16

Limits

CASIO
Essential Materials

Introduction

These teaching materials were created with the hope of conveying to many teachers and students the appeal of scientific calculators.

(1) Change awareness (emphasizing the thinking process) and boost efficiency in learning mathematics

- By reducing the time spent on manual calculations, we can have learning with a focus on the thinking process that is more efficient.
- This reduces the aversion to mathematics caused by complicated calculations, and allows students to experience the joy of thinking, which is the essence of mathematics.

(2) Diversification of learning materials and problem-solving methods

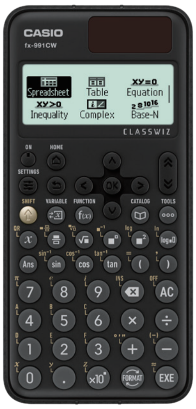
- Making it possible to do difficult calculations manually allows for diversity in learning materials and problem-solving methods.

(3) Promoting understanding of mathematical concepts

- By using the various functions of the scientific calculator in creative ways, students are able to deepen their understanding of mathematical concepts through calculations and discussions from different perspectives than before.
- This allows for exploratory learning through easy trial and error of questions.
- Listing and graphing of numerical values by means of tables allows students to discover laws and to understand visually.

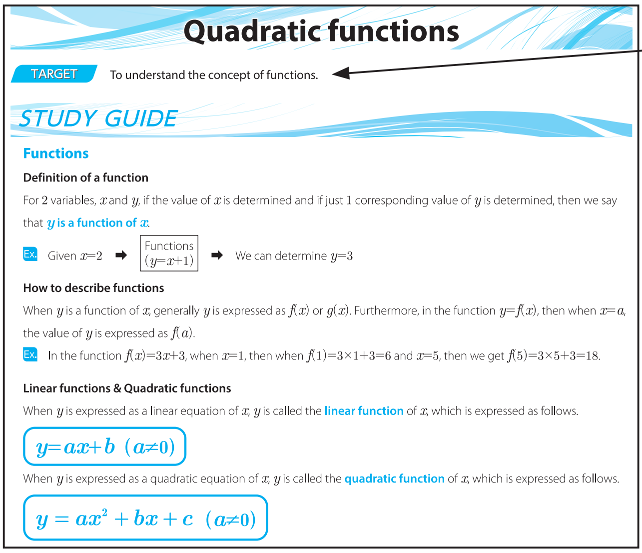
Features of this book

- As well as providing first-time scientific calculator users with opportunities to learn basic scientific calculator functions from the ground up, the book also has material to show people who already use scientific calculators the appeal of scientific calculators described above.
- You can also learn about functions and techniques that are not available on conventional Casio models or other brands of scientific calculators.
- This book covers many units of high school mathematics, allowing students to learn how to use the scientific calculator as they study each topic.
- This book can be used in a variety of situations, from classroom activities to independent study and homework by students.



Better Mathematics Learning with Scientific Calculator

Structure



Quadratic functions

TARGET To understand the concept of functions.

STUDY GUIDE

Functions

Definition of a function

For 2 variables, x and y, if the value of x is determined and if just 1 corresponding value of y is determined, then we say that **y is a function of x.**

Ex. Given $x=2$ ➡ Functions ($y=x+1$) ➡ We can determine $y=3$

How to describe functions

When y is a function of x, generally y is expressed as $f(x)$ or $g(x)$. Furthermore, in the function $y=f(x)$, then when $x=a$, the value of y is expressed as $f(a)$.

Ex. In the function $f(x)=3x+3$, when $x=1$, then when $f(1)=3\times1+3=6$ and $x=5$, then we get $f(5)=3\times5+3=18$.

Linear functions & Quadratic functions

When y is expressed as a linear equation of x, y is called the **linear function** of x, which is expressed as follows.

$$\boxed{y=ax+b\ \ (a\neq0)}$$

When y is expressed as a quadratic equation of x, y is called the **quadratic function** of x, which is expressed as follows.

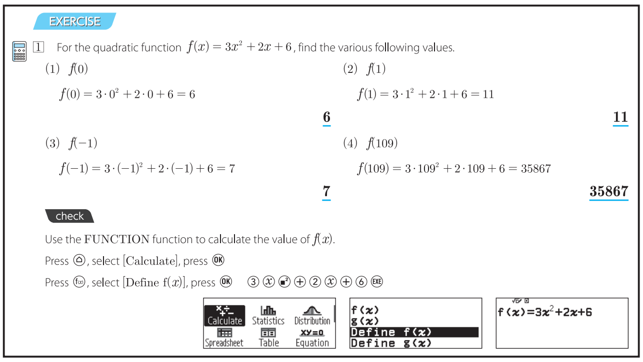
$$\boxed{y=ax^2+bx+c\ \ (a\neq0)}$$

TARGET

Students can identify the objective to learn in each section.

STUDY GUIDE

Mathematical theorems and concepts are explained in detail. A scientific calculator is used to check and derive formulas according to the topic.



EXERCISE

1 For the quadratic function $f(x)=3x^2+2x+6$, find the various following values.

(1) $f(0)$

$f(0)=3\cdot0^2+2\cdot0+6=6$

6

(2) $f(1)$

$f(1)=3\cdot1^2+2\cdot1+6=11$

11

(3) $f(-1)$

$f(-1)=3\cdot(-1)^2+2\cdot(-1)+6=7$

7

(4) $f(109)$

$f(109)=3\cdot109^2+2\cdot109+6=35867$

35867

check

Use the FUNCTION function to calculate the value of $f(x)$.

Press ⌂, select [Calculate], press OK

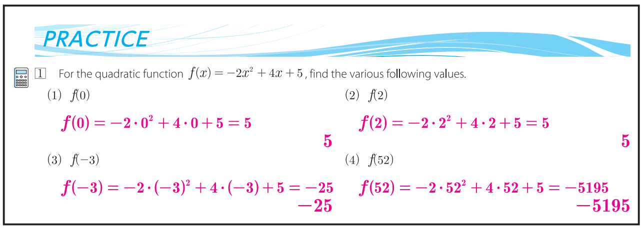
Press f(x), select [Define f(x)], press OK ③ (x) (■²) ⊕ ② (x) ⊕ ⑥ (EXE)

EXERCISE

Students learn basic examples based on the explanation in Study Guide.

check

Explains how to use the scientific calculator to solve problems and check answers.



PRACTICE

1 For the quadratic function $f(x)=-2x^2+4x+5$, find the various following values.

(1) $f(0)$

$f(0)=-2\cdot0^2+4\cdot0+5=5$

5

(2) $f(2)$

$f(2)=-2\cdot2^2+4\cdot2+5=5$

5

(3) $f(-3)$

$f(-3)=-2\cdot(-3)^2+4\cdot(-3)+5=-25$

-25

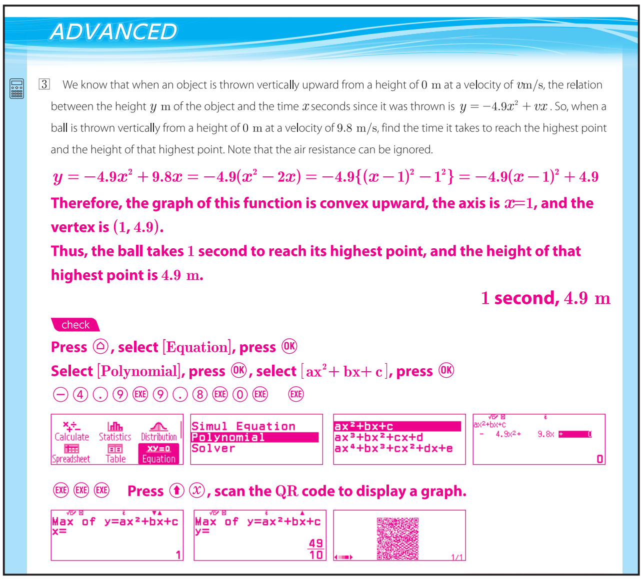
(4) $f(52)$

$f(52)=-2\cdot52^2+4\cdot52+5=-5195$

-5195

PRACTICE

Students can do practice problems similar to those in EXERCISE. They can also practice using the scientific calculator as they learned to in Check.



ADVANCED

3 We know that when an object is thrown vertically upward from a height of 0 m at a velocity of vm/s, the relation between the height y m of the object and the time x seconds since it was thrown is $y=-4.9x^2+vx$. So, when a ball is thrown vertically from a height of 0 m at a velocity of 9.8 m/s, find the time it takes to reach the highest point and the height of that highest point. Note that the air resistance can be ignored.

$y=-4.9x^2+9.8x=-4.9(x^2-2x)=-4.9\{(x-1)^2-1^2\}=-4.9(x-1)^2+4.9$

Therefore, the graph of this function is convex upward, the axis is $x=1$, and the vertex is (1, 4.9).

Thus, the ball takes 1 second to reach its highest point, and the height of that highest point is 4.9 m.

1 second, 4.9 m

check

Press ⌂, select [Equation], press OK

Select [Polynomial], press OK, select [ax^2+bx+c], press OK

⊖ ④ ⊙ ⑨ (EXE) ⑨ ⊙ ⑧ (EXE) ⓪ (EXE) (EXE)

(EXE) (EXE) (EXE) **Press ⇧ (x), scan the QR code to display a graph.**

ADVANCED

Practical problems have been included in several topics. Solutions using a scientific calculator are also presented as necessary.

Other marks

Ex.	Simple examples on how to apply equations and theorems
explanation	Formulas and their supplementary explanations
proof	Proofs and checks of mathematical formulas
EXTRA Info.	Knowledge and information on formulas and other supplementary information in other units
OTHER METHODS	Alternative solutions and different verification methods for previously presented problems

Calculator mark

🖩	Where to use the scientific calculator

Colors of fonts in the teaching materials

- In STUDY GUIDE, important mathematical terms and formulas are printed in blue.
- In PRACTICE and ADVANCED the answers are printed in red.
  (Separate data is also available without the red parts, so it can be used for exercises.)

Applicable models

The applicable model is fx-991CW.
(Instructions on how to do input are for the fx-991CW, but in many cases similar calculations can be done on other models.)

Related Links

- Information and educational materials relevant to scientific calculators can be viewed on the following site.
  https://edu.casio.com
- The following video can be viewed to learn about the multiple functions of scientific calculators.
  https://www.youtube.com/playlist?list=PLRgxo9AwbIZLurUCZnrbr4cLfZdqY6aZA

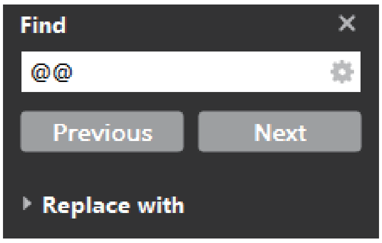
How to use PDF data

About types of data

- Data for all unit editions and data for each unit are available.
- For the above data, the PRACTICE and ADVANCED data without the answers in red is also available.

How to find where the scientific calculator is used

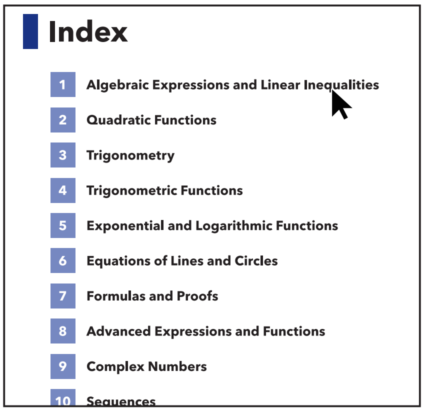
(1) Open a search window in the PDF Viewer.
(2) Type in "@@" as a search term.
(3) You can sequentially check where the calculator marks appear in the data.



@@

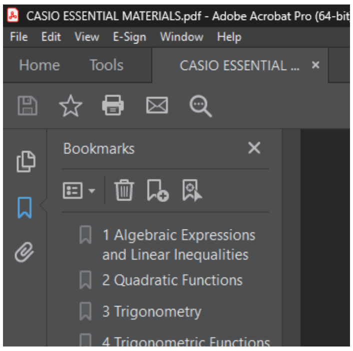
How to search for a unit and section

(1) Search for units of data in all unit editions
- The data in all unit editions has a unit table of contents.
- Selecting a unit in the table of contents lets you jump to the first page of that unit.
- There is a bookmark on the first page of each unit, so you can jump from there also.



Index

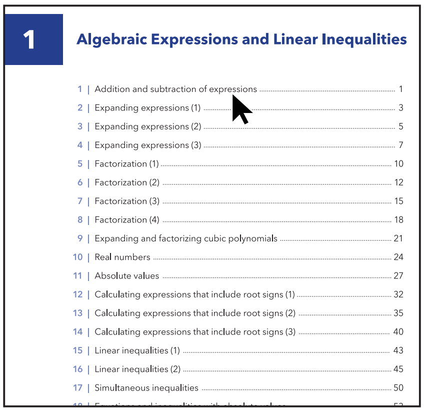
Table of contents of unit



1 Algebraic Expressions and Linear Inequalities
2 Quadratic Functions
3 Trigonometry
4 Trigonometric Functions

Bookmark of unit

(2) Search for sections
- There are tables of contents for sections on the first page of units.
- Selecting a section in the table of contents takes you to the first page of that section.



1 **Algebraic Expressions and Linear Inequalities**

Table of contents of section

Limits of sequences (convergence, divergence, and oscillation of infinite sequences)

TARGET To understand the convergence, divergence, and oscillation of infinite sequences.

STUDY GUIDE

Limits of sequences

Convergence of infinite sequences

A sequence of numbers with an infinite number of terms is called an **infinite sequence**. When n is infinitely large in an infinite sequence $\{a_n\}$ and approaches a certain value α (a finite definite value) that has a_n, we say that, "the sequence $\{a_n\}$ is **converging** on α, and write it as " $\lim\limits_{n\to\infty} a_n = \alpha$ " or "when $n \to \infty$, then $a_n \to \alpha$". Also, we call α the **limiting value** of the sequence $\{a_n\}$.

Ex. $\lim\limits_{n\to\infty} \dfrac{1}{n} = 0$

Divergence of infinite sequences

When a sequence $\{a_n\}$ does not converge, then the sequence $\{a_n\}$ is said to **diverge**.

When $n \to \infty$, then a_n also increases infinitely, and we say that "sequence $\{a_n\}$ is **diverging to positive infinity**", and write it as "$\lim\limits_{n\to\infty} a_n = \infty$" or "when $n \to \infty$, then $a_n \to \infty$". On the other hand, when $n \to \infty$, a_n takes a negative value and its absolute value increases, then we say that "sequence $\{a_n\}$ is **diverging to negative infinity**", and write it as "$\lim\limits_{n\to\infty} a_n = -\infty$" or "when $n \to \infty$, then $a_n \to -\infty$".

This means that there is no limiting value.

Ex. $\lim\limits_{n\to\infty} n^2 = \infty$

Also, when a sequence $\{a_n\}$ does not converge but does not diverge to positive infinity or negative infinity, then we say that the "sequence $\{a_n\}$ is **oscillating**. This means that there is no limit $\lim\limits_{n\to\infty} a_n$.

Ex. $\lim\limits_{n\to\infty} (-2)^n$ does not exist. (The sequence $\{(-2)^n\}$ is oscillating.)

Limits of sequences

(1) **Converging ··· Has a limit value α** $\lim\limits_{n\to\infty} a_n = \alpha$

(2) **Diverging**
- **Diverges to positive infinity** $\lim\limits_{n\to\infty} a_n = \infty$
- **Diverges to negative infinity** $\lim\limits_{n\to\infty} a_n = -\infty$
- **Oscillates ························ Has no limit**

(Converging and diverging to positive or negative infinity:) **Has a limit**

Properties of limits of sequences

When the sequence $\{a_n\},\{b_n\}$ is converging and $\lim\limits_{n\to\infty} a_n = \alpha, \lim\limits_{n\to\infty} b_n = \beta$, then the following holds.

(1) $\lim\limits_{n\to\infty} ka_n = k\alpha$ (k **is a constant**)

(2) $\lim\limits_{n\to\infty}(a_n \pm b_n) = \alpha \pm \beta$ (**double sign same order**)

(3) $\lim\limits_{n\to\infty} a_nb_n = \alpha\beta$

(4) $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{\alpha}{\beta}$ $(\beta\neq 0)$

In particular, from (1) and (2), we get $\lim\limits_{n\to\infty}(ka_n \pm lb_n) = k\alpha \pm l\beta$ (k, l are constants with double signs in same order).

Note that for sequences $\{a_n\},\{b_n\}$, when $\lim\limits_{n\to\infty} a_n = \infty, \lim\limits_{n\to\infty} b_n = 0$, it is not always the case that $\lim\limits_{n\to\infty} a_nb_n = 0$.

Ex. When $a_n = n^2, b_n = \dfrac{1}{n}$, we get $\lim\limits_{n\to\infty} a_n = \infty, \lim\limits_{n\to\infty} b_n = 0$, but it becomes $\lim\limits_{n\to\infty} a_nb_n = \lim\limits_{n\to\infty} n = \infty$.

How to find the limits of indeterminate forms

When determining the limits of sequences, we say that forms like $\dfrac{\infty}{\infty}, \infty-\infty, 0\times\infty, \dfrac{0}{0}$, etc. are **indeterminate forms**.

Usually, we transform and calculate them as follows.

(1) For the fractional expression $\left(\dfrac{\infty}{\infty}\right)$, we divide the denominator and numerator by the highest order term of the denominator.

Ex. $$\lim_{n\to\infty}\frac{3n^2-n}{2n^2+1} = \lim_{n\to\infty}\frac{3-\dfrac{1}{n}}{2+\dfrac{1}{n^2}} = \frac{3}{2}$$

(2) For the polynomial expression $(\infty-\infty)$, we factor it out by using the highest order term.

Ex. $$\lim_{n\to\infty}(n^2-2n+4) = \lim_{n\to\infty} n^2\left(1-\frac{2}{n}+\frac{4}{n^2}\right) = \infty$$

(3) For the irrational expression $(\infty-\infty)$, we rationalize the numerator or the denominator.

Ex. To rationalize the numerator $$\lim_{n\to\infty}(\sqrt{n+2}-\sqrt{n}) = \lim_{n\to\infty}\frac{n+2-n}{\sqrt{n+2}+\sqrt{n}} = \lim_{n\to\infty}\frac{2}{\sqrt{n+2}+\sqrt{n}} = 0$$

EXERCISE

◆ Find the limits of the following.

(1) $\lim_{n\to\infty}\frac{4n^2+2n-1}{n^2-n-3}$

$$=\lim_{n\to\infty}\frac{4+\frac{2}{n}-\frac{1}{n^2}}{1-\frac{1}{n}-\frac{3}{n^2}}=4$$

$\underline{4}$

(2) $\lim_{n\to\infty}\frac{n-4n^2}{5n^2-3n}$

$$=\lim_{n\to\infty}\frac{\frac{1}{n}-4}{5-\frac{3}{n}}=-\frac{4}{5}$$

$\underline{-\frac{4}{5}}$

(3) $\lim_{n\to\infty}\frac{2n^2-5n}{(n+2)(n+3)}$

$$=\lim_{n\to\infty}\frac{2-\frac{5}{n}}{\left(1+\frac{2}{n}\right)\left(1+\frac{3}{n}\right)}=2$$

$\underline{2}$

(4) $\lim_{n\to\infty}\frac{\sqrt{2n}}{\sqrt{n+3}}$

$$=\lim_{n\to\infty}\sqrt{\frac{2}{1+\frac{3}{n}}}=\sqrt{2}$$

$\underline{\sqrt{2}}$

(5) $\lim_{n\to\infty}\frac{1}{n^2}\sum_{k=1}^{n}(8k-3)$

$$=\lim_{n\to\infty}\frac{1}{n^2}\left\{8\cdot\frac{1}{2}n(n+1)-3n\right\}=\lim_{n\to\infty}\frac{4n^2+n}{n^2}$$

$$=\lim_{n\to\infty}\left(4+\frac{1}{n}\right)=4$$

$\underline{4}$

(6) $\lim_{n\to\infty}(5n-4n^2)$

$$=\lim_{n\to\infty}n^2\left(\frac{5}{n}-4\right)=-\infty$$

$\underline{-\infty}$

(7) $\lim_{n\to\infty}(2n-\sqrt{4n^2+n})$

$$=\lim_{n\to\infty}\frac{4n^2-(4n^2+n)}{2n+\sqrt{4n^2+n}}=\lim_{n\to\infty}\frac{-n}{2n+\sqrt{4n^2+n}}$$

$$=\lim_{n\to\infty}\frac{-1}{2+\sqrt{4+\frac{1}{n}}}=-\frac{1}{4}$$

$\underline{-\frac{1}{4}}$

(8) $\lim_{n\to\infty}\frac{2}{\sqrt{n+3}-\sqrt{n}}$

$$=\lim_{n\to\infty}\frac{2(\sqrt{n+3}+\sqrt{n})}{n+3-n}=\lim_{n\to\infty}\frac{2}{3}(\sqrt{n+3}+\sqrt{n})$$

$$=\infty$$

$\underline{\infty}$

PRACTICE

◆ Find the limits of the following.

(1) $\lim\limits_{n\to\infty}\dfrac{n^3-5n^2+2}{6n-1}$

$$=\lim_{n\to\infty}\frac{n^2-5n+\frac{2}{n}}{6-\frac{1}{n}}$$

$$=\lim_{n\to\infty}\frac{n^2\left(1-\frac{5}{n}+\frac{2}{n^3}\right)}{6-\frac{1}{n}}=\infty$$

∞

(2) $\lim\limits_{n\to\infty}\dfrac{5n^2+1}{(n+2)(n+4)}$

$$=\lim_{n\to\infty}\frac{5+\frac{1}{n^2}}{\left(1+\frac{2}{n}\right)\left(1+\frac{4}{n}\right)}=5$$

5

(3) $\lim\limits_{n\to\infty}\{\log_2(n+3)-\log_2(4n-1)\}$

$$=\lim_{n\to\infty}\log_2\frac{n+3}{4n-1}=\lim_{n\to\infty}\log_2\frac{1+\frac{3}{n}}{4-\frac{1}{n}}$$

$$=\log_2\frac{1}{4}=-2$$

-2

(4) $\lim\limits_{n\to\infty}\dfrac{\sqrt{2n-1}}{n+5}$

$$=\lim_{n\to\infty}\frac{\sqrt{\frac{2}{n}-\frac{1}{n^2}}}{1+\frac{5}{n}}=0$$

0

(5) $\lim\limits_{n\to\infty}\dfrac{1}{3n^3}\sum\limits_{k=1}^{n}k^2$

$$=\lim_{n\to\infty}\left\{\frac{1}{3n^3}\cdot\frac{1}{6}n(n+1)(2n+1)\right\}$$

$$=\lim_{n\to\infty}\frac{(n+1)(2n+1)}{18n^2}$$

$$=\lim_{n\to\infty}\frac{\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)}{18}=\frac{1}{9}$$

$\dfrac{1}{9}$

(6) $\lim\limits_{n\to\infty}(8+3n-n^2)$

$$=\lim_{n\to\infty}n^2\left(\frac{8}{n^2}+\frac{3}{n}-1\right)$$

$$=-\infty$$

$-\infty$

(7) $\lim\limits_{n\to\infty}(\sqrt{n^2+3n}-\sqrt{n^2-n})$

$$=\lim_{n\to\infty}\frac{n^2+3n-(n^2-n)}{\sqrt{n^2+3n}+\sqrt{n^2-n}}$$

$$=\lim_{n\to\infty}\frac{4n}{\sqrt{n^2+3n}+\sqrt{n^2-n}}$$

$$=\lim_{n\to\infty}\frac{4}{\sqrt{1+\frac{3}{n}}+\sqrt{1-\frac{1}{n}}}=2$$

2

(8) $\lim\limits_{n\to\infty}\dfrac{1}{\sqrt{n}(\sqrt{n+2}-\sqrt{n-1})}$

$$=\lim_{n\to\infty}\frac{\sqrt{n+2}+\sqrt{n-1}}{\sqrt{n}\{n+2-(n-1)\}}$$

$$=\lim_{n\to\infty}\frac{\sqrt{n+2}+\sqrt{n-1}}{3\sqrt{n}}$$

$$=\lim_{n\to\infty}\frac{1}{3}\left(\sqrt{1+\frac{2}{n}}+\sqrt{1-\frac{1}{n}}\right)=\frac{2}{3}$$

$\dfrac{2}{3}$

EXTRA Info.

Use the scientific calculator to confirm the limits of sequences.

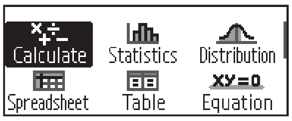
This section introduces a method to confirm the limits of infinite sequences by using the VARIABLE function of the scientific calculator.

After inputting the general term of the sequence and substituting a value with a large number of digits, we can find the same results by using regular calculations to derive the limits.

Ex. Use the scientific calculator to confirm the limit of $\lim_{n\to\infty}\frac{6n^2+3n-4}{2n^2-5n+1}=3$.

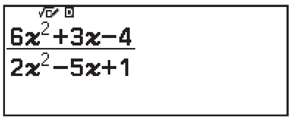
Confirm by substituting 10^{11} for x in the formula $\frac{6x^2+3x-4}{2x^2-5x+1}$.

Press (⌂), select [Calculate], press (OK)



Input $\frac{6x^2+3x-4}{2x^2-5x+1}$.

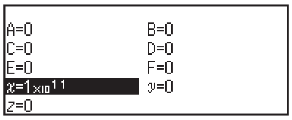
(⊟) ⑥ ⓧ (■²) ⊕ ③ ⓧ ⊖ ④ (∨) ② ⓧ (■²) ⊖ ⑤ ⓧ ⊕ ①



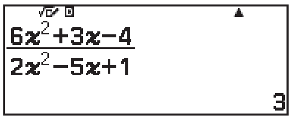
$\frac{6x^2+3x-4}{2x^2-5x+1}$

In the VARIABLE screen, input [x=10^{11} (*)].

(⇄x) (∨) (∨) (∨) ① ⓪ (■□) ① ① (EXE) (↶) (EXE)



A=0 B=0
C=0 D=0
E=0 F=0
x=1×10^11 y=0
z=0



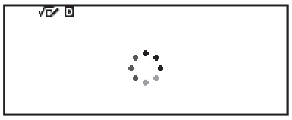
$\frac{6x^2+3x-4}{2x^2-5x+1}$

3

* If you set too large a value to be substituted, the scientific calculator may stop processing the operation.

We recommend using the values given as examples in this manual while doing the exercises.

(If processing stops, press the clear button (⏻) to recover.)

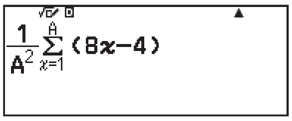


Ex. Use the scientific calculator to confirm the limit of $\lim_{n\to\infty}\frac{1}{n^2}\sum_{k=1}^{n}(8k-4)=4$.

Confirm by substituting 10^3 for A in the formula $\frac{1}{\mathrm{A}^2}\sum_{x=1}^{\mathrm{A}}(8x-4)$.

Input $\frac{1}{\mathrm{A}^2}\sum_{x=1}^{\mathrm{A}}(8x-4)$.

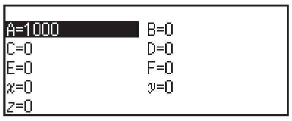
① (⊟) (⬆) ④ (■²) (>) (📖) (OK) (∨) (∨) (OK) ⑧ ⓧ ⊖ ④ (∨) ① (∧) (⬆) ④



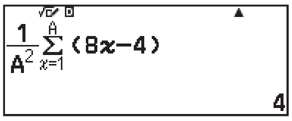
$\frac{1}{\mathrm{A}^2}\sum_{x=1}^{\mathrm{A}}(8x-4)$

In the VARIABLE screen, input [A=10^3].

(⇄x) ① ⓪ (■□) ③ (EXE) (↶) (EXE)



A=1000 B=0
C=0 D=0
E=0 F=0
x=0 y=0
z=0



$\frac{1}{\mathrm{A}^2}\sum_{x=1}^{\mathrm{A}}(8x-4)$

4

Squeeze theorem

TARGET To understand the squeeze theorem and how to use it.

STUDY GUIDE

Squeeze theorem

For the converging sequence $\{a_n\},\{b_n\}$, when we have $\lim_{n\to\infty} a_n=\alpha, \lim_{n\to\infty} b_n=\beta$, then the following holds.

Limit values and magnitude relationship of sequences

$$a_n \leq b_n \quad \Rightarrow \quad \alpha \leq \beta$$

Squeeze theorem

$a_n \leq c_n \leq b_n$ **and** $\alpha=\beta \quad \Rightarrow$ **sequence** $\{c_n\}$ **also converges,**

for $\lim_{n\to\infty} c_n=\alpha$

Even **when it is difficult to directly find the limit** of a sequence $\{c_n\}$, by using the **squeeze theorem**, we can find the limit value of the sequence $\{c_n\}$ from the limit values of the sequences $\{a_n\},\{b_n\}$.

EXERCISE

◆ Find the limits of the following when n is a natural number.

(1) $\lim_{n\to\infty} \frac{1}{n}\sin n\pi$

From $-1\leq \sin n\pi \leq 1$, by dividing each side by $n(n>0)$, we get $-\frac{1}{n} \leq \frac{1}{n}\sin n\pi \leq \frac{1}{n}$.

Now, because $\lim_{n\to\infty}\left(-\frac{1}{n}\right)=0, \lim_{n\to\infty}\frac{1}{n}=0$, the squeeze theorem gives us $\lim_{n\to\infty}\frac{1}{n}\sin n\pi=0$.

$$\boldsymbol{\lim_{n\to\infty}\frac{1}{n}\sin n\pi=0}$$

(2) $\lim_{n\to\infty}\frac{(-1)^n}{n}$

When n is even $(-1)^n=1$ or when n is odd, because $(-1)^n=-1$, we get $-1\leq(-1)^n\leq 1$.

By dividing each side by $n(n>0)$, we get $-\frac{1}{n}\leq\frac{(-1)^n}{n}\leq\frac{1}{n}$.

Now, because $\lim_{n\to\infty}\left(-\frac{1}{n}\right)=0, \lim_{n\to\infty}\frac{1}{n}=0$, the squeeze theorem gives us $\lim_{n\to\infty}\frac{(-1)^n}{n}=0$.

$$\boldsymbol{\lim_{n\to\infty}\frac{(-1)^n}{n}=0}$$

(3) $\lim\limits_{n\to\infty}\dfrac{3^n}{n!}$

When $n\geq3$, we get $\dfrac{3^n}{n!}=\dfrac{3\cdot3\cdots3}{1\cdot2\cdots n}\leq\dfrac{3\cdot3\cdot\cancel{3\cdots3}\cdot3}{1\cdot2\cdot\cancel{3\cdots3}\cdot n}=\dfrac{3^3}{2n}$. Therefore, we get $0<\dfrac{3^n}{n!}\leq\dfrac{3^3}{2n}$.

Now, because $\lim\limits_{n\to\infty}\dfrac{3^3}{2n}=0$, the squeeze theorem gives us $\lim\limits_{n\to\infty}\dfrac{3^n}{n!}=0$.

$$\lim_{n\to\infty}\frac{3^n}{n!}=0$$

PRACTICE

◆ Find the limits of the following when n is a natural number.

(1) $\lim\limits_{n\to\infty}\dfrac{1}{n}\cos\dfrac{3n\pi}{4}$

From $-1\leq\cos\dfrac{3n\pi}{4}\leq1$**, by dividing each side by** $n(n>0)$**, we get** $-\dfrac{1}{n}\leq\dfrac{1}{n}\cos\dfrac{3n\pi}{4}\leq\dfrac{1}{n}$.

Now, because $\lim\limits_{n\to\infty}\left(-\dfrac{1}{n}\right)=0$, $\lim\limits_{n\to\infty}\dfrac{1}{n}=0$**, from the squeeze theorem,**

we get $\lim\limits_{n\to\infty}\dfrac{1}{n}\cos\dfrac{3n\pi}{4}=0$.

$$\lim_{n\to\infty}\frac{1}{n}\cos\frac{3n\pi}{4}=0$$

(2) $\lim\limits_{n\to\infty}\dfrac{4^n}{n!}$

When $n\geq4$**, we get** $\dfrac{4^n}{n!}=\dfrac{4\cdot4\cdots4}{1\cdot2\cdots n}\leq\dfrac{4\cdot4\cdot4\cdot\cancel{4\cdots4}\cdot4}{1\cdot2\cdot3\cdot\cancel{4\cdots4}\cdot n}=\dfrac{4^4}{6n}$.

Therefore, we get $0<\dfrac{4^n}{n!}\leq\dfrac{4^4}{6n}$.

Now, because $\lim\limits_{n\to\infty}\dfrac{4^4}{6n}=0$**, the squeeze theorem gives us** $\lim\limits_{n\to\infty}\dfrac{4^n}{n!}=0$.

$$\lim_{n\to\infty}\frac{4^n}{n!}=0$$

(3) $\lim\limits_{n\to\infty}\dfrac{1}{n}\tan\dfrac{2n\pi}{3}$

Because n **is a natural number, for the value of** $\tan\dfrac{2n\pi}{3}$ **we get** 0 **when** $n=3m(m=1, 2, 3, \ldots)$;
we get $-\sqrt{3}$ **when** $n=3m+1(m=0, 1, 2, \ldots)$; **and we get** $\sqrt{3}$ **when** $n=3m+2(m=0, 1, 2, \ldots)$.

Therefore, from $-\sqrt{3}\leq\tan\dfrac{2n\pi}{3}\leq\sqrt{3}$**, by dividing each side by** $n(n>0)$**,**

we get $-\dfrac{\sqrt{3}}{n}\leq\dfrac{1}{n}\tan\dfrac{2n\pi}{3}\leq\dfrac{\sqrt{3}}{n}$.

Now, because $\lim\limits_{n\to\infty}\left(-\dfrac{\sqrt{3}}{n}\right)=0$, $\lim\limits_{n\to\infty}\dfrac{\sqrt{3}}{n}=0$**, from the squeeze theorem,**

we get $\lim\limits_{n\to\infty}\dfrac{1}{n}\tan\dfrac{2n\pi}{3}=0$.

$$\lim_{n\to\infty}\frac{1}{n}\tan\frac{2n\pi}{3}=0$$

Limits of infinite geometric sequences

TARGET To understand how to find the limits of infinite geometric sequences and the limits of sequences expressed as geometric sequences.

STUDY GUIDE

Limits of infinite geometric sequences

The sequence $a, ar, ar^2, \cdots, ar^{n-1}, \cdots$ is called a **geometric sequence** of a, the first term, with a common ratio of r.

The limits of an infinite geometric sequence $\{r^n\}$ with a first term of r and a common ratio of r are outlined below.

When $r>1$, then $\lim\limits_{n\to\infty} r^n = \infty$.

When $r=1$, then $\lim\limits_{n\to\infty} r^n = 1$.

When $|r|<1(-1<r<1)$, then $\lim\limits_{n\to\infty} r^n = 0$.

When $r\leqq-1$, then the sequence $\{r^n\}$ oscillates.

Ex. $2>1$, so $\lim\limits_{n\to\infty} 2^n = \infty$

Since $\left|-\dfrac{2}{3}\right| < 1$, then $\lim\limits_{n\to\infty}\left(-\dfrac{2}{3}\right)^n = 0$.

$-2<-1$, so $\lim\limits_{n\to\infty}(-2)^n$ does not exist. (The sequence $\{(-2)^n\}$ is oscillating.)

Conditions for convergence of geometric sequences

From the limits of infinite geometric sequences described above, the following holds.

The sequence $\{r^n\}$ converges $\Leftrightarrow$ $-1<r\leqq1$

The sequence $\{ar^{n-1}\}$ converges $\Leftrightarrow$ $a=0$ or $-1<r\leqq1$

EXERCISE

1 Find the limits of the following.

(1) $\lim\limits_{n\to\infty}\dfrac{2^{n+1}-3^{n+1}}{3^n-2^n}$

We can find the limits by dividing the denominator and numerator by 3^n, which has a large base, such that the denominator converges,

to give us $\lim\limits_{n\to\infty}\dfrac{2^{n+1}-3^{n+1}}{3^n-2^n} = \lim\limits_{n\to\infty}\dfrac{2\left(\dfrac{2}{3}\right)^n-3}{1-\left(\dfrac{2}{3}\right)^n} = -3$.

-3

(2) $\lim\limits_{n\to\infty}(3^n - 2^n)$

Factor out 3^n to find the limit, so we get $\lim\limits_{n\to\infty}(3^n - 2^n) = \lim\limits_{n\to\infty} 3^n\left\{1 - \left(\frac{2}{3}\right)^n\right\} = \infty$.

∞

(3) $\lim\limits_{n\to\infty}\dfrac{(-5)^n}{3^n - 1}$

Divide the denominator and numerator by 3^n, to get $\lim\limits_{n\to\infty}\dfrac{(-5)^n}{3^n-1} = \lim\limits_{n\to\infty}\dfrac{\left(-\frac{5}{3}\right)^n}{1-\left(\frac{1}{3}\right)^n}$.

Now, when $n\to\infty$, the sequence $\left\{\left(-\frac{5}{3}\right)^n\right\}$ oscillates, so there is no limit.

Has no limit

(4) $\lim\limits_{n\to\infty}\dfrac{\sqrt{4^n - 3^n}}{2^n - \sqrt{3^n}}$

Divide the denominator and numerator by 2^n to find the limit, so we get

$$\lim_{n\to\infty}\frac{\sqrt{4^n-3^n}}{2^n-\sqrt{3^n}} = \lim_{n\to\infty}\frac{\frac{\sqrt{4^n-3^n}}{2^n}}{1-\frac{\sqrt{3^n}}{2^n}} = \lim_{n\to\infty}\frac{\sqrt{1-\left(\frac{3}{4}\right)^n}}{1-\sqrt{\left(\frac{3}{4}\right)^n}} = 1\,.$$

1

2 Find the ranges of values x, such that the following infinite geometric sequences converge.

(1) $\left\{\left(\frac{x}{3}\right)^n\right\}$

Since the common ratio is $\frac{x}{3}$, conditions for convergence are $-1<\frac{x}{3}\leq 1$, so we find the range of values for x is $-3<x\leq 3$.

$-3<x\leq 3$

(2) $\{x(x-4)^n\}$

Since the first term is $x(x-4)$ and the common ratio is $x-4$, the conditions for convergence are $x(x-4)=0$ or $-1<x-4\leq 1$.

From $x(x-4)=0$, we get $x=0, 4$ …(i)

From $-1<x-4\leq 1$, we get $3<x\leq 5$ …(ii)

From (i) and (ii), the range of values we find for x is $x=0$ or $3<x\leq 5$.

$x=0, 3<x\leq 5$

PRACTICE

$\boxed{1}$ Find the limits of the following.

(1) $\lim\limits_{n\to\infty}\dfrac{4^{n+1}-3^n}{2^{2n}+1}$

$$=\lim_{n\to\infty}\frac{4^{n+1}-3^n}{4^n+1}=\lim_{n\to\infty}\frac{4-\left(\frac{3}{4}\right)^n}{1+\left(\frac{1}{4}\right)^n}=4$$

4

(2) $\lim\limits_{n\to\infty}\dfrac{5^n-8^n}{2^{2n}}$

$$=\lim_{n\to\infty}\frac{8^n\left\{\left(\frac{5}{8}\right)^n-1\right\}}{4^n}=\lim_{n\to\infty}2^n\left\{\left(\frac{5}{8}\right)^n-1\right\}=-\infty$$

$-\infty$

(3) $\lim\limits_{n\to\infty}\dfrac{(\sqrt{5}-\sqrt{3})^n}{\sqrt{5^n}-\sqrt{3^n}}$

$$=\lim_{n\to\infty}\frac{\left(\frac{\sqrt{5}-\sqrt{3}}{\sqrt{5}}\right)^n}{1-\sqrt{\left(\frac{3}{5}\right)^n}}=\lim_{n\to\infty}\frac{\left(1-\sqrt{\frac{3}{5}}\right)^n}{1-\sqrt{\left(\frac{3}{5}\right)^n}}=0$$

0

(4) $\lim\limits_{n\to\infty}\dfrac{1-(-2)^n}{1+2^n}$

$$=\lim_{n\to\infty}\frac{\left(\frac{1}{2}\right)^n-(-1)^n}{\left(\frac{1}{2}\right)^n+1}$$

Now, when $n\to\infty$, the sequence $\{(-1)^n\}$ oscillates, so there is no limit.

Has no limit

$\boxed{2}$ Find the ranges of values x, such that the infinite geometric sequence $\{(x^2-x-1)^n\}$ converges.

Since the common ratio is x^2-x-1, the condition for convergence is $-1<x^2-x-1\leq 1$.

From $-1<x^2-x-1$, we get $x^2-x>0, x(x-1)>0$, so $x<0$ and $1<x$ …(i)

From $x^2-x-1\leq 1$, we get $x^2-x-2\leq 0$ or $(x+1)(x-2)\leq 0$, so $-1\leq x\leq 2$ …(ii)

From (i) and (ii), the range of values we find for x is $-1\leq x<0$ or $1<x\leq 2$.

$-1\leq x<0,\ 1<x\leq 2$

EXTRA Info.

Use the scientific calculator to confirm the limits of infinite geometric sequences.

Confirm the limits of infinite geometric sequences by using the VARIABLE function of the scientific calculator.

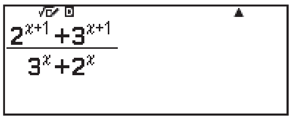
Ex. Use the scientific calculator to confirm the limit of $\lim_{n\to\infty}\dfrac{2^{n+1}+3^{n+1}}{3^n+2^n}=3$.

Confirm by substituting 10^2 for x in the formula $\dfrac{2^{x+1}+3^{x+1}}{3^x+2^x}$.

Press ⌂, select [Calculate], press OK

Input $\dfrac{2^{x+1}+3^{x+1}}{3^x+2^x}$.

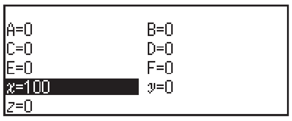
⊟ ② ■□ 𝑥 ⊕ ① > ⊕ ③ ■□ 𝑥 ⊕ ① ∨ ③ ■□ 𝑥 > ⊕ ② ■□ 𝑥



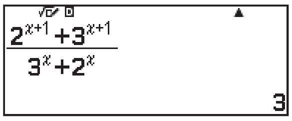
$\dfrac{2^{x+1}+3^{x+1}}{3^x+2^x}$

In the VARIABLE screen, input $[x=10^2]$.

⇄𝑥 ∨ ∨ ∨ ① ⓪ ■² EXE ↺ EXE



A=0 B=0
C=0 D=0
E=0 F=0
x=100 y=0
z=0



$\dfrac{2^{x+1}+3^{x+1}}{3^x+2^x}$

3

Recurrence formula and limits

TARGET To understand how to find the limits of sequences defined by the recurrence formula.

STUDY GUIDE

Recurrence formula and general terms

The general terms of a sequence defined by the recurrence formula between 2 adjacent terms is found mainly by the following methods.

(a) Use the characteristic equation to transform the recurrence formula and solve as a geometric sequence.

(b) Use a sequence of differences.

(c) Predict general terms and prove them by mathematical induction.

Recurrence formula and limits

The limit of a sequence defined by the recurrence formula is found by determining the limit of the general terms found by the above methods.

EXERCISE

◆ Find the limits of the sequences $\{a_n\}$ defined as follows. Provided that n is a natural number.

(1) $a_1=1, a_{n+1}=\frac{1}{2}a_n-2$

From the characteristic equation $\alpha=\frac{1}{2}\alpha-2$, we can get $\frac{1}{2}\alpha=-2, \alpha=-4$.

Therefore, we can transform the recurrence formula to $a_{n+1}+4=\frac{1}{2}(a_n+4)$.

The sequence $\{a_n+4\}$ has a first term of $(a_1+4=)5$ and is a geometric sequence with a common ratio of $\frac{1}{2}$, so we get $a_n+4=5\left(\frac{1}{2}\right)^{n-1}$.

Therefore, since $a_n=5\left(\frac{1}{2}\right)^{n-1}-4$, we get $\lim_{n\to\infty}a_n=\lim_{n\to\infty}\left\{5\left(\frac{1}{2}\right)^{n-1}-4\right\}=-4$.

$\underline{-4}$

(2) $a_1=1,\ 4^n a_{n+1}=4^n a_n+12$

From $4^n a_{n+1}=4^n a_n+12$, we get $a_{n+1}=a_n+3\left(\frac{1}{4}\right)^{n-1}, a_{n+1}-a_n=3\left(\frac{1}{4}\right)^{n-1}$.

Thus, $a_n=a_1+\sum_{k=1}^{n-1}(a_{k+1}-a_k)=1+\sum_{k=1}^{n-1}3\left(\frac{1}{4}\right)^{k-1}=1+\frac{3\left\{1-\left(\frac{1}{4}\right)^{n-1}\right\}}{1-\frac{1}{4}}=1+4\left\{1-\left(\frac{1}{4}\right)^{n-1}\right\}=5-\left(\frac{1}{4}\right)^{n-2}$ $(n\geq2)$.

Therefore, we get $\lim_{n\to\infty}a_n=\lim_{n\to\infty}\left\{5-\left(\frac{1}{4}\right)^{n-2}\right\}=5$.

$\underline{5}$

PRACTICE

1 Find the limits of the sequences $\{a_n\}$ defined as follows. Provided that n is a natural number.

(1) $a_1=5,\ a_{n+1}=-\frac{1}{3}a_n+4$

From the characteristic equation $\alpha=-\frac{1}{3}\alpha+4$, we can get $\frac{4}{3}\alpha=4, \alpha=3$.

Therefore, we can transform the recurrence formula to $a_{n+1}-3=-\frac{1}{3}(a_n-3)$.

The sequence $\{a_n-3\}$ has a first term of $(5-3=)2$ and is a geometric sequence with a common ratio of $-\frac{1}{3}$, so we get $a_n-3=2\left(-\frac{1}{3}\right)^{n-1}$.

Therefore, since $a_n=2\left(-\frac{1}{3}\right)^{n-1}+3$, we get $\lim_{n\to\infty}a_n=\lim_{n\to\infty}\left\{2\left(-\frac{1}{3}\right)^{n-1}+3\right\}=3$.

3

(2) $a_1=4,\ 3^n a_{n+1}=3^n a_n-2^n$

From $3^n a_{n+1}=3^n a_n-2^n$, we get $a_{n+1}=a_n-\left(\frac{2}{3}\right)^n$, $a_{n+1}-a_n=-\left(\frac{2}{3}\right)^n$.

$$a_n=a_1+\sum_{k=1}^{n-1}(a_{k+1}-a_k)=4-\sum_{k=1}^{n-1}\left(\frac{2}{3}\right)^k=4-\frac{\frac{2}{3}\left\{1-\left(\frac{2}{3}\right)^{n-1}\right\}}{1-\frac{2}{3}}=4-2\left\{1-\left(\frac{2}{3}\right)^{n-1}\right\}$$

$$=2+3\left(\frac{2}{3}\right)^n \quad (n\geq 2)$$

Therefore, we get $\lim_{n\to\infty}a_n=\lim_{n\to\infty}\left\{2+3\left(\frac{2}{3}\right)^n\right\}=2$.

2

2 Find the limit value $\lim_{n\to\infty}a_n$, given $b_n=\log_2 a_n$, of the sequence $\{a_n\}$ defined as $a_1=32,\ a_{n+1}=4\sqrt[3]{a_n}$.

Because $a_1>0$, then inductively from $a_{n+1}=4\sqrt[3]{a_n}$, for all natural numbers n we get $a_n>0$.

Then, we take the logarithm using as a base the 2 on both sides of the recurrence formula.

$$\log_2 a_{n+1}=\log_2 4\sqrt[3]{a_n}=\log_2 4+\log_2\sqrt[3]{a_n}=\log_2 a_n^{\frac{1}{3}}+\log_2 2^2=\frac{1}{3}\log_2 a_n+2$$

Now, given $b_n=\log_2 a_n$, we can get $b_{n+1}=\frac{1}{3}b_n+2$ ...(i)

From the characteristic equation $\alpha=\frac{1}{3}\alpha+2$, we can get $\frac{2}{3}\alpha=2, \alpha=3$.

From this, we can transform the recurrence formula (i) to $b_{n+1}-3=\frac{1}{3}(b_n-3)$.

Since the sequence $\{b_n-3\}$ is a geometric sequence with a first term of $(b_1-3=\log_2 32-3=5-3=)2$ and a common ratio of $\frac{1}{3}$,

then from $b_n-3=2\left(\frac{1}{3}\right)^{n-1}$, $b_n=2\left(\frac{1}{3}\right)^{n-1}+3$, we can get $\lim_{n\to\infty}b_n=\lim_{n\to\infty}\left\{2\left(\frac{1}{3}\right)^{n-1}+3\right\}=3$.

From $b_n=\log_2 a_n$, so $a_n=2^{b_n}$, which gives us $\lim_{n\to\infty}a_n=\lim_{n\to\infty}2^{b_n}=2^3=8$.

8

Convergence and divergence of infinite series

TARGET To understand the convergence and divergence of infinite series and how to find their sums.

STUDY GUIDE

Convergence and divergence of infinite series

Infinite series

An expression of an infinite sequence $\{a_n\}$, in which each term is connected by the "+" symbol $a_1 + a_2 + a_3 + \cdots + a_n + \cdots$ and $\sum_{n=1}^{\infty} a_n$, is called an **infinite series**, and a sum S_n of its terms from the first term to the n-th term is called a **partial sum**.

$$S_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \cdots + a_n$$

Sums of infinite series

The sequence $\{ S_n \}$ of the partial sum of an infinite series $\sum_{n=1}^{\infty} a_n$ converges, such that its limit value is S, giving us $S = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{k=1}^{n} a_k = \sum_{n=1}^{\infty} a_n$, and we call this S the **sum of the infinite series**.

Convergence and divergence of infinite series

$$(1) \quad \sum_{n=1}^{\infty} a_n \textbf{ converges} \Leftrightarrow \textbf{Sequence } \{S_n\} \textbf{ converges}$$

$$(2) \quad \sum_{n=1}^{\infty} a_n \textbf{ diverges} \Leftrightarrow \textbf{Sequence } \{S_n\} \textbf{ diverges}$$

Convergence and divergence of infinite series and limits of sequences

$$(3) \quad \sum_{n=1}^{\infty} a_n \textbf{ converges} \Rightarrow \lim_{n\to\infty} a_n = 0$$

$$(4) \quad \lim_{n\to\infty} a_n \neq 0 \Rightarrow \sum_{n=1}^{\infty} a_n \textbf{ diverges}$$

explanation

Proof of (3)

Given that $\sum_{n=1}^{\infty} a_n$ converges and its sum is S, then $\lim_{n\to\infty} a_n = \lim_{n\to\infty}(S_n - S_{n-1}) = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = S - S = 0$.

Converse of (3) and (4)

The converse of (3) and (4) **does not hold**.

EXERCISE

◆ Determine whether the following infinite series converge or diverge, and if they converge, find their sum.

(1) $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{n(n+1)}+\cdots$

By transformation $\frac{1}{n(n+1)}=\frac{1}{n}-\frac{1}{n+1}$, so

$$S_n=\left(1-\cancel{\frac{1}{2}}\right)+\left(\cancel{\frac{1}{2}}-\cancel{\frac{1}{3}}\right)+\left(\cancel{\frac{1}{3}}-\cancel{\frac{1}{4}}\right)+\cdots+\left(\cancel{\frac{1}{n}}-\frac{1}{n+1}\right)=1-\frac{1}{n+1}$$

Therefore, because $\lim_{n\to\infty}S_n=\lim_{n\to\infty}\left(1-\frac{1}{n+1}\right)=1$, this series converges and the sum is 1.

Converges and the sum is 1

(2) $\frac{1}{1+\sqrt{3}}+\frac{1}{\sqrt{2}+2}+\frac{1}{\sqrt{3}+\sqrt{5}}+\cdots+\frac{1}{\sqrt{n}+\sqrt{n+2}}+\cdots$

By transformation $\frac{1}{\sqrt{n}+\sqrt{n+2}}=\frac{\sqrt{n+2}-\sqrt{n}}{n+2-n}=\frac{1}{2}(\sqrt{n+2}-\sqrt{n})$, so

$$S_n=\frac{1}{2}\{(\cancel{\sqrt{3}}-1)+(\cancel{2}-\sqrt{2})+(\cancel{\sqrt{5}}-\cancel{\sqrt{3}})+\cdots+(\sqrt{n+2}-\cancel{\sqrt{n}})\}=\frac{1}{2}(\sqrt{n+2}+\sqrt{n+1}-\sqrt{2}-1)$$

Therefore, because $\lim_{n\to\infty}S_n=\lim_{n\to\infty}\frac{1}{2}(\sqrt{n+2}+\sqrt{n+1}-\sqrt{2}-1)=\infty$, this series diverges to positive infinity.

Diverges to positive infinity

(3) $\frac{2}{3}+\frac{2}{15}+\frac{2}{35}+\cdots+\frac{2}{4n^2-1}+\cdots$

By transformation $\frac{2}{4n^2-1}=\frac{2}{(2n-1)(2n+1)}=\frac{1}{2n-1}-\frac{1}{2n+1}$, so

$$S_n=\left(1-\cancel{\frac{1}{3}}\right)+\left(\cancel{\frac{1}{3}}-\cancel{\frac{1}{5}}\right)+\left(\cancel{\frac{1}{5}}-\cancel{\frac{1}{7}}\right)+\cdots+\left(\cancel{\frac{1}{2n-1}}-\frac{1}{2n+1}\right)=1-\frac{1}{2n+1}$$

Therefore, because $\lim_{n\to\infty}S_n=\lim_{n\to\infty}\left(1-\frac{1}{2n+1}\right)=1$, this series converges and the sum is 1.

Converges and the sum is 1

PRACTICE

◆ Determine whether the following infinite series converge or diverge, and if they converge, find their sum.

(1) $\frac{6}{2\cdot 4}+\frac{6}{3\cdot 5}+\frac{6}{4\cdot 6}+\cdots+\frac{6}{(n+1)(n+3)}+\cdots$

By transformation $\frac{6}{(n+1)(n+3)}=3\left(\frac{1}{n+1}-\frac{1}{n+3}\right)$**, so**

$$S_n=3\left\{\left(\frac{1}{2}-\cancel{\frac{1}{4}}\right)+\left(\frac{1}{3}-\cancel{\frac{1}{5}}\right)+\left(\cancel{\frac{1}{4}}-\cancel{\frac{1}{6}}\right)+\cdots+\left(\cancel{\frac{1}{n+1}}-\frac{1}{n+3}\right)\right\}$$
$$=3\left\{\frac{5}{6}-\left(\frac{1}{n+2}+\frac{1}{n+3}\right)\right\}$$

Therefore, because $\lim_{n\to\infty}S_n=\lim_{n\to\infty}3\left\{\frac{5}{6}-\left(\frac{1}{n+2}+\frac{1}{n+3}\right)\right\}=\frac{5}{2}$**, this series converges and the sum is** $\frac{5}{2}$**.**

Converges and the sum is $\frac{5}{2}$

(2) $\frac{1}{\sqrt{2}+2}+\frac{1}{2+\sqrt{6}}+\frac{1}{\sqrt{6}+2\sqrt{2}}+\cdots+\frac{1}{\sqrt{2n}+\sqrt{2n+2}}+\cdots$

By transformation $\frac{1}{\sqrt{2n}+\sqrt{2n+2}}=\frac{\sqrt{2n+2}-\sqrt{2n}}{2n+2-2n}=\frac{\sqrt{2}}{2}(\sqrt{n+1}-\sqrt{n})$**, so**

$$S_n=\frac{\sqrt{2}}{2}\{(\cancel{\sqrt{2}}-1)+(\cancel{\sqrt{3}}-\cancel{\sqrt{2}})+(\cancel{2}-\cancel{\sqrt{3}})+\cdots+(\sqrt{n+1}-\cancel{\sqrt{n}})\}$$
$$=\frac{\sqrt{2}}{2}(\sqrt{n+1}-1)$$

Therefore, because $\lim_{n\to\infty}S_n=\lim_{n\to\infty}\frac{\sqrt{2}}{2}(\sqrt{n+1}-1)=\infty$**, this series diverges to positive infinity.**

Diverges to positive infinity

(3) $\frac{1}{3}+\frac{1}{3+5}+\frac{1}{3+5+7}+\cdots+\frac{1}{3+5+7+\cdots+(2n+1)}+\cdots$

By transformation $\frac{1}{3+5+7+\cdots+(2n+1)}=\frac{1}{\frac{1}{2}n(2n+4)}=\frac{1}{2}\left(\frac{1}{n}-\frac{1}{n+2}\right)$**, so**

$$S_n=\frac{1}{2}\left\{\left(1-\cancel{\frac{1}{3}}\right)+\left(\frac{1}{2}-\cancel{\frac{1}{4}}\right)+\left(\cancel{\frac{1}{3}}-\cancel{\frac{1}{5}}\right)+\cdots+\left(\cancel{\frac{1}{n}}-\frac{1}{n+2}\right)\right\}$$
$$=\frac{1}{2}\left\{\frac{3}{2}-\left(\frac{1}{n+1}+\frac{1}{n+2}\right)\right\}$$

Therefore, because $\lim_{n\to\infty}S_n=\lim_{n\to\infty}\frac{1}{2}\left\{\frac{3}{2}-\left(\frac{1}{n+1}+\frac{1}{n+2}\right)\right\}=\frac{3}{4}$**, this series converges and the sum is** $\frac{3}{4}$**.**

Converges and the sum is $\frac{3}{4}$

EXTRA Info.

Use the scientific calculator to confirm the sums of infinite series.

Confirm the sums of infinite series by using the VARIABLE function of the scientific calculator.

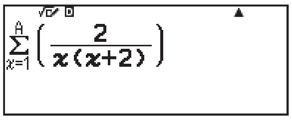
Ex. Use the scientific calculator to confirm $\frac{2}{1\cdot 3}+\frac{2}{2\cdot 4}+\frac{2}{3\cdot 5}+\cdots+\frac{2}{n(n+2)}+\cdots=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{2}{k(k+2)}=\frac{3}{2}$.

Confirm by substituting 10^3 for A in the formula $\sum_{x=1}^{\mathrm{A}}\frac{2}{x(x+2)}$.

Press ⌂, select [Calculate], press OK

Input $\sum_{x=1}^{\mathrm{A}}\frac{2}{x(x+2)}$.

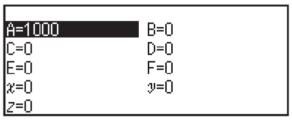
📖 OK ∨ ∨ OK ② ▤ 𝑥 (𝑥 ＋ ②) ∨ ① ∧ ⬆ ④



$\sum_{x=1}^{A}\left(\frac{2}{x(x+2)}\right)$

In the VARIABLE screen, input [A$=10^3$].

𝑥 ① ⓪ ■□ ③ EXE ↺ EXE



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E=0 F=0
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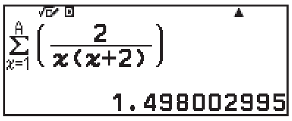

$\sum_{x=1}^{A}\left(\frac{2}{x(x+2)}\right)$

1.498002995

Ex. Use the scientific calculator to confirm

$$\frac{1}{1}+\frac{1}{1+2}+\frac{1}{1+2+3}+\cdots+\frac{1}{1+2+3+\cdots+n}+\cdots=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{1+2+3+\cdots+k}=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{2}{k(k+1)}=2.$$

Confirm by substituting 10^3 for A in the formula $\sum_{x=1}^{\mathrm{A}}\frac{2}{x(x+1)}$.

Input $\sum_{x=1}^{\mathrm{A}}\frac{2}{x(x+1)}$.

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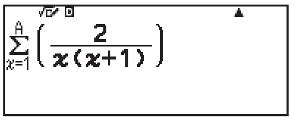
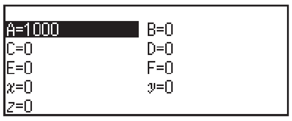


$\sum_{x=1}^{A}\left(\frac{2}{x(x+1)}\right)$

In the VARIABLE screen, input [A$=10^3$].

𝑥 ① ⓪ ■□ ③ EXE ↺ EXE



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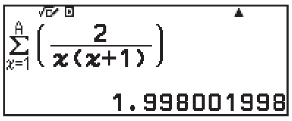


$\sum_{x=1}^{A}\left(\frac{2}{x(x+1)}\right)$

1.998001998

Convergence and divergence of infinite geometric series

TARGET To understand the convergence and divergence of infinite geometric series and their sums.

STUDY GUIDE

Convergence and divergence of infinite geometric series

From the convergence and divergence of the infinite geometric series $\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots + ar^{n-1} + \cdots$, we can derive the following.

If $a \neq 0$

When $|r| < 1$, the series converges, and the sum is given by

$$\frac{a}{1-r} \quad \left(\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}\right).$$

When $|r| \geq 1$, the series diverges.

If $a = 0$

The series converges and the sum is 0.

Infinite geometric series and recurring decimals

We can express recurring decimals as fractions by using the concept of infinite geometric series.

Ex. $$0.\dot{3}\dot{6} = 0.36 + 0.0036 + 0.000036 + \cdots = \frac{\frac{36}{100}}{1 - \frac{1}{100}} = \frac{36}{99} = \frac{4}{11}$$

EXERCISE

1 Determine whether the following infinite geometric series converge or diverge, and if they converge, find their sum.

(1) $\sqrt{2} - 2 + 2\sqrt{2} - 4 + \cdots$

The first term is not 0, the common ratio is $\frac{-2}{\sqrt{2}} = -\sqrt{2}$, and because $|-\sqrt{2}\,| \geq 1$, it diverges.

Diverges

(2) $1 + \frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \cdots$

The first term is 1, the common ratio is $\frac{2}{3}$, and because $\left|\frac{2}{3}\right| < 1$, it converges.

So the sum is $\frac{1}{1 - \frac{2}{3}} = 3$.

Converges and the sum is 3

2 Find the ranges of values of x, such that the following infinite geometric series converge.

(1) $x+x(2-x)+x(2-x)^2+\cdots$

Since the first term is x and the common ratio is $2-x$, the conditions for convergence are $x=0$ or $|2-x|<1$.

From $|2-x|<1$, we get $-1<2-x<1$ and $1<x<3$.

Therefore, the range we find is $x=0$ and $1<x<3$.

$$\boldsymbol{x=0,\ 1<x<3}$$

(2) $1+\log_2 x+(\log_2 x)^2+\cdots$

Since the first term is 1 and the common ratio is $\log_2 x$, the condition for convergence is $|\log_2 x|<1$.

Therefore, from $-1<\log_2 x<1$ we get $2^{-1}<x<2$. Also, this satisfies the anti-logarithm condition $x>0$.

Therefore, the range we find is $\frac{1}{2}<x<2$.

$$\boldsymbol{\frac{1}{2}<x<2}$$

PRACTICE

1 Determine whether the following infinite geometric series converge or diverge, and if they converge, find their sum.

(1) $\frac{1}{2}+\frac{1}{3}+\frac{2}{9}+\frac{4}{27}+\cdots$

The first term is $\frac{1}{2}$, the common ratio is $\frac{1}{3}\div\frac{1}{2}=\frac{2}{3}$, and because $\left|\frac{2}{3}\right|<1$, it converges.

So the sum is $\dfrac{\frac{1}{2}}{1-\frac{2}{3}}=\frac{3}{2}$.

Converges and the sum is $\frac{3}{2}$

(2) $12-4\sqrt{3}+4-\frac{4\sqrt{3}}{3}+\cdots$

The first term is 12, the common ratio is $\frac{-4\sqrt{3}}{12}=-\frac{\sqrt{3}}{3}$, and because $\left|-\frac{\sqrt{3}}{3}\right|<1$, it converges.

So the sum is $\dfrac{12}{1+\frac{\sqrt{3}}{3}}=18-6\sqrt{3}$.

Converges and the sum is $18-6\sqrt{3}$

(3) $-3+3-3+3-\cdots$

The first term is not 0, the common ratio is -1, so because $|-1|\geq 1$, it diverges.

Diverges

(4) $(2+\sqrt{3})-(3+\sqrt{3})+6-(18-6\sqrt{3})+\cdots$

The first term is not 0, the common ratio is $-\dfrac{3+\sqrt{3}}{2+\sqrt{3}}=-3+\sqrt{3}$, and because $|-3+\sqrt{3}|\geq 1$, it diverges.

Diverges

$\boxed{2}$ Find the ranges of values of x, such that the following infinite geometric series converge.

(1) $1+x(x+2)+x^2(x+2)^2+\cdots$

Since the first term is 1 and the common ratio is $x(x+2)$, the condition for convergence is $|x(x+2)|<1$, which gives us $-1<x(x+2)<1$.

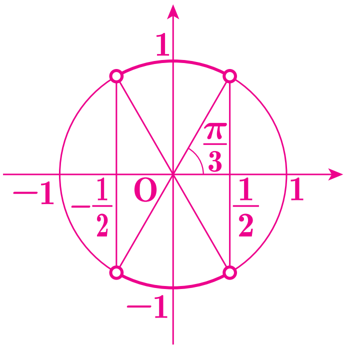
From $-1<x(x+2)$, we get $x^2+2x+1>0, (x+1)^2>0, x\neq -1$ ...(i)

From $x(x+2)<1$, we get $x^2+2x-1<0, -1-\sqrt{2}<x<-1+\sqrt{2}$ ...(ii)

From (i) and (ii), the range we find is $-1-\sqrt{2}<x<-1, -1<x<-1+\sqrt{2}$.

$$-1-\sqrt{2}<x<-1, -1<x<-1+\sqrt{2}$$

(2) $1-2\cos x+4\cos^2 x-8\cos^3 x+\cdots$ $(0\leq x<2\pi)$



The first term is 1, the common ratio is $-2\cos x$, so the condition for convergence is $|-2\cos x|<1$, so

$-1<-2\cos x<1$,

to give us $-\dfrac{1}{2}<\cos x<\dfrac{1}{2}$.

Because $0\leq x<2\pi$, the range we find is

$\dfrac{\pi}{3}<x<\dfrac{2}{3}\pi, \dfrac{4}{3}\pi<x<\dfrac{5}{3}\pi$.

$$\frac{\pi}{3}<x<\frac{2}{3}\pi, \frac{4}{3}\pi<x<\frac{5}{3}\pi$$

EXTRA Info.

Use the scientific calculator to confirm the sums of infinite geometric series.

Confirm the sums of infinite geometric series by using the $\mathrm{VARIABLE}$ function of the scientific calculator.

Ex. Use the scientific calculator to confirm the infinite geometric series $1+\frac{1}{2}+\frac{1}{4}+\cdots=\sum_{n=1}^{\infty}1\cdot\left(\frac{1}{2}\right)^{n-1}=2$.

Confirm by substituting 10^2 for A in the formula $\sum_{x=1}^{\mathrm{A}}1\cdot\left(\frac{1}{2}\right)^{x-1}$.

Press ⌂, select [Calculate], press OK

Input $\sum_{x=1}^{\mathrm{A}}1\cdot\left(\frac{1}{2}\right)^{x-1}$.

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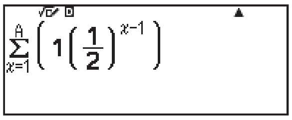
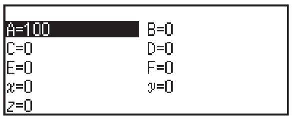


$\sum_{x=1}^{A}\left(1\left(\frac{1}{2}\right)^{x-1}\right)$

In the $\mathrm{VARIABLE}$ screen, input $[\mathrm{A}=10^2]$.

⇄x ① ⓪ ■² EXE ↩ EXE



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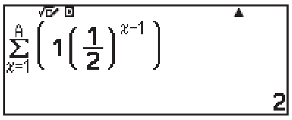


$\sum_{x=1}^{A}\left(1\left(\frac{1}{2}\right)^{x-1}\right)$

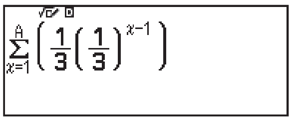
2

Ex. Use the scientific calculator to confirm the infinite geometric series $\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\cdots=\sum_{n=1}^{\infty}\frac{1}{3}\cdot\left(\frac{1}{3}\right)^{n-1}=\frac{1}{2}$.

Confirm by substituting 10^2 for A in the formula $\sum_{x=1}^{\mathrm{A}}\frac{1}{3}\cdot\left(\frac{1}{3}\right)^{x-1}$.

Input $\sum_{x=1}^{\mathrm{A}}\frac{1}{3}\cdot\left(\frac{1}{3}\right)^{x-1}$.

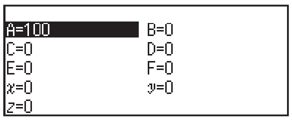
📖 OK ∨ ∨ OK ① ⊟ ③ > (① ⊟ ③ >) ■□ 𝑥 ⊖ ① ∨ ① ∧ ⇧ ④



$\sum_{x=1}^{A}\left(\frac{1}{3}\left(\frac{1}{3}\right)^{x-1}\right)$

In the $\mathrm{VARIABLE}$ screen, input $[\mathrm{A}=10^2]$.

⇄x ① ⓪ ■² EXE ↩ EXE



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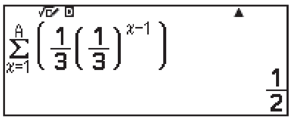


$\sum_{x=1}^{A}\left(\frac{1}{3}\left(\frac{1}{3}\right)^{x-1}\right)$

$\frac{1}{2}$

Properties of infinite series

TARGET To understand how to find sums by using the properties of infinite series.

STUDY GUIDE

Properties of sums of infinite series

Given that the infinite series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge such that $\sum_{n=1}^{\infty} a_n = S$ and $\sum_{n=1}^{\infty} b_n = T$. Then, for the sums of infinite series, we can derive the following from the properties of limits of sequences.

(1) $\sum_{n=1}^{\infty} ka_n = kS$ **(k is a constant)**

(2) $\sum_{n=1}^{\infty} (a_n \pm b_n) = S \pm T$ **(double sign same order)**

From (1) and (2): $\sum_{n=1}^{\infty} (ka_n \pm lb_n) = kS \pm lT$

(k and l are constant and double sign same order)

EXERCISE

◆ Find the sums of the following infinite series.

(1) $\sum_{n=1}^{\infty} \left(\frac{1}{2^{n-1}} - \frac{1}{5^{n-1}} \right)$

$$= \sum_{n=1}^{\infty} \left\{ \left(\frac{1}{2}\right)^{n-1} - \left(\frac{1}{5}\right)^{n-1} \right\}$$

Now, because $\left|\frac{1}{2}\right| < 1, \left|\frac{1}{5}\right| < 1$, both the infinite geometric series, which are their common ratios, converge.

Therefore, $\sum_{n=1}^{\infty} \left\{ \left(\frac{1}{2}\right)^{n-1} - \left(\frac{1}{5}\right)^{n-1} \right\} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} - \sum_{n=1}^{\infty} \left(\frac{1}{5}\right)^{n-1} = \frac{1}{1-\frac{1}{2}} - \frac{1}{1-\frac{1}{5}} = 2 - \frac{5}{4} = \frac{3}{4}$

$\frac{3}{4}$

(2) $\displaystyle\sum_{n=1}^{\infty}\frac{4^n-3^n}{5^n}$

$$=\sum_{n=1}^{\infty}\left\{\left(\frac{4}{5}\right)^n-\left(\frac{3}{5}\right)^n\right\}$$

Now, with these infinite geometric series, for $\displaystyle\sum_{n=1}^{\infty}\left(\frac{4}{5}\right)^n$, the first term is $\frac{4}{5}$ and the common ratio is $\frac{4}{5}$; and for $\displaystyle\sum_{n=1}^{\infty}\left(\frac{3}{5}\right)^n$, the first term is $\frac{3}{5}$ and the common ratio is $\frac{3}{5}$.

Both of their common ratios are $\left|\frac{4}{5}\right|<1$ and $\left|\frac{3}{5}\right|<1$, so both of these infinite geometric series converge.

Therefore, $\displaystyle\sum_{n=1}^{\infty}\left\{\left(\frac{4}{5}\right)^n-\left(\frac{3}{5}\right)^n\right\}=\sum_{n=1}^{\infty}\left(\frac{4}{5}\right)^n-\sum_{n=1}^{\infty}\left(\frac{3}{5}\right)^n=\frac{\frac{4}{5}}{1-\frac{4}{5}}-\frac{\frac{3}{5}}{1-\frac{3}{5}}=4-\frac{3}{2}=\frac{5}{2}$

$$\mathbf{\frac{5}{2}}$$

(3) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{3^n}\sin\frac{n\pi}{2}$

Consider 4 separate cases depending on the value given to n in $\sin\frac{n\pi}{2}$, given m is a natural number.

If $n=4m-3$ then $\sin\frac{n\pi}{2}=1$; if $n=4m-2$ or $4m$ then $\sin\frac{n\pi}{2}=0$; and if $n=4m-1$ then $\sin\frac{n\pi}{2}=-1$.

Therefore, $\displaystyle\sum_{n=1}^{\infty}\frac{1}{3^n}\sin\frac{n\pi}{2}=\frac{1}{3}\cdot1+\frac{1}{3^2}\cdot0+\frac{1}{3^3}\cdot(-1)+\frac{1}{3^4}\cdot0+\frac{1}{3^5}\cdot1+\frac{1}{3^6}\cdot0+\frac{1}{3^7}\cdot(-1)+\frac{1}{3^8}\cdot0+\cdots$

$$=\frac{1}{3}-\frac{1}{3^3}+\frac{1}{3^5}-\frac{1}{3^7}+\cdots$$

This is an infinite geometric series with a first term of $\frac{1}{3}$ and a common ratio of $-\left(\frac{1}{3}\right)^2$, and it converges because $\left|-\left(\frac{1}{3}\right)^2\right|<1$.

Therefore, the sum we find is $\displaystyle\sum_{n=1}^{\infty}\frac{1}{3^n}\sin\frac{n\pi}{2}=\frac{\frac{1}{3}}{1+\left(\frac{1}{3}\right)^2}=\frac{3}{9+1}=\frac{3}{10}$.

$$\mathbf{\frac{3}{10}}$$

PRACTICE

◆ Find the sums of the following infinite series.

(1) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{3^n}-\frac{1}{2^{n-1}}\right)$

$$=\sum_{n=1}^{\infty}\left\{\left(\frac{1}{3}\right)^n-\left(\frac{1}{2}\right)^{n-1}\right\}$$

Now, because $\left|\frac{1}{3}\right|<1$, $\left|\frac{1}{2}\right|<1$, both the infinite geometric series, which are their common ratios, converge.

Therefore, $\displaystyle\sum_{n=1}^{\infty}\left\{\left(\frac{1}{3}\right)^n-\left(\frac{1}{2}\right)^{n-1}\right\}=\sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^n-\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{n-1}=\frac{\frac{1}{3}}{1-\frac{1}{3}}-\frac{1}{1-\frac{1}{2}}=\frac{1}{2}-2=-\frac{3}{2}$

$$\mathbf{-\frac{3}{2}}$$

(2) $\sum_{n=1}^{\infty}\frac{2^{n-1}+4^{n+1}}{8^{n}}$

$$=\sum_{n=1}^{\infty}\left\{\frac{1}{8}\left(\frac{1}{4}\right)^{n-1}+2\left(\frac{1}{2}\right)^{n-1}\right\}$$

Now, because $\left|\frac{1}{4}\right|<1, \left|\frac{1}{2}\right|<1$, both the infinite geometric series, which are their common ratios, converge.

Therefore, $\sum_{n=1}^{\infty}\left\{\frac{1}{8}\left(\frac{1}{4}\right)^{n-1}+2\left(\frac{1}{2}\right)^{n-1}\right\}=\sum_{n=1}^{\infty}\frac{1}{8}\left(\frac{1}{4}\right)^{n-1}+\sum_{n=1}^{\infty}2\left(\frac{1}{2}\right)^{n-1}=\frac{1}{8}\cdot\frac{1}{1-\frac{1}{4}}+2\cdot\frac{1}{1-\frac{1}{2}}=\frac{1}{6}+4=\frac{25}{6}$.

$$\frac{25}{6}$$

(3) $\sum_{n=1}^{\infty}\frac{3^{n-1}+(-1)^{n}}{5^{n}}$

$$=\sum_{n=1}^{\infty}\left\{\frac{1}{5}\left(\frac{3}{5}\right)^{n-1}+\left(-\frac{1}{5}\right)^{n}\right\}$$

Now, because $\left|\frac{3}{5}\right|<1, \left|-\frac{1}{5}\right|<1$, both the infinite geometric series, which are their common ratios, converge.

Therefore, $\sum_{n=1}^{\infty}\left\{\frac{1}{5}\left(\frac{3}{5}\right)^{n-1}+\left(-\frac{1}{5}\right)^{n}\right\}=\sum_{n=1}^{\infty}\frac{1}{5}\left(\frac{3}{5}\right)^{n-1}+\sum_{n=1}^{\infty}\left(-\frac{1}{5}\right)^{n}=\frac{1}{5}\cdot\frac{1}{1-\frac{3}{5}}-\frac{1}{5}\cdot\frac{1}{1+\frac{1}{5}}=\frac{1}{2}-\frac{1}{6}=\frac{1}{3}$

$$\frac{1}{3}$$

(4) $\sum_{n=1}^{\infty}\frac{1}{2^{n}}\cos n\pi$

Consider 2 separate cases depending on the value given to n in $\cos n\pi$, given m is a natural number.

If $n=2m-1$ then $\cos n\pi=-1$; and if $n=2m$ then $\cos n\pi=1$.

Therefore, $\sum_{n=1}^{\infty}\frac{1}{2^{n}}\cos n\pi=\frac{1}{2}\cdot(-1)+\frac{1}{2^{2}}\cdot 1+\frac{1}{2^{3}}\cdot(-1)+\frac{1}{2^{4}}\cdot 1+\cdots$

$$=-\frac{1}{2}+\frac{1}{2^{2}}-\frac{1}{2^{3}}+\frac{1}{2^{4}}-\cdots$$

This is an infinite geometric series with a first term of $-\frac{1}{2}$ and a common ratio of $-\frac{1}{2}$, and it converges because $\left|-\frac{1}{2}\right|<1$.

Therefore, the sum we find is $\sum_{n=1}^{\infty}\frac{1}{2^{n}}\cos n\pi=\frac{-\frac{1}{2}}{1+\frac{1}{2}}=\frac{-1}{2+1}=-\frac{1}{3}$.

$$-\frac{1}{3}$$

EXTRA Info.

Use the scientific calculator to confirm the properties of infinite series.

Confirm the properties of sums of infinite series by using the VARIABLE function of the scientific calculator.

Ex. Use the scientific calculator to confirm the sum of the infinite series

$$\sum_{n=1}^{\infty}\left(\frac{2}{3^{n-1}}+\frac{3}{4^{n-1}}\right)=\sum_{n=1}^{\infty}\left\{2\cdot\left(\frac{1}{3}\right)^{n-1}+3\cdot\left(\frac{1}{4}\right)^{n-1}\right\}=7 .$$

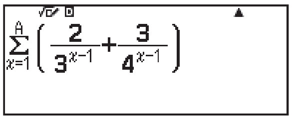
Confirm by substituting 10^2 for A in the formula $\sum_{x=1}^{\mathrm{A}}\left(\frac{2}{3^{x-1}}+\frac{3}{4^{x-1}}\right)$.

Press ⌂, select [Calculate], press OK

Input $\sum_{x=1}^{\mathrm{A}}\left(\frac{1}{3^{x-1}}+\frac{1}{4^{x-1}}\right)$.

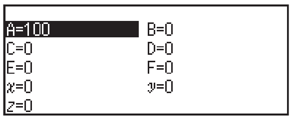
📖 OK ∨ ∨ OK

② ⊟ ③ ■□ 𝑥 ⊖ ① > > ⊕ ③ ⊟ ④ ■□ 𝑥 ⊖ ① ∨ ① ∧ ⬆ ④


$\sum_{x=1}^{A}\left(\frac{2}{3^{x-1}}+\frac{3}{4^{x-1}}\right)$

In the VARIABLE screen, input $[\mathrm{A}=10^2]$.

⇄𝑥 ① ⓪ ■² EXE ⮌ EXE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E=0 F=0
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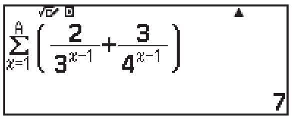
z=0

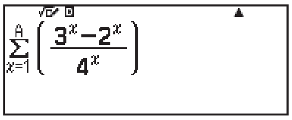

$\sum_{x=1}^{A}\left(\frac{2}{3^{x-1}}+\frac{3}{4^{x-1}}\right)$
7

Ex. Use the scientific calculator to confirm the sum of the infinite series $\sum_{n=1}^{\infty}\frac{3^n-2^n}{4^n}=\sum_{n=1}^{\infty}\left\{\frac{3}{4}\cdot\left(\frac{3}{4}\right)^{n-1}-\frac{1}{2}\cdot\left(\frac{1}{2}\right)^{n-1}\right\}=2$.

Confirm by substituting 10^2 for A in the formula $\sum_{x=1}^{\mathrm{A}}\frac{3^x-2^x}{4^x}$.

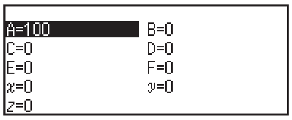
Input $\sum_{x=1}^{\mathrm{A}}\frac{3^x-2^x}{4^x}$.

📖 OK ∨ ∨ OK ⊟ ③ ■□ 𝑥 > ⊖ ② ■□ 𝑥 ∨ ④ ■□ 𝑥 ∨ ① ∧ ⬆ ④


$\sum_{x=1}^{A}\left(\frac{3^x-2^x}{4^x}\right)$

In the VARIABLE screen, input $[\mathrm{A}=10^2]$.

⇄𝑥 ① ⓪ ■² EXE ⮌ EXE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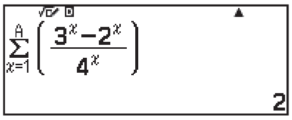
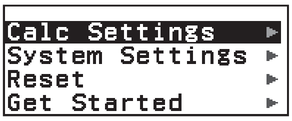

$\sum_{x=1}^{A}\left(\frac{3^x-2^x}{4^x}\right)$
2

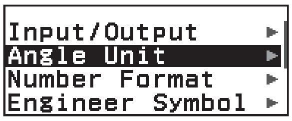
Ex. Use the scientific calculator to confirm the sum of the infinite series $\sum_{n=1}^{\infty}\frac{\cos n\pi}{3^n}=-\frac{1}{3}+\left(-\frac{1}{3}\right)^2+\cdots=-\frac{1}{4}$.

Confirm by substituting 10^2 for A in the formula $\sum_{x=1}^{\mathrm{A}}\frac{\cos x\pi}{3^x}$.

Set the angle display to Radian.

Press ☰, select $[\mathrm{Calc\ Settings}]$, press OK, select $[\mathrm{Angle\ Unit}]$, press OK, select $[\mathrm{Radian}]$, press OK, press AC


Calc Settings ▸
System Settings ▸
Reset ▸
Get Started ▸


Input/Output ▸
Angle Unit ▸
Number Format ▸
Engineer Symbol ▸


○Degree
◉Radian
○Gradian

Input $\sum_{x=1}^{\mathrm{A}}\frac{\cos x\pi}{3^x}$.

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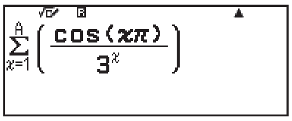
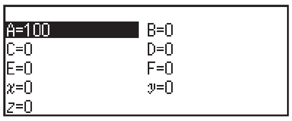

$\sum_{x=1}^{A}\left(\frac{\cos(x\pi)}{3^x}\right)$

In the $\mathrm{VARIABLE}$ screen, input $[\mathrm{A}=10^2]$.

⇄𝑥 1 0 ■² EXE ↶ EXE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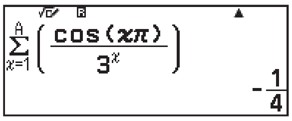

$\sum_{x=1}^{A}\left(\frac{\cos(x\pi)}{3^x}\right)$

$-\frac{1}{4}$

Limits of functions

TARGET To understand the limits of functions (converging and diverging) and how to find them.

STUDY GUIDE

Limits of functions

The limits of functions are outlined below.

$$\lim_{x\to a} f(x)=\left.\left\{\begin{array}{l}\alpha\text{(a finite definite value)}\cdots\cdots\text{Converges}\\ \left.\begin{array}{l}\infty\\ -\infty\end{array}\right\}\cdots\text{Diverges}\end{array}\right.\right\}\text{Has a limit}$$

$\lim_{x\to a} f(x)$ **does not exist** · **Has no limit**

Given α as a limit value, **it is not always the case that $f(a)=\alpha$.**

Properties of limits of functions

Given $\lim_{x\to a} f(x)=\alpha, \lim_{x\to a} g(x)=\beta$ (and that α and β are constants), then the following properties hold.

(1) $\lim_{x\to a} kf(x)=k\alpha$ (k **is a constant**)

(2) $\lim_{x\to a}\{f(x)\pm g(x)\}=\alpha\pm\beta$ (**double sign same order**)

Namely, (1) and (2) give us

$$\lim_{x\to a}\{kf(x)\pm lg(x)\}=k\alpha\pm l\beta$$

(k **and** l **are constant and double sign same order**)

(3) $\lim_{x\to a} f(x)g(x)=\alpha\beta$ $\qquad \lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\alpha}{\beta}$ $(\beta\neq 0)$

For "$x\to a$" in the limits above, it is acceptable as either "$x\to\infty$" or "$x\to-\infty$".

EXERCISE

◆ Find the limits of the following.

(1) $\lim_{x\to 2}(x^2+3x-5)$

$=4+6-5=5$

$\underline{5}$

(2) $\lim_{x\to 1}\frac{5x^2-3}{4x+1}$

$=\frac{5-3}{4+1}=\frac{2}{5}$

$\underline{\frac{2}{5}}$

(3) $\lim_{x\to 0}\left(-\frac{1}{x^2}\right)$

So for $\lim_{x\to 0}x^2=0$, we get $x^2>0$.

$\lim_{x\to 0}\left(-\frac{1}{x^2}\right)=-\infty$

$\underline{-\infty}$

(4) $\lim_{x\to -\infty}(2-x^3)$

So, we get $\lim_{x\to -\infty}x^3=-\infty$.

$\lim_{x\to -\infty}(2-x^3)=\infty$

$\underline{\infty}$

(5) $\lim_{x\to -3}2^x$

$=2^{-3}=\frac{1}{8}$

$\underline{\frac{1}{8}}$

(6) $\lim_{x\to 1}\log_5 x$

$=\log_5 1=0$

$\underline{0}$

PRACTICE

◆ Find the limits of the following.

(1) $\lim_{x\to 3}(x-1)(2x^2-7)$

$=2\cdot 11=22$

22

(2) $\lim_{x\to -2}\frac{x+8}{(x+1)(x^2-7)}$

$=\frac{6}{-1\cdot(-3)}=\frac{6}{3}=2$

2

(3) $\lim_{x\to \infty}\frac{1}{x^2-3}$

So, we get $\lim_{x\to\infty}x^2=\infty$.

$\lim_{x\to\infty}\frac{1}{x^2-3}=0$

0

(4) $\lim_{x\to -3}\left\{1+\frac{1}{(x+3)^2}\right\}$

So for $\lim_{x\to -3}(x+3)^2=0$, we get $(x+3)^2>0$.

$\lim_{x\to -3}\left\{1+\frac{1}{(x+3)^2}\right\}=\infty$

∞

(5) $\lim_{x\to 3}\log_2(x^2-x+10)$

$=\log_2(9-3+10)=\log_2 16=4$

4

(6) $\lim_{x\to 3}\frac{x}{\sqrt{x^2-5}}$

$=\frac{3}{\sqrt{4}}=\frac{3}{2}$

$\frac{3}{2}$

Limits of functions (indeterminate form)

TARGET To understand how to calculate the limits of functions that have indeterminate form.

STUDY GUIDE

How to find the limits of functions that have indeterminate form

When determining the limits of functions, we say that forms like $\frac{0}{0}$, $\frac{\infty}{\infty}$, and $\infty-\infty$ are **indeterminate forms**. For these forms, we transform them into forms from which we can find their limits and then calculate them, the same as the limits of sequences.

Main ways to transform indeterminate forms

(a) For $\frac{0}{0}$

Fractional functions → Factorize the denominator and numerator, then reduce.

Irrational function → Rationalize the numerator or the denominator, and then reduce.

(b) For $\frac{\infty}{\infty}$

Fractional functions → Divide the denominator and the numerator by the highest order term of the denominator.

(c) For $\infty-\infty$

Polynomial functions → Factor it out by using the highest order term.

Irrational function → Rationalize the numerator or the denominator.

EXERCISE

$\boxed{1}$ Find the limits of the following.

(1) $\lim_{x\to1}\frac{x^2+4x-5}{x^2-4x+3}$

$$=\lim_{x\to1}\frac{(x-1)(x+5)}{(x-1)(x-3)}=\lim_{x\to1}\frac{x+5}{x-3}=\frac{6}{-2}=-3$$

$\underline{\mathbf{-3}}$

(2) $\lim_{x\to3}\frac{3-\sqrt{x+6}}{x-3}$

$$=\lim_{x\to3}\frac{9-(x+6)}{(x-3)(3+\sqrt{x+6})}=\lim_{x\to3}\frac{-(x-3)}{(x-3)(3+\sqrt{x+6})}=\lim_{x\to3}\frac{-1}{3+\sqrt{x+6}}=-\frac{1}{3+3}=-\frac{1}{6}$$

$\underline{\mathbf{-\frac{1}{6}}}$

(3) $\lim_{x\to\infty}\frac{8x^2-x+3}{4x^2}$

$$=\lim_{x\to\infty}\frac{8-\frac{1}{x}+\frac{3}{x^2}}{4}=\frac{8}{4}=2$$

$\underline{\mathbf{2}}$

(4) $\lim_{x\to\infty}(3x^3-x^2)$

$$=\lim_{x\to\infty}x^3\left(3-\frac{1}{x}\right)=\infty$$

∞

(5) $\lim_{x\to\infty}\sqrt{x}(\sqrt{x+1}-\sqrt{x})$

$$=\lim_{x\to\infty}\frac{\sqrt{x}(x+1-x)}{\sqrt{x+1}+\sqrt{x}}=\lim_{x\to\infty}\frac{1}{\sqrt{1+\frac{1}{x}}+1}=\frac{1}{1+1}=\frac{1}{2}$$

$\frac{1}{2}$

(6) $\lim_{x\to-\infty}\frac{\sqrt{x^2-2}-1}{x}$

Let $x=-t$, then when $x\to-\infty$, we get $t\to\infty$.

$$\lim_{x\to-\infty}\frac{\sqrt{x^2-2}-1}{x}=\lim_{t\to\infty}\frac{\sqrt{t^2-2}-1}{-t}=\lim_{t\to\infty}\left(-\sqrt{1-\frac{2}{t^2}}+\frac{1}{t}\right)=-1$$

-1

$\boxed{2}$ When $\lim_{x\to-1}\frac{\sqrt{x+a}-b}{x+1}=\frac{1}{4}$ is true, find the values of constants a and b.

Because $\lim_{x\to-1}(x+1)=0$, then we must get $\lim_{x\to-1}(\sqrt{x+a}-b)=0$. Therefore, we get $\sqrt{a-1}-b=0, b=\sqrt{a-1}$. Substitute this into the left side, then rationalize the numerator to find the limits.

$$\lim_{x\to-1}\frac{\sqrt{x+a}-b}{x+1}=\lim_{x\to-1}\frac{\sqrt{x+a}-\sqrt{a-1}}{x+1}=\lim_{x\to-1}\frac{x+a-(a-1)}{(x+1)(\sqrt{x+a}+\sqrt{a-1})}=\lim_{x\to-1}\frac{1}{\sqrt{x+a}+\sqrt{a-1}}=\frac{1}{2\sqrt{a-1}}$$

Therefore, from $\frac{1}{2\sqrt{a-1}}=\frac{1}{4}$ we can get $\sqrt{a-1}=2, a=5, b=\sqrt{a-1}=\sqrt{4}=2$.

$a=5$, $b=2$

PRACTICE

$\boxed{1}$ Find the limits of the following.

(1) $\lim_{x\to-2}\frac{x^2-4x-12}{x^2+6x+8}$

$$=\lim_{x\to-2}\frac{(x+2)(x-6)}{(x+2)(x+4)}=\lim_{x\to-2}\frac{x-6}{x+4}=\frac{-8}{2}=-4$$

-4

(2) $\lim_{x\to0}\frac{3x}{\sqrt{2-x}-\sqrt{2+x}}$

$$=\lim_{x\to0}\frac{3x(\sqrt{2-x}+\sqrt{2+x})}{2-x-(2+x)}=\lim_{x\to0}\left\{-\frac{3}{2}(\sqrt{2-x}+\sqrt{2+x})\right\}=-3\sqrt{2}$$

$-3\sqrt{2}$

(3) $\lim_{x\to\infty} \dfrac{\sqrt{3x^2+x}-1}{x+2}$

$$= \lim_{x\to\infty} \frac{\sqrt{3+\frac{1}{x}}-\frac{1}{x}}{1+\frac{2}{x}} = \sqrt{3}$$

$\sqrt{3}$

(4) $\lim_{x\to-\infty} (x^2+x+2)$

Let $x=-t$, then when $x\to-\infty$, we get $t\to\infty$.

$$\lim_{x\to-\infty} (x^2+x+2) = \lim_{t\to\infty}(t^2-t+2) = \lim_{t\to\infty} t^2\left(1-\frac{1}{t}+\frac{2}{t^2}\right) = \infty$$

∞

$\boxed{2}$ When $\lim_{x\to 2} \dfrac{a\sqrt{x+1}-b}{x-2} = 1$ is true, find the values of constants a and b.

Because $\lim_{x\to 2}(x-2)=0$, then we must get $\lim_{x\to 2}(a\sqrt{x+1}-b)=0$.

Therefore, we get $\sqrt{3}a-b=0, b=\sqrt{3}a$.

Substitute this into the left side, then rationalize the numerator to find the limits.

$$\lim_{x\to 2} \frac{a\sqrt{x+1}-b}{x-2} = \lim_{x\to 2} \frac{a\sqrt{x+1}-\sqrt{3}a}{x-2} = \lim_{x\to 2} \frac{a(x+1-3)}{(x-2)(\sqrt{x+1}+\sqrt{3})}$$

$$= \lim_{x\to 2} \frac{a}{\sqrt{x+1}+\sqrt{3}} = \frac{a}{2\sqrt{3}}$$

Therefore, from $\dfrac{a}{2\sqrt{3}} = 1$ we get $a = 2\sqrt{3}, b = \sqrt{3}a = 6$.

$a = 2\sqrt{3}, b = 6$

EXTRA Info.

Use the scientific calculator to confirm the limit of functions (indeterminate form).

Confirm the limits of functions that have indeterminate forms by using the VARIABLE function of the scientific calculator.

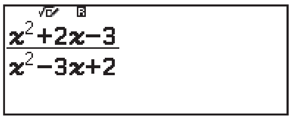
Ex. Use the scientific calculator to confirm the limit of $\lim_{x\to 1}\frac{x^2+2x-3}{x^2-3x+2}=\lim_{x\to 1}\frac{(x-1)(x+3)}{(x-1)(x-2)}=-4$.

Confirm by substituting $1+10^{-8}$ for x in the formula $\frac{x^2+2x-3}{x^2-3x+2}$.

Press ⌂, select [Calculate], press OK

Input $\frac{x^2+2x-3}{x^2-3x+2}$.

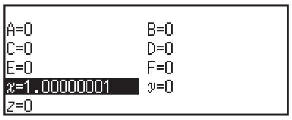
$\frac{\blacksquare}{\square}$ ⓧ $\blacksquare^2$ ⊕ ② ⓧ ⊖ ③ ∨ ⓧ $\blacksquare^2$ ⊖ ③ ⓧ ⊕ ②



$\frac{x^2+2x-3}{x^2-3x+2}$

In the VARIABLE screen, input $[x=1+10^{-8}]$.

⇄x ∨ ∨ ∨ ① ⊕ ① ⓪ $\blacksquare^{\square}$ ⊖ ⑧ EXE ↶ EXE



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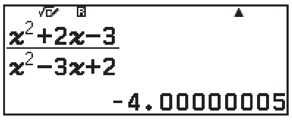


$\frac{x^2+2x-3}{x^2-3x+2}$

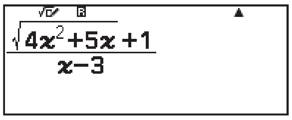
-4.00000005

Ex. Use the scientific calculator to confirm the limit of $\lim_{x\to\infty}\frac{\sqrt{4x^2+5x+1}}{x-3}=\lim_{x\to\infty}\frac{\sqrt{4+\frac{5}{x}+\frac{1}{x}}}{1-\frac{3}{x}}=2$.

Confirm by substituting 10^8 for x in the formula $\frac{\sqrt{4x^2+5x+1}}{x-3}$.

Input $\frac{\sqrt{4x^2+5x+1}}{x-3}$.

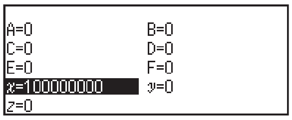
$\frac{\blacksquare}{\square}$ $\sqrt{\blacksquare}$ ④ ⓧ $\blacksquare^2$ ⊕ ⑤ ⓧ > ⊕ ① ∨ ⓧ ⊖ ③



$\frac{\sqrt{4x^2+5x+1}}{x-3}$

In the VARIABLE screen, input $[x=10^8]$.

⇄x ∨ ∨ ∨ ① ⓪ $\blacksquare^{\square}$ ⑧ EXE ↶ EXE



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z=0
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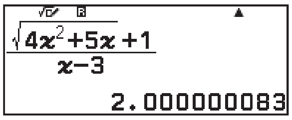


$\frac{\sqrt{4x^2+5x+1}}{x-3}$

2.000000083

One-sided limits

TARGET To understand one-sided limits and the existence of limits.

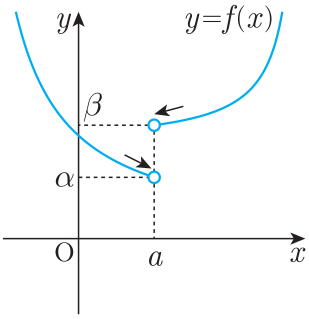
STUDY GUIDE

Right-sided limits and left-sided limits

Let x have a larger value than a, then as it approaches a, we express it as $x \to a+0$; let x have a smaller value than a, then as it approaches a, we express it as $x \to a-0$. Furthermore, the limits for $f(x)$ at this time are called the **right-sided (right) limit** and the **left-sided (left) limit**, which we express as $\lim\limits_{x\to a+0} f(x)$ and $\lim\limits_{x\to a-0} f(x)$. In particular, when a=0 then we omit a and write x→0+0 as x→+0, and write x→0−0 as x→−0.

Right-sided limit $\lim\limits_{x\to a+0} f(x) = \beta$

Left-sided limit $\lim\limits_{x\to a-0} f(x) = \alpha$



Existence of limits

For $\lim\limits_{x\to a} f(x) = \alpha$, the right-sided limit and the left-sided limit are both α, specifically, this means that $\lim\limits_{x\to a+0} f(x) = \lim\limits_{x\to a-0} f(x) = \alpha$. If $\lim\limits_{x\to a+0} f(x) \neq \lim\limits_{x\to a-0} f(x)$, then $\lim\limits_{x\to a} f(x)$ **does not exist**.

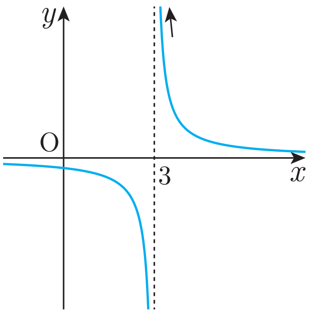
EXERCISE

$\boxed{1}$ Find the limits of the following.

(1) $\lim\limits_{x\to 3+0} \dfrac{1}{x-3}$

When x→3+0, because x−3>0, we get $\lim\limits_{x\to 3+0} \dfrac{1}{x-3} = \infty$.

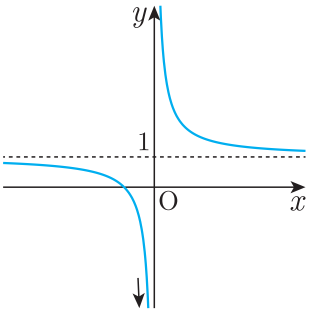
$\underline{\infty}$



(2) $\lim\limits_{x\to -0} \dfrac{x+1}{x}$

When x→−0, because x<0 and x+1>0, we get $\lim\limits_{x\to -0} \dfrac{x+1}{x} = -\infty$.

$\underline{-\infty}$



(3) $\lim\limits_{x\to3+0}\dfrac{x^2-9}{|x-3|}$

$$\lim_{x\to3+0}\frac{x^2-9}{|x-3|}=\lim_{x\to3+0}\frac{(x+3)(x-3)}{|x-3|}$$

When $x\to3+0$, because $x-3>0$, we get $|x-3|=x-3$.

Therefore, we get $\lim\limits_{x\to3+0}\dfrac{(x+3)(x-3)}{|x-3|}=\lim\limits_{x\to3+0}\dfrac{(x+3)(x-3)}{x-3}=\lim\limits_{x\to3+0}(x+3)=6$.

6

(4) $\lim\limits_{x\to4-0}\dfrac{|x-4|}{\sqrt{x}-2}$

$$\lim_{x\to4-0}\frac{|x-4|}{\sqrt{x}-2}=\lim_{x\to4-0}\frac{|x-4|\,(\sqrt{x}+2)}{x-4}$$

When $x\to4-0$, because $x-4<0$, we get $|x-4|=-(x-4)$.

Therefore, we get $\lim\limits_{x\to4-0}\dfrac{|x-4|\,(\sqrt{x}+2)}{x-4}=\lim\limits_{x\to4-0}\dfrac{-(x-4)(\sqrt{x}+2)}{x-4}=\lim\limits_{x\to4-0}\{-(\sqrt{x}+2)\}=-4$.

-4

2 Determine whether the limit $\lim\limits_{x\to0}\dfrac{x^2-3x}{|x|}$ exists.

When $x\to+0$, because $x>0$, we get $|x|=x$.

Therefore, we get $\lim\limits_{x\to+0}\dfrac{x^2-3x}{|x|}=\lim\limits_{x\to+0}\dfrac{x^2-3x}{x}=\lim\limits_{x\to+0}(x-3)=-3$.

When $x\to-0$, because $x<0$, we get $|x|=-x$.

Therefore, we get $\lim\limits_{x\to-0}\dfrac{x^2-3x}{|x|}=\lim\limits_{x\to-0}\dfrac{x^2-3x}{-x}=\lim\limits_{x\to-0}\{-(x-3)\}=3$.

Therefore, since $\lim\limits_{x\to+0}\dfrac{x^2-3x}{|x|}\neq\lim\limits_{x\to-0}\dfrac{x^2-3x}{|x|}$, there are no limits $\lim\limits_{x\to0}\dfrac{x^2-3x}{|x|}$.

There are no limits

PRACTICE

1 Find the limits of the following.

(1) $\lim\limits_{x\to1+0}\dfrac{2x}{x-1}$

When $x\to1+0$, because $x-1>0$ and $2x>0$, we get $\lim\limits_{x\to1+0}\dfrac{2x}{x-1}=\infty$.

∞

(2) $\lim\limits_{x\to-0}\dfrac{x-1}{x^2-4x}$

$$\lim_{x\to-0}\frac{x-1}{x^2-4x}=\lim_{x\to-0}\frac{x-1}{x(x-4)}$$

When $x\to-0$, because $x<0$, $x-4<0$, and $x-1<0$, we get $\lim\limits_{x\to-0}\dfrac{x-1}{x^2-4x}=-\infty$.

$-\infty$

(3) $\lim\limits_{x\to-2-0}\dfrac{x^2+3x+2}{|3x+6|}$

$$\lim_{x\to-2-0}\frac{x^2+3x+2}{|3x+6|}=\lim_{x\to-2-0}\frac{(x+1)(x+2)}{3|x+2|}$$

When $x\to-2-0$, because $x+2<0$, we get $|x+2|=-(x+2)$.

Therefore, we get $\lim\limits_{x\to-2-0}\dfrac{(x+1)(x+2)}{3|x+2|}=\lim\limits_{x\to-2-0}\dfrac{(x+1)(x+2)}{-3(x+2)}=\lim\limits_{x\to-2-0}\left(-\dfrac{x+1}{3}\right)=\dfrac{1}{3}$.

$\dfrac{1}{3}$

(4) $\lim\limits_{x\to1-0}\dfrac{\sqrt{x^2-2x+1}}{x^3-1}$

$$\lim_{x\to1-0}\frac{\sqrt{x^2-2x+1}}{x^3-1}=\lim_{x\to1-0}\frac{\sqrt{(x-1)^2}}{x^3-1}=\lim_{x\to1-0}\frac{|x-1|}{(x-1)(x^2+x+1)}$$

When $x\to1-0$, because $x-1<0$, we get $|x-1|=-(x-1)$.

Therefore, we get $\lim\limits_{x\to1-0}\dfrac{-(x-1)}{(x-1)(x^2+x+1)}=\lim\limits_{x\to1-0}\left(-\dfrac{1}{x^2+x+1}\right)=-\dfrac{1}{3}$.

$-\dfrac{1}{3}$

$\boxed{2}$ Determine whether the limit $\lim\limits_{x\to1}\dfrac{2x^2-2x}{|x-1|}$ exists.

$$\lim_{x\to1}\frac{2x^2-2x}{|x-1|}=\lim_{x\to1}\frac{2x(x-1)}{|x-1|}$$

When $x\to1+0$, because $x-1>0$, we get $|x-1|=x-1$.

Therefore, we get $\lim\limits_{x\to1+0}\dfrac{2x(x-1)}{|x-1|}=\lim\limits_{x\to1+0}\dfrac{2x(x-1)}{x-1}=\lim\limits_{x\to1+0}2x=2$.

When $x\to1-0$, because $x-1<0$, we get $|x-1|=-(x-1)$.

Therefore, we get $\lim\limits_{x\to1-0}\dfrac{2x(x-1)}{|x-1|}=\lim\limits_{x\to1-0}\dfrac{2x(x-1)}{-(x-1)}=\lim\limits_{x\to1-0}(-2x)=-2$.

Therefore, since $\lim\limits_{x\to1+0}\dfrac{2x^2-2x}{|x-1|}\neq\lim\limits_{x\to1-0}\dfrac{2x^2-2x}{|x-1|}$**, there are no limits** $\lim\limits_{x\to1}\dfrac{2x^2-2x}{|x-1|}$.

There are no limits

EXTRA Info.

Use the scientific calculator to confirm one-sided limits.

Confirm the one-sided limit by using the VARIABLE function of the scientific calculator.

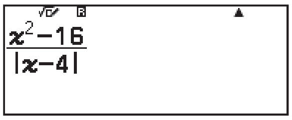
Ex. Use the scientific calculator to confirm the limit of $\lim_{x\to4+0}\frac{x^2-16}{|x-4|}=\lim_{x\to4+0}\frac{(x+4)(x-4)}{x-4}=8$.

Confirm by substituting $4+10^{-8}$ for x in the formula $\frac{x^2-16}{|x-4|}$.

Press ⌂, select [Calculate], press OK

Input $\frac{x^2-16}{|x-4|}$.

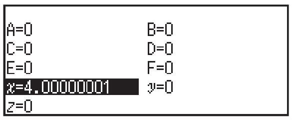
⊟ ⓧ $\blacksquare^2$ ⊖ ① ⑥ ∨ 📖 ∨ ∨ OK OK ⓧ ⊖ ④



$\frac{x^2-16}{|x-4|}$

In the VARIABLE screen, input [x= $4+10^{-8}$].

⇄x ∨ ∨ ∨ ④ ⊕ ① ⓪ $\blacksquare^{\square}$ ⊖ ⑧ EXE ↺ EXE



```
A=0          B=0
C=0          D=0
E=0          F=0
x=4.00000001 y=0
z=0
```

$\frac{x^2-16}{|x-4|}$

8.00000001

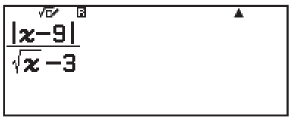
Ex. Use the scientific calculator to confirm the limit of

$$\lim_{x\to9-0}\frac{|x-9|}{\sqrt{x}-3}=\lim_{x\to9-0}\frac{|x-9|}{\sqrt{x}-3}\cdot\frac{\sqrt{x}+3}{\sqrt{x}+3}=\lim_{x\to9-0}\frac{-(x-9)(\sqrt{x}+3)}{x-9}=-6\,.$$

Confirm by substituting $9-10^{-8}$ for x in the formula $\frac{|x-9|}{\sqrt{x}-3}$.

Input $\frac{|x-9|}{\sqrt{x}-3}$.

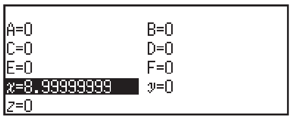
⊟ 📖 ∨ ∨ OK OK ⓧ ⊖ ⑨ ∨ √■ ⓧ 〉 ⊖ ③



$\frac{|x-9|}{\sqrt{x}-3}$

In the VARIABLE screen, input [x= $9-10^{-8}$].

⇄x ∨ ∨ ∨ ⑨ ⊖ ① ⓪ $\blacksquare^{\square}$ ⊖ ⑧ EXE ↺ EXE



```
A=0          B=0
C=0          D=0
E=0          F=0
x=8.99999999 y=0
z=0
```

$\frac{|x-9|}{\sqrt{x}-3}$

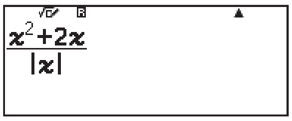
-5.999999998

Ex. Use the scientific calculator to confirm whether the limit of $\lim_{x\to 0}\frac{x^2+2x}{|x|}$ exists.

(1) Confirm the value of $\lim_{x\to +0}\frac{x^2+2x}{|x|}$ by substituting 10^{-10} for x in $\frac{x^2+2x}{|x|}$.

Input $\frac{x^2+2x}{|x|}$.

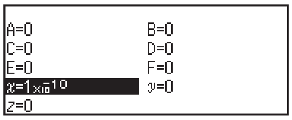
$\frac{\blacksquare}{\square}$ x $\blacksquare^2$ + 2 x ∨ 📖 ∨ ∨ OK OK x



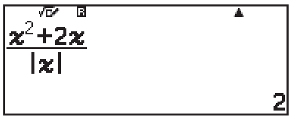
```
x²+2x
 |x|
```

In the VARIABLE screen, input $[x=10^{-10}]$.

⇄x ∨ ∨ ∨ 1 0 $\blacksquare^{\square}$ − 1 0 EXE ↩ EXE



```
A=0          B=0
C=0          D=0
E=0          F=0
x=1×10⁻¹⁰    y=0
z=0
```

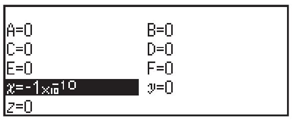


```
x²+2x
 |x|
                2
```

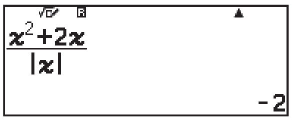
(2) Confirm the value of $\lim_{x\to -0}\frac{x^2+2x}{|x|}$ by substituting -10^{-10} for x in $\frac{x^2+2x}{|x|}$.

In the VARIABLE screen, input $[x=-10^{-10}]$.

⇄x ∨ ∨ ∨ − 1 0 $\blacksquare^{\square}$ − 1 0 EXE ↩ EXE



```
A=0          B=0
C=0          D=0
E=0          F=0
x=-1×10⁻¹⁰   y=0
z=0
```



```
x²+2x
 |x|
               -2
```

Since $\lim_{x\to +0}\frac{x^2+2x}{|x|}=2$ and $\lim_{x\to -0}\frac{x^2+2x}{|x|}=-2$, so the limit of $\lim_{x\to 0}\frac{x^2+2x}{|x|}$ does not exist.

Limits of various functions and Napier's number e

TARGET To understand the limits of trigonometric functions, exponential functions, and logarithmic functions.

STUDY GUIDE

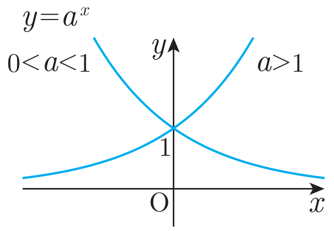
Limits of exponential functions and logarithmic functions

Limits of exponential functions

The limits of exponential functions are outlined below.

When $a>1$ $\lim_{x\to\infty} a^x = \infty,\ \lim_{x\to-\infty} a^x = 0$

When $0<a<1$ $\lim_{x\to\infty} a^x = 0,\ \lim_{x\to-\infty} a^x = \infty$

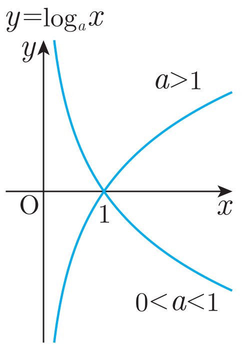


Limits of logarithmic functions

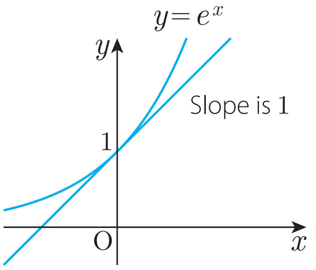
The limits of logarithmic functions are outlined below.

When $a>1$ $\lim_{x\to\infty} \log_a x = \infty,\ \lim_{x\to+0} \log_a x = -\infty$

When $0<a<1$ $\lim_{x\to\infty} \log_a x = -\infty,\ \lim_{x\to+0} \log_a x = \infty$



Definition of Napier's number e



The graph of the exponential function $y=a^x$, from $a^0=1$, is a curve that passes through the fixed point $(0, 1)$. The slope of the tangent at this point $(0, 1)$ changes corresponding to the value of the base a, and the value of a when the **slope** is **1**, as shown in the figure on the right, is defined as e, which is called **Napier's number**. Napier's number e is defined by one of the following 3 expressions, giving a value of about $2.7182...$ Furthermore, a logarithm with base e is called the **natural logarithm**, and is usually abbreviated as e.

$$e=\lim_{h\to0}(1+h)^{\frac{1}{h}} \qquad e=\lim_{x\to\pm\infty}\left(1+\frac{1}{x}\right)^x \qquad \lim_{x\to0}\frac{e^x-1}{x}=1$$

explanation

For $e=\lim_{h\to0}(1+h)^{\frac{1}{h}}$, let $h=\frac{1}{x}$, then when $h\to0$, since $x\to\pm\infty$, we get $e=\lim_{x\to\pm\infty}\left(1+\frac{1}{x}\right)^x$.

Also, if we let $h=e^x-1$, because $e^x=1+h$, then by taking the natural logarithms of both sides, we get $x=\log(1+h)$.

Now, when $x\to0$, since $h\to0$, we get $\lim_{x\to0}\frac{e^x-1}{x}=\lim_{h\to0}\frac{h}{\log(1+h)}=\lim_{h\to0}\frac{1}{\frac{1}{h}\log(1+h)}=\lim_{h\to0}\frac{1}{\log(1+h)^{\frac{1}{h}}}=\frac{1}{\log e}=1$.

EXTRA Info.

Use the scientific calculator to confirm Napier's number.

Confirm the value of Napier's number by using the various functions of the scientific calculator.

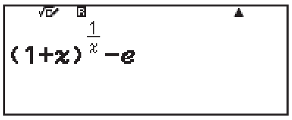
Ex. Use the scientific calculator to confirm Napier's number $e(=2.71828\cdots)$.

Substitute 10^{-20} for x in the formula $(1+x)^{\frac{1}{x}}-e$.

Press ⌂, select [Calculate], press OK

Input $(1+x)^{\frac{1}{x}}-e$.

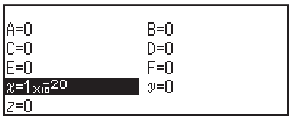
(1 + x) ■□ 1 ⊟ x > > − ⬆ 8



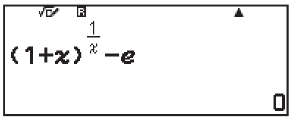
$(1+x)^{\frac{1}{x}}-e$

In the VARIABLE screen, input $[x=10^{-20}]$.

⇄x ∨ ∨ ∨ 1 0 ■□ − 2 0 EXE ↩ EXE



```
A=0          B=0
C=0          D=0
E=0          F=0
x=1×10^-20   y=0
z=0
```



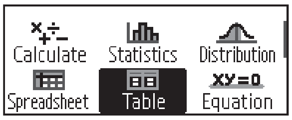
$(1+x)^{\frac{1}{x}}-e$

0

Since $\lim\limits_{x\to 0}\left\{(1+x)^{\frac{1}{x}}-e\right\}=0$, we can confirm that $\lim\limits_{x\to 0}(1+x)^{\frac{1}{x}}=e$.

We can also confirm by using Table as follows.

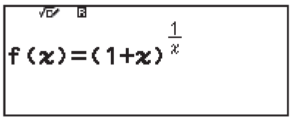
Press ⌂, select [Table], press OK, then clear the previous data by pressing ⏻



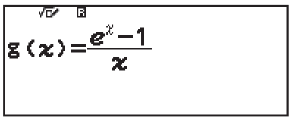
Press ⋯, select [Define f(x)/g(x)], press OK, select [Define f(x)], press OK

After inputting f(x)$=(1+x)^{\frac{1}{x}}$, press EXE

In the same way, input g(x)$=\dfrac{e^x-1}{x}$.

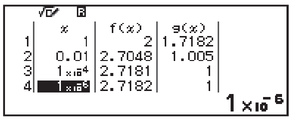


$f(x)=(1+x)^{\frac{1}{x}}$



$g(x)=\dfrac{e^x-1}{x}$

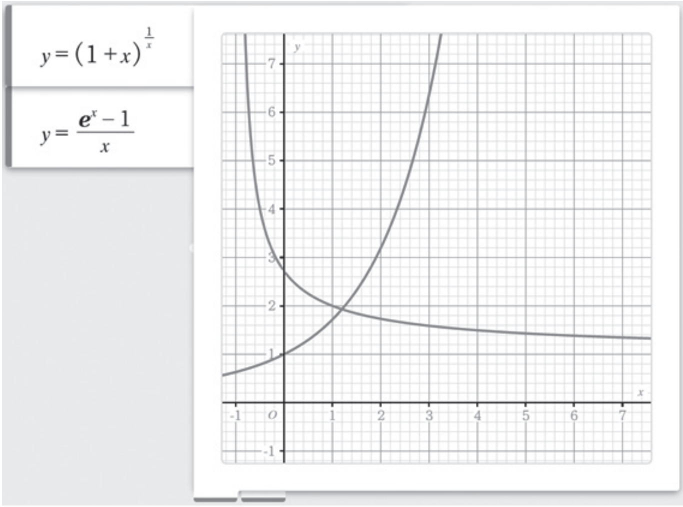
After inputting $[x1{:}1,\ x2{:}\,10^{(-2)},\ x3{:}\,10^{(-4)}$, and $x4{:}\,10^{(-6)}]$ in the table, press EXE



	x	f(x)	g(x)
1	1	2	1.7182
2	0.01	2.7048	1.005
3	1×10^{-4}	2.7181	1
4	1×10^{-6}	2.7182	1

1×10^{-6}

Press ⬆ x, scan the QR code to display a graph.



Limits and magnitude relationship of functions

Given $\lim_{x\to a} f(x)=\alpha, \lim_{x\to a} g(x)=\beta$ (and that α and β are constants), then the following holds.

(1) By approaching $x=a$, then always $f(x)\leq g(x) \Rightarrow \alpha\leq\beta$

(2) By approaching $x=a$, then always $f(x)\leq h(x)\leq g(x)$ and $\alpha=\beta \Rightarrow \lim_{x\to a} h(x)=\alpha$

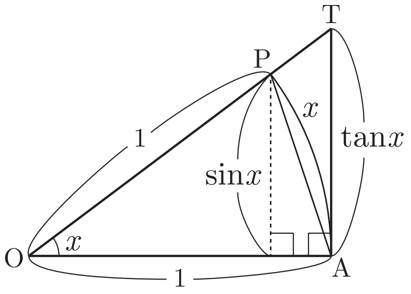
(Squeeze theorem)

Limits of trigonometric functions

$$\lim_{x\to 0}\frac{\sin x}{x}=1$$

Let radians be the units of the angles in trigonometric functions.

explanation



In the figure on the right, when $0<x<\frac{\pi}{2}$, then for the area of the shape,

we can derive that $\triangle$OAP< sector OAP<$\triangle$OAT.

Therefore, we get $\frac{1}{2}\sin x<\frac{1}{2}x<\frac{1}{2}\tan x, \sin x<x<\tan x$.

Then, by dividing each side by $\sin x (>0)$ and taking the inverse, we get $\cos x<\frac{\sin x}{x}<1$.

Now, since $\lim_{x\to +0}\cos x=1$, from the squeeze theorem, we get $\lim_{x\to +0}\frac{\sin x}{x}=1$.

Furthermore, when $x\to -0$ too, by letting $x=-t$, we can get $t\to +0$, which lets us show it in the same way.

EXTRA Info.

Generally, in the above figure, when $x\simeq 0$, then $\sin x\simeq x\simeq \tan x$.

Therefore, when $x\to 0$, for the ratio of these 2 numbers, the values $\frac{\sin x}{x}, \frac{\tan x}{x}, \frac{\sin x}{\tan x}$ and their inverse $\frac{x}{\sin x}, \frac{x}{\tan x}, \frac{\tan x}{\sin x}$, always have a limit value of 1.

EXTRA Info.

Use the scientific calculator to confirm the limit of trigonometric functions.

Confirm the limits of trigonometric functions by using the various functions of the scientific calculator.

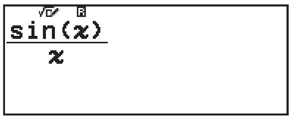
Ex. Use the scientific calculator to confirm the limit of the trigonometric function $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$.

Substitute 10^{-10} for x in $\frac{\sin x}{x}$.

Press ⌂, select $[\mathrm{Calculate}]$, press OK

Input $\frac{\sin x}{x}$.

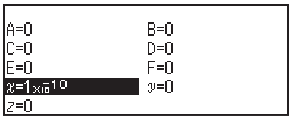
▤ sin x) ∨ x



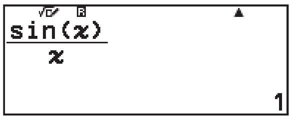
$\frac{\sin(x)}{x}$

In the $\mathrm{VARIABLE}$ screen, input $[x = 10^{-10}]$.

⇄x ∨ ∨ ∨ 1 0 ■□ − 1 0 EXE ↶ EXE



A=0 B=0
C=0 D=0
E=0 F=0
x=1×10⁻¹⁰ y=0
z=0



$\frac{\sin(x)}{x}$

1

We can confirm that $\lim_{x \to 0} \frac{\sin x}{x} = 1$.

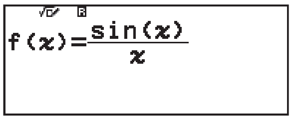
We can also confirm by using Table as follows.

Press ⌂, select $[\mathrm{Table}]$, press OK, then clear the previous data by pressing ⏻

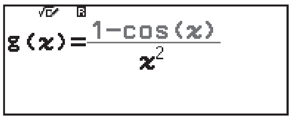
Press ⋯, select $[\mathrm{Define\ f}(x)/\mathrm{g}(x)]$, press OK, select $[\mathrm{Define\ f}(x)]$, press OK

After inputting $\mathrm{f}(x) = \frac{\sin x}{x}$, press EXE

In the same way, input $\mathrm{g}(x) = \frac{1 - \cos x}{x^2}$.

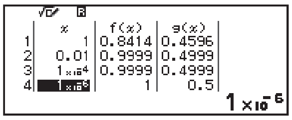


$\mathrm{f}(x) = \frac{\sin(x)}{x}$



$\mathrm{g}(x) = \frac{1-\cos(x)}{x^2}$

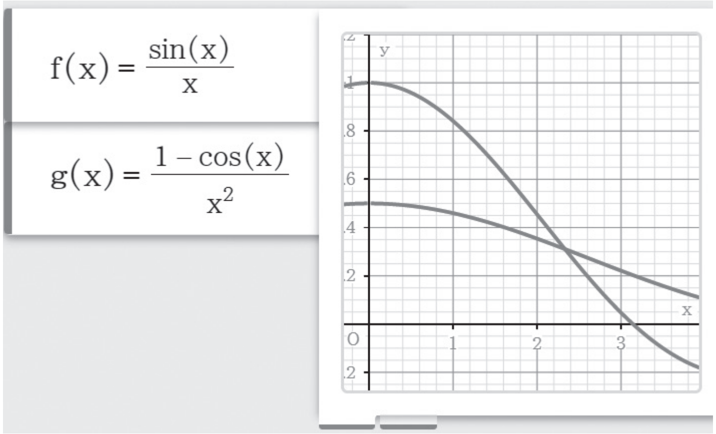
After inputting $[x1{:}1,\ x2{:}\,10^{(-2)}$, $x3{:}\,10^{(-4)}$, and $x4{:}\,10^{(-6)}]$ in the table, press EXE



	x	f(x)	g(x)
1	1	0.8414	0.4596
2	0.01	0.9999	0.4999
3	1×10^{-4}	0.9999	0.4999
4	1×10^{-6}	1	0.5

1×10^{-6}

Press ⇧ x, scan the QR code to display a graph.



$\mathrm{f}(\mathrm{x}) = \frac{\sin(\mathrm{x})}{\mathrm{x}}$

$\mathrm{g}(\mathrm{x}) = \frac{1-\cos(\mathrm{x})}{\mathrm{x}^2}$

The formula $\lim_{x \to 0} \frac{1 - \cos x}{x^2} = \frac{1}{2}$ is also important.

EXERCISE

◆ Find the limits of the following.

(1) $\lim\limits_{x\to\infty}\dfrac{1-2^x}{3^x-4^x}$

We can find the limits by dividing the denominator and numerator by 4^x, which has a large base, such that the denominator converges.

$$=\lim_{x\to\infty}\frac{\left(\frac{1}{4}\right)^x-\left(\frac{1}{2}\right)^x}{\left(\frac{3}{4}\right)^x-1}=0$$

$\mathbf{0}$

(2) $\lim\limits_{x\to-\infty}\left\{\left(\dfrac{1}{5}\right)^x-\left(\dfrac{1}{2}\right)^x\right\}$

$$=\lim_{x\to-\infty}\left(\frac{1}{5}\right)^x\left\{1-\left(\frac{5}{2}\right)^x\right\}=\infty$$

OTHER METHODS

Let $x=-t$, then when $x\to-\infty$, we get $t\to\infty$.

$$=\lim_{t\to\infty}\left\{\left(\frac{1}{5}\right)^{-t}-\left(\frac{1}{2}\right)^{-t}\right\}=\lim_{t\to\infty}(5^t-2^t)=\lim_{t\to\infty}5^t\left\{1-\left(\frac{2}{5}\right)\right\}^t=\infty$$

$\boldsymbol{\infty}$

(3) $\lim\limits_{x\to-\infty}\left(1-\dfrac{1}{x}\right)^{2x}$

Let $-\dfrac{1}{x}=t$, then when $x\to-\infty$, we get $t\to+0$.

$$=\lim_{t\to+0}(1+t)^{-\frac{2}{t}}=\lim_{t\to+0}\{(1+t)^{\frac{1}{t}}\}^{-2}=e^{-2}$$

$\boldsymbol{e^{-2}}$

(4) $\lim\limits_{x\to+0}\log_{\frac{1}{5}}\dfrac{1}{x}$

$$=\lim_{x\to+0}(-\log_{\frac{1}{5}}x)=-\infty$$

$\boldsymbol{-\infty}$

(5) $\lim\limits_{x\to0}\dfrac{\log(1+x)}{x}$

$$=\lim_{x\to0}\log(1+x)^{\frac{1}{x}}=\log e=1$$

$\mathbf{1}$

(6) $\lim\limits_{x\to0}\dfrac{\sin5x}{\sin4x}$

$$=\lim_{x\to0}\left(\frac{\sin5x}{5x}\cdot\frac{4x}{\sin4x}\cdot\frac{5}{4}\right)=\frac{5}{4}$$

$\boldsymbol{\frac{5}{4}}$

(7) $\lim\limits_{x\to0}\dfrac{1-\cos x}{x\sin x}$

$$=\lim_{x\to0}\frac{1-\cos^2x}{x\sin x(1+\cos x)}=\lim_{x\to0}\frac{\sin^2x}{x\sin x(1+\cos x)}$$

$$=\lim_{x\to0}\left(\frac{\sin x}{x}\cdot\frac{1}{1+\cos x}\right)=\frac{1}{2}$$

$\boldsymbol{\frac{1}{2}}$

(8) $\lim\limits_{x\to0}\dfrac{\sin(\sin x)}{\sin x}$

Let $\sin x=t$, then when $x\to0$, we get $t\to0$.

$$=\lim_{t\to0}\frac{\sin t}{t}=1$$

$\mathbf{1}$

(9) $\lim\limits_{x\to-\infty}x\sin\dfrac{3}{x}$

Let $\dfrac{3}{x}=t$, then when $x\to-\infty$, we get $t\to-0$.

$$=\lim_{t\to-0}\frac{3}{t}\sin t=3$$

$\mathbf{3}$

PRACTICE

◆ Find the limits of the following.

(1) $\lim\limits_{x\to-\infty}\dfrac{3^x-3^{-x}}{3^x+3^{-x}}$

$$=\lim_{x\to-\infty}\frac{(3^x)^2-1}{(3^x)^2+1}=-1$$

-1

OTHER METHODS

Let $-x=t$, then when $x\to-\infty$, we get $t\to\infty$.

$$=\lim_{t\to\infty}\frac{3^{-t}-3^t}{3^{-t}+3^t}=\lim_{t\to\infty}\frac{3^{-2t}-1}{3^{-2t}+1}=-1$$

(2) $\lim\limits_{x\to\infty}(4^x-7^x)$

$$=\lim_{x\to\infty}7^x\left\{\left(\frac{4}{7}\right)^x-1\right\}=-\infty$$

$-\infty$

(3) $\lim\limits_{x\to\infty}\left(1+\dfrac{3}{x}\right)^x$

Let $\dfrac{3}{x}=t$, then when $x\to\infty$, we get $t\to+0$.

$$=\lim_{t\to+0}(1+t)^{\frac{3}{t}}=\lim_{t\to+0}\{(1+t)^{\frac{1}{t}}\}^3=e^3$$

e^3

(4) $\lim\limits_{x\to\infty}\log_{\frac{1}{4}}\dfrac{1}{x}$

$$=\lim_{x\to\infty}(-\log_{\frac{1}{4}}x)=-(-\infty)=\infty$$

∞

(5) $\lim\limits_{x\to0}\dfrac{6x}{\log(1+x)}$

$$=\lim_{x\to0}\frac{6}{\frac{1}{x}\log(1+x)}=\lim_{x\to0}\frac{6}{\log(1+x)^{\frac{1}{x}}}$$

$$=\frac{6}{\log e}=6$$

6

(6) $\lim\limits_{x\to0}\dfrac{\sin 5x}{\tan x}$

$$=\lim_{x\to0}\left(\frac{\sin 5x}{5x}\cdot\frac{x}{\sin x}\cdot5\cos x\right)=5$$

5

(7) $\lim\limits_{x\to\frac{\pi}{2}}\dfrac{1-\sin x}{x\cos^2 x}$

$$=\lim_{x\to\frac{\pi}{2}}\frac{1-\sin x}{x(1-\sin^2 x)}=\lim_{x\to\frac{\pi}{2}}\frac{1}{x(1+\sin x)}$$

$$=\frac{1}{\frac{\pi}{2}\cdot2}=\frac{1}{\pi}$$

$\dfrac{1}{\pi}$

(8) $\lim\limits_{x\to\infty}x^2\sin\dfrac{4}{x^2}$

Let $\dfrac{4}{x^2}=t$, then when $x\to\infty$, we get $t\to0$.

$$=\lim_{t\to0}\frac{4\sin t}{t}=4$$

4

(9) $\lim\limits_{x\to0}\dfrac{3\sin^2 x}{x\log(1+x)}$

$$=\lim_{x\to0}\left\{\frac{3}{\frac{1}{x}\log(1+x)}\cdot\left(\frac{\sin x}{x}\right)^2\right\}=\lim_{x\to0}\left\{\frac{3}{\log(1+x)^{\frac{1}{x}}}\cdot\left(\frac{\sin x}{x}\right)^2\right\}=\frac{3}{\log e}=3$$

3

EXTRA Info.

Examine Napier's number by using the scientific calculator.

Examine the properties of Napier's number by using the VARIABLE function of the scientific calculator.

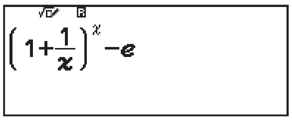
Ex. Use the scientific calculator to confirm the limit of $\lim_{x \to \pm\infty} \left(1+\frac{1}{x}\right)^x = e$.

Confirm by substituting 10^{20} and -10^{20} for x in the formula $\left(1+\frac{1}{x}\right)^x - e$.

Press ⌂, select [Calculate], press OK

Input $\left(1+\frac{1}{x}\right)^x - e$.

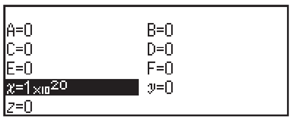
(1 + 1 ▤ x >) ▪ x > − ⇧ 8



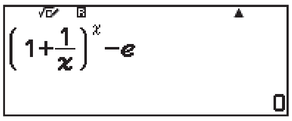
$\left(1+\frac{1}{x}\right)^x - e$

In the VARIABLE screen, input [$x = 10^{20}$].

⇄x ∨ ∨ ∨ 1 0 ▪ 2 0 EXE ↶ EXE



```
A=0            B=0
C=0            D=0
E=0            F=0
x=1×10^20      y=0
z=0
```

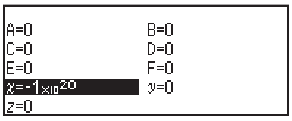


$\left(1+\frac{1}{x}\right)^x - e$

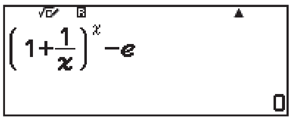
0

In the VARIABLE screen, input [$x = -10^{20}$].

⇄x ∨ ∨ ∨ − 1 0 ▪ 2 0 EXE ↶ EXE



```
A=0            B=0
C=0            D=0
E=0            F=0
x=-1×10^20     y=0
z=0
```



$\left(1+\frac{1}{x}\right)^x - e$

0

Since $\lim_{x \to \pm\infty} \left\{\left(1+\frac{1}{x}\right)^x - e\right\} = 0$, we can confirm that $\lim_{x \to \pm\infty} \left(1+\frac{1}{x}\right)^x = e$.

ADVANCED

PRACTICE

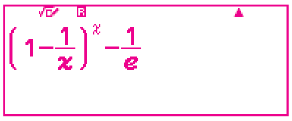
1 Use the scientific calculator to calculate $\lim\limits_{x\to\infty}\left(1-\frac{1}{x}\right)^x-\frac{1}{e}$, and then predict $\lim\limits_{x\to\infty}\left(1-\frac{1}{x}\right)^x$. Also, prove the results.

Substitute 10^{20} for x in the formula $\left(1-\frac{1}{x}\right)^x-\frac{1}{e}$.

Press ⌂, select [Calculate], press OK

Input $\left(1-\frac{1}{x}\right)^x-\frac{1}{e}$.

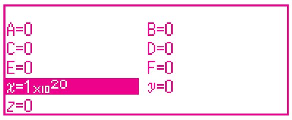
(1 − 1 ▤ x >) ■ x > − 1 ▤ ⇧ 8



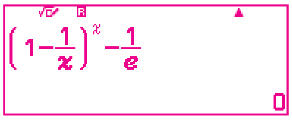
$\left(1-\frac{1}{x}\right)^x-\frac{1}{e}$

In the VARIABLE screen, input $[x=10^{20}]$.

⇄x ∨ ∨ ∨ 1 0 ■ 2 0 EXE ↺ EXE



A=0 B=0
C=0 D=0
E=0 F=0
x=1×10^20 y=0
z=0



$\left(1-\frac{1}{x}\right)^x-\frac{1}{e}$
0

From the above, we can infer $\lim\limits_{x\to\infty}\left(1-\frac{1}{x}\right)^x=\frac{1}{e}$.

[Proof]

From $\lim\limits_{x\to\infty}\left(1-\frac{1}{x}\right)^x$, then let $x=-X$, such that

$$\lim_{x\to\infty}\left(1-\frac{1}{x}\right)^x=\lim_{X\to-\infty}\left(1+\frac{1}{X}\right)^{-X}=\left\{\lim_{X\to-\infty}\left(1+\frac{1}{X}\right)^X\right\}^{-1}=e^{-1}=\frac{1}{e}$$

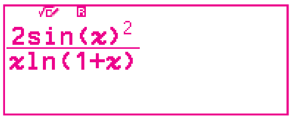
$\frac{1}{e}$

2 Use the scientific calculator to predict $\lim_{x \to 0} \frac{2\sin^2 x}{x\log(1+x)}$. Also, prove the results.

Substitute 10^{-10} for x in the formula $\frac{2\sin^2 x}{x\log(1+x)}$.

Input $\frac{2\sin^2 x}{x\log(1+x)}$.

② sin 𝑥) ■² ⊟ 𝑥 ⇧ log■□ ① + 𝑥)



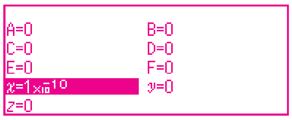
√□/ R

$\frac{2\sin(x)^2}{x\ln(1+x)}$

(The calculator screen shows $\sin^2 x \to \sin(x)^2, \log(1+x) \to \ln(1+x)$.)

In the VARIABLE screen, input $[x = 10^{-10}]$.

⇄𝑥 ∨ ∨ ∨ ① ⓪ ■□ − ① ⓪ EXE ⮌ EXE



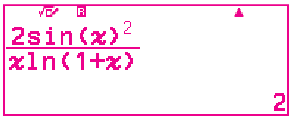
A=0 B=0

C=0 D=0

E=0 F=0

$x=1\times_{10}{}^{-10}$ $y=0$

$z=0$



√□/ R ▲

$\frac{2\sin(x)^2}{x\ln(1+x)}$

2

From the above, we can infer $\lim_{x \to 0} \frac{2\sin^2 x}{x\log(1+x)} = 2$.

[Proof]

$$\lim_{x\to 0}\frac{2\sin^2 x}{x\log(1+x)} = \lim_{x\to 0}\frac{2\sin^2 x}{x^2\cdot\frac{1}{x}\log(1+x)} = \lim_{x\to 0} 2\cdot\left(\frac{\sin x}{x}\right)^2\cdot\frac{1}{\log(1+x)^{\frac{1}{x}}}$$

$$= 2\cdot 1^2\cdot\frac{1}{\log e} = 2$$

2

Continuity of functions

TARGET To understand whether functions are continuous or discontinuous and such conditions.

STUDY GUIDE

Continuity of functions

Continuity and discontinuity of functions

When a function $f(x)$ meets the following conditions for every value a of x in its domain, then the function $f(x)$ is **continuous** when $x=a$.

$$\boxed{\lim_{x\to a+0} f(x) = \lim_{x\to a-0} f(x) = f(a)}$$

The above conditions summarize the following 3 conditions.

(1) $f(a)$ exists.

(2) The limit $\lim\limits_{x\to a} f(x)$ exists. (Exists as a finite value.)

(3) The values of $\lim\limits_{x\to a} f(x)$ and $f(a)$ are equal.

If any 1 of these 3 conditions is not satisfied, then the function $f(x)$ is **discontinuous** when $x=a$.

Continuous functions

When a function $f(x)$ is continuous for all the values in its domain, then that function $f(x)$ is a **continuous function**.

Therefore, trigonometric functions, exponential functions, and logarithmic functions are continuous functions.

Furthermore, if a function $f(x)$ is continuous for $g(x)$ when $x=a$, then the next function is also continuous when $x=a$.

$\boldsymbol{kf(x)+lg(x)}$ (k and l are constants), $\boldsymbol{f(x)g(x)}$, $\boldsymbol{\dfrac{f(x)}{g(x)}}$ $(g(x)\neq 0)$

Therefore, polynomial functions and rational functions are continuous functions.

EXERCISE

$\boxed{1}$ Determine if the following function $f(x)$ is continuous when $x=0$.

(1) $f(x)=-x\ (x<0),\ f(x)=x^2-1\ (x\geq 0)$

From $\lim\limits_{x\to +0} f(x) = \lim\limits_{x\to +0}(x^2-1) = -1$, $\lim\limits_{x\to -0} f(x) = \lim\limits_{x\to -0}(-x) = 0$, we get $\lim\limits_{x\to +0} f(x) \neq \lim\limits_{x\to -0} f(x)$.

Therefore, because $\lim\limits_{x\to 0} f(x)$ does not exist, it is not continuous when $x=0$.

Not continuous when $x=0$

(2) $f(x)=x^2-2x-1\ (x<0, 0<x),\ f(x)=1\ (x=0)$

From $\lim\limits_{x\to +0} f(x) = \lim\limits_{x\to +0}(x^2-2x-1) = -1$, $\lim\limits_{x\to -0} f(x) = \lim\limits_{x\to -0}(x^2-2x-1) = -1$, we get $\lim\limits_{x\to 0} f(x) = -1$.

Whereas, because $f(0)=1$, we get $\lim\limits_{x\to 0} f(x) \neq f(0)$.

Therefore, it is not continuous when $x=0$.

Not continuous when $x=0$

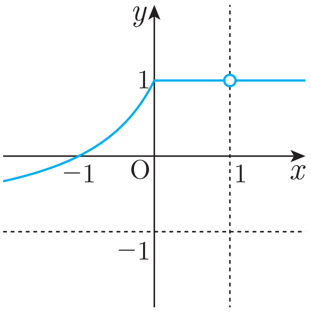
$\boxed{2}$ Determine the continuity of the function $f(x)=\dfrac{|x|-1}{x-1}$, and draw its graph.

When $x\geq0$ and $x\neq1$, because $|x|-1=x-1$, we get $f(x)=\dfrac{x-1}{x-1}=1$.

When $x=1$, $f(1)$ does not exist.

When $x<0$, because $|x|-1=-x-1$, we get $f(x)=\dfrac{-x-1}{x-1}=-1-\dfrac{2}{x-1}$.

Therefore, it is not continuous when $x=1$, and the graph is as shown on the right.



PRACTICE

$\boxed{1}$ Determine if the following function $f(x)$ is continuous when $x=1$.

(1) $f(x)=-x^2+3x\ (x\leq1),\ f(x)=x+2\ (x>1)$

From $\lim\limits_{x\to1+0}f(x)=\lim\limits_{x\to1+0}(x+2)=3,\ \lim\limits_{x\to1-0}f(x)=\lim\limits_{x\to1-0}(-x^2+3x)=2$, we get $\lim\limits_{x\to1+0}f(x)\neq\lim\limits_{x\to1-0}f(x)$.

Therefore, because $\lim\limits_{x\to1}f(x)$ does not exist, it is not continuous when $x=1$.

Not continuous when $x=1$

(2) $f(x)=x^2+2x-4\ (x<1),\ f(x)=4x-5\ (x\geq1)$

From $\lim\limits_{x\to1+0}f(x)=\lim\limits_{x\to1+0}(4x-5)=-1,\ \lim\limits_{x\to1-0}f(x)=\lim\limits_{x\to1-0}(x^2+2x-4)=-1$, we get $\lim\limits_{x\to1}f(x)=-1$.

Furthermore, since $f(1)=4-5=-1$, we get $\lim\limits_{x\to1}f(x)=f(1)$.

Therefore, it is continuous when $x=1$.

Continuous when $x=1$

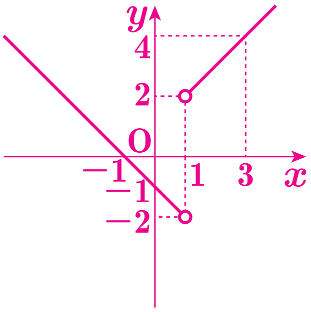
$\boxed{2}$ Determine the continuity of the function $f(x)=\dfrac{|x-1|(x+1)}{x-1}$, and draw its graph.

When $x>1$, because $|x-1|=x-1$, we get $f(x)=\dfrac{(x-1)(x+1)}{x-1}=x+1$.

When $x<1$, because $|x-1|=-(x-1)$, we get $f(x)=\dfrac{-(x-1)(x+1)}{x-1}=-x-1$.

When $x=1$, $f(1)$ does not exist.

Therefore, it is not continuous when $x=1$, and the graph is as shown on the right.



ADVANCED

EXTRA Info.

Examine the continuity of functions by using the scientific calculator.

Use the scientific calculator to work on examining problems related to the continuity of functions.

◆ Use the scientific calculator to show that the equation $\cos x = 2x$ has at least 1 real solution in the range of $0 < x < 1$. Also, estimate what value the solution is.

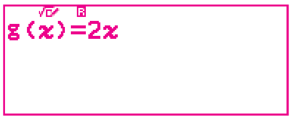
Press ⌂, select [Table], press OK, then clear the previous data by pressing ⏻

Press ⋯, select [Define $f(x)/g(x)$], press OK, select [Define $f(x)$], press OK

After inputting $f(x) = \cos x$, press EXE


```
f(x)=cos(x)
```

In the same way, input $g(x) = 2x$.

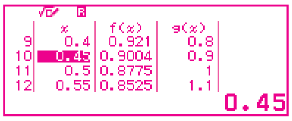

```
g(x)=2x
```

Press ⋯, select [Table Range], press OK

After inputting [Start:0, End:1, and Step:0.05], select [Execute], press EXE


```
Table Range
 Start:0
 End  :1
 Step :0.05
```

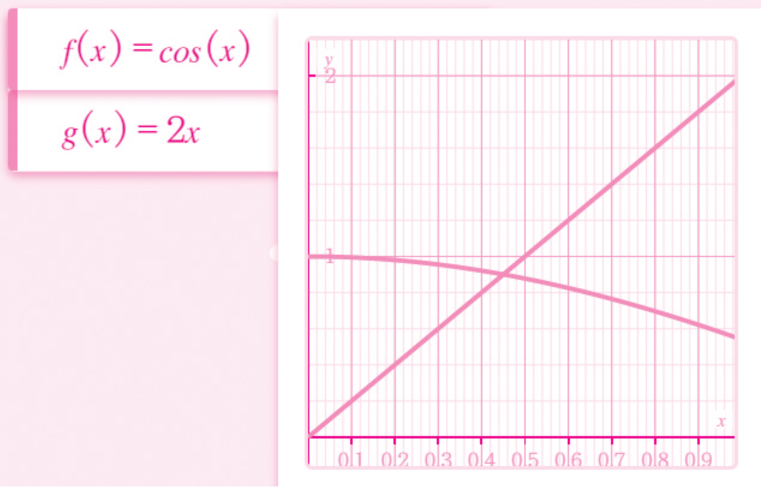
We can confirm that the magnitudes of $f(x)$ and $g(x)$ alternate in the range of $0.45 < x < 0.5$.



	x	$f(x)$	$g(x)$
9	0.4	0.921	0.8
10	0.45	0.9004	0.9
11	0.5	0.8775	1
12	0.55	0.8525	1.1

0.45

Press ⇧ ⓧ, scan the QR code to display a graph.



Intermediate-value theorem

TARGET To understand the intermediate-value theorem and how to use it.

STUDY GUIDE

Open intervals and closed intervals

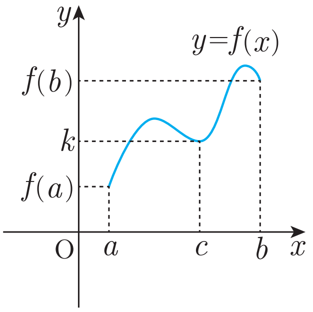
When considering a domain, such as a range of letters of possible values, we say that an interval that contains no end point is an **open interval**, and an interval that contains an endpoint is called a **closed interval**. Furthermore, we express an open interval $a<x<b$ as (a, b) and a closed interval $a\leq x\leq b$ as $[a, b]$.

Properties of continuous functions

The following theorems hold for continuous functions with closed intervals.

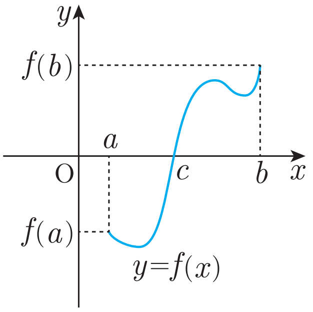
Intermediate-value theorem

If the function $f(x)$ is continuous over a closed interval $[a, b]$, and when $f(a)\neq f(b)$, then for any value k between $f(a)$ and $f(b)$, there is at least 1 real number c that satisfies $f(c) = k, a < c < b$.



Specifically, the equation $f(x)=k$ has at least 1 real root between a and b.

In particular, if $f(a)$ and $f(b)$ have different signs, then the equation $f(x)=0$ has at least 1 real root between a and b.



Maximum and minimum values existence theorem

A continuous function in a closed interval has maximum and minimum values in the closed interval.

EXERCISE

1 Show that the following equation has at least 1 real root within the range ( ).

(1) $x^3+x^2-x+1=0 \quad (-2<x<-1)$

Let $f(x)=x^3+x^2-x+1$, then $f(x)$ is continuous over the interval $[-2, -1]$.

Also, we get $f(-2)=-8+4+2+1=-1<0, f(-1)=-1+1+1+1=2>0$.

Therefore, since $f(-2)$ and $f(-1)$ have different signs, the intermediate-value theorem gives us at least 1 real root over the interval $(-2, -1)$.

(2) $2^x+2^{-x}=7 \quad (2<x<3)$

Let $f(x)=2^x+2^{-x}$, then $f(x)$ is continuous over the interval $[2, 3]$.

Also, from $f(2)=4+\dfrac{1}{4}=\dfrac{17}{4}, f(3)=8+\dfrac{1}{8}=\dfrac{65}{8}$, we get $f(2)<7<f(3)$.

Therefore, from the intermediate-value theorem, there is at least 1 real root over the interval $(2, 3)$.

$\boxed{2}$ Show that the equation $x^3 - x^2 - 2x + 1 = 0$ has 3 different real roots.

Given $f(x) = x^3 - x^2 - 2x + 1$, then $f(x)$ is a continuous function. Then, by substituting a whole number for x, we can find the sign of $f(x)$.

We get $f(-2) = -8 - 4 + 4 + 1 = -7 < 0$, $f(0) = 1 > 0$, $f(1) = 1 - 1 - 2 + 1 = -1 < 0$, and $f(2) = 8 - 4 - 4 + 1 = 1 > 0$.

Therefore, the equation $x^3 - x^2 - 2x + 1 = 0$ has 3 real roots because there are different real roots in the intervals $(-2, 0)$, $(0, 1)$, and $(1, 2)$ respectively.

PRACTICE

$\boxed{1}$ Show that the following equation has at least 1 real root within the range ( ).

(1) $x^4 - 4x^2 + 2 = 0 \quad (-2 < x < -1)$

Let $f(x) = x^4 - 4x^2 + 2$, then $f(x)$ is continuous over the interval $[-2, -1]$.
Also, we get $f(-2) = 16 - 16 + 2 = 2 > 0$, $f(-1) = 1 - 4 + 2 = -1 < 0$.
Therefore, since $f(-2)$ and $f(-1)$ have different signs, the intermediate-value theorem gives us at least 1 real root over the interval $(-2, -1)$.

(2) $\log x - (3 - x) = 0 \quad (2 < x < 3)$

Let $f(x) = \log x - (3 - x)$, then $f(x)$ is continuous over the interval $[2, 3]$.
Also, we get $f(2) = \log 2 - 1$, $f(3) = \log 3$.
Now, from $2 < e < 3$, since $\log 2 < 1 < \log 3$, we get $f(2) < 0 < f(3)$.
Therefore, since $f(2)$ and $f(3)$ have different signs, the intermediate-value theorem gives us at least 1 real root over the interval $(2, 3)$.

$\boxed{2}$ Show that the equation $x^3 - 3x^2 + 1 = 0$ has 3 different real roots.

Given $f(x) = x^3 - 3x^2 + 1$, then $f(x)$ is a continuous function. Then, by substituting a whole number for x, we can find the sign of $f(x)$.
This gives us $f(-1) = -1 - 3 + 1 = -3 < 0$, $f(0) = 1 > 0$,
$f(1) = 1 - 3 + 1 = -1 < 0$, $f(3) = 27 - 27 + 1 = 1 > 0$.
Therefore, the equation $x^3 - 3x^2 + 1 = 0$ has 3 different real roots because there are different real roots in the intervals $(-1, 0)$, $(0, 1)$, and $(1, 3)$.

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