)^2$

$$\begin{aligned} y' &= 2(\log |x|) \cdot \frac{1}{x} \\ &= \frac{2\log |x|}{x} \end{aligned}$$

$$y' = \frac{2\log |x|}{x}$$

(5) $y = \log |\log x|$

$$\begin{aligned} y' &= \frac{1}{\log x} \cdot \frac{1}{x} \\ &= \frac{1}{x \log x} \end{aligned}$$

$$y' = \frac{1}{x \log x}$$

(6) $y = e^{3x} \log x$

$$\begin{aligned} y' &= e^{3x} \cdot 3 \cdot \log x + e^{3x} \cdot \frac{1}{x} \\ &= \frac{e^{3x}}{x}(3x \log x + 1) \end{aligned}$$

$$y' = \frac{e^{3x}}{x}(3x \log x + 1)$$

Differentiation of exponential functions a^x

TARGET To understand how to find the derivatives of functions expressed using exponential functions a^x.

STUDY GUIDE

Derivative of the exponential function a^x

The derivative of the exponential function a^x is expressed as shown below. Provided that $a>0$ and $a\neq 1$.

$$(a^x)' = a^x \log a$$

explanation

The definition of a logarithm gives us $a = b^y \Leftrightarrow y = \log_b a$, which can be expressed as $a = b^{\log_b a}$. Now, given $b=e$, then since $a = e^{\log a}$, we get $a^x = (e^{\log a})^x = e^{x\log a}$. Therefore, by using the fact that the derivative of the exponential function a^x is $(e^x)' = e^x$, we can derive the following.

$$(a^x)' = (e^{x\log a})' = e^{x\log a}\cdot \log a = a^x \log a$$

EXERCISE

◆ Differentiate the following functions.

(1) $y = 4^x$

$$\begin{aligned} y' &= 4^x \log 4 \\ &= 2^{2x}\cdot 2\log 2 \\ &= 2^{2x+1}\log 2 \end{aligned}$$

$$\underline{y' = 2^{2x+1}\log 2}$$

(2) $y = 7^{3x+1}$

$$\begin{aligned} y' &= 7^{3x+1}(\log 7)\cdot 3 \\ &= 3\cdot 7^{3x+1}\log 7 \end{aligned}$$

$$\underline{y' = 3\cdot 7^{3x+1}\log 7}$$

(3) $y = 2^{x^2}$

$$\begin{aligned} y' &= 2^{x^2}(\log 2)\cdot 2x \\ &= 2^{x^2+1}x\log 2 \end{aligned}$$

$$\underline{y' = 2^{x^2+1}x\log 2}$$

(4) $y = \left(\dfrac{1}{3}\right)^{3x}$

$$\begin{aligned} y' &= \left(\frac{1}{3}\right)^{3x}\cdot\left(\log\frac{1}{3}\right)\cdot 3 \\ &= 3^{-3x}(-\log 3)\cdot 3 \\ &= -3^{-3x+1}\log 3 \end{aligned}$$

$$\underline{y' = -3^{-3x+1}\log 3}$$

(5) $y = 3^x\cos x$

$$\begin{aligned} y' &= 3^x\log 3\cos x + 3^x\cdot(-\sin x) \\ &= 3^x(\log 3\cos x - \sin x) \end{aligned}$$

$$\underline{y' = 3^x(\log 3\cos x - \sin x)}$$

(6) $y = x^3\cdot 2^x$

$$\begin{aligned} y' &= 3x^2\cdot 2^x + x^3\cdot 2^x\log 2 \\ &= x^2\cdot 2^x(3 + x\log 2) \end{aligned}$$

$$\underline{y' = x^2\cdot 2^x(3 + x\log 2)}$$

PRACTICE

◆ Differentiate the following functions.

(1) $y = 9^x$

$$y' = 9^x \log 9$$
$$= 3^{2x} \cdot 2\log 3$$
$$= 2 \cdot 3^{2x} \log 3$$

$$y' = 2 \cdot 3^{2x} \log 3$$

(2) $y = \left(\frac{1}{5}\right)^{5x}$

$$y' = \left(\frac{1}{5}\right)^{5x} \cdot \left(\log \frac{1}{5}\right) \cdot 5$$
$$= 5^{-5x} \cdot (-\log 5) \cdot 5$$
$$= -5^{-5x+1} \log 5$$

$$y' = -5^{-5x+1} \log 5$$

(3) $y = 2^x \sin x$

$$y' = 2^x \log 2 \sin x + 2^x \cos x$$
$$= 2^x (\log 2 \sin x + \cos x)$$

$$y' = 2^x (\log 2 \sin x + \cos x)$$

(4) $y = x^2 \cdot 5^x$

$$y' = 2x \cdot 5^x + x^2 \cdot 5^x \log 5$$
$$= x \cdot 5^x (2 + x \log 5)$$

$$y' = x \cdot 5^x (2 + x \log 5)$$

(5) $y = \dfrac{2^x - 1}{2^x + 1}$

$$y' = \frac{2^x \log 2 \cdot (2^x + 1) - (2^x - 1) \cdot 2^x \log 2}{(2^x + 1)^2}$$
$$= \frac{2^x \log 2 \cdot 2}{(2^x + 1)^2}$$
$$= \frac{2^{x+1} \log 2}{(2^x + 1)^2}$$

$$y' = \frac{2^{x+1} \log 2}{(2^x + 1)^2}$$

Differentiation of the function x^p

TARGET To understand how to find the derivative when the exponent of x^n is expanded to a real number.

STUDY GUIDE

Derivative of the function x^p

When α is a rational number, we get $(x^\alpha)' = \alpha x^{\alpha-1}$. So, we can consider expanding the exponent α to an irrational number.

If the exponent is a rational number ($x>0$, and r is a rational number)

$$(x^r)' = rx^{r-1}$$

Ex. $(x^2)' = 2x, (x^{\frac{1}{3}})' = \frac{1}{3}x^{-\frac{2}{3}}$

If the exponent is an irrational number ($x>0$, and p is an irrational number)

$$(x^p)' = px^{p-1}$$

explanation

From the definition of a logarithm, when $x>0$, since $x = e^{\log x}$, we can get $x^p = (e^{\log x})^p = e^{p\log x}$.

Therefore, we get $(x^p)' = (e^{p\log x})' = e^{p\log x} \cdot \frac{p}{x} = x^p \cdot \frac{p}{x} = px^{p-1}$.

Or, by using $y = x^p$, because both sides are positive, we can get $\log y = \log x^p, \log y = p\log x$.

By differentiating both sides by x, we find $\frac{dy}{dx}$, which gives us $\frac{1}{y} \cdot \frac{dy}{dx} = \frac{p}{x}, \frac{dy}{dx} = \frac{p}{x}y = \frac{p}{x}x^p = px^{p-1}$.

Specifically, we can also derive $(x^p)' = px^{p-1}$.

From the above, when the exponent α is a real number and $x>0$, then the following holds.

$$(x^\alpha)' = \alpha x^{\alpha-1}$$

EXERCISE

◆ Differentiate the following functions.

(1) $y = (x^{\sqrt{2}} + 1)(x^{\sqrt{2}} - 4)$

$$y' = \sqrt{2}x^{\sqrt{2}-1}(x^{\sqrt{2}} - 4) + (x^{\sqrt{2}} + 1) \cdot \sqrt{2}x^{\sqrt{2}-1} = \sqrt{2}x^{\sqrt{2}-1}\{(x^{\sqrt{2}} - 4) + (x^{\sqrt{2}} + 1)\} = \sqrt{2}x^{\sqrt{2}-1}(2x^{\sqrt{2}} - 3)$$

$$\boldsymbol{y' = \sqrt{2}x^{\sqrt{2}-1}(2x^{\sqrt{2}} - 3)}$$

(2) $y = x(x^e + 2)$

$y' = 1 \cdot (x^e + 2) + x \cdot ex^{e-1} = x^e + 2 + ex^e = (e+1)x^e + 2$

$$y' = (e+1)x^e + 2$$

(3) $y = \frac{1}{\sqrt{x}} + \sqrt[3]{x^2}$

$y = x^{-\frac{1}{2}} + x^{\frac{2}{3}}$

$y' = -\frac{1}{2}x^{-\frac{3}{2}} + \frac{2}{3}x^{-\frac{1}{3}} = -\frac{1}{2\sqrt{x^3}} + \frac{2}{3\sqrt[3]{x}}$

$$y' = -\frac{1}{2\sqrt{x^3}} + \frac{2}{3\sqrt[3]{x}}$$

PRACTICE

◆ Differentiate the following functions.

(1) $y = (3x^\pi + 2)(x^\pi - 1)$

$$\begin{aligned} y' &= 3\pi x^{\pi-1}(x^\pi - 1) + (3x^\pi + 2) \cdot \pi x^{\pi-1} \\ &= \pi x^{\pi-1}\{3(x^\pi - 1) + (3x^\pi + 2)\} \\ &= \pi x^{\pi-1}(6x^\pi - 1) \end{aligned}$$

$$y' = \pi x^{\pi-1}(6x^\pi - 1)$$

(2) $y = x^{\sqrt{3}}(x - 2)$

$$\begin{aligned} y' &= \sqrt{3}x^{\sqrt{3}-1}(x - 2) + x^{\sqrt{3}} \cdot 1 \\ &= \sqrt{3}x^{\sqrt{3}} - 2\sqrt{3}x^{\sqrt{3}-1} + x^{\sqrt{3}} \\ &= (\sqrt{3} + 1)x^{\sqrt{3}} - 2\sqrt{3}x^{\sqrt{3}-1} \end{aligned}$$

$$y' = (\sqrt{3} + 1)x^{\sqrt{3}} - 2\sqrt{3}x^{\sqrt{3}-1}$$

(3) $y = 5\sqrt{x^3} + \frac{1}{\sqrt[3]{x}}$

$y = 5x^{\frac{3}{2}} + x^{-\frac{1}{3}}$

$y' = \frac{15}{2}x^{\frac{1}{2}} - \frac{1}{3}x^{-\frac{4}{3}} = \frac{15}{2}\sqrt{x} - \frac{1}{3\sqrt[3]{x^4}}$

$$y' = \frac{15}{2}\sqrt{x} - \frac{1}{3\sqrt[3]{x^4}}$$

High-order derivatives

TARGET To understand how to derive new derivatives by further differentiating derivatives that have been derived by differentiation.

STUDY GUIDE

High-order derivatives

2nd-order derivatives

When the derivative $f'(x)$ of the function $y=f(x)$ is differentiable by x, we consider the derivative of $f'(x)$ and call it the **2nd-order derivative** of $y=f(x)$.

2nd-order derivatives are expressed as $y'', f''(x), \dfrac{d^2y}{dx^2}, \dfrac{d^2}{dx^2}f(x)$.

n-th-order derivatives

In general, when a function $y=f(x)$ is differentiable n times, the function derived by differentiating $f(x)$ n times is called an **n-th-order derivative** of $y=f(x)$.

n-th-order derivatives are expressed as $y^{(n)}, f^{(n)}(x), \dfrac{d^ny}{dx^n}, \dfrac{d^n}{dx^n}f(x)$. Furthermore, derivatives of the 2nd or higher order are collectively called **high-order derivatives**.

EXERCISE

1 Find the 2nd order derivatives and 3rd order derivatives of the following functions.

(1) $y = x^3 - 4x^2 + 3x - 1$

$$y' = 3x^2 - 8x + 3$$
$$y'' = 6x - 8$$
$$y''' = 6$$

$$\boldsymbol{y'' = 6x - 8}$$
$$\boldsymbol{y''' = 6}$$

(2) $y = \cos x$

$$y' = -\sin x$$
$$y'' = -\cos x$$
$$y''' = -(-\sin x) = \sin x$$

$$\boldsymbol{y'' = -\cos x}$$
$$\boldsymbol{y''' = \sin x}$$

(3) $y = e^{3x}$

$$y' = e^{3x}\cdot 3 = 3e^{3x}$$
$$y'' = 3\cdot 3e^{3x} = 9e^{3x}$$
$$y''' = 9\cdot 3e^{3x} = 27e^{3x}$$

$$\boldsymbol{y'' = 9e^{3x}}$$
$$\boldsymbol{y''' = 27e^{3x}}$$

(4) $y = x^2 \log x$

$$y' = 2x\log x + x^2\cdot\frac{1}{x} = x(2\log x + 1)$$
$$y'' = 1\cdot(2\log x + 1) + x\cdot\frac{2}{x} = 2\log x + 3$$
$$y''' = 2\cdot\frac{1}{x} = \frac{2}{x}$$

$$\boldsymbol{y'' = 2\log x + 3}$$
$$\boldsymbol{y''' = \frac{2}{x}}$$

2 Estimate the n-th order derivatives of the following functions.

(1) $y = \sin x$

$$y' = \cos x = \sin\left(x + \frac{\pi}{2}\right)$$
$$y'' = \cos\left(x + \frac{\pi}{2}\right) = \sin\left\{\left(x + \frac{\pi}{2}\right) + \frac{\pi}{2}\right\} = \sin(x+\pi)$$
$$y''' = \cos(x+\pi) = \sin\left\{(x+\pi) + \frac{\pi}{2}\right\} = \sin\left(x + \frac{3}{2}\pi\right)$$
$$y^{(n)} = \sin\left(x + \frac{n}{2}\pi\right)$$

$$\boldsymbol{y^{(n)} = \sin\left(x + \frac{n}{2}\pi\right)}$$

(2) $y = xe^x$

$$y' = 1\cdot e^x + xe^x = (1+x)e^x$$
$$y'' = 1\cdot e^x + (1+x)e^x = (2+x)e^x$$
$$y''' = 1\cdot e^x + (2+x)e^x = (3+x)e^x$$
$$y^{(n)} = (n+x)e^x$$

$$\boldsymbol{y^{(n)} = (n+x)e^x}$$

PRACTICE

1 Find the 2nd order derivatives and 3rd order derivatives of the following functions.

(1) $y=-2x^3+x^2+5x+1$

$y'=-6x^2+2x+5$
$y''=-12x+2$
$y'''=-12$

$$y''=-12x+2$$
$$y'''=-12$$

(2) $y=\log x$

$y'=\dfrac{1}{x}=x^{-1}$
$y''=-x^{-2}=-\dfrac{1}{x^2}$
$y'''=2x^{-3}=\dfrac{2}{x^3}$

$$y''=-\frac{1}{x^2}$$
$$y'''=\frac{2}{x^3}$$

(3) $y=a^x$ $(a>0,\ a\neq 1)$

$y'=a^x\log a$
$y''=(a^x\log a)\log a=a^x(\log a)^2$
$y'''=(a^x\log a)(\log a)^2=a^x(\log a)^3$

$$y''=a^x(\log a)^2$$
$$y'''=a^x(\log a)^3$$

(4) $y=e^x\sin x$

$y'=e^x\sin x+e^x\cos x=e^x(\sin x+\cos x)$
$y''=e^x(\sin x+\cos x)+e^x(\cos x-\sin x)=2e^x\cos x$
$y'''=2\{e^x\cos x+e^x\cdot(-\sin x)\}=2e^x(\cos x-\sin x)$

$$y''=2e^x\cos x$$
$$y'''=2e^x(\cos x-\sin x)$$

2 Estimate the n-th order derivatives of the following functions.

(1) $y=\cos x$

$y'=-\sin x=\cos\left(x+\dfrac{\pi}{2}\right)$
$y''=-\sin\left(x+\dfrac{\pi}{2}\right)=\cos\left\{\left(x+\dfrac{\pi}{2}\right)+\dfrac{\pi}{2}\right\}=\cos(x+\pi)$
$y'''=-\sin(x+\pi)=\cos\left\{(x+\pi)+\dfrac{\pi}{2}\right\}=\cos\left(x+\dfrac{3}{2}\pi\right)$
$y^{(n)}=\cos\left(x+\dfrac{n}{2}\pi\right)$

$$y^{(n)}=\cos\left(x+\frac{n}{2}\pi\right)$$

(2) $y=e^{2x}+e^{-x}$

$y'=e^{2x}\cdot 2+e^{-x}\cdot(-1)=2e^{2x}-e^{-x}$
$y''=2\cdot 2e^{2x}-(-e^{-x})=2^2e^{2x}+(-1)^2e^{-x}$
$y'''=2^2\cdot 2e^{2x}+(-1)^2\cdot(-e^{-x})=2^3e^{2x}+(-1)^3e^{-x}$
$y^{(n)}=2^ne^{2x}+(-1)^ne^{-x}$

$$y^{(n)}=2^ne^{2x}+(-1)^ne^{-x}$$

Differentiation of implicit functions

TARGET To understand how to find the derivatives of functions expressed in the form of implicit functions.

STUDY GUIDE

Implicit functions

A function expressed as $F(x, y)=0$ for x and y is called an **implicit function**, while a function expressed as $y=f(x)$ is called an **explicit function**.

Implicit function expression $F(x, y)=0$
Explicit function expression $y=f(x)$

Differentiation of functions expressed as implicit functions

When it is difficult to solve the implicit function $F(x, y)=0$ for y, or **when it becomes a complicated equation** even when it is solved, consider y as a function of x, and differentiate with respect to x leaving both sides of $F(x, y)=0$ as they are to find the derivatives expressed as x and y.

When finding a derivative $\dfrac{dy}{dx}$ from an implicit function $F(x, y)=0$, differentiate the part expressed as y as follows.

$$\boxed{\frac{d}{dx}f(y)=\frac{d}{dy}f(y)\frac{dy}{dx}=f'(y)\frac{dy}{dx}}$$

EXERCISE

◆ Express $\dfrac{dy}{dx}$ as an x, y equation for the following functions.

(1) $x^2-y^2=4$

$$2x-2y\frac{dy}{dx}=0$$
$$y\frac{dy}{dx}=x$$
$$\frac{dy}{dx}=\frac{x}{y}$$

$$\boldsymbol{\frac{dy}{dx}=\frac{x}{y}}$$

(2) $y^2=3(x-2)$

$$2y\frac{dy}{dx}=3$$
$$\frac{dy}{dx}=\frac{3}{2y}$$

$$\boldsymbol{\frac{dy}{dx}=\frac{3}{2y}}$$

(3) $\sqrt{x}+\sqrt{y}=9$

$$x^{\frac{1}{2}}+y^{\frac{1}{2}}=9$$
$$\frac{1}{2}x^{-\frac{1}{2}}+\frac{1}{2}y^{-\frac{1}{2}}\frac{dy}{dx}=0$$
$$\frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{y}}\frac{dy}{dx}=0$$
$$\frac{1}{\sqrt{y}}\frac{dy}{dx}=-\frac{1}{\sqrt{x}}$$
$$\frac{dy}{dx}=-\sqrt{\frac{y}{x}}$$

$$\boldsymbol{\frac{dy}{dx}=-\sqrt{\frac{y}{x}}}$$

(4) $x^2-4xy+5y^2=1$

$$2x-4\left(y+x\frac{dy}{dx}\right)+5\cdot 2y\frac{dy}{dx}=0$$
$$(x-2y)-(2x-5y)\frac{dy}{dx}=0$$
$$\frac{dy}{dx}=\frac{x-2y}{2x-5y}$$

$$\boldsymbol{\frac{dy}{dx}=\frac{x-2y}{2x-5y}}$$

PRACTICE

◆ Express $\dfrac{dy}{dx}$ as an x, y equation for the following functions.

(1) $\dfrac{x^2}{4}+\dfrac{y^2}{9}=1$

$$\frac{1}{2}x+\frac{2}{9}y\frac{dy}{dx}=0$$
$$\frac{2}{9}y\frac{dy}{dx}=-\frac{1}{2}x$$
$$\frac{dy}{dx}=-\frac{9x}{4y}$$

$$\frac{dy}{dx}=-\frac{9x}{4y}$$

(2) $2x^3y^2=1$

$$2\left(3x^2y^2+x^3\cdot 2y\frac{dy}{dx}\right)=0$$
$$2x^3y\frac{dy}{dx}=-3x^2y^2$$
$$\frac{dy}{dx}=-\frac{3y}{2x}$$

$$\frac{dy}{dx}=-\frac{3y}{2x}$$

(3) $x^2-6xy+10y^2=4$

$$2x-6\left(y+x\frac{dy}{dx}\right)+10\cdot 2y\frac{dy}{dx}=0$$
$$x-3y-(3x-10y)\frac{dy}{dx}=0$$
$$(3x-10y)\frac{dy}{dx}=x-3y$$
$$\frac{dy}{dx}=\frac{x-3y}{3x-10y}$$

$$\frac{dy}{dx}=\frac{x-3y}{3x-10y}$$

Differentiation of functions expressed as parameters

TARGET To understand how to find the derivatives of functions expressed using parameters.

STUDY GUIDE

Differentiation of functions expressed as parameters

When y is a function of x, and x and y are expressed as $x=f(t)$ and $y=g(t)$ by using parameter t, then we can find the derivative $\frac{dy}{dx}$ as follows.

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\quad\left(\frac{dx}{dt}\neq 0\right)$$

EXERCISE

◆ Express $\frac{dy}{dx}$ in terms of t for the following functions expressed as parameters.

(1) $x=t^2+1, y=2t-5$

$$\frac{dx}{dt}=2t$$
$$\frac{dy}{dt}=2$$
$$\frac{dy}{dx}=\frac{2}{2t}=\frac{1}{t}$$

$$\frac{dy}{dx}=\frac{1}{t}$$

(2) $x=3t^2-1, y=t^3+7$

$$\frac{dx}{dt}=6t$$
$$\frac{dy}{dt}=3t^2$$
$$\frac{dy}{dx}=\frac{3t^2}{6t}=\frac{1}{2}t$$

$$\frac{dy}{dx}=\frac{1}{2}t$$

(3) $x=4(t-\sin t), y=4(1-\cos t)$

$$\frac{dx}{dt}=4(1-\cos t)$$
$$\frac{dy}{dt}=4\sin t$$
$$\frac{dy}{dx}=\frac{4\sin t}{4(1-\cos t)}=\frac{\sin t}{1-\cos t}$$

$$\frac{dy}{dx}=\frac{\sin t}{1-\cos t}$$

(4) $x=t-\frac{1}{t}, y=t+\frac{1}{t}$

$$\frac{dx}{dt}=1+\frac{1}{t^2}$$
$$\frac{dy}{dt}=1-\frac{1}{t^2}$$
$$\frac{dy}{dx}=\frac{1-\frac{1}{t^2}}{1+\frac{1}{t^2}}=\frac{t^2-1}{t^2+1}$$

$$\frac{dy}{dx}=\frac{t^2-1}{t^2+1}$$

PRACTICE

◆ Express $\frac{dy}{dx}$ in terms of t for the following functions expressed as parameters.

(1) $x = t^3 - 2t^2, y = \frac{1}{2}t^2$

$$\frac{dx}{dt} = 3t^2 - 4t$$
$$\frac{dy}{dt} = t$$
$$\frac{dy}{dx} = \frac{t}{3t^2 - 4t} = \frac{1}{3t - 4}$$

$$\frac{dy}{dx} = \frac{1}{3t - 4}$$

(2) $x = \sqrt{1 - t^2}, y = 3t^2 - 2$

$$\frac{dx}{dt} = \frac{1}{2}(1 - t^2)^{-\frac{1}{2}} \cdot (-2t) = -t(1 - t^2)^{-\frac{1}{2}}$$
$$\frac{dy}{dt} = 6t$$
$$\frac{dy}{dx} = -\frac{6t}{t(1 - t^2)^{-\frac{1}{2}}} = -6(1 - t^2)^{\frac{1}{2}} = -6\sqrt{1 - t^2}$$

$$\frac{dy}{dx} = -6\sqrt{1 - t^2}$$

(3) $x = -4\sin t, y = 1 + \cos 2t$

$$\frac{dx}{dt} = -4\cos t$$
$$\frac{dy}{dt} = (-\sin 2t) \cdot 2 = -4\sin t\cos t$$
$$\frac{dy}{dx} = \frac{-4\sin t\cos t}{-4\cos t} = \sin t$$

$$\frac{dy}{dx} = \sin t$$

(4) $x = t + \frac{1}{t}, y = t^2 + \frac{1}{t^2}$

$$\frac{dx}{dt} = 1 - t^{-2} = 1 - \frac{1}{t^2}$$
$$\frac{dy}{dt} = 2t - 2t^{-3} = 2t - \frac{2}{t^3}$$
$$\frac{dy}{dx} = \frac{2t - \frac{2}{t^3}}{1 - \frac{1}{t^2}} = \frac{2(t^4 - 1)}{t^3 - t} = \frac{2(t^2 + 1)(t^2 - 1)}{t(t^2 - 1)} = \frac{2(t^2 + 1)}{t}$$

$$\frac{dy}{dx} = \frac{2(t^2 + 1)}{t}$$

Differentiation of logarithms

TARGET To understand how to find derivatives by using the properties of logarithms.

STUDY GUIDE

Differentiation of logarithms

When a function to be differentiated is in the form of a **complex product, quotient,** or **exponent**, the derivative can be easily found by using the following properties of logarithms.

Properties of logarithms

Products and quotients of anti-logarithms $\rightarrow$ Sums and differences of logarithms

Exponents of anti-logarithms $\rightarrow$ Constant multiple of logarithms

In general, for the function $y=f(x)$, the method to easily find $\{\log|f(x)|\}'$, by taking the natural log of the absolute value of both sides, and then differentiating, is called **differentiation of logarithms**. When y is a function of x, by differentiating $\log|y|$ by x, gives us $\frac{d}{dx}(\log|y|)=\frac{d}{dy}(\log|y|)\cdot\frac{dy}{dx}=\frac{y'}{y}$, so we can state the following.

Product **When $y=f_1(x)\cdot f_2(x)$,**

$$\log|y|=\log|f_1(x)\cdot f_2(x)|=\log|f_1(x)|+\log|f_2(x)|$$

Therefore, differentiating both sides by x gives us

$$\frac{y'}{y}=\frac{\{f_1(x)\}'}{f_1(x)}+\frac{\{f_2(x)\}'}{f_2(x)}$$

Quotient **When $y=\frac{f_1(x)}{f_2(x)}$,**

$$\log|y|=\log\left|\frac{f_1(x)}{f_2(x)}\right|=\log|f_1(x)|-\log|f_2(x)|$$

Therefore, differentiating both sides by x gives us

$$\frac{y'}{y}=\frac{\{f_1(x)\}'}{f_1(x)}-\frac{\{f_2(x)\}'}{f_2(x)}$$

EXERCISE

◆ Use logarithmic differentiation to differentiate the following functions.

(1) $y=(2x-3)^2(3x+1)^3$

Take the natural logarithms of the absolute values of both sides to get

$\log|y|=\log|(2x-3)^2(3x+1)^3|=2\log|2x-3|+3\log|3x+1|$.

By differentiating both sides by x, we find y'.

$$\frac{y'}{y}=\frac{2\cdot 2}{2x-3}+\frac{3\cdot 3}{3x+1}=\frac{4(3x+1)+9(2x-3)}{(2x-3)(3x+1)}=\frac{30x-23}{(2x-3)(3x+1)}$$

$$y'=y\cdot\frac{30x-23}{(2x-3)(3x+1)}=(2x-3)^2(3x+1)^3\cdot\frac{30x-23}{(2x-3)(3x+1)}=(2x-3)(3x+1)^2(30x-23)$$

$$\boldsymbol{y'=(2x-3)(3x+1)^2(30x-23)}$$

(2) $y=\dfrac{(2x+1)^3}{(x-2)^2}$

Take the natural logarithms of the absolute values of both sides to get

$\log|y|=\log\left|\dfrac{(2x+1)^3}{(x-2)^2}\right|=3\log|2x+1|-2\log|x-2|$.

By differentiating both sides by x, we find y'.

$$\frac{y'}{y}=\frac{3\cdot 2}{2x+1}-\frac{2}{x-2}=\frac{2}{(2x+1)(x-2)}\{3(x-2)-(2x+1)\}=\frac{2(x-7)}{(2x+1)(x-2)}$$

$$y'=y\cdot\frac{2(x-7)}{(2x+1)(x-2)}=\frac{(2x+1)^3}{(x-2)^2}\cdot\frac{2(x-7)}{(2x+1)(x-2)}=\frac{2(2x+1)^2(x-7)}{(x-2)^3}$$

$$\boldsymbol{y'=\frac{2(2x+1)^2(x-7)}{(x-2)^3}}$$

PRACTICE

◆ Use logarithmic differentiation to differentiate the following functions.

(1) $y=\dfrac{x^2(2x-1)^3}{(x-3)^4}$

Take the natural logarithms of the absolute values of both sides to get

$\log|y|=\log\left|\dfrac{x^2(2x-1)^3}{(x-3)^4}\right|=2\log|x|+3\log|2x-1|-4\log|x-3|$.

By differentiating both sides by x, we find y'.

$$\frac{y'}{y}=\frac{2}{x}+\frac{3\cdot 2}{2x-1}-\frac{4}{x-3}$$

$$=\frac{2}{x(2x-1)(x-3)}\{(2x-1)(x-3)+3x(x-3)-2x(2x-1)\}=\frac{2(x^2-14x+3)}{x(2x-1)(x-3)}$$

$$y'=y\cdot\frac{2(x^2-14x+3)}{x(2x-1)(x-3)}=\frac{x^2(2x-1)^3}{(x-3)^4}\cdot\frac{2(x^2-14x+3)}{x(2x-1)(x-3)}$$

$$=\frac{2x(2x-1)^2(x^2-14x+3)}{(x-3)^5}$$

$$\boldsymbol{y'=\frac{2x(2x-1)^2(x^2-14x+3)}{(x-3)^5}}$$

(2) $y=x^{2x+1}$ $(x>0)$

From $x>0$, we get $x^{2x+1}>0, y>0$, so by taking the natural logarithms of both sides we can get $\log y=\log x^{2x+1}=(2x+1)\log x$.

By differentiating both sides by x, we find y'.

$$\frac{y'}{y}=2\log x+(2x+1)\cdot\frac{1}{x}=2\log x+2+\frac{1}{x}$$

$$y'=y\cdot\left(2\log x+2+\frac{1}{x}\right)=x^{2x+1}\left(2\log x+2+\frac{1}{x}\right)=x^{2x}(2x\log x+2x+1)$$

$$y'=x^{2x}(2x\log x+2x+1)$$

(3) $y=x^{\sqrt{x}}$ $(x>0)$

From $x>0$, we get $x^{\sqrt{x}}>0, y>0$, so by taking the natural logarithms of both sides we can get $\log y=\log x^{\sqrt{x}}=\sqrt{x}\log x$.

$$\frac{y'}{y}=\frac{1}{2\sqrt{x}}\log x+\sqrt{x}\cdot\frac{1}{x}=\frac{1}{2\sqrt{x}}(\log x+2)$$

$$y'=y\cdot\frac{1}{2\sqrt{x}}(\log x+2)=x^{\sqrt{x}}\cdot\frac{1}{2\sqrt{x}}(\log x+2)=\frac{1}{2}x^{\sqrt{x}-\frac{1}{2}}(\log x+2)$$

$$y'=\frac{1}{2}x^{\sqrt{x}-\frac{1}{2}}(\log x+2)$$

Equation for a tangent

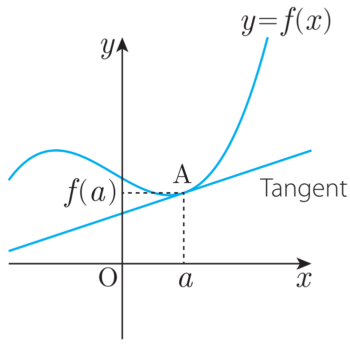
TARGET To understand how to find the equations for tangents drawn on the graphs of various functions.

STUDY GUIDE

Equation for a tangent

The equation for a tangent at a point $\mathrm{A}\left(a, f(a)\right)$ on the curve $y=f(x)$, is shown below.

$$\boxed{y - f(a) = f'(a)(x-a)}$$



EXERCISE

1 Find the equation of a tangent at point P on the following curve.

(1) $y = \dfrac{2x}{x+1}, \mathrm{P}(1,1)$

Given $f(x) = \dfrac{2x}{x+1}$, from $f'(x) = \dfrac{2(x+1)-2x}{(x+1)^2} = \dfrac{2}{(x+1)^2}$ we can get $f'(1) = \dfrac{2}{(1+1)^2} = \dfrac{1}{2}$.

Therefore, from $y - 1 = \dfrac{1}{2}(x-1)$, the equation for the tangent is $y = \dfrac{1}{2}x + \dfrac{1}{2}$.

$$\boldsymbol{y = \frac{1}{2}x + \frac{1}{2}}$$

(2) $y = \sin 3x + x, \mathrm{P}(0,0)$

Given $f(x) = \sin 3x + x$, from $f'(x) = 3\cos 3x + 1$ we can get $f'(0) = 3 + 1 = 4$.

Therefore, the equation for the tangent is $y = 4x$.

$$\boldsymbol{y = 4x}$$

2 Find the equation of a tangent drawn from a point $\mathrm{Q}(0,0)$ on a curve $y = 2\log x + 1$.

$y' = \dfrac{2}{x}$

If the x coordinate of the contact point is t, the equation for the tangent is $y - (2\log t + 1) = \dfrac{2}{t}(x - t)$.

Because this passes through point Q, then from $-(2\log t + 1) = -2$, we get $t = \sqrt{e}$.

Therefore, from $y - 2 = \dfrac{2}{\sqrt{e}}(x - \sqrt{e})$, the equation for the tangent is $y = \dfrac{2}{\sqrt{e}}x$.

$$\boldsymbol{y = \frac{2}{\sqrt{e}}x}$$

PRACTICE

1 Find the equation of a tangent at point P on the following curve.

(1) $y=\frac{x^2+2}{x+2}$, P(1,1)

$f(x)=\frac{x^2+2}{x+2}, f'(x)=\frac{2x(x+2)-(x^2+2)}{(x+2)^2}=\frac{x^2+4x-2}{(x+2)^2}, f'(1)=\frac{1+4-2}{(1+2)^2}=\frac{1}{3}$

Therefore, from $y-1=\frac{1}{3}(x-1)$, the equation for the tangent is $y=\frac{1}{3}x+\frac{2}{3}$.

$y=\frac{1}{3}x+\frac{2}{3}$

(2) $y=\sqrt{3x+1}$, P(1,2)

$f(x)=\sqrt{3x+1}, f'(x)=\frac{3}{2\sqrt{3x+1}}, f'(1)=\frac{3}{2\sqrt{3+1}}=\frac{3}{4}$

Therefore, from $y-2=\frac{3}{4}(x-1)$, the equation for the tangent is $y=\frac{3}{4}x+\frac{5}{4}$.

$y=\frac{3}{4}x+\frac{5}{4}$

(3) $y=\frac{1}{e^{2x}}$, P$(-1,e^2)$

$f(x)=\frac{1}{e^{2x}}, f'(x)=-\frac{2}{e^{2x}}, f'(-1)=-\frac{2}{e^{-2}}=-2e^2$

Therefore, from $y-e^2=-2e^2(x+1)$, the equation for the tangent is $y=-2e^2x-e^2$.

$y=-2e^2x-e^2$

2 Find the equation of a tangent drawn from a point Q(1, 0) on a curve $y=\sqrt{x^2+1}$.

$y'=\frac{2x}{2\sqrt{x^2+1}}=\frac{x}{\sqrt{x^2+1}}$

If the x coordinate of the contact point is t, the equation for the tangent is

$y-\sqrt{t^2+1}=\frac{t}{\sqrt{t^2+1}}(x-t)$.

Because this passes through point Q, then from $-\sqrt{t^2+1}=\frac{t}{\sqrt{t^2+1}}(1-t)$, we get $t=-1$.

Therefore, from $y-\sqrt{2}=-\frac{1}{\sqrt{2}}(x+1)$,

the equation for the tangent is

$y=-\frac{1}{\sqrt{2}}x-\frac{1}{\sqrt{2}}+\sqrt{2}, y=-\frac{1}{\sqrt{2}}x-\frac{1}{\sqrt{2}}+\frac{2}{\sqrt{2}}, y=-\frac{1}{\sqrt{2}}x+\frac{1}{\sqrt{2}}$.

$y=-\frac{1}{\sqrt{2}}x+\frac{1}{\sqrt{2}}$

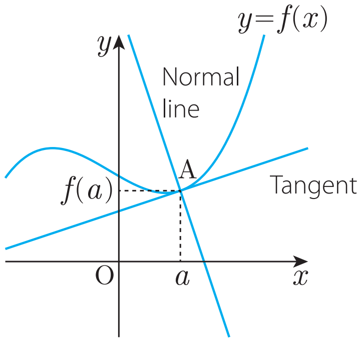
Equation for a normal line

TARGET To understand how to find the equations for normal lines drawn on the graphs of various functions.

STUDY GUIDE

Equation for a normal line

The **normal line** is a straight line passing through a fixed point on a curve and perpendicular to the tangent at that point.
The equation for a normal line at a point $\mathrm{A}\left(a, f(a)\right)$ on the curve $y=f(x)$, is shown below.



When $f'(a) \neq 0$, we get $y - f(a) = -\dfrac{1}{f'(a)}(x-a)$

When $f'(a) = 0$, we get $x = a$

EXERCISE

1 Find the equation of a normal line at point P on the following curve.

(1) $y = 3x - \dfrac{1}{x}, \mathrm{P}(-1, -2)$

Given $f(x) = 3x - \dfrac{1}{x}$, from $f'(x) = 3 + \dfrac{1}{x^2}$ we can get $f'(-1) = 3 + \dfrac{1}{(-1)^2} = 4$.

Therefore, from $y + 2 = -\dfrac{1}{4}(x+1)$, the equation for the normal line is $y = -\dfrac{1}{4}x - \dfrac{9}{4}$.

$$\boldsymbol{y = -\frac{1}{4}x - \frac{9}{4}}$$

(2) $y = -\cos\dfrac{x}{2}, \mathrm{P}(3\pi, 0)$

Given $f(x) = -\cos\dfrac{x}{2}$, from $f'(x) = \dfrac{1}{2}\sin\dfrac{x}{2}$ we can get $f'(3\pi) = \dfrac{1}{2}\sin\dfrac{3}{2}\pi = -\dfrac{1}{2}$.

Therefore, from $y = 2(x - 3\pi)$, the equation for the normal line is $y = 2x - 6\pi$.

$$\boldsymbol{y = 2x - 6\pi}$$

2 Find the equation of a normal line drawn from a point $\mathrm{Q}(3, 0)$ on a curve $y = \sqrt{2x+5}\ \left(x > -\dfrac{5}{2}\right)$.

$y' = \dfrac{2}{2\sqrt{2x+5}} = \dfrac{1}{\sqrt{2x+5}}$

If the x coordinate of the intersection of the curve and the normal line is t, the equation for the normal line is $y - \sqrt{2t+5} = -\sqrt{2t+5}(x-t)$.

Because this passes through point Q, we get $-\sqrt{2t+5} = -\sqrt{2t+5}(3-t), (t-2)\sqrt{2t+5} = 0$.

From $t > -\dfrac{5}{2}$, since $2t+5 \neq 0$, we get $t=2$.

Therefore, from $y - 3 = -3(x-2)$, the equation for the normal line is $y = -3x + 9$.

$$\boldsymbol{y = -3x + 9}$$

PRACTICE

1 Find the equation of a normal line at point P on the following curve.

(1) $y=\frac{x-1}{4x-1}, \mathrm{P}(0,1)$

$$f(x)=\frac{x-1}{4x-1}, f'(x)=\frac{4x-1-4(x-1)}{(4x-1)^2}=\frac{3}{(4x-1)^2}, f'(0)=\frac{3}{(-1)^2}=3$$

Therefore, from $y-1=-\frac{1}{3}x$, the equation for the normal line is $y=-\frac{1}{3}x+1$.

$$y=-\frac{1}{3}x+1$$

(2) $y=2e^{3x}, \mathrm{P}(0,2)$

$$f(x)=2e^{3x}, f'(x)=2e^{3x}\cdot 3=6e^{3x}, f'(0)=6\cdot 1=6$$

Therefore, from $y-2=-\frac{1}{6}x$, the equation for the normal line is $y=-\frac{1}{6}x+2$.

$$y=-\frac{1}{6}x+2$$

2 Find the equation of a normal line drawn from a point $\mathrm{Q}(4,0)$ on a curve $y=\sqrt{3x^2-2}\left(|x|>\sqrt{\frac{2}{3}}\right)$.

$$y'=\frac{6x}{2\sqrt{3x^2-2}}=\frac{3x}{\sqrt{3x^2-2}}$$

If the x coordinate of the intersection of the curve and the normal line is t, the equation for the normal line is $y-\sqrt{3t^2-2}=-\frac{\sqrt{3t^2-2}}{3t}(x-t)$.

Because this passes through point Q, we get

$$-\sqrt{3t^2-2}=-\frac{\sqrt{3t^2-2}}{3t}(4-t), 4(t-1)\sqrt{3t^2-2}=0.$$

From $|t|>\sqrt{\frac{2}{3}}$, since $3t^2-2\neq 0$, we get $t=1$.

Therefore, from $y-1=-\frac{1}{3}(x-1)$, the equation for the normal line is
$y=-\frac{1}{3}x+\frac{4}{3}$.

$$y=-\frac{1}{3}x+\frac{4}{3}$$

Function increase/decrease and derivative signs

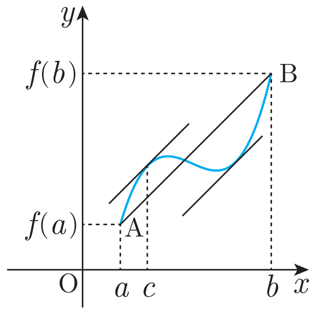
TARGET To understand the relationship between the increase and decrease of functions and the sign of 1st-order derivatives.

STUDY GUIDE

Mean-value theorem

Many of the various theorems and properties of calculus, including the relationship between increasing/decreasing of functions and the sign of 1st-order derivatives, are proved based on the following **mean-value theorem**.

If the function $f(x)$ is continuous in a closed interval $[a, b]$, and is differentiable in an open interval (a, b), then there is at least 1 real number c that satisfies $\dfrac{f(b)-f(a)}{b-a}=f'(c), a<c<b$.



As shown in the figure on the upper right, the mean-value theorem shows that "on the graph of the function $y=f(x)$ $(a\leq x\leq b)$, there is at least 1 point where the slope of the tangent at that point is equal to the slope of the straight line AB".

Increasing and decreasing functions

With respect to the increase and decrease of the function $y=f(x)$, the following holds.

In the open interval (a, b), if always $f'(x)>0$
$\Rightarrow$ then $f(x)$ increases monotonically in the closed interval $[a, b]$.
In the open interval (a, b), if always $f'(x)<0$
$\Rightarrow$ then $f(x)$ decreases monotonically in the closed interval $[a, b]$.
In the open interval (a, b), if always $f'(x)=0$
$\Leftrightarrow$ then $f(x)$ is constant in the closed interval $[a, b]$.

explanation

Consider 2 numbers x_1 and x_2 that satisfy $a<x_1<x_2<b$, where $f'(x)>0$ is always satisfied in the open interval (a, b).

From the mean-value theorem, there exists a real number c, which satisfies $\dfrac{f(x_2)-f(x_1)}{x_2-x_1}=f'(c)$, $x_1<c<x_2$.

Now, since $f'(x)>0$ always gives us $f'(c)>0$, then $x_2-x_1>0$, such that from $f(x_2)-f(x_1)>0$, we get $f(x_1)<f(x_2)$. Similarly, when $f'(x)<0$ is always true, then from $f'(c)<0$ we can derive $f(x_1)>f(x_2)$ and when $f'(x)=0$ is always true, then from $f'(c)=0$ we can derive $f(x_1)=f(x_2)$.

EXERCISE

◆ Determine whether the function $f(x)=\dfrac{2x+1}{x^2+2}$ is increasing or decreasing.

$$f'(x)=\frac{2(x^2+2)-(2x+1)\cdot 2x}{(x^2+2)^2}=-\frac{2(x^2+x-2)}{(x^2+2)^2}=-\frac{2(x+2)(x-1)}{(x^2+2)^2}$$

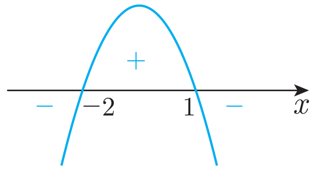
When $f'(x)=0$, then $x=-2, 1$.

Therefore, the sign chart is as shown on the right.

Therefore, increasing when $-2\le x\le 1$, and decreasing when $x\le -2,\ 1\le x$.

Increasing when $-2\le x\le 1$, and decreasing when $x\le -2,\ 1\le x$

Changing of sign of $f'(x)$



x	$\cdots$	-2	$\cdots$	1	$\cdots$
$f'(x)$	$-$	0	$+$	0	$-$
$f(x)$	↘		↗		↘

Because $(x^2+2)^2>0$, changing the sign of $f'(x)$ coincides with changing the sign of $-2(x+2)(x-1)$, as shown in the graph above.

PRACTICE

◆ Determine whether the next functions are increasing or decreasing.

(1) $f(x)=(2x+1)e^x$

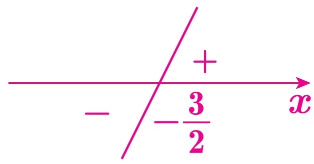
$$f'(x)=2e^x+(2x+1)e^x=(2x+3)e^x$$

When $f'(x)=0$, then $x=-\dfrac{3}{2}$.

Therefore, the sign chart is as shown on the right.

Therefore, increasing when $x\ge -\dfrac{3}{2}$, and decreasing when $x\le -\dfrac{3}{2}$.

Changing of sign of $f'(x)$



x	$\cdots$	$-\frac{3}{2}$	$\cdots$
$f'(x)$	$-$	0	$+$
$f(x)$	↘		↗

Increasing when $x\ge -\dfrac{3}{2}$, and decreasing when $x\le -\dfrac{3}{2}$

(2) $f(x)=x^4-6x^2+8x-5$

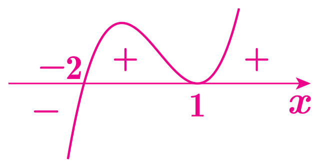
$$f'(x)=4x^3-12x+8=4(x^3-3x+2)=4(x-1)^2(x+2)$$

When $f'(x)=0$, then $x=-2, 1$.

Therefore, the sign chart is as shown on the right.

Therefore, increasing when $x\ge -2$, and decreasing when $x\le -2$.

Changing of sign of $f'(x)$



x	$\cdots$	-2	$\cdots$	1	$\cdots$
$f'(x)$	$-$	0	$+$	0	$+$
$f(x)$	↘		↗		↗

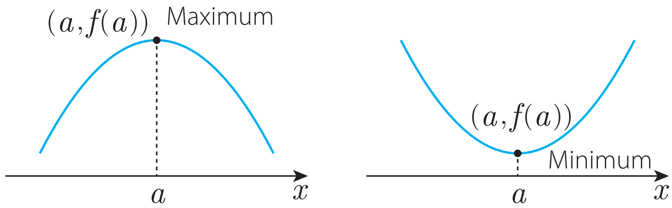
Increasing when $x\ge -2$, and decreasing when $x\le -2$

Maximums and minimums of functions

TARGET To understand the conditions in which functions have extrema.

STUDY GUIDE

Definitions of maximum and minimum



For a continuous function $y=f(x)$, when the function $f(x)$ changes from increasing before to decreasing after $x=a$, we say that **$x=a$ is the maximum**, and we say that $f(a)$ is the **local maximum**.

Furthermore, when it changes from decreasing before to increasing after $x=a$, we say that **$x=a$ is the minimum**, and we say that $f(a)$ is the **local minimum**.

The maxima and minima are collectively called **extrema**.

Maxima and minima of functions and changing the sign of derivatives

When the function $y=f(x)$ is continuous and differentiable with $x=a$, the sign of $f'(x)$ changes as follows before and after the point of minimum and maximum.

For a maximum $x=a$
$\Leftrightarrow$ The sign of $f'(x)$ changes from positive before $x=a$ to negative after.
For a minimum $x=a$
$\Leftrightarrow$ The sign of $f'(x)$ changes from negative before $x=a$ to positive after.
Therefore, for the extrema of $x=a \Rightarrow f'(a)=0$

EXERCISE

1 Find the values of the constants a and b when the function $f(x) = -\dfrac{x^2+ax+b}{x^2+1}$ has a local minimum of -2 at $x=2$.

$$f'(x) = -\frac{(2x+a)(x^2+1)-(x^2+ax+b)\cdot 2x}{(x^2+1)^2} = \frac{ax^2+2(b-1)x-a}{(x^2+1)^2}$$

Since the local minimum is -2 at $x=2$, then $f(2)=-2$ and $f'(2)=0$.

From $f(2) = -\dfrac{4+2a+b}{5} = -2$, we get $2a+b=6$ ...(i)

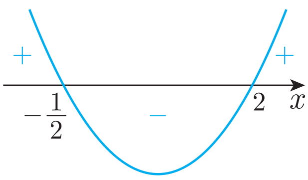
From $f'(2) = \dfrac{4a+4(b-1)-a}{5^2} = 0$, we get $3a+4b=4$ ...(ii)

Thus, by using simultaneous equations to solve for (i) and (ii), we can get $a=4$ and $b=-2$.

Conversely, now we get $f'(x) = \dfrac{4x^2-6x-4}{(x^2+1)^2} = \dfrac{2(2x+1)(x-2)}{(x^2+1)^2}$, and the sign of $f'(x)$ changes from negative before $x=2$ to positive after.

Therefore, for $x=2$ the local minimum is -2, such that to satisfy the problem, we get $a=4$ and $b=-2$.

Changing of sign of $f'(x)$



$a=4$, $b=-2$

2 Find the value of the constant a when the local maximum of the function $f(x)=2x+\frac{a}{x}$ is -2.

Because the denominator $\neq 0$, the domain is all real numbers, except for $x=0$.

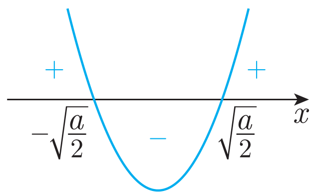
$$f'(x)=2-\frac{a}{x^2}=\frac{2x^2-a}{x^2}$$

When $a\leq 0$, then $f'(x)>0$ is always true, and since $f(x)$ has no extrema, we know $a>0$.

Therefore, when $f'(x)=0$, then from $2x^2-a=0$, we get $x=\pm\sqrt{\frac{a}{2}}$.

Since the sign of $f'(x)$ changes from positive before $x=-\sqrt{\frac{a}{2}}$ to negative after, it has a local maximum of $x=-\sqrt{\frac{a}{2}}$.

Changing of sign of $f'(x)$



Therefore, from $f\left(-\sqrt{\frac{a}{2}}\right)=-\sqrt{2a}-\sqrt{2a}=-2\sqrt{2a}=-2$, we get $a=\frac{1}{2}$.

$$\boldsymbol{a=\frac{1}{2}}$$

PRACTICE

◆ Find the value of the constant a when the local maximum of the function $f(x)=\frac{x^2-x+a}{x-1}$ $(a>0)$ is 0.

Because the denominator $\neq 0$, the domain is all real numbers, except for $x=1$.

$$f'(x)=\frac{(2x-1)(x-1)-(x^2-x+a)}{(x-1)^2}=\frac{x^2-2x-a+1}{(x-1)^2}$$

Assuming a local maximum at $x=\alpha(\alpha\neq 1)$, then we get $f(\alpha)=0$ and $f'(\alpha)=0$.

From $f(\alpha)=\frac{\alpha^2-\alpha+a}{\alpha-1}=0$, we get $\alpha^2-\alpha+a=0$ $\quad$ ...(i)

From $f'(\alpha)=\frac{\alpha^2-2\alpha-a+1}{(\alpha-1)^2}=0$, we get $\alpha^2-2\alpha-a+1=0$ $\quad$ ...(ii)

We eliminate a from (i) and (ii), such that from
$2\alpha^2-3\alpha+1=0, (2\alpha-1)(\alpha-1)=0$, $\alpha\neq 1$, we get $\alpha=\frac{1}{2}$.

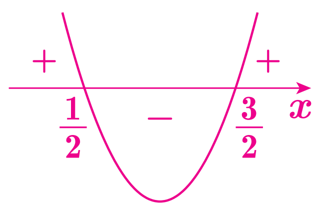
Therefore, by substituting $\alpha=\frac{1}{2}$ into (i), we get $a=-\left(\frac{1}{2}\right)^2+\frac{1}{2}=\frac{1}{4}$.

Conversely, at this point we have

$$f'(x)=\frac{x^2-2x+\frac{3}{4}}{(x-1)^2}=\frac{4x^2-8x+3}{4(x-1)^2}=\frac{(2x-3)(2x-1)}{4(x-1)^2}$$

and the sign of $f'(x)$ changes from positive before $x=\frac{1}{2}$ to negative after.

Changing of sign of $f'(x)$



Therefore, for $x=\frac{1}{2}$ the local maximum is 0, such that to satisfy the problem, we get $a=\frac{1}{4}$.

$$\boldsymbol{a=\frac{1}{4}}$$

Maximum and minimum values of functions

TARGET To understand how to find the maximum and minimum values of a function using a sign chart.

STUDY GUIDE

How to find maximum values and minimum values

We can find the maximum and minimum values of a function by examining changes in the function. Since the maximum value and the minimum value of the continuous function in the closed interval are either the **local maximum or local minimum**, or **the value of the function at the end point of the domain**, it is good to examine them by drawing a **sign chart** and **graphs**.

If the domain of the function $f(x)$ is all real numbers, we must examine the extrema as well as $\lim_{x\to-\infty} f(x)$ and $\lim_{x\to\infty} f(x)$.

In the case of a discontinuous function, it is necessary to examine the behavior of the function around the point of discontinuity by finding the limits.

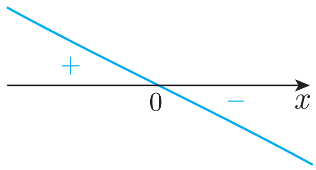
It should be noted that the maximum and minimum values **do not always coincide with the local maximum and local minimum** in either case.

EXERCISE

◆ Find the maximum values and minimum values of the following functions.

(1) $f(x) = \sqrt{1+x} + \sqrt{1-x}$

Changing of sign of $f'(x)$



From $1+x \geq 0$ and $1-x \geq 0$, we get a domain of $-1 \leq x \leq 1$.

$$f'(x) = \frac{1}{2\sqrt{1+x}} - \frac{1}{2\sqrt{1-x}} = \frac{\sqrt{1-x}-\sqrt{1+x}}{2\sqrt{1-x^2}} = \frac{-x}{\sqrt{1-x^2}(\sqrt{1-x}+\sqrt{1+x})}$$

When $f'(x)=0$, then $x=0$.

$f(0) = 2, f(-1) = \sqrt{2}, f(1) = \sqrt{2}$ give us the sign chart as shown on the right.

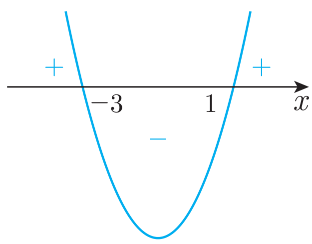
x	-1	$\cdots$	0	$\cdots$	1
$f'(x)$	/	$+$	0	$-$	/
$f(x)$	$\sqrt{2}$	↗	2	↘	$\sqrt{2}$

Therefore, when $x=0$, the maximum value is 2, and when $x=\pm 1$ the minimum value is $\sqrt{2}$.

When $x=0$, the maximum value is 2, and when $x=\pm 1$, the minimum value is $\sqrt{2}$

(2) $f(x) = (x^2-3)e^x \quad (-2 \leq x \leq 2)$

Changing of sign of $f'(x)$



$f'(x) = 2xe^x + (x^2-3)e^x = (x^2+2x-3)e^x = (x-1)(x+3)e^x$

When $f'(x)=0$, then from $(x-1)(x+3)=0$, we get $x=1$ and -3.

$f(1) = -2e, f(-2) = \frac{1}{e^2}, f(2) = e^2$ give us the sign chart as shown on the right.

x	-2	$\cdots$	1	$\cdots$	2
$f'(x)$		$-$	0	$+$	
$f(x)$	$\frac{1}{e^2}$	↘	$-2e$	↗	e^2

Therefore, when $x=2$, the maximum value is e^2 , and when $x=1$ the minimum value is $-2e$.

When $x=2$, the maximum value is e^2 , and when $x=1$, the minimum value is $-2e$

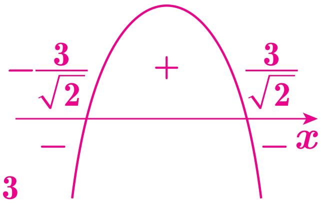
PRACTICE

◆ Find the maximum values and minimum values of the following functions.

(1) $f(x)=x\sqrt{9-x^2}$

From $9-x^2\geq 0$, and since $(x-3)(x+3)\leq 0$, we get a domain of $-3\leq x\leq 3$.

Changing of sign of $f'(x)$



$$f'(x)=\sqrt{9-x^2}+x\cdot\frac{-2x}{2\sqrt{9-x^2}}=-\frac{2x^2-9}{\sqrt{9-x^2}}$$

When $f'(x)=0$, then from $2x^2-9=0$, we can get $x=\pm\frac{3}{\sqrt{2}}$.

$f\left(-\frac{3}{\sqrt{2}}\right)=-\frac{9}{2}, f\left(\frac{3}{\sqrt{2}}\right)=\frac{9}{2}$,

$f(-3)=f(3)=0$

give us the sign chart as shown on the right.

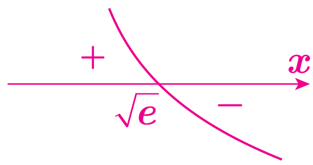
x	-3	$\cdots$	$-\frac{3}{\sqrt{2}}$	$\cdots$	$\frac{3}{\sqrt{2}}$	$\cdots$	3
$f'(x)$	/	$-$	0	$+$	0	$-$	/
$f(x)$	0	$\searrow$	$-\frac{9}{2}$	$\nearrow$	$\frac{9}{2}$	$\searrow$	0

Therefore, when $x=\frac{3\sqrt{2}}{2}$, the maximum value is $\frac{9}{2}$, and when $x=-\frac{3\sqrt{2}}{2}$ the minimum value is $-\frac{9}{2}$.

When $x=\frac{3\sqrt{2}}{2}$, the maximum value is $\frac{9}{2}$,
and when $x=-\frac{3\sqrt{2}}{2}$, the minimum value is $-\frac{9}{2}$

(2) $f(x)=\frac{\log x}{x^2}$ $\left(\frac{1}{e}\leq x\leq e^2\right)$

Changing of sign of $f'(x)$



$$f'(x)=\frac{\frac{1}{x}\cdot x^2-(\log x)\cdot 2x}{x^4}=\frac{1-2\log x}{x^3}$$

When $f'(x)=0$, then from $1-2\log x=0$ we get $x=\sqrt{e}$.

$f(\sqrt{e})=\frac{1}{2e}, f\left(\frac{1}{e}\right)=-e^2, f(e^2)=\frac{2}{e^4}$

give us the sign chart as shown on the right.

x	$\frac{1}{e}$	$\cdots$	$\sqrt{e}$	$\cdots$	e^2
$f'(x)$		$+$	0	$-$	
$f(x)$	$-e^2$	$\nearrow$	$\frac{1}{2e}$	$\searrow$	$\frac{2}{e^4}$

Therefore, when $x=\sqrt{e}$, the maximum value is $\frac{1}{2e}$, and when $x=\frac{1}{e}$, the minimum value is $-e^2$.

When $x=\sqrt{e}$, the maximum value is $\frac{1}{2e}$,
and when $x=\frac{1}{e}$, the minimum value is $-e^2$

Concavity, convexity and inflection points of curves

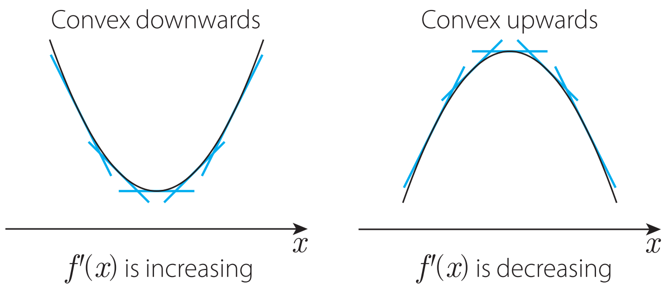
TARGET To understand how to determine whether a curve is concave or convex by using 2nd order derivatives.

STUDY GUIDE

Concavity and convexity of curves

For a given interval, if the slope of the tangent of a curve $y=f(x)$ increases, specifically $f'(x)$ increases as x gets larger, then the curve $y=f(x)$ is said to be **convex downwards** in that interval. Conversely, when it gets smaller, the curve is said to be **convex upwards** in that interval.

Furthermore, when $f''(x)$ exists, the sign of $f''(x)$ corresponds to the increase or decrease of $f'(x)$, such that the following holds.

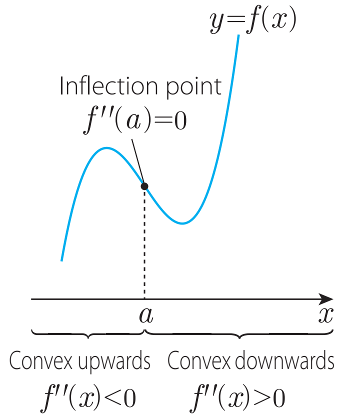


> **Given an interval**
>
> **Always, $f''(x)>0$ $\Leftrightarrow$ $f'(x)$ (slope of tangent) is increasing**
> **$\Leftrightarrow$ Curve $y=f(x)$ is convex downwards**
>
> **Always, $f''(x)<0$ $\Leftrightarrow$ $f'(x)$ (slope of tangent) is decreasing**
> **$\Leftrightarrow$ Curve $y=f(x)$ is convex upwards**

Inflection point

A boundary point where the concavity/convexity of the curve $y=f(x)$ changes is called an **inflection point**. When points $(a, f(a))$ are inflection points of curve $y=f(x)$, we can state the following.

> **$f''(a)=0$, and the sign of $f''(x)$ changes before and after $x=a$.**



EXERCISE

◆ Find the concavity/convexity and inflection points of the graphs of the functions below.

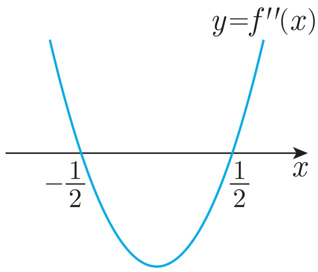
(1) $f(x)=x^4-\dfrac{3}{2}x^2+1$

$f'(x)=4x^3-3x, f''(x)=12x^2-3=3(4x^2-1)=3(2x+1)(2x-1)$

When $f''(x)=0$, then $x=\pm\dfrac{1}{2}$.

Since the sign of $f''(x)$ changes before and after these points, these points are inflection points.

From $f\left(\dfrac{1}{2}\right)=\dfrac{11}{16}, f\left(-\dfrac{1}{2}\right)=\dfrac{11}{16}$, we get inflection points of $\left(-\dfrac{1}{2}, \dfrac{11}{16}\right), \left(\dfrac{1}{2}, \dfrac{11}{16}\right)$.

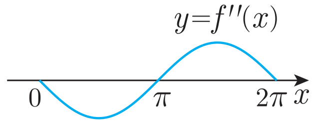


Also, for $x<-\dfrac{1}{2}$ and $\dfrac{1}{2}<x$, it is convex downwards, and for $-\dfrac{1}{2}<x<\dfrac{1}{2}$, it is convex upwards.

For $x<-\dfrac{1}{2}$ and $\dfrac{1}{2}<x$, it is convex downwards, and for $-\dfrac{1}{2}<x<\dfrac{1}{2}$, it is convex upwards, and the inflection points are $\left(-\dfrac{1}{2}, \dfrac{11}{16}\right), \left(\dfrac{1}{2}, \dfrac{11}{16}\right)$

(2) $f(x) = 3x + \sin x \quad (0 \le x \le 2\pi)$

$f'(x) = 3 + \cos x, f''(x) = -\sin x$



When $f''(x)=0$, we get $x=0, \pi, 2\pi$.

Since the sign of $f''(x)$ changes before and after $x=\pi$, from $f(\pi)=3\pi$, we get inflection points of $(\pi, 3\pi)$.

Also, for $0<x<\pi$, it is convex downwards, and for $\pi<x<2\pi$, it is convex upwards.

For $0<x<\pi$, it is convex upwards, for $\pi<x<2\pi$, it is convex downwards, and the inflection points are $(\pi$ and $3\pi)$

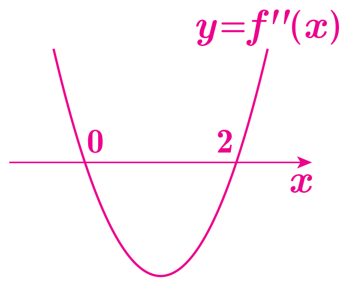
PRACTICE

◆ Find the concavity/convexity and inflection points of the graphs of the functions below.

(1) $f(x) = x^3(x-4)$

$f'(x) = 3x^2(x-4) + x^3 = 4x^3 - 12x^2, f''(x) = 12x^2 - 24x = 12x(x-2)$

When $f''(x)=0$, then $x=0, 2$.



Since the sign of $f''(x)$ changes before and after these points, these points are inflection points.

From $f(0) = 0, f(2) = -16$, we get deflection points of $(0, 0)$ and $(2, -16)$.

Also, for $x<0$ and $2<x$, it is convex downwards, and for $0<x<2$, it is convex upwards.

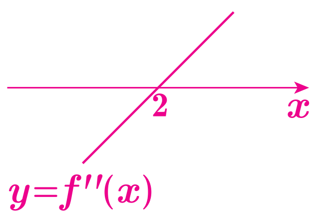
For $x<0$ and $2<x$, it is convex downwards, and for $0<x<2$ it is convex upwards, and the inflection points are $(0, 0)$ and $(2, -16)$

(2) $f(x) = 2xe^{-x}$

$f'(x) = 2\{e^{-x} + x \cdot (-e^{-x})\} = 2(1-x)e^{-x}$

$f''(x) = 2\{-e^{-x} + (1-x) \cdot (-e^{-x})\} = 2(x-2)e^{-x}$

Since $e^{-x} > 0$, when $f''(x)=0$, we get $x=2$.



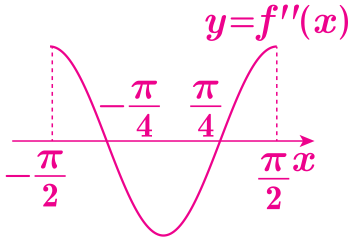
The sign of $f''(x)$ changes before and after this, so from $f(2) = 4e^{-2}$, we get inflection points of $\left(2, \frac{4}{e^2}\right)$.

Also, for $x<2$, it is convex upwards, and for $x>2$ it is convex downwards.

For $x<2$, it is convex upwards, and for $x>2$, it is convex downwards, and the inflection points are $\left(2, \frac{4}{e^2}\right)$

(3) $f(x)=\cos^2 x+1\ \left(-\frac{\pi}{2}\leq x\leq\frac{\pi}{2}\right)$

$f'(x)=2\cos x(-\sin x)=-2\sin x\cos x=-\sin 2x$, $f''(x)=-2\cos 2x$



From $-\frac{\pi}{2}\leq x\leq\frac{\pi}{2}$, because $-\pi\leq 2x\leq\pi$,

when $f''(x)=0$, then from $2x=\pm\frac{\pi}{2}$, we get $x=\pm\frac{\pi}{4}$.

Since the sign of $f''(x)$ changes before and after

these points, these points are inflection points.

From $f\left(\pm\frac{\pi}{4}\right)=\frac{3}{2}$, we get inflection points of $\left(-\frac{\pi}{4},\frac{3}{2}\right),\left(\frac{\pi}{4},\frac{3}{2}\right)$.

Also, for $-\frac{\pi}{2}\leq x<-\frac{\pi}{4}$ and $\frac{\pi}{4}<x\leq\frac{\pi}{2}$, it is convex downwards, and for $-\frac{\pi}{4}<x<\frac{\pi}{4}$, it is convex upwards.

For $-\frac{\pi}{2}\leq x<-\frac{\pi}{4}$ and $\frac{\pi}{4}<x\leq\frac{\pi}{2}$, it is convex downwards,

and for $-\frac{\pi}{4}<x<\frac{\pi}{4}$, it is convex upwards,

and the inflection points are $\left(-\frac{\pi}{4},\frac{3}{2}\right)$ and $\left(\frac{\pi}{4},\frac{3}{2}\right)$

How to draw graphs of functions

TARGET To understand how to draw an approximate graph of a function.

STUDY GUIDE

Procedure to draw graphs of functions

More accurate graphs can be drawn by confirming and determining the following items.

(1) Domain and range ······ Be careful of discontinuities while confirming.

(2) Existence of symmetry and periodicity ······ Determine x-axis symmetry, y-axis symmetry (even functions), origin symmetry (odd functions), and the cycle, etc.

(3) Increase/decrease and maxima/minima of functions ······ Determine change in sign of $f'(x)$.

(4) Concavity/convexity and inflection points of curves ······ Determine change in sign of $f''(x)$.

(5) Existence of asymptote ······ Determine polarity, such as when $x\to\pm\infty$.

(6) Intersection of coordinates ······ Substitute $x=0$ and $y=0$ into equation.

Asymptotes and how to find them

A straight line, whose distance to a given curve approaches 0, but does not touch the curve is called an **asymptote** of the curve. We can find asymptotes by confirming the following.

(1) Asymptote perpendicular to the x axis

When $\lim_{x\to a+0} f(x)=\pm\infty$ or $\lim_{x\to a-0} f(x)=\pm\infty$, the line $x=a$ is an asymptote

(2) Asymptote not perpendicular to the x axis

When $\lim_{x\to\infty}\{f(x)-(mx+n)\}=0$, or $\lim_{x\to-\infty}\{f(x)-(mx+n)\}=0$, then the line $y=mx+n$ is an asymptote

Here, we can find m and n from $m=\lim_{x\to\pm\infty}\dfrac{f(x)}{x}$, $n=\lim_{x\to\pm\infty}\{f(x)-mx\}$ (double sign same order)

explanation

Regarding (2), from $\lim_{x\to\pm\infty}\{f(x)-(mx+n)\}=0$, we can get $\lim_{x\to\pm\infty} x\left\{\dfrac{f(x)}{x}-\left(m+\dfrac{n}{x}\right)\right\}=0$.

Therefore, given $\lim_{x\to\pm\infty}\left\{\dfrac{f(x)}{x}-\left(m+\dfrac{n}{x}\right)\right\}=0$, such that $\lim_{x\to\pm\infty}\dfrac{n}{x}=0$, gives us $m=\lim_{x\to\pm\infty}\dfrac{f(x)}{x}$.

Further, by using the m that we found, we can get $n=\lim_{x\to\pm\infty}\{f(x)-mx\}$ from $\lim_{x\to\pm\infty}\{f(x)-(mx+n)\}=0$.

EXERCISE

◆ Determine whether the function $f(x)=\dfrac{2x^2-x+1}{x-1}$ is concave or convex, its extrema and asymptotes, and then draw a graph.

Because the denominator $x-1\neq 0$, we get $x\neq 1$, so the domain is all real numbers, except for $x=1$.

Because $f(x)=\dfrac{2x^2-x+1}{x-1}=2x+1+\dfrac{2}{x-1}$, then

$$f'(x)=2-\frac{2}{(x-1)^2}=\frac{2(x-1)^2-2}{(x-1)^2}=\frac{2x(x-2)}{(x-1)^2}$$

$$f''(x)=\left\{2-\frac{2}{(x-1)^2}\right\}'=\frac{2\cdot 2(x-1)}{(x-1)^4}=\frac{4}{(x-1)^3}$$

Therefore, when $f'(x)=0$, then $x=0, 2$.

$f(0)=-1, f(2)=2\cdot 2^2-2+1=7$

For this domain, since $f''(x)\neq 0$, there are no inflection points.

From the above, we get the following sign chart.

x	$\cdots$	0	$\cdots$	1	$\cdots$	2	$\cdots$
$f'(x)$	$+$	0	$-$	/	$-$	0	$+$
$f''(x)$	$-$	$-$	$-$	/	$+$	$+$	$+$
$f(x)$	$\nearrow$	-1	$\searrow$	/	$\searrow$	7	$\nearrow$

$$\lim_{x\to 1+0} f(x)=\lim_{x\to 1+0}\left(2x+1+\frac{2}{x-1}\right)=\infty,\ \lim_{x\to 1-0} f(x)=-\infty$$

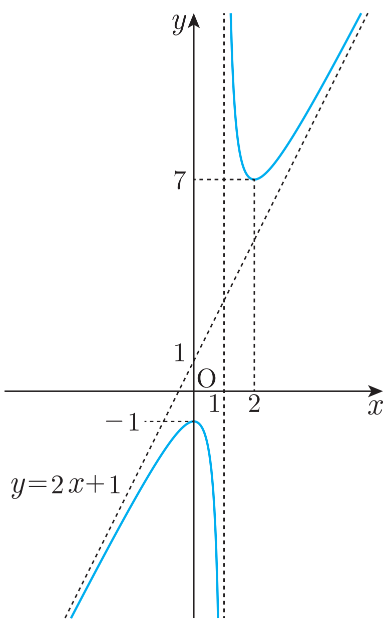
Therefore, the line $x=1$ is an asymptote.

$$\lim_{x\to\pm\infty}\frac{f(x)}{x}=\lim_{x\to\pm\infty}\left\{\frac{2x+1}{x}+\frac{2}{x(x-1)}\right\}=\lim_{x\to\pm\infty}\left\{2+\frac{1}{x}+\frac{2}{x(x-1)}\right\}=2$$

$$\lim_{x\to\pm\infty}\{f(x)-2x\}=\lim_{x\to\pm\infty}\left\{\left(2x+1+\frac{2}{x-1}\right)-2x\right\}=\lim_{x\to\pm\infty}\left(1+\frac{2}{x-1}\right)=1$$

Therefore, the line $y=2x+1$ is an asymptote.

From the above, the graph is as shown on the right.



explanation

In the sign chart for $f(x)$, $\nearrow$ means convex downwards and increasing monotonically, and $\nearrow$ means convex upwards and increasing monotonically. In addition, $\searrow$ means convex downwards and decreasing monotonically, and $\searrow$ means convex upwards and decreasing monotonically.

PRACTICE

◆ Determine whether the function $f(x) = x + \dfrac{1}{x}$ is concave or convex, its extrema and asymptotes, and then draw a graph.

Because the denominator $x \neq 0$, the domain is all real numbers, except for $x=0$.

From $f(x) = x + \dfrac{1}{x}$, we get

$$f'(x) = 1 - \frac{1}{x^2} = \frac{x^2-1}{x^2} = \frac{(x+1)(x-1)}{x^2}$$

$$f''(x) = \left(1 - \frac{1}{x^2}\right)' = \frac{2}{x^3}$$

Therefore, when $f'(x)=0$, then $x=\pm 1$.

$f(1)=1+1=2$, $f(-1)=-1-1=-2$

For this domain, since $f''(x) \neq 0$, there are no inflection points.

From the above, we get the following sign chart.

x	$\cdots$	-1	$\cdots$	0	$\cdots$	1	$\cdots$
$f'(x)$	$+$	0	$-$	/	$-$	0	$+$
$f''(x)$	$-$	$-$	$-$	/	$+$	$+$	$+$
$f(x)$	↗	-2	↘	/	↘	2	↗

$$\lim_{x \to +0} f(x) = \lim_{x \to +0}\left(x + \frac{1}{x}\right) = \infty, \quad \lim_{x \to -0} f(x) = -\infty$$

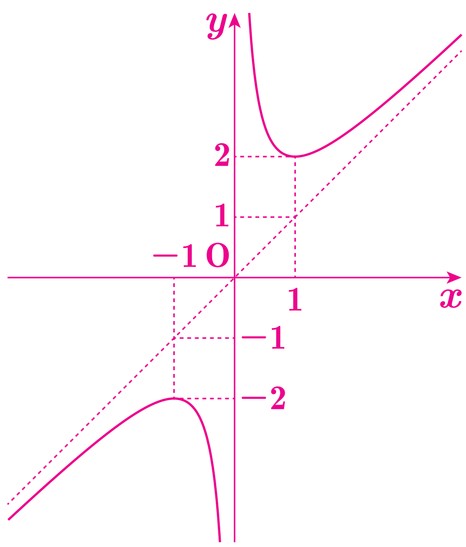
Therefore, the line $x=0$, that is to say, the y axis is an asymptote.

$$\lim_{x \to \pm\infty} \frac{f(x)}{x} = \lim_{x \to \pm\infty}\left(1 + \frac{1}{x^2}\right) = 1$$

$$\lim_{x \to \pm\infty} \{f(x) - x\} = \lim_{x \to \pm\infty}\left\{\left(x + \frac{1}{x}\right) - x\right\} = \lim_{x \to \pm\infty} \frac{1}{x} = 0$$

Therefore, the line $y=x$ is also an asymptote.

From the above, the graph is as shown on the right.



2nd-order derivatives and extrema

TARGET To understand how to use 2nd-order derivatives to determine maximums and minimums.

STUDY GUIDE

Determining the maximums and minimums of functions

Usually, we determine the maximums and minimums of functions by using the change in signs of 1st-order derivatives, but if we can easily find a 2nd-order derivative, then we can also determine them by using the **signs of 2nd-order derivatives**.

When a function $f(x)$ is continuous, and $f'(x)$ and $f''(x)$ exist

$f'(a)=0$ and $f''(a)>0$

$\Rightarrow$ The sign of $f'(x)$ changes from negative before $x=a$ to positive after.

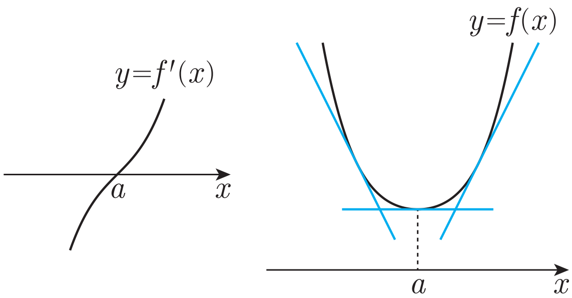
$\Leftrightarrow$ The function $f(x)$ has a local minimum at $x=a$.

$f'(a)=0$ and $f''(a)<0$

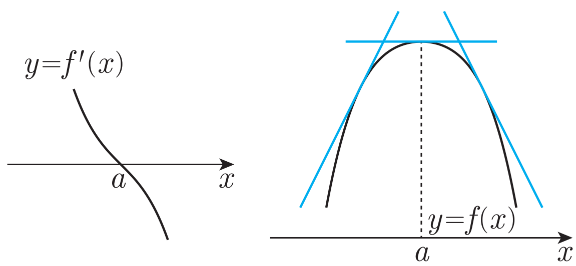
$\Rightarrow$ The sign of $f'(x)$ changes from positive before $x=a$ to negative after.

$\Leftrightarrow$ The function $f(x)$ has a local maximum at $x=a$.

When the local minimum is at $x=a$



When the local maximum is at $x=a$



If $f'(a)$ does not exist or $f''(a)=0$, they cannot be determined, so be careful when applying this.

EXERCISE

◆ Use a 2nd-order derivative to find the extrema of the following functions.

(1) $f(x)=x^3+x^2-x-1$

$f'(x)=3x^2+2x-1=(3x-1)(x+1),\ f''(x)=6x+2$

When $f'(x)=0$, then $x=-1, \dfrac{1}{3}$.

Therefore, from $f''(-1)=-6+2=-4<0$ and $f(-1)=-1+1+1-1=0$, we can find that at $x=-1$, the local maximum is 0.

Also, from $f''\left(\dfrac{1}{3}\right)=\dfrac{6}{3}+2=4>0,\ f\left(\dfrac{1}{3}\right)=\dfrac{1}{27}+\dfrac{1}{9}-\dfrac{1}{3}-1=-\dfrac{32}{27}$, we can find that at $x=\dfrac{1}{3}$, the local minimum is $-\dfrac{32}{27}$.

When $x=-1$ the local maximum is 0, when $x=\dfrac{1}{3}$ the local minimum is $-\dfrac{32}{27}$

(2) $f(x) = x + 2\sin x \quad (0<x<2\pi)$

$f'(x) = 1 + 2\cos x, f''(x) = -2\sin x$

When $f'(x)=0$, then $\cos x = -\dfrac{1}{2}$, and since $0<x<2\pi$, we get $x = \dfrac{2}{3}\pi, \dfrac{4}{3}\pi$.

Therefore, from $f''\left(\dfrac{2}{3}\pi\right) = -2\sin\dfrac{2}{3}\pi = -2\cdot\dfrac{\sqrt{3}}{2} = -\sqrt{3} < 0, f\left(\dfrac{2}{3}\pi\right) = \dfrac{2}{3}\pi + 2\sin\dfrac{2}{3}\pi = \dfrac{2}{3}\pi + \sqrt{3}$,

we have a local maximum $\dfrac{2}{3}\pi + \sqrt{3}$ at $x=\dfrac{2}{3}\pi$.

Also, from $f''\left(\dfrac{4}{3}\pi\right) = -2\sin\dfrac{4}{3}\pi = -2\cdot\left(-\dfrac{\sqrt{3}}{2}\right) = \sqrt{3} > 0, f\left(\dfrac{4}{3}\pi\right) = \dfrac{4}{3}\pi + 2\sin\dfrac{4}{3}\pi = \dfrac{4}{3}\pi - \sqrt{3}$,

we have a local minimum of $\dfrac{4}{3}\pi - \sqrt{3}$ at $x=\dfrac{4}{3}\pi$.

When $x=\dfrac{2}{3}\pi$, the local maximum is $\dfrac{2}{3}\pi + \sqrt{3}$,

when $x=\dfrac{4}{3}\pi$, the local minimum is $\dfrac{4}{3}\pi - \sqrt{3}$

PRACTICE

◆ Use a 2nd-order derivative to find the extrema of the following functions.

(1) $f(x) = x^3 - \dfrac{1}{2}x^2 - 2x - 1$

$f'(x) = 3x^2 - x - 2 = (3x+2)(x-1), f''(x) = 6x - 1$

When $f'(x)=0$, then $x = -\dfrac{2}{3}, 1$.

Therefore, from $f''\left(-\dfrac{2}{3}\right) = -4 - 1 = -5 < 0$ and $f\left(-\dfrac{2}{3}\right) = -\dfrac{8}{27} - \dfrac{2}{9} + \dfrac{4}{3} - 1 = -\dfrac{5}{27}$,

we have a local maximum $-\dfrac{5}{27}$ at $x = -\dfrac{2}{3}$.

Also, from $f''(1) = 6 - 1 = 5 > 0$ and $f(1) = 1 - \dfrac{1}{2} - 2 - 1 = -\dfrac{5}{2}$,

we have a local minimum $-\dfrac{5}{2}$ at $x=1$.

When $x = -\dfrac{2}{3}$, the local maximum is $-\dfrac{5}{27}$,

when $x=1$, the local minimum is $-\dfrac{5}{2}$

(2) $f(x) = 3\sin^2 x$ $(0 < x < \pi)$

$f'(x) = 3 \cdot 2 \sin x \cos x = 3 \sin 2x$, $f''(x) = (3 \cos 2x) \cdot 2 = 6 \cos 2x$

When $f'(x)=0$**, then** $\sin 2x=0$**, since** $0<x<\pi$**, from** $0<2x<2\pi$**, we get** $2x=\pi$ **and** $x=\frac{\pi}{2}$**.**

Therefore, from $f''\left(\frac{\pi}{2}\right) = 6 \cos \pi = -6 < 0$, $f\left(\frac{\pi}{2}\right) = 3 \cdot 1^2 = 3$**, we have a local maximum** 3 **at** $x=\frac{\pi}{2}$**.**

When $x=\frac{\pi}{2}$**, the local maximum is** 3

Proving inequalities

TARGET To understand how to prove inequalities by determining the increase or decrease of the values of functions.

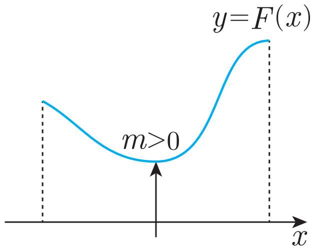
STUDY GUIDE

Proving inequalities

To prove the inequality $f(x)>g(x)$, we set $F(x)=f(x)-g(x)$ and prove $F(x)>0$ by showing the following.

For a minimum value m of a function $F(x)$, then $m>0$.

To do this proof, we use derivatives to determine the change in the value of the function $F(x)$.



Minimum value $=m$

EXERCISE

◆ Prove the following inequalities.

(1) $x\log\dfrac{x}{2}>x-3 \quad (x>0)$

[Proof]

Given $f(x)=x\log\dfrac{x}{2}-(x-3)$, we can get $f'(x)=\log\dfrac{x}{2}+x\cdot\dfrac{2}{x}\cdot\dfrac{1}{2}-1=\log\dfrac{x}{2}$.

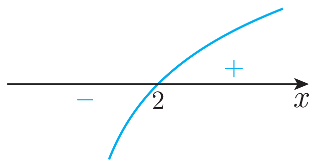
When $f'(x)=0$, then from $\dfrac{x}{2}=1$, we get $x=2$.

We can also get $f(2)=2\log1+1=1$.

Thus, as the sign chart on the right shows, when $x>0$, we get $f(x)\geq1>0$.

Therefore, from $x\log\dfrac{x}{2}-(x-3)>0$, when $x>0$, then $x\log\dfrac{x}{2}>x-3$.

Changing of sign of $f'(x)$



x	0	$\cdots$	2	$\cdots$
$f'(x)$		$-$	0	$+$
$f(x)$		$\searrow$	1	$\nearrow$

(2) $e^x-1>\sin x \quad (x>0)$

[Proof]

Given $f(x)=e^x-1-\sin x$, we can get $f'(x)=e^x-\cos x$.

Now, given $x>0$, since $-1\leq\cos x\leq1<e^x$, then $f'(x)>0$, so the function $f(x)$ is increasing monotonically.

We can also get $f(0)=1-1-0=0$.

Thus, when $x>0$, we get $f(x)>f(0)=0$.

Therefore, from $e^x-1-\sin x>0$, when $x>0$, then $e^x-1>\sin x$.

PRACTICE

◆ Prove the following inequalities.

(1) $x^2 + \sin x \geq x\cos x \quad (x \geq 0)$

[Proof]

Given $f(x) = x^2 + \sin x - x\cos x$,
we can get $f'(x) = 2x + \cos x - (\cos x - x\sin x) = x(2 + \sin x)$.
Now, from $-1 \leq \sin x \leq 1$, since $1 \leq 2 + \sin x \leq 3$, we can get $2 + \sin x > 0$.
Thus, when $x \geq 0$, then $f'(x) \geq 0$, so the function $f(x)$ is increasing monotonically.
We can also get $f(0) = 0$.
Therefore, when $x \geq 0$, we get $f(x) \geq f(0) = 0$.
Accordingly, from $x^2 + \sin x - x\cos x \geq 0$, when $x \geq 0$, then $x^2 + \sin x \geq x\cos x$. Furthermore, the equality sign is achieved from $f(0) = 0$ when $x = 0$.

(2) $\sqrt{2x} > \log x \quad (x > 0)$

[Proof]

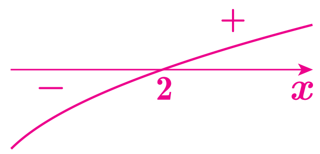
Given $f(x) = \sqrt{2x} - \log x$, we can get $f'(x) = \dfrac{2}{2\sqrt{2x}} - \dfrac{1}{x} = \dfrac{\sqrt{x} - \sqrt{2}}{\sqrt{2x}}$.

When $f'(x) = 0$, then from $\sqrt{x} - \sqrt{2} = 0$, we get $x = 2$.

We can also get $f(2) = \sqrt{4} - \log 2 = 2 - \log 2$.
Therefore, the sign chart is as shown on the right.
Now, from $e > 2$, since $\log e > \log 2$, we can get $f(2) = 2 - \log 2 > 2 - \log e = 2 - 1 = 1 > 0$.
Therefore, when $x > 0$, we get $f(x) \geq 2 - \log 2 > 0$.

Accordingly, from $\sqrt{2x} - \log x > 0$, when $x > 0$, then $\sqrt{2x} > \log x$.

Changing of sign of $f'(x)$



x	0	$\cdots$	2	$\cdots$
$f'(x)$		$-$	0	$+$
$f(x)$		$\searrow$	$2-\log 2$ (>0)	$\nearrow$

Number of real roots of equations

TARGET To understand how to find the number of roots of equations by using the common points of graphs.

STUDY GUIDE

Number of real roots of equations

Since the **real roots of an equation** correspond to the x **coordinates** of **common points in** 2 **graphs**, the number of real roots can be found by determining the number of the following common points.

(a) Number of real roots of equation $f(x)=0$

Determine the number of common points on the graph of $y=f(x)$ and the x-axis.

(b) Number of real roots of equation $f(x)=k$ (k is constant)

Determine the number of common points on the graph of $y=f(x)$ and the line $y=k$.

(c) Number of real roots of equation $f(x)=g(x)$

Determine the number of common points on the graph of $y=f(x)$ and the graph of $y=g(x)$.

In general, if the equation $f(x)=0$ contains the character constant k, we transform it to $g(x)=k$ and determine the common points between the graphs of $y=g(x)$ and the line $y=k$.

EXTRA Info.

By placing conditions on the sign of $f'(x)$, as shown below, the **intermediate-value theorem** can be used to determine the number of shared points on the x axis.

When the function $f(x)$ is continuous in a closed interval $[a, b]$, then in this interval $f'(x)$ has a definite sign and $f(a)f(b)<0$ $\Leftrightarrow$ So in this interval there is only 1 real root of equation $f(x)=0$.

EXERCISE

◆ Find the number of different real roots of the equation $\log x = \frac{1}{2}x - 1$.

From the anti-logarithm conditions, we get $x>0$.

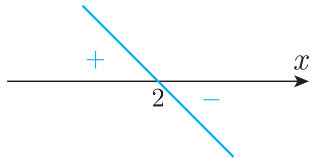
Given $f(x) = \log x - \frac{1}{2}x + 1$, we can get $f'(x) = \frac{1}{x} - \frac{1}{2} = \frac{2-x}{2x}$.

When $f'(x)=0$, then from $2-x=0$, we get $x=2$.

$f(2)=\log 2-1+1=\log 2>0$, $\lim_{x\to+0} f(x) = -\infty$,

$f(e^2) = \log e^2 - \frac{1}{2}e^2 + 1 = 3 - \frac{1}{2}e^2 = \frac{1}{2}(6-e^2) < 0$

Changing of sign of $f'(x)$

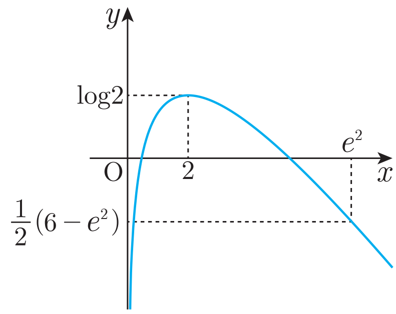


x	0	$\cdots$	2	$\cdots$	e^2	$\cdots$
$f'(x)$		$+$	0	$-$	$-$	$-$
$f(x)$	$(-\infty)$	$\nearrow$	$\log 2$ (>0)	$\searrow$	$\frac{1}{2}(6-e^2)$ (<0)	$\searrow$

Thus, as the sign chart of function $f(x)$ on the right shows, we understand from the graph that there is 1 intersection each in the intervals $(0, 2)$ and $(2, e^2)$.

Therefore, as there are 2 intersections of the graph of function $f(x)$ and the x axis, we can also find there are 2 real roots.

2



PRACTICE

◆ Find the number of different real roots of the following equation.

(1) $x^{\frac{3}{2}} - 3x + 2 = 0$

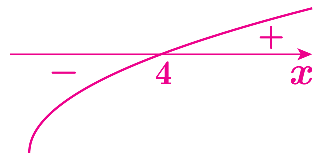
From $x^{\frac{3}{2}} = \sqrt{x^3}$, since $x^3 \geq 0$, we can get $x \geq 0$.

Given $f(x) = x^{\frac{3}{2}} - 3x + 2$, we can get $f'(x) = \frac{3}{2}x^{\frac{1}{2}} - 3 = \frac{3}{2}(\sqrt{x} - 2)$.

When $f'(x)=0$, then from $\sqrt{x} = 2$, we get $x=4$.

$f(0) = 2$, $f(4) = 4^{\frac{3}{2}} - 12 + 2 = -2$, $\lim_{x\to\infty} f(x) = \infty$

Changing of sign of $f'(x)$

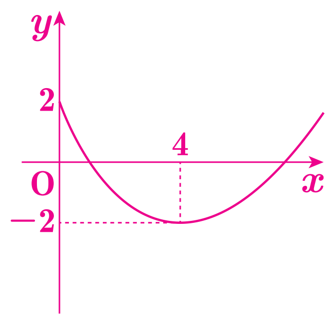


Thus, the sign chart of function $f(x)$ is shown on the right.

x	0	$\cdots$	4	$\cdots$	∞
$f'(x)$		$-$	0	$+$	
$f(x)$	2	$\searrow$	-2	$\nearrow$	∞

Therefore, as there are 2 intersections of the graph of function $f(x)$ and the x axis, we can also find there are 2 real roots.

2



(2) $1+x-\tan x=0$ $(0 \leq x \leq 2\pi)$

Given $f(x)=1+x-\tan x$**,**

we can get $f'(x)=1-\dfrac{1}{\cos^2 x}=\dfrac{\cos^2 x-1}{\cos^2 x}=-\dfrac{\sin^2 x}{\cos^2 x}=-\tan^2 x \leq 0$.

When $f'(x)=0$**, we get** $x=0, \pi, 2\pi$.

$$f(0)=1+0-0=1, f(\pi)=1+\pi-0=1+\pi, f(2\pi)=1+2\pi+0=1+2\pi$$

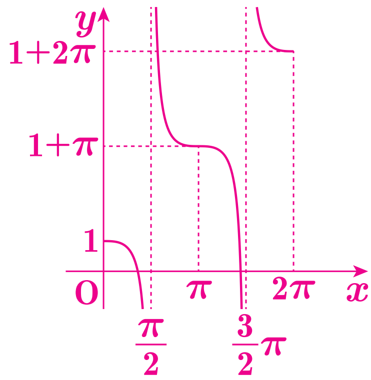
$$\lim_{x\to\frac{\pi}{2}-0} f(x)=-\infty,\ \lim_{x\to\frac{3\pi}{2}-0} f(x)=-\infty$$

Thus, the sign chart of function $f(x)$ **is shown below.**

x	0	$\cdots$	$\frac{\pi}{2}$	$\cdots$	π	$\cdots$	$\frac{3}{2}\pi$	$\cdots$	2π
$f'(x)$	0	$-$		$-$	0	$-$		$-$	0
$f(x)$	1	$\searrow$	$(-\infty)$	$\searrow$	$1+\pi$	$\searrow$	$(-\infty)$	$\searrow$	$1+2\pi$

Therefore, as there are 2 **intersections of the graph of function** $f(x)$ **and the** x **axis, we can also find there are** 2 **real roots.**

2



1st order approximation of a function $f(x)$

TARGET To understand 1st-order approximation of functions and how to use that to find approximate values.

STUDY GUIDE

1st order approximation of a function

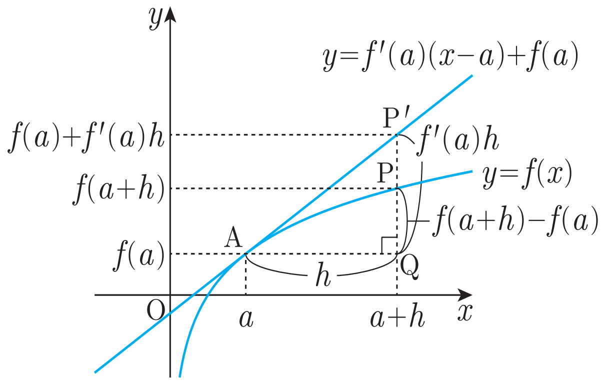
The differential coefficient $f'(a)$ at $x=a$ of the function $f(x)$ is defined as $f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$. Thus, when $|h|$ is sufficiently small, then from $\frac{f(a+h)-f(a)}{h}\simeq f'(a)$, we get $f(a+h)-f(a)\simeq f'(a)h$, $\boldsymbol{f(a+h)\simeq f(a)+f'(a)h}$.

Furthermore, by letting $a=0$ and $h=x$, we can get $\boldsymbol{f(x)\simeq f(0)+f'(0)x}$, so we can derive the 1st-order approximation of the function $f(x)$. Such approximations are outlined below.

When $|h|$ is sufficiently small $\quad f(a+h)\simeq f(a)+f'(a)h$
When $|x|$ is sufficiently small $\quad f(x)\simeq f(0)+f'(0)x$

Graphical meaning of approximations

In the diagram on the right, from $y-f(a)=f'(a)(x-a)$, we find the equation of tangent AP' at point A is $y=f'(a)(x-a)+f(a)$. Thus, the coordinates of point P' are expressed as $(a+h, f(a)+f'(a)h)$. Now, since $f(a+h)-f(a)=\mathrm{PQ}$ and $f'(a)h=\mathrm{P'Q}$, the approximation $\boldsymbol{f(a+h)-f(a)\simeq f'(a)h}$ indicates that when $|h|$ is sufficiently small, then $\mathbf{PQ\simeq P'Q}$. Therefore, it can be understood that the approximation $f(a+h)\simeq f(a)+f'(a)h$ has the following meaning expressed graphically.



$|h|$ is sufficiently small

Curved line AP $\longrightarrow$ Tangent of AP' at point A
Approximation
y coordinates of point P, $f(a+h)$ $\longrightarrow$ y coordinates of point P', $f(a)+f'(a)h$

Derivation of 1st-order approximations by using the mean-value theorem

When the function $f(x)$ is continuous in a closed interval $[0, x]$, and is differentiable in the open interval $(0, x)$, then from the **mean-value theorem**, there exists a real number c that satisfies $\frac{f(x)-f(0)}{x-0}=f'(c)$, $0<c<x$. Now, when the $|x|$ is sufficiently small, from $0<c<x$, c is a value closer to 0, and we can consider that $f'(c)\simeq f'(0)$. Therefore, from $\frac{f(x)-f(0)}{x}\simeq f'(0)$, we can derive $\boldsymbol{f(x)\simeq f(0)+f'(0)x}$.

1st-order approximation of $(1+x)^r$

By using the approximation $f(x) \simeq f(0)+f'(0)x$, we can derive the following approximations in regards to $(1+x)^r$ (r is a rational number).

When $|x|$ is sufficiently small, $(1+x)^r \simeq 1+rx$ (r is a rational number)

explanation

Given that $f(x)=(1+x)^r$ (r is a rational number), then $f'(x)=r(1+x)^{r-1}$, so we can get $f(0)=1$, $f'(0)=r$.
Therefore, from $f(x) \simeq f(0)+f'(0)x$, when $|x|$ is sufficiently small, we can get $(1+x)^r \simeq 1+rx$.

EXERCISE

◆ Use 1st-order approximation to find an approximation of $\sin 61°$ to the 3rd decimal place. Provided that $\sqrt{3}=1.732, \pi=3.141$.

When $|h|$ is sufficiently small, $f(a+h)$ can be approximated by a 1st-order expression of h, such that $f(a+h) \simeq f(a)+f'(a)h$.
Therefore, given $f(x)=\sin x$, then from $f'(x)=\cos x$, we can get $\sin(a+h) \simeq \sin a + h\cos a$.

Now, consider $\left|\frac{\pi}{180}\right|$ to be sufficiently small as regards $61° = \frac{\pi}{3}+\frac{\pi}{180}$.

$$\sin 61° = \sin\left(\frac{\pi}{3}+\frac{\pi}{180}\right) \simeq \sin\frac{\pi}{3}+\frac{\pi}{180}\cos\frac{\pi}{3} = \frac{\sqrt{3}}{2}+\frac{\pi}{180}\times\frac{1}{2} = \frac{1.732}{2}+\frac{3.141}{180}\times\frac{1}{2} \simeq 0.875$$

Therefore, we get $\sin 61° \simeq 0.875$.

0.875

OTHER METHODS

When the $|x|$ is sufficiently small, $f(x)$ can be approximated by a 1st-order expression of x, such that $f(x) \simeq f(0)+f'(0)x$.

Since $61° = \frac{\pi}{3}+\frac{\pi}{180}$, given $f(x)=\sin\left(\frac{\pi}{3}+x\right)$, from $f'(x)=\cos\left(\frac{\pi}{3}+x\right)$,

we can get $f(0)=\sin\frac{\pi}{3}=\frac{\sqrt{3}}{2}, f'(0)=\cos\frac{\pi}{3}=\frac{1}{2}$.

Therefore, we get $\sin\left(\frac{\pi}{3}+x\right) \simeq \frac{\sqrt{3}}{2}+\frac{1}{2}x$.

Now, consider $\left|\frac{\pi}{180}\right|$ to be sufficiently small.

$$\sin 61° = \sin\left(\frac{\pi}{3}+\frac{\pi}{180}\right) \simeq \frac{\sqrt{3}}{2}+\frac{1}{2}\times\frac{\pi}{180} = \frac{1.732}{2}+\frac{1}{2}\times\frac{3.141}{180} \simeq 0.875$$

Therefore, we get $\sin 61° \simeq 0.875$.

PRACTICE

◆ Use 1st-order approximation to find an approximation of $\tan 46°$ to the 3rd decimal place. Provided that $\pi=3.141$.

When $|h|$ is sufficiently small, $f(a+h)$ can be approximated by a 1st-order expression of h, such that $f(a+h)\simeq f(a)+f'(a)h$.

Therefore, given $f(x)=\tan x$, then from $f'(x)=\dfrac{1}{\cos^2 x}$, we can get

$$\tan(a+h)\simeq \tan a+\frac{h}{\cos^2 a}.$$

Now, consider $\left|\dfrac{\pi}{180}\right|$ to be sufficiently small as regards $46°=\dfrac{\pi}{4}+\dfrac{\pi}{180}$.

$$\tan 46° = \tan\left(\frac{\pi}{4}+\frac{\pi}{180}\right)\simeq \tan\frac{\pi}{4}+\frac{\frac{\pi}{180}}{\cos^2\frac{\pi}{4}}=1+\frac{3.141}{180}\times 2\simeq 1.035$$

Therefore, we get $\tan 46°\simeq 1.035$.

1.035

OTHER METHODS

When the $|x|$ is sufficiently small, $f(x)$ can be approximated by a 1st-order expression of x, such that $f(x)\simeq f(0)+f'(0)x$.

Since $46°=\dfrac{\pi}{4}+\dfrac{\pi}{180}$, given $f(x)=\tan\left(\dfrac{\pi}{4}+x\right)$, from $f'(x)=\dfrac{1}{\cos^2\left(\dfrac{\pi}{4}+x\right)}$,

we can get $f(0)=\tan\dfrac{\pi}{4}=1,\ f'(0)=\dfrac{1}{\cos^2\frac{\pi}{4}}=2$.

Therefore, we get $\tan\left(\dfrac{\pi}{4}+x\right)\simeq 1+2x$.

Now, consider $\left|\dfrac{\pi}{180}\right|$ to be sufficiently small.

$$\tan 46° = \tan\left(\frac{\pi}{4}+\frac{\pi}{180}\right)\simeq 1+2\times\frac{\pi}{180}=1+2\times\frac{3.141}{180}\simeq 1.035$$

Therefore, we get $\tan 46°\simeq 1.035$.

CASIO Essential Materials

Publisher: CASIO Institute for Educational Development

Date of Publication: 2023/12/22 (1st edition)

https://edu.casio.com