15 Differential Calculus and Integral Calculus

CASIO
Essential Materials

Introduction

These teaching materials were created with the hope of conveying to many teachers and students the appeal of scientific calculators.

(1) Change awareness (emphasizing the thinking process) and boost efficiency in learning mathematics

- By reducing the time spent on manual calculations, we can have learning with a focus on the thinking process that is more efficient.
- This reduces the aversion to mathematics caused by complicated calculations, and allows students to experience the joy of thinking, which is the essence of mathematics.

(2) Diversification of learning materials and problem-solving methods

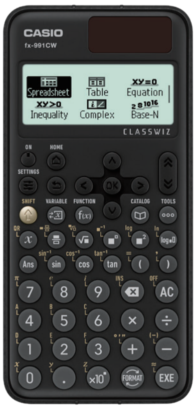
- Making it possible to do difficult calculations manually allows for diversity in learning materials and problem-solving methods.

(3) Promoting understanding of mathematical concepts

- By using the various functions of the scientific calculator in creative ways, students are able to deepen their understanding of mathematical concepts through calculations and discussions from different perspectives than before.
- This allows for exploratory learning through easy trial and error of questions.
- Listing and graphing of numerical values by means of tables allows students to discover laws and to understand visually.

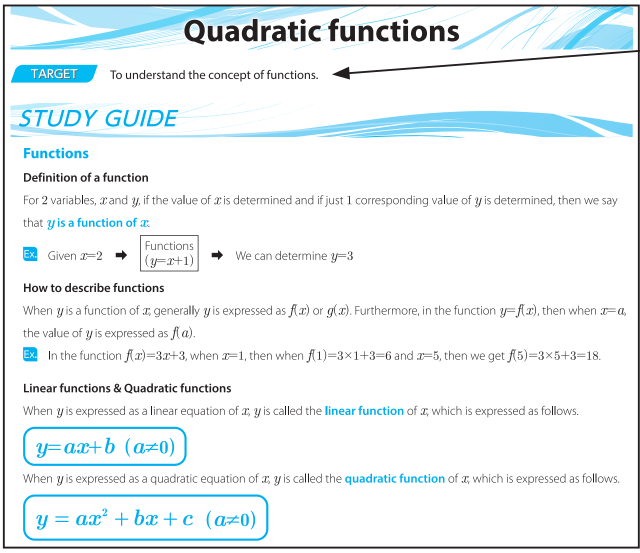
Features of this book

- As well as providing first-time scientific calculator users with opportunities to learn basic scientific calculator functions from the ground up, the book also has material to show people who already use scientific calculators the appeal of scientific calculators described above.
- You can also learn about functions and techniques that are not available on conventional Casio models or other brands of scientific calculators.
- This book covers many units of high school mathematics, allowing students to learn how to use the scientific calculator as they study each topic.
- This book can be used in a variety of situations, from classroom activities to independent study and homework by students.



Better Mathematics Learning with Scientific Calculator

Structure



Quadratic functions

TARGET To understand the concept of functions.

STUDY GUIDE

Functions

Definition of a function

For 2 variables, x and y, if the value of x is determined and if just 1 corresponding value of y is determined, then we say that **y is a function of x**.

Ex. Given $x=2$ ➡ Functions ($y=x+1$) ➡ We can determine $y=3$

How to describe functions

When y is a function of x, generally y is expressed as $f(x)$ or $g(x)$. Furthermore, in the function $y=f(x)$, then when $x=a$, the value of y is expressed as $f(a)$.

Ex. In the function $f(x)=3x+3$, when $x=1$, then when $f(1)=3\times1+3=6$ and $x=5$, then we get $f(5)=3\times5+3=18$.

Linear functions & Quadratic functions

When y is expressed as a linear equation of x, y is called the **linear function** of x, which is expressed as follows.

$$y=ax+b \quad (a\neq0)$$

When y is expressed as a quadratic equation of x, y is called the **quadratic function** of x, which is expressed as follows.

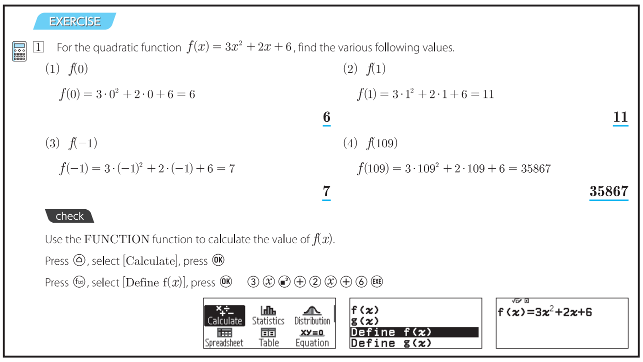
$$y=ax^2+bx+c \quad (a\neq0)$$

TARGET

Students can identify the objective to learn in each section.

STUDY GUIDE

Mathematical theorems and concepts are explained in detail. A scientific calculator is used to check and derive formulas according to the topic.



EXERCISE

1 For the quadratic function $f(x)=3x^2+2x+6$, find the various following values.

(1) $f(0)$

$f(0)=3\cdot0^2+2\cdot0+6=6$ — **6**

(2) $f(1)$

$f(1)=3\cdot1^2+2\cdot1+6=11$ — **11**

(3) $f(-1)$

$f(-1)=3\cdot(-1)^2+2\cdot(-1)+6=7$ — **7**

(4) $f(109)$

$f(109)=3\cdot109^2+2\cdot109+6=35867$ — **35867**

check

Use the FUNCTION function to calculate the value of $f(x)$.

Press ⓞ, select [Calculate], press OK

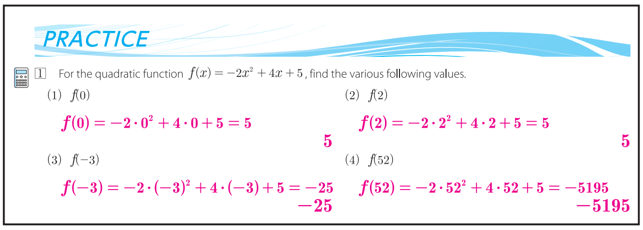
Press f(x), select [Define f(x)], press OK ③ x x² + ② x + ⑥ EXE

EXERCISE

Students learn basic examples based on the explanation in Study Guide.

check

Explains how to use the scientific calculator to solve problems and check answers.



PRACTICE

1 For the quadratic function $f(x)=-2x^2+4x+5$, find the various following values.

(1) $f(0)$

$f(0)=-2\cdot0^2+4\cdot0+5=5$ — **5**

(2) $f(2)$

$f(2)=-2\cdot2^2+4\cdot2+5=5$ — **5**

(3) $f(-3)$

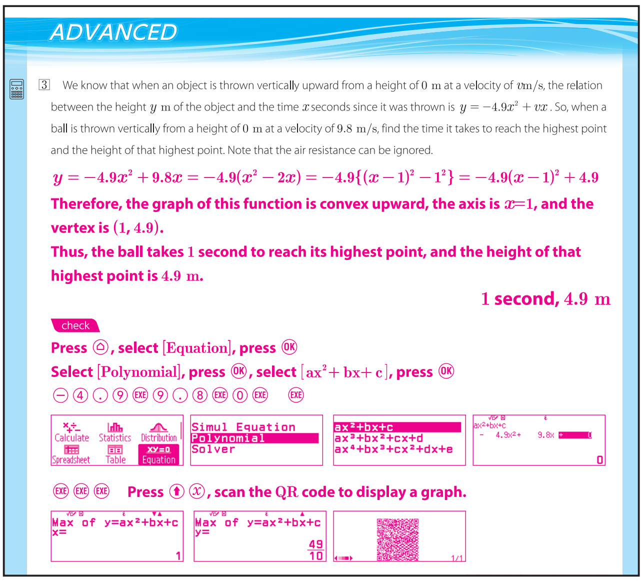
$f(-3)=-2\cdot(-3)^2+4\cdot(-3)+5=-25$ — **−25**

(4) $f(52)$

$f(52)=-2\cdot52^2+4\cdot52+5=-5195$ — **−5195**

PRACTICE

Students can do practice problems similar to those in EXERCISE. They can also practice using the scientific calculator as they learned to in Check.



ADVANCED

3 We know that when an object is thrown vertically upward from a height of 0 m at a velocity of v m/s, the relation between the height y m of the object and the time x seconds since it was thrown is $y=-4.9x^2+vx$. So, when a ball is thrown vertically from a height of 0 m at a velocity of 9.8 m/s, find the time it takes to reach the highest point and the height of that highest point. Note that the air resistance can be ignored.

$y=-4.9x^2+9.8x=-4.9(x^2-2x)=-4.9\{(x-1)^2-1^2\}=-4.9(x-1)^2+4.9$

Therefore, the graph of this function is convex upward, the axis is $x=1$, and the vertex is (1, 4.9).

Thus, the ball takes 1 second to reach its highest point, and the height of that highest point is 4.9 m.

1 second, 4.9 m

check

Press ⓞ, select [Equation], press OK

Select [Polynomial], press OK, select [ax²+bx+c], press OK

Press ⇧ x, scan the QR code to display a graph.

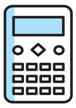
ADVANCED

Practical problems have been included in several topics. Solutions using a scientific calculator are also presented as necessary.

Other marks

Ex.	Simple examples on how to apply equations and theorems
explanation	Formulas and their supplementary explanations
proof	Proofs and checks of mathematical formulas
EXTRA Info.	Knowledge and information on formulas and other supplementary information in other units
OTHER METHODS	Alternative solutions and different verification methods for previously presented problems

Calculator mark



Where to use the scientific calculator

Colors of fonts in the teaching materials

- In STUDY GUIDE, important mathematical terms and formulas are printed in blue.
- In PRACTICE and ADVANCED the answers are printed in red.
  (Separate data is also available without the red parts, so it can be used for exercises.)

Applicable models

The applicable model is fx-991CW.
(Instructions on how to do input are for the fx-991CW, but in many cases similar calculations can be done on other models.)

Related Links

- Information and educational materials relevant to scientific calculators can be viewed on the following site.
  https://edu.casio.com
- The following video can be viewed to learn about the multiple functions of scientific calculators.
  https://www.youtube.com/playlist?list=PLRgxo9AwblZLurUCZnrbr4cLfZdqY6aZA

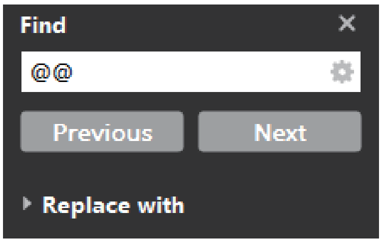
How to use PDF data

About types of data

- Data for all unit editions and data for each unit are available.
- For the above data, the PRACTICE and ADVANCED data without the answers in red is also available.

How to find where the scientific calculator is used

(1) Open a search window in the PDF Viewer.
(2) Type in "@@" as a search term.
(3) You can sequentially check where the calculator marks appear in the data.



How to search for a unit and section

(1) Search for units of data in all unit editions
- The data in all unit editions has a unit table of contents.
- Selecting a unit in the table of contents lets you jump to the first page of that unit.
- There is a bookmark on the first page of each unit, so you can jump from there also.

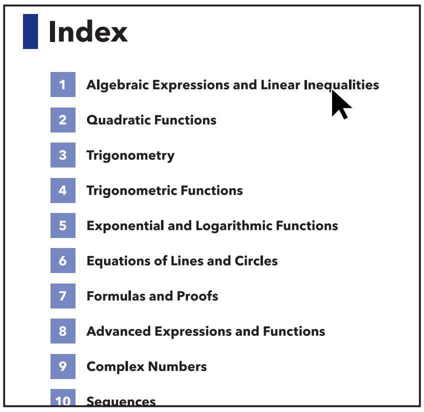
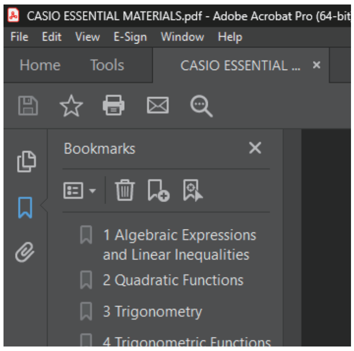


Table of contents of unit



Bookmark of unit

(2) Search for sections
- There are tables of contents for sections on the first page of units.
- Selecting a section in the table of contents takes you to the first page of that section.

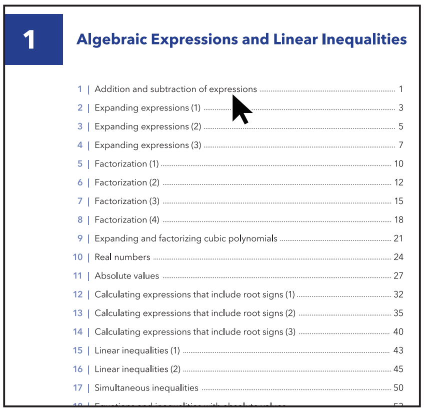


Table of contents of section

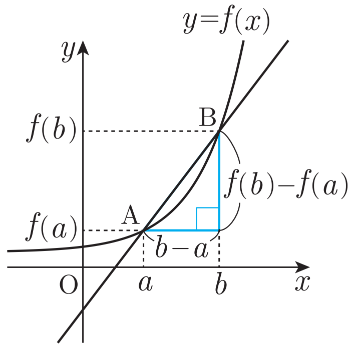
Average rate of change

TARGET To understand the average rate of change, which is the proportion of change on average, and how to find it.

STUDY GUIDE

Average rate of change

In the function $y=f(x)$, when the value of x changes from a to b, the ratio of the amount of change of x, which is $b-a$, to the amount of change of y, which is $f(b)-f(a)$, is called the **average rate of change** of the function $y=f(x)$ when x changes from a to b. The average rate of change is shown by the slope of the straight line AB in the figure on the right.



$$\textbf{(Average rate of change)} = \frac{f(b)-f(a)}{b-a}$$

EXERCISE

◆ For the function $y=2x^2-1$, find the following average rates of change.

(1) Average rate of change from $x=2$ to $x=4$

When $x=2$, the value of y is $y=2\times2^2-1=7$.

When $x=4$, the value of y is $y=2\times4^2-1=31$.

The average rate of change is $\dfrac{31-7}{4-2}=\dfrac{24}{2}=12$.

12

(2) Average rate of change from $x=a$ to $x=b$

When $x=a$, the value of y is $y=2a^2-1$.

When $x=b$, the value of y is $y=2b^2-1$.

The average rate of change is $\dfrac{2b^2-1-(2a^2-1)}{b-a}=\dfrac{2(b^2-a^2)}{b-a}=\dfrac{2(b-a)(b+a)}{b-a}=2(a+b)$.

$2(a+b)$

(3) Average rate of change from $x=1$ to $x=1+h$

When $x=1$, the value of y is $y=2\times1^2-1=1$.

When $x=1+h$, the value of y is $y=2\times(1+h)^2-1=1+4h+2h^2$.

The average rate of change is $\dfrac{1+4h+2h^2-1}{1+h-1}=\dfrac{2h(2+h)}{h}=2(2+h)$.

$2(2+h)$

PRACTICE

◆ For the function $y=-3x^2$, find the following average rates of change.

(1) Average rate of change from $x=-1$ to $x=3$

When $x=-1$, the value of y is $y=-3\times(-1)^2=-3$.

When $x=3$, the value of y is $y=-3\times3^2=-27$.

The average rate of change is $\dfrac{-27-(-3)}{3-(-1)}=\dfrac{-24}{4}=-6$.

-6

(2) Average rate of change from $x=a$ to $x=b$

When $x=a$, the value of y is $y=-3a^2$.

When $x=b$, the value of y is $y=-3b^2$.

The average rate of change is

$$\frac{-3b^2-(-3a^2)}{b-a}=\frac{-3(b^2-a^2)}{b-a}=\frac{-3(b-a)(b+a)}{b-a}=-3(a+b).$$

$-3(a+b)$

(3) Average rate of change from $x=1$ to $x=1+h$

When $x=1$, the value of y is $y=-3\times1^2=-3$.

When $x=1+h$, the value of y is $y=-3(1+h)^2=-3-6h-3h^2$.

The average rate of change is $\dfrac{-3-6h-3h^2-(-3)}{1+h-1}=\dfrac{-3h(2+h)}{h}=-3(2+h)$.

$-3(2+h)$

Limit values

TARGET To understand limit values and how to find them.

STUDY GUIDE

Limit values

In the function $f(x)$, when the value x is infinitely approaching a, while remaining a different value than a, then the value of $f(x)$ is infinitely approaching b, which is expressed as $\lim_{x\to a} f(x)=b$. This value b is called the **limit value** of $f(x)$ when $x\to a$. lim is the abbreviated symbol for limit, and is read as "the limit of".

Limit values

$$\lim_{x\to a} f(x)=b \text{ or } f(x)\to b \text{ when } x\to a$$

Ex. In the function $f(x)=x+4$, when the value x is infinitely approaching 3, the limit value is $\lim_{x\to 3}(x+4)=7$.

EXERCISE

◆ Find the limit values of the following.

(1) $\lim_{x\to 5}(x^2-3x+2)$

When the value of x infinitely approaches 5, x^2-3x+2 approaches $5^2-3\times5+2=12$.

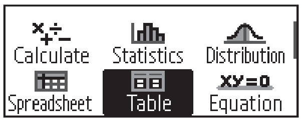
$\lim_{x\to 5}(x^2-3x+2)=12$

12

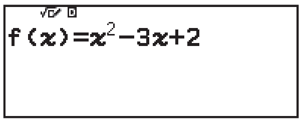
check

Use Table (to create a function table) to see how limit values are approached.

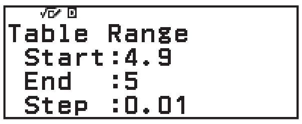
Press ⌂, select [Table], press OK, then clear the previous data by pressing ⏻

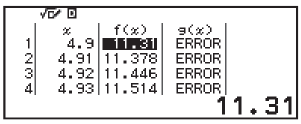


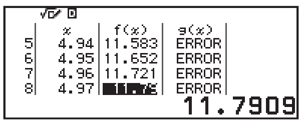
Press ⋯, select [Define f(x)/g(x)], press OK, select [Define f(x)], press OK, after inputting f(x)= x^2-3x+2, press EXE

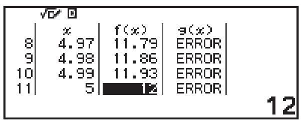


Press ⋯, select [Table Range], press OK, after inputting [Start:4.9, End:5, Step:0.01], select [Execute], press EXE









From the table, we can confirm that the value of $f(x)$ approaches 12 as the value of x approaches 5 from 4.9.

In this case, the value approaches 5 while remaining smaller than 5, but we get similar results even though it approaches while remaining a larger value.

(2) $\lim_{x \to 2} \frac{3x+10}{x+2}$

When the value of x infinitely approaches 2, $\frac{3x+10}{x+2}$ approaches $\frac{3 \times 2+10}{2+2} = \frac{16}{4} = 4$.

$$\lim_{x \to 2} \frac{3x+10}{x+2} = 4$$

4

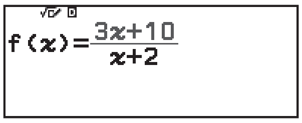
check

Use Table (to create a function table) to see how limit values are approached.

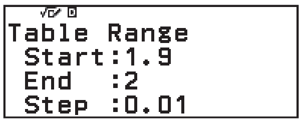
Press (HOME), select [Table], press (OK), then clear the previous data by pressing (⏻)

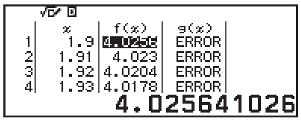
Press (MENU), select [Define f(x)/g(x)], press (OK), select [Define f(x)], press (OK),

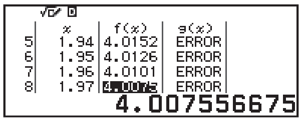
after inputting f(x)= $\frac{3x+10}{x+2}$, press (EXE)

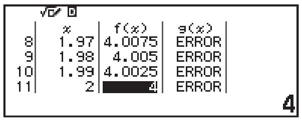


Press (MENU), select [Table Range], press (OK), after inputting [Start:1.9, End:2, Step:0.01], select [Execute], press (EXE)









From the table, we can confirm that the value of $f(x)$ approaches 4 as the value of x approaches 2 from 1.9.

(3) $\lim_{h \to 0} \frac{h^2+3h}{h}$

$$\lim_{h \to 0} \frac{h^2+3h}{h} = \lim_{h \to 0} \frac{h(h+3)}{h} = \lim_{h \to 0}(h+3)$$

When the value of h infinitely approaches 0, $h+3$ approaches $0+3=3$.

$$\lim_{h \to 0} \frac{h^2+3h}{h} = 3$$

3

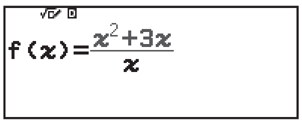
check

Use Table (to create a function table) to see how limit values are approached.

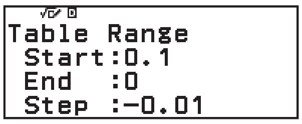
Press (HOME), select [Table], press (OK), then clear the previous data by pressing (⏻)

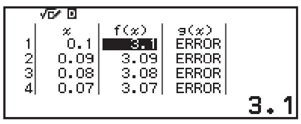
Press (MENU), select [Define f(x)/g(x)], press (OK), select [Define f(x)], press (OK),

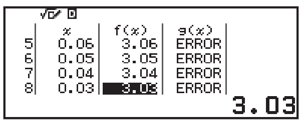
after inputting f(x)= $\frac{x^2+3x}{x}$, press (EXE)

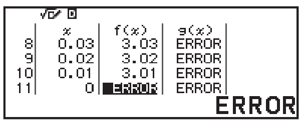


Press (MENU), select [Table Range], press (OK), after inputting [Start:0.1, End:0, Step:−0.01], select [Execute], press (EXE)









From the table, we can confirm that the value of $f(x)$ approaches 3 as the value of x approaches 0 from 0.1.
In the table, x=0 appears as ERROR because there is no value.

PRACTICE

◆ Find the limit values of the following.

(1) $\lim_{x\to4}(x^2-5x+2)$

When the value of x infinitely approaches 4, x^2-5x+2 approaches

$4^2-5\times4+2=-2$.

$\lim_{x\to4}(x^2-5x+2)=-2$

-2

(2) $\lim_{x\to1}\frac{7x+1}{x+3}$

When the value of x infinitely approaches 1, $\frac{7x+1}{x+3}$ approaches $\frac{7\times1+1}{1+3}=\frac{8}{4}=2$.

$\lim_{x\to1}\frac{7x+1}{x+3}=2$

2

(3) $\lim_{h\to0}\frac{-2h^2+8h}{h}$

$\lim_{h\to0}\frac{-2h^2+8h}{h}=\lim_{h\to0}\frac{-2h(h-4)}{h}=\lim_{h\to0}\{-2(h-4)\}$

When the value of h infinitely approaches 0, $-2(h-4)$ approaches $-2(0-4)=8$.

$\lim_{h\to0}\frac{-2h^2+8h}{h}=8$

8

ADVANCED

(4) $\lim_{x\to-2}\frac{x^2-4}{x+2}$

$\lim_{x\to-2}\frac{x^2-4}{x+2}=\lim_{x\to-2}\frac{(x+2)(x-2)}{x+2}=\lim_{x\to-2}(x-2)$

When the value of x infinitely approaches -2, $x-2$ approaches $(-2-2)=-4$.

$\lim_{x\to-2}\frac{x^2-4}{x+2}=-4$

-4

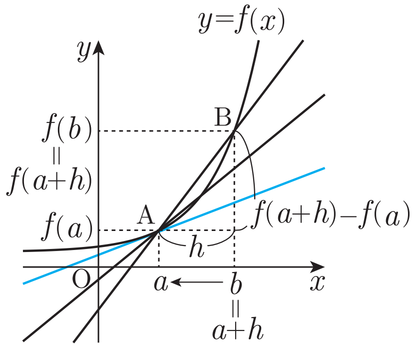
Differential coefficients

TARGET To understand differential coefficients expressed by an instantaneous average rate of change and how to find them.

STUDY GUIDE

Differential coefficients

In the average rate of change $\dfrac{f(b)-f(a)}{b-a}$ of the function $f(x)$, when the b is infinitely approaching a, the limit value is $\lim\limits_{b\to a}\dfrac{f(b)-f(a)}{b-a}$. At this point, as $b-a=h$ and given that the rate of change of x is h, we can express the limit value as follows. This value is called the **differential coefficient** of the function $f(x)$ for $x=a$, and is expressed as $f'(a)$.



$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

EXERCISE

◆ Find the following differential coefficients.

(1) The differential coefficient of the function $f(x)=3x-5$ when $x=4$

$$f'(4)=\lim_{h\to 0}\frac{f(4+h)-f(4)}{h}=\lim_{h\to 0}\frac{\{3(4+h)-5\}-(3\times 4-5)}{h}=\lim_{h\to 0}\frac{3h}{h}=3$$

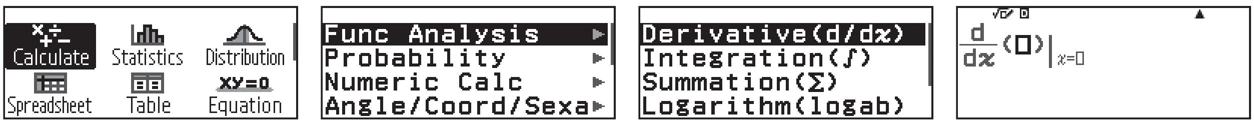
3

check

Confirm this by using the function to calculate differential coefficients.

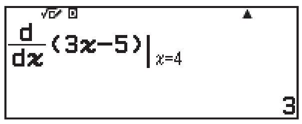
Press ⌂, select [Calculate], press OK

Press ⊡, select [Func Analysis], press OK, select [Derivative(d/dx)], press OK



On the screen that is displayed, you can input the function and the value of x to confirm the differential coefficient.

③ ⓧ ⊖ ⑤ ⊳ ④ EXE

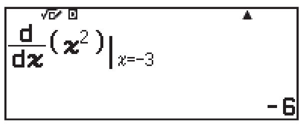


(2) The differential coefficient $f'(-3)$ of the function $f(x)=x^2$

$$f'(-3)=\lim_{h\to 0}\frac{f(-3+h)-f(-3)}{h}=\lim_{h\to 0}\frac{(-3+h)^2-(-3)^2}{h}=\lim_{h\to 0}\frac{h(-6+h)}{h}=-6$$

−6

⊡ OK OK ⓧ ■² ⊳ ⊖ ③ EXE



PRACTICE

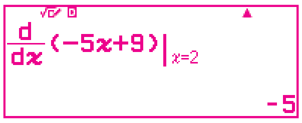
◆ Find the limit values of the following.

(1) The differential coefficient of the function $f(x)=-5x+9$ when $x=2$

$$f'(2)=\lim_{h\to 0}\frac{f(2+h)-f(2)}{h}=\lim_{h\to 0}\frac{\{-5(2+h)+9\}-(-5\times 2+9)}{h}=\lim_{h\to 0}\frac{-5h}{h}=-5$$

−5

[📖] [OK] [OK] [−] [5] [x] [+] [9] [>] [2] [EXE]

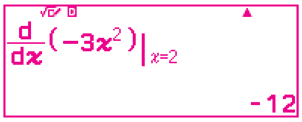


(2) The differential coefficient $f'(2)$ of the function $f(x)=-3x^2$

$$f'(2)=\lim_{h\to 0}\frac{f(2+h)-f(2)}{h}=\lim_{h\to 0}\frac{-3(2+h)^2-(-3\times 2^2)}{h}=\lim_{h\to 0}\frac{-3h(4+h)}{h}=-12$$

−12

[📖] [OK] [OK] [−] [3] [x] [■²] [>] [2] [EXE]



Derivatives

TARGET To understand derivatives and how to find them.

STUDY GUIDE

Derivatives

In the function $f(x)$, 1 new function is obtained by mapping the differential coefficient $f'(a)$ to each value a of x. This function is called the **derivative** of $f(x)$ and is expressed as follows using $f'(x)$.

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

In the above expression, h is called the increment of x, which is written as Δx. Further, $f(x+h)-f(x)$ is called the increment of y, and is expressed by the symbol Δy. Now, we can express the derivative as follows.

$$f'(x)=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

In addition to $f'(x)$, there are other symbols for derivatives, such as y', $\frac{dy}{dx}$, and $\frac{d}{dx}f(x)$. When we find the derivative $f'(x)$ from the function $f(x)$ of x, we say we are **differentiating** the function $f(x)$.

EXERCISE

◆ Find the following derivatives.

(1) Derivative of the function $f(x)=x^2$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{(x+h)^2-x^2}{h}=\lim_{h\to 0}\frac{h(2x+h)}{h}=\lim_{h\to 0}(2x+h)=2x$$

$$f'(x)=2x$$

(2) Derivative of the function $f(x)=5$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{5-5}{h}=\lim_{h\to 0}0=0$$

$$f'(x)=0$$

PRACTICE

◆ Find the following derivatives.

(1) Derivative of the function $f(x) = 7x + 9$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\{7(x+h)+9\} - (7x+9)}{h} = \lim_{h \to 0} \frac{7h}{h} = \lim_{h \to 0} 7 = 7$$

$$f'(x) = 7$$

(2) Derivative of the function $f(x) = -3x^2$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{-3(x+h)^2 - (-3x^2)}{h} = \lim_{h \to 0} \frac{-3h(2x+h)}{h}$$
$$= \lim_{h \to 0} \{-3(2x+h)\} = -6x$$

$$f'(x) = -6x$$

ADVANCED

(3) Derivative of the function $f(x) = x^3$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h} = \lim_{h \to 0} \frac{h(3x^2 + 3xh + h^2)}{h}$$
$$= \lim_{h \to 0} (3x^2 + 3xh + h^2) = 3x^2$$

$$f'(x) = 3x^2$$

Properties of derivatives

TARGET To learn the properties of derivatives and to understand how to find various derivatives.

STUDY GUIDE

Properties of derivatives

To find the derivative $f'(x)$ from the function $f(x)$, we use the following formulas which are derived from the definitions.

(1) $y = x^n$ (n being a natural number) $\Rightarrow$ $y' = nx^{n-1}$

(2) $y = c$ (c being a constant) $\Rightarrow$ $y' = 0$

(3) $y = kf(x)$ (k being a constant) $\Rightarrow$ $y' = kf'(x)$

(4) $y = f(x) \pm g(x)$ $\Rightarrow$ $y' = f'(x) \pm g'(x)$

Ex.

(1) When $y = x^4$, then $y' = 4x^3$.

(2) When $y = -6$, then $y' = 0$.

(3) When $y = 4x^3$, then $y' = 4 \times 3x^2 = 12x^2$.

(4) When $y = 3x^5 + 2x^3$, then $y' = 3 \times 5x^4 + 2 \times 3x^2 = 15x^4 + 6x^2$.

By substituting a for x in the derivative $f'(x)$ of the function $f(x)$, we can find the derivative $f'(a)$ for $x=a$.

Derivatives of variables other than x

The function $y=f(x)$ expresses a function using the variables x and y, but we can choose any variables to express a function. For example, the relation $S = \pi r^2$ for the radius r of a circle and the area S of a circle is also a function of the variables r and S.

Now, when **S is differentiated by r**, we get $\frac{dS}{dr} = 2\pi r$ where we see the formula for finding the circumference of a circle.

Furthermore, in the relation $V = \frac{4}{3}\pi r^3$ for the radius r of a sphere and the volume V of a sphere, when **V is differentiated by r**, we get $\frac{dV}{dr} = 4\pi r^2$ where we see the formula for finding the surface area of a sphere.

EXERCISE

◆ Solve the following problems.

(1) Differentiate the function $y = -7x^3 - 5x^2 + 4x - 8$.

$$y' = (-7x^3)' - (5x^2)' + (4x)' - (8)' = -7(x^3)' - 5(x^2)' + 4(x)' - (8)' = -7 \cdot 3x^2 - 5 \cdot 2x + 4 \cdot 1 - 0$$
$$= -21x^2 - 10x + 4$$

$$y' = -21x^2 - 10x + 4$$

(2) Differentiate the function $f(x)=x(x-2)(x+3)$.

$f(x)=x(x-2)(x+3)=x^3+x^2-6x$

$f'(x)=(x^3)'+(x^2)'-(6x)'=(x^3)'+(x^2)'-6(x)'=3x^2+2x-6\cdot 1=3x^2+2x-6$

$$f'(x)=3x^2+2x-6$$

(3) Use the derivative of the function $f(x)=-4x^2+7x-10$ to find the derivative of $f(x)$ when $x=5$.

$f'(x)=(-4x^2)'+(7x)'-(10)'=-4(x^2)'+7(x)'-(10)'=-4\cdot 2x+7\cdot 1-0=-8x+7$

$f'(5)=-8\cdot 5+7=-33$

$$-33$$

[book] [OK] [OK] [−] [4] [x] [■²] [+] [7] [x] [−] [1] [0] [>] [5] [EXE]

$\frac{d}{dx}(-4x^2+7x-10)|_{x=5}$

-33

PRACTICE

◆ Solve the following problems.

(1) Differentiate the function $y=-2x^3+4x^2-7x+13$.

$y'=(-2x^3)'+(4x^2)'-(7x)'+(13)'=-2(x^3)'+4(x^2)'-7(x)'+(13)'$

$=-2\cdot 3x^2+4\cdot 2x-7\cdot 1+0=-6x^2+8x-7$

$$y'=-6x^2+8x-7$$

(2) Differentiate the function $f(x)=3x(x+1)(x-1)$.

$f(x)=3x(x+1)(x-1)=3x^3-3x$

$f'(x)=(3x^3)'-(3x)'=3(x^3)'-3(x)'=3\cdot 3x^2-3\cdot 1=9x^2-3$

$$f'(x)=9x^2-3$$

(3) Use the derivative of the function $f(x)=5x^2-9x+8$ to find the derivative of $x=-2$ when $f(x)$.

$f'(x)=(5x^2)'-(9x)'+(8)'=5(x^2)'-9(x)'+(8)'=5\cdot 2x-9\cdot 1+0=10x-9$

$f'(-2)=10\cdot(-2)-9=-29$

$$-29$$

[book] [OK] [OK] [5] [x] [■²] [−] [9] [x] [+] [8] [>] [−] [2] [EXE]

$\frac{d}{dx}(5x^2-9x+8)|_{x=-2}$

-29

Equations of tangents

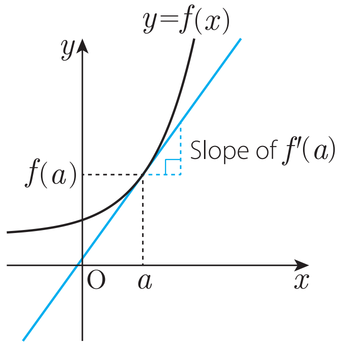
TARGET To understand how to find tangent equations from graphs of various functions.

STUDY GUIDE

Equations of tangents

The **differential coefficient $f'(a)$** of the function $y=f(x)$ where $x=a$ is the **slope of the tangent** of the point $(a, f(a))$ on the graph.

The **equation of the tangent** for the point $(a, f(a))$ on the graph of the function $y=f(x)$ is expressed as follows.



$$y - f(a) = f'(a)(x-a)$$

EXTRA Info.

Equation for a straight line passing through a point (x_1, y_1) and having a slope of m

$y - y_1 = m(x - x_1)$

EXERCISE

1 Let m be the slope of the tangent l for the point $\mathrm{P}(5, 10)$ on the graph of the function $y = x^2 - 4x + 5$. Now, solve the following problems.

(1) Find the value of m.

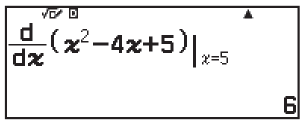
Given $f(x) = x^2 - 4x + 5$, we can get $m=f'(5)$.

By differentiating $f(x)$, we get $f'(x) = (x^2 - 4x + 5)' = 2x - 4$.

Therefore, we get $m = f'(5) = 2 \cdot 5 - 4 = 6$.

6

[MENU] [OK] [OK] [x] [■²] [−] [4] [x] [+] [5] [▷] [5] [EXE]



(2) Find the equation of tangent l.

The tangent l is a straight line passing through the point $\mathrm{P}(5, 10)$ with a slope of $m=6$.

Therefore, from $y - 10 = 6(x-5)$, we get $y = 6x - 20$.

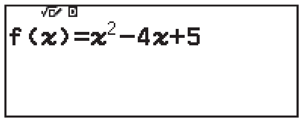
$$y = 6x - 20$$

check

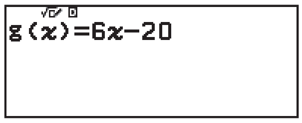
To better understand, use Table and the QR code to calculate and draw the relation of the tangent and graph of the function.

Press ⌂, select [Table], press OK, then clear the previous data by pressing ⏻

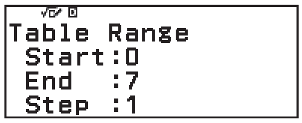
Press ⋯, select [Define $\mathrm{f}(x)/\mathrm{g}(x)$], press OK, select [Define $\mathrm{f}(x)$], press OK, after inputting $\mathrm{f}(x)= x^2 - 4x + 5$, press EXE



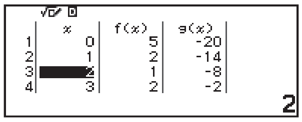
In the same way, input $\mathrm{g}(x)= 6x - 20$.

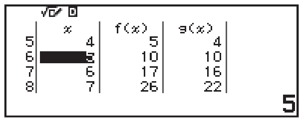


Press ⋯, select [Table Range], press OK, after inputting [Start:0, End:7, Step:1], select [Execute], press EXE



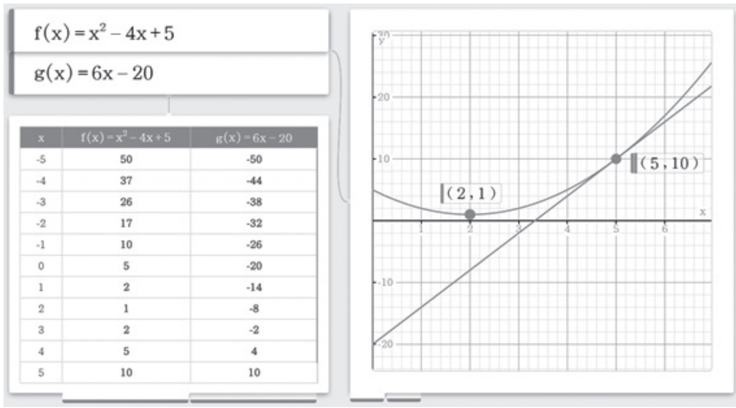
Press ⇧ ⓧ, scan the QR code to display a graph.


	x	f(x)	g(x)
1	0	5	-20
2	1	2	-14
3	2	1	-8
4	3	2	-2


	x	f(x)	g(x)
5	4	5	4
6	5	10	10
7	6	17	16
8	7	26	22

From the table and graph, we can see that the minimum value for $f(x)$ is x=2.

$f(x)$ and $g(x)$ share a point (5, 10).



x	f(x) = x² − 4x + 5	g(x) = 6x − 20
-5	50	-50
-4	37	-44
-3	26	-38
-2	17	-32
-1	10	-26
0	5	-20
1	2	-14
2	1	-8
3	2	-2
4	5	4
5	10	10

2 The graph of the function $y = x^2 + 1$ has 2 tangents drawn from the origin. Find the equations of these tangents.

Given $f(x) = x^2 + 1$, we can get $f'(x) = 2x$.

Given the coordinates of the tangent is $(a, a^2 + 1)$, we find the slope of the tangent is $f'(a) = 2a$.

The equation of the tangent comes from $y - (a^2 + 1) = 2a(x - a)$ to give us $y = 2ax - a^2 + 1$.

This line passes through the origin, so from $0 = 2a \cdot 0 - a^2 + 1$, we get a=±1.

When a=1, then $y = 2x$.

When a=−1, then $y = -2x$.

$y = 2x, y = -2x$

PRACTICE

1 Let m be the slope of the tangent l for the point $\mathrm{P}(-1, 9)$ on the graph of the function $y=-x^2-6x+4$. Now, solve the following problems.

(1) Find the value of m.

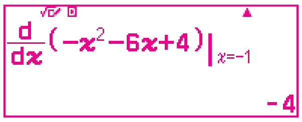
Given $f(x)=-x^2-6x+4$, we can get $m=f'(-1)$.

By differentiating $f(x)$, we get $f'(x)=(-x^2-6x+4)'=-2x-6$.

Therefore, we get $m=f'(-1)=-2\cdot(-1)-6=-4$.

$$-4$$

[MATH] [OK] [OK] [−] [x] [■²] [−] [6] [x] [+] [4] [>] [−] [1] [EXE]



(2) Find the equation of tangent l.

The tangent l is a straight line passing through the point $\mathrm{P}(-1, 9)$ with a slope of $m=-4$.

Therefore, from $y-9=-4\{x-(-1)\}$, we get $y=-4x+5$.

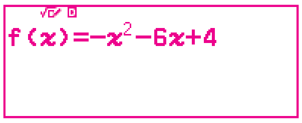
$$y=-4x+5$$

check

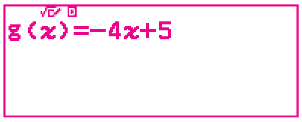
To better understand, use Table and the QR code to calculate and draw the relation of the tangent and graph of the function.

Press [HOME], select [Table], press [OK], then clear the previous data by pressing [AC]

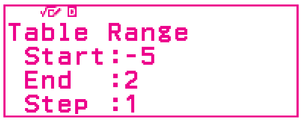
Press [TOOLS], select [Define f(x)/g(x)], press [OK], select [Define f(x)], press [OK], after inputting $\mathrm{f}(x)=-x^2-6x+4$, press [EXE]



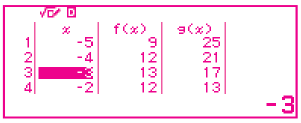
In the same way, input $\mathrm{g}(x)=-4x+5$.



Press [TOOLS], select [Table Range], press [OK], after inputting [Start:−5, End:2, Step:1], select [Execute], press [EXE]

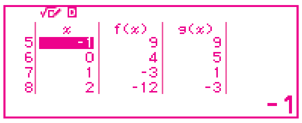


Press [SHIFT] [x], scan the QR code to display a graph.



	x	f(x)	g(x)
1	-5	9	25
2	-4	12	21
3	-3	13	17
4	-2	12	13

-3

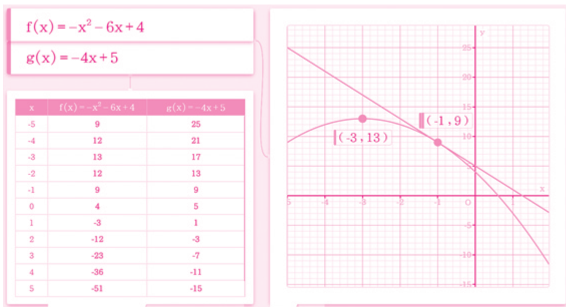


	x	f(x)	g(x)
5	-1	9	9
6	0	4	5
7	1	-3	1
8	2	-12	-3

-1

From the table and graph, we can see that the maximum value for $f(x)$ is $x=-3$.

$f(x)$ and $g(x)$ share a point $(-1, 9)$.



x	f(x) = −x² − 6x + 4	g(x) = −4x + 5
-5	9	25
-4	12	21
-3	13	17
-2	12	13
-1	9	9
0	4	5
1	-3	1
2	-12	-3
3	-23	-7
4	-36	-11
5	-51	-15

2 The graph of the function $y=3x^2+2$ has 2 tangents drawn from the point $(0, -10)$. Find the equations of these tangents.

Given $f(x)=3x^2+2$, we can get $f'(x)=6x$.

Given the coordinates of the tangent is $(a, 3a^2+2)$, we find the slope of the tangent is $f'(a)=6a$.

The equation of the tangent comes from $y-(3a^2+2)=6a(x-a)$ to give us $y=6ax-3a^2+2$.

This line passes through the point $(0, -10)$, so from $-10=6a\cdot 0-3a^2+2$, we get $a=\pm 2$.

When $a=2$, then $y=12x-10$.

When $a=-2$, then $y=-12x-10$.

$$y=12x-10, y=-12x-10$$

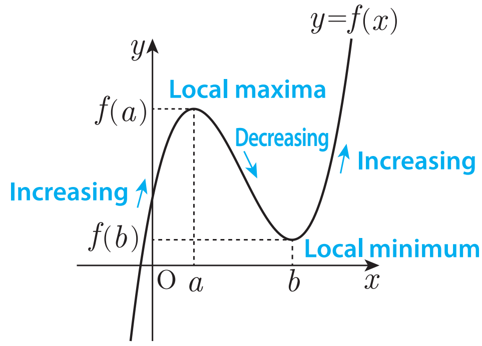
Increasing/decreasing of functions and their local maximum/minimum (1)

TARGET To understand the basics of increasing/decreasing functions and their local maxima/minima.

STUDY GUIDE

Increasing/decreasing of functions and their local maxima/minima

The differential coefficient expresses **the slope of the tangent at the point of tangency**. From this, we can consider the increase and decrease of the function $f(x)$. If $f'(x)>0$ within a certain range, the tangent has a positive slope, so that the graph of the function $f(x)$ is increasing. Similarly, if $f'(x)<0$ within a certain range, the tangent has a negative slope, so that the graph of the function $f(x)$ is decreasing.



For the function $f(x)$, within a range:

If $f'(x)>0$, then $f(x)$ is increasing within that range

If $f'(x)<0$, then $f(x)$ is decreasing within that range

When the function $f(x)$ changes from increasing to decreasing at $x=a$, the function $f(x)$ has a **local maximum** at $x=a$, so we say $f(a)$ is the **local maximum value**.

When the function $f(x)$ changes from increasing to decreasing at $x=b$, the function $f(x)$ has a **local minimum** at $x=b$, so we say $f(b)$ is the **local minimum value**.

Collectively the local maximum and minimum values are called **extrema**.

To effectively organize the increases and decreases of the function $f(x)$, use an increase/decrease table like the one on the right.

x	$\cdots$	a	$\cdots$	b	$\cdots$
$f'(x)$	$+$	0	$-$	0	$+$
$f(x)$	↗	Local maximum value	↘	Local minimum value	↗

For $f'(a)=0$, from before to after $x=a$, the sign of $f'(x)$ changes from positive to negative.

⇒ When $x=a$ is the local maximum, then $f(a)$ is the local maximum value.

For $f'(b)=0$, from before to after $x=b$, the sign of $f'(x)$ changes from negative to positive.

⇒ When $x=b$ is the local minimum, then $f(b)$ is the local minimum value.

How to draw a graph of the function $y=f(x)$

(1) Differentiate $y=f(x)$ to get $f'(x)=0$ to find x.

(2) Determine the sign of $(y'=)f'(x)$ as regards $f'(x)=0$ for a range of values other than x.

(3) In the increase/decrease table, organize the increase and decrease of y and the changes in the sign of y'.

(4) Find the extrema from the increase/decrease table, and draw a graph.

EXERCISE

1 Solve the following problems with regard to the function $y = x^3 - 3x$.

(1) Find the derivative of y'. Also, given $y'=0$, find the value of x.

$$y' = (x^3 - 3x)' = 3x^2 - 3$$
$$3x^2 - 3 = 0, x^2 = 1, x = \pm 1$$

$$\boldsymbol{y' = 3x^2 - 3, x = \pm 1}$$

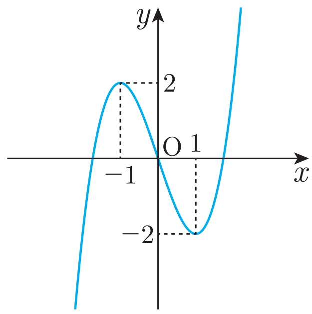
(2) Complete the increase/decrease table on the right.

When $x=-1$, then $y=2$, which is the local maximum value.

When $x=1$, then $y=-2$, which is the local minimum value.

x	…	-1	…	1	…
y'	$+$	0	$-$	0	$+$
y	↗	2	↘	-2	↗

(3) Draw a graph of the function $y = x^3 - 3x$.



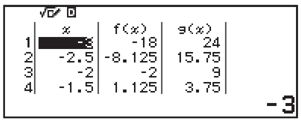
check

(2) Press (Home), select [Table], press (OK), then clear the previous data by pressing (Power)

Press (⋯), select [Define f(x)/g(x)], press (OK), select [Define f(x)], press (OK), after inputting f(x)$= x^3 - 3x$, press (EXE)

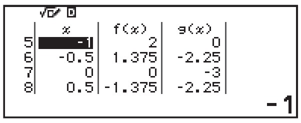
In the same way, input g(x)$= 3x^2 - 3$.

Press (⋯), select [Table Range], press (OK), after inputting [Start:-3, End:3, Step:0.5], select [Execute], press (EXE)



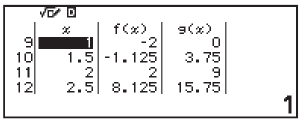
	x	f(x)	g(x)
1	-3	-18	24
2	-2.5	-8.125	15.75
3	-2	-2	9
4	-1.5	1.125	3.75

-3



	x	f(x)	g(x)
5	-1	2	0
6	-0.5	1.375	-2.25
7	0	0	-3
8	0.5	-1.375	-2.25

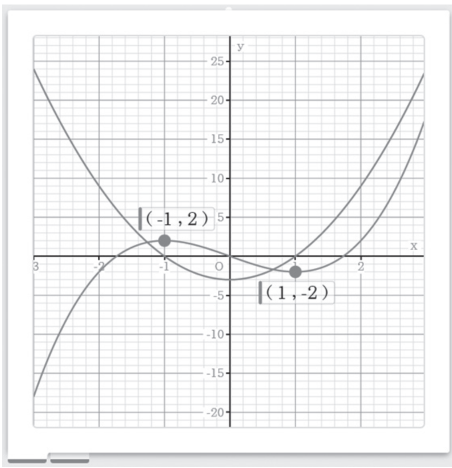
-1



	x	f(x)	g(x)
9	1	-2	0
10	1.5	-1.125	3.75
11	2	2	9
12	2.5	8.125	15.75

1

(3) Press (↑) (x), scan the QR code to display a graph.



In the region where the derivative $g(x)$ is positive, $f(x)$ increases. It decreases in negative regions.

2 Determine whether the function $y=2x^3-3x^2$ is increasing or decreasing, draw a graph, and find the local maximum and minimum values.

$y'=(2x^3-3x^2)'=6x^2-6x$

$6x^2-6x=0, 6x(x-1)=0, x=0,1$

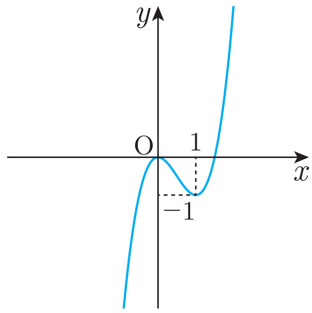
The increase/decrease table is shown on the right.

When x=0, then y=0, which is the local maximum value.

When x=1, then y=−1, which is the local minimum value.

The graph is in the diagram on the right.

x	…	0	…	1	…
y'	+	0	−	0	+
y	↗	0	↘	−1	↗



Local maximum value …0, local minimum value …−1

check

Press (home), select [Table], press (OK), then clear the previous data by pressing (power)

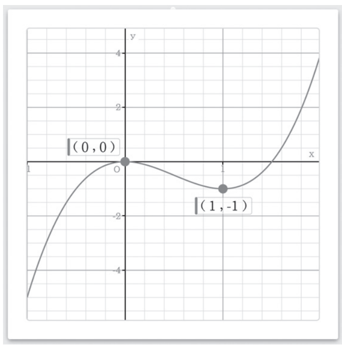
Press (⋯), select [Define f(x)/g(x)], press (OK), select [Define f(x)], press (OK),

after inputting f(x)=$2x^3-3x^2$, press (EXE)

Press (⋯), select [Table Range], press (OK), after inputting

[Start:−1, End:2, Step:1], select [Execute], press (EXE)

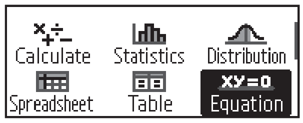
Press (shift) (x), scan the QR code to display a graph.



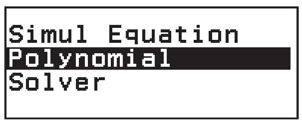
OTHER METHODS

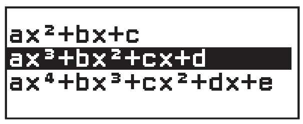
Use Equation to directly find the local maximum and minimum values.

Press (home), select [Equation], press (OK)



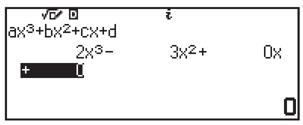
Select [Polynomial], press (OK), select [ax^3+bx^2+cx+d], press (OK)



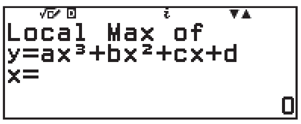


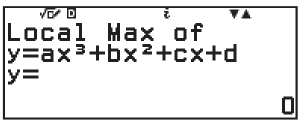
Input a=2, b=−3, c=0, and d=0.

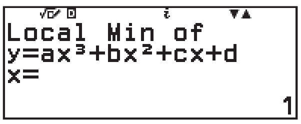
(2) (EXE) (−) (3) (EXE) (0) (EXE) (0) (EXE)

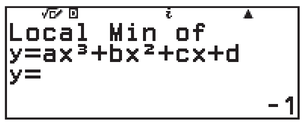


Displayed in order by pressing (EXE) (EXE) for intersections with the x axis, then pressing (EXE) (EXE) for local maximum values, then pressing (EXE) (EXE) for local minimum values.









PRACTICE

1 Solve the following problems with regard to the function $y=-x^3+3x^2-3$.

(1) Find the derivative of y'. Also, given $y'=0$, find the value of x.

$y'=(-x^3+3x^2-3)'=-3x^2+6x$

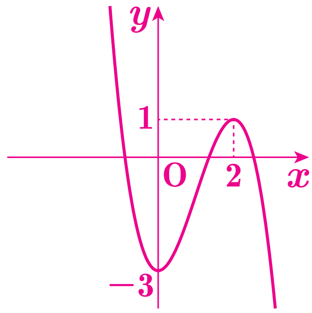
$-3x^2+6x=0, -3x(x-2)=0, x=0, 2$

$y'=-3x^2+6x, x=0, 2$

(2) Complete the increase/decrease table on the right.

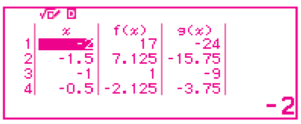
x	…	0	…	2	…
y'	−	0	+	0	−
y	↘	−3	↗	1	↘

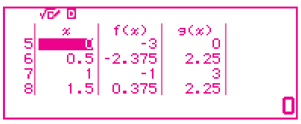
(3) Draw a graph of the function $y=-x^3+3x^2-3$.

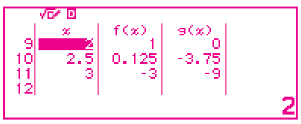


check

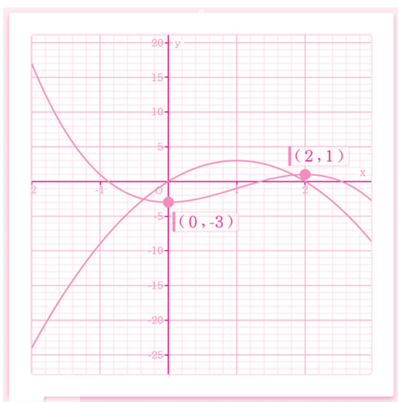
(2) Press ⌂, select [Table], press OK, then clear the previous data by pressing ⏻
Press ⋯, select [Define f(x)/g(x)], press OK, select [Define f(x)], press OK, after inputting f(x)$=-x^3+3x^2-3$, press EXE
In the same way, input g(x)$=-3x^2+6x$.
Press ⋯, select [Table Range], press OK, after inputting [Start:−2, End:3, Step:0.5], select [Execute], press EXE







(3) Press ⇧ x, scan the QR code to display a graph.
In the region where the derivative $g(x)$ is positive, $f(x)$ increases.
It decreases in negative regions.



2 Determine whether the function $y=-x^3+12x$ is increasing or decreasing, draw a graph, and find the local maximum and minimum values.

$y'=(-x^3+12x)'=-3x^2+12$

$-3x^2+12=0, x^2=4, x=\pm 2$

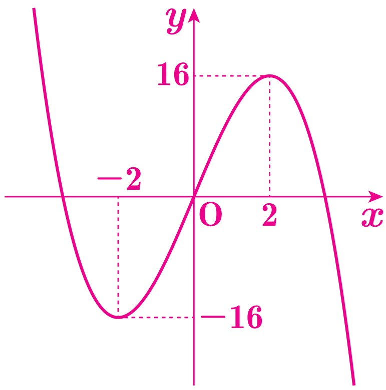
The increase/decrease table is shown on the right.

x	$\cdots$	-2	$\cdots$	2	$\cdots$
y'	$-$	0	$+$	0	$-$
y	↘	-16	↗	16	↘

When $x=-2$, then $y=-16$, which is the local minimum value.

When $x=2$, then $y=16$, which is the local maximum value.

The graph is in the diagram on the right.



Local maximum value …16,
local minimum value …-16

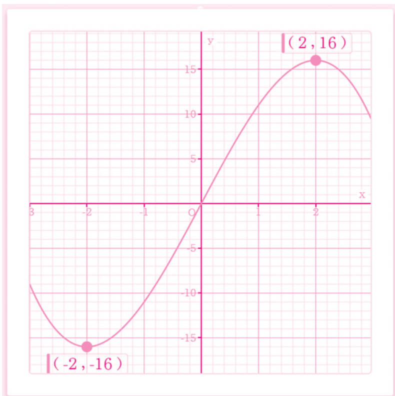
check

Press ⌂, select [Table], press OK, then clear the previous data by pressing ⏻

Press ⋯, select [Define f(x)/g(x)], press OK, select [Define f(x)], press OK, after inputting f(x)$=-x^3+12x$, press EXE

Press ⋯, select [Table Range], press OK, after inputting [Start:-3, End:3, Step:1], select [Execute], press EXE

Press ⇧ x, scan the QR code to display a graph.



Increasing/decreasing of functions and their local maximum/minimum (2)

TARGET To understand differential coefficients and extrema, and monotonic increase and monotonic decrease.

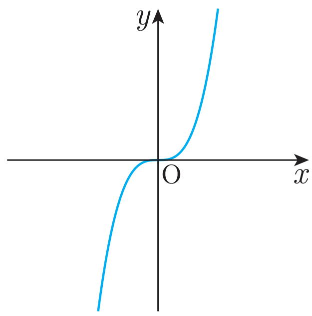
STUDY GUIDE

Differential coefficients and extrema

If the function $f(x)$ has an extreme value at $x=a$, then we can derive $f'(a)=0$; however, even if $f'(a)=0$, we cannot absolutely derive that the function $f(x)$ has an extreme value at $x=a$.

Ex. For the function $f(x)=x^3$, from $f'(x)=3x^2$ we can get $f'(0)=0$, but we cannot get the extrema as shown in the increase/decrease table on the right. This is because before and after $x=0$, it is always $f'(x)>0$, so the value only increases.

x	…	0	…
$f'(x)$	+	0	+
$f(x)$	↗	0	↗



Monotonic increase & monotonic decrease

The following holds regarding the increase and decrease of the function $f(x)$.

(1) Function $f(x)$ has extrema at $x=a \Rightarrow f'(a)=0$

(2) For function $f(x)$, only if $f'(x)>0$ or $f'(x)\geq 0 \Rightarrow$ Monotonic increase

For function $f(x)$, only if $f'(x)<0$ or $f'(x)\leq 0 \Rightarrow$ Monotonic decrease

Note that in (2) above, a constant increase is called a **monotonic increase**, and a constant decrease is called a **monotonic decrease**.

EXERCISE

1 Solve the following problems with regard to the function $f(x)=x^3+ax^2+bx$.

(1) Determine the values of the constants a and b to get the extrema $x=-1$ and $x=3$.

$f'(x)=3x^2+2ax+b$

The function $f(x)$ has extrema at $x=-1$ and $x=3$, so we get $f'(-1)=0$ and $f'(3)=0$.

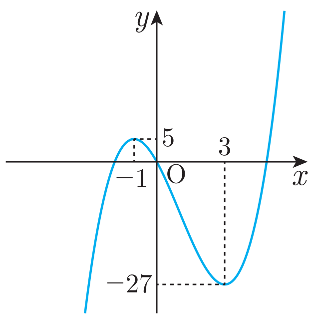
$f'(-1)=3\cdot(-1)^2+2a\cdot(-1)+b=0, 2a-b=3 \quad \ldots\text{(i)}$

$f'(3)=3\cdot 3^2+2a\cdot 3+b=0, 6a+b=-27 \quad \ldots\text{(ii)}$

Solving for (i) and (ii), gives us $a=-3$ and $b=-9$.

$a=-3$, $b=-9$

x	…	-1	…	3	…
$f'(x)$	+	0	−	0	+
$f(x)$	↗	5	↘	-27	↗



(2) Find the extrema for (1).

The increase/decrease table for $f(x)$ function is shown on the right.

Therefore, when $x=-1$, the local maximum value is 5 and when $x=3$, the local minimum value is -27.

When $x=-1$, the local maximum value is 5, and when $x=3$, the local minimum value is -27

check

Press ⌂, select [Table], press OK, then clear the previous data by pressing ⏻

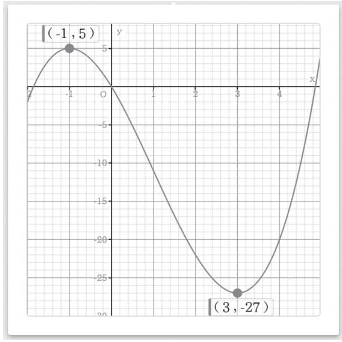
Press ⋯, select [Define $f(x)/g(x)$], press OK, select [Define $f(x)$], press OK,

after inputting $f(x)= x^3 - 3x^2 - 9x$, press EXE

Press ⋯, select [Table Range], press OK, after inputting

[Start:-2, End:5, Step:1], select [Execute], press EXE

Press ⇧ x, scan the QR code to display a graph.



2 Given the function $f(x) = x^3 + x - 4$, prove that the value of $f(x)$ always increases for all ranges of x.

[Proof]

$f'(x) = 3x^2 + 1$

Since for all values of x, $x^2 \geq 0$, then from $3x^2 + 1 \geq 1 > 0$, we get $f'(x)>0$.

Therefore, given the function $f(x) = x^3 + x - 4$, the value of $f(x)$ always increases for all ranges of x.

PRACTICE

1 Determine the value of the constants a and b such that the function $f(x)=-x^3+ax+b$ has a local maximum at 4 when $x=1$.

$f'(x)=-3x^2+a$

The function $f(x)$ has a local maximum at 4 when $x=1$, so we get $f'(1)=0$ and $f(1)=4$.

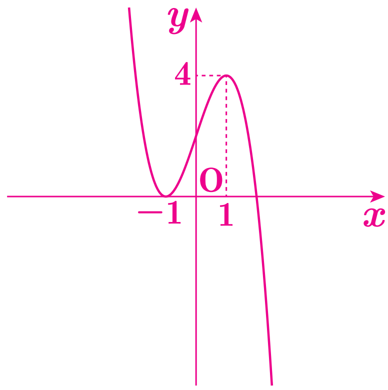
$f'(1)=-3\cdot 1^2+a=0, a=3$ ...(i)

$f(1)=-1^3+a\cdot 1+b=4, a+b=5$ ...(ii)

Solving for (i) and (ii) gives us $a=3$ and $b=2$.

The increase/decrease table and graph are shown on the right.

x	$\cdots$	-1	$\cdots$	1	$\cdots$
$f'(x)$	$-$	0	$+$	0	$-$
$f(x)$	↘	0	↗	4	↘



$a=3$, $b=2$

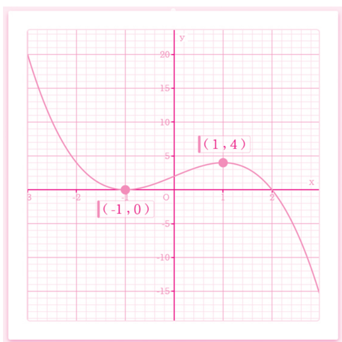
check

Press ⌂, select [Table], press OK, then clear the previous data by pressing ⏻

Press ⋯, select [Define f(x)/g(x)], press OK, select [Define f(x)], press OK, after inputting f(x)$=-x^3+3x+2$, press EXE

Press ⋯, select [Table Range], press OK, after inputting [Start:−3, End:3, Step:1], select [Execute], press EXE

Press ⇧ ⓧ, scan the QR code to display a graph.



2 Given the function $f(x)=2x^3+5x+7$, prove that the value of $f(x)$ always increases for all ranges of x.

[Proof]

$f'(x)=6x^2+5$

Since $x^2\geq 0$ for all values of x, from $6x^2+5\geq 5>0$, we get $f'(x)>0$.

Therefore, given the function $f(x)=2x^3+5x+7$, the value of $f(x)$ always increases for all ranges of x.

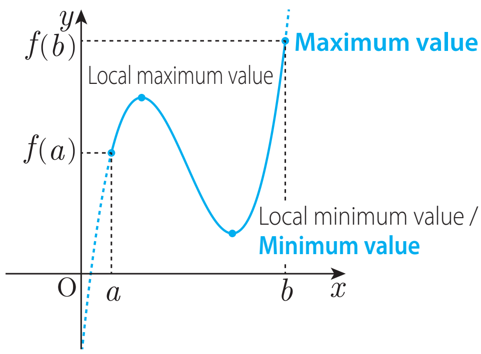
Increase/decrease of functions and applicable graphs (1)

TARGET To understand the maximum and minimum of a function within a given range.

STUDY GUIDE

Maximum/minimum of functions

The maximum and minimum values of the function $f(x)$ as defined by $a \leq x \leq b$ may not always be equal to the local maximum and minimum values.
The maximum and minimum are the maximum and minimum when considering the entire domain.
The local maximum and local minimum are the maximum and minimum, respectively, within a range sufficiently close to those points.
Therefore, to find the maximum value and minimum value, we look at the following things.



(1) Find the derivative $f'(x)$ to solve $f'(x)=0$.
(2) Draw an increase/decrease table and graph.
(3) Compare the values of both ends of the domain and the maxima to find the maximum value and minimum value.

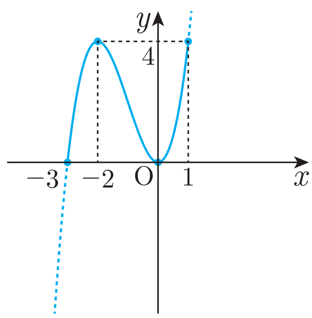
EXERCISE

◆ Solve the following problems with regard to the function $y = x^3 + 3x^2$.

(1) Find the maximum and minimum values for $-3 \leq x \leq 1$.

From $y' = 3x^2 + 6x$, we can get
$3x^2 + 6x = 0, 3x(x+2) = 0, x = 0, -2$.
The increase/decrease table and graph are shown on the right.
When $x=-2$, the local maximum value is 4, which is also the maximum value.
When $x=0$, the local minimum value is 0, which is also the minimum value.
When $x=-3$, the minimum value is 0.
When $x=1$, the maximum value is 4.

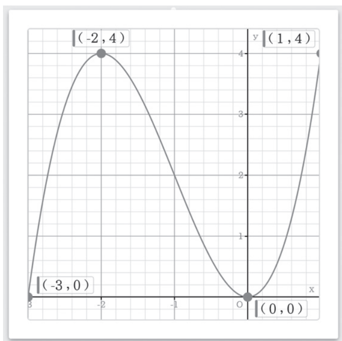
x	-3	…	-2	…	0	…	1
y'		$+$	0	$-$	0	$+$	
y	0	↗	4	↘	0	↗	4



When $x=-2$ or 1, the maximum value is 4, and when $x=-3$ or 0, the minimum value is 0

check

Press ⌂, select [Table], press OK, then clear the previous data by pressing ⏻
Press ⋯, select [Define f(x)/g(x)], press OK, select [Define f(x)], press OK, after inputting f(x)$= x^3 + 3x^2$, press EXE
Press ⋯, select [Table Range], press OK, after inputting [Start:-3, End:1, Step:1], select [Execute], press EXE
Press ⬆ ⓧ, scan the QR code to display a graph.



(2) Find the maximum and minimum values for $-4 \leq x \leq 2$.

From $y' = 3x^2 + 6x$, we can get

$3x^2 + 6x = 0, 3x(x+2) = 0, x = 0, -2$.

The increase/decrease table and graph are shown on the right.

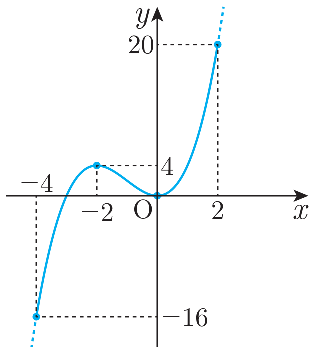
When $x=-2$, the local maximum value is 4.

When $x=0$, the local minimum value is 0.

When $x=-4$, the minimum value is -16.

When $x=2$, the maximum value is 20.

x	-4	$\cdots$	-2	$\cdots$	0	$\cdots$	2
y'		$+$	0	$-$	0	$+$	
y	-16	↗	4	↘	0	↗	20



When $x=2$, the maximum value is 20, and when $x=-4$ the minimum value is -16

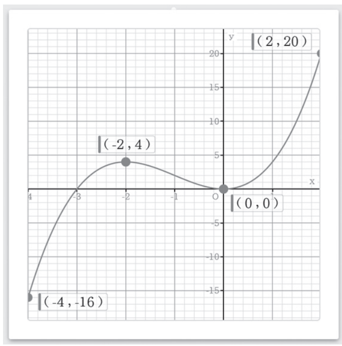
check

Press ⓐ, select [Table], press OK, then clear the previous data by pressing ⏻

Press ⋯, select [Define f(x)/g(x)], press OK, select [Define f(x)], press OK, after inputting f(x)$= x^3 + 3x^2$, press EXE

Press ⋯, select [Table Range], press OK, after inputting [Start:−4, End:2, Step:1], select [Execute], press EXE

Press ⇧ ⓧ, scan the QR code to display a graph.



PRACTICE

◆ Solve the following problems with regard to the function $y = -2x^3 + 3x^2$.

(1) Find the maximum and minimum values for $0 \leq x \leq 1$.

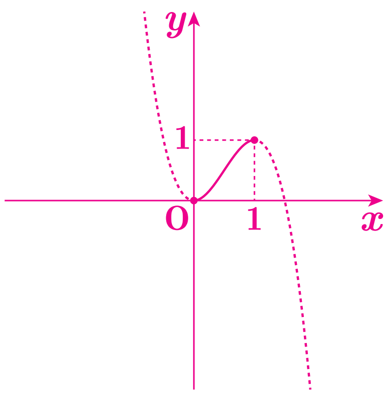
From $y' = -6x^2 + 6x$, we get $-6x^2 + 6x = 0, -6x(x-1) = 0, x = 0, 1$.

The increase/decrease table and graph are shown on the right.

When $x=0$, the local minimum value is 0, which is also the minimum value.

When $x=1$, the local maximum value is 1, which is also the maximum value.

x	$\cdots$	0	$\cdots$	1	$\cdots$
y'	$-$	0	$+$	0	$-$
y	↘	0	↗	1	↘



When $x=1$, the maximum value is 1, and when $x=0$, the minimum value is 0

(2) Find the maximum and minimum values for $-1 \leq x \leq 2$.

From $y' = -6x^2 + 6x$, we can get $-6x^2 + 6x = 0, -6x(x-1) = 0, x = 0, 1$.

The increase/decrease table and graph are shown on the right.

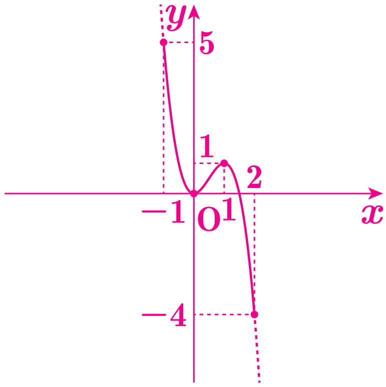
x	-1	$\cdots$	0	$\cdots$	1	$\cdots$	2
y'		$-$	0	$+$	0	$-$	
y	5	↘	0	↗	1	↘	-4

When $x=0$, the local minimum value is 0.

When $x=1$, the local maximum value is 1.

When $x=-1$, the maximum value is 5.

When $x=2$, the minimum value is -4.



When $x=-1$, the maximum value is 5, and when $x=2$ the minimum value is -4

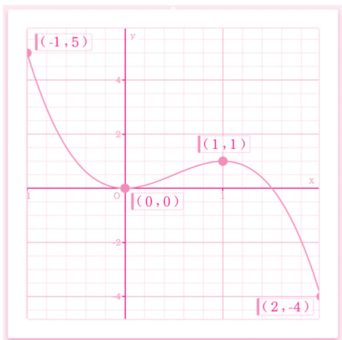
check

Press ⌂, select [Table], press OK, then clear the previous data by pressing ⏻

Press ⋯, select [Define f(x)/g(x)], press OK, select [Define f(x)], press OK, after inputting f(x)$=-2x^3+3x^2$, press EXE

Press ⋯, select [Table Range], press OK, after inputting [Start:-1, End:2, Step:1], select [Execute], press EXE

Press ⇧ x, scan the QR code to display a graph.



Increase/decrease of functions and applicable graphs (2)

TARGET To understand how to use graphs to find the number of real roots in an equation.

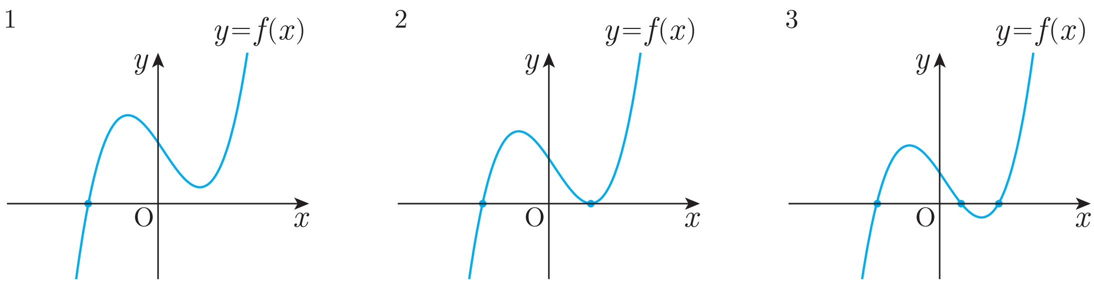
STUDY GUIDE

Number of real roots in the equation

The number of real roots in the equation $f(x)=0$ is the x-coordinates with common points on the x-axis of the graph of the function $y=f(x)$ x. Therefore, by determining how the function $y=f(x)$ increases and decreases, and drawing the graph, we can determine the number of real roots of the equation $f(x)=0$.

(Number of real roots in the equation $f(x)=0$)
$\Leftrightarrow$ (Graph of the function $y=f(x)$ and the number of common points on the x axis)

Number of real roots in $f(x)=0$



EXERCISE

◆ Find the number of different real roots in the equation $x^3-3x+1=0$.

For the function $y=x^3-3x+1$, we get $y'=3x^2-3=3(x-1)(x+1)$.

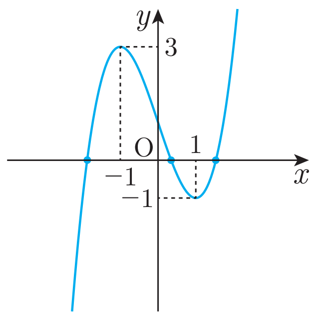
When $y'=0$, we get $x=\pm1$.

The increase/decrease table and graph are shown on the right.

Therefore, the number of real roots is **3**.

x	…	-1	…	1	…
y'	+	0	−	0	+
y	↗	3	↘	-1	↗

3



check

Press ⌂, select [Table], press OK, then clear the previous data by pressing ⏻

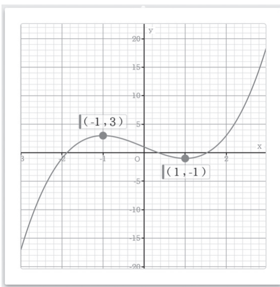
Press ⋯, select [Define f(x)/g(x)], press OK, select [Define f(x)], press OK,

after inputting f(x)= x^3-3x+1, press EXE

Press ⋯, select [Table Range], press OK, after inputting

[Start:−3, End:3, Step:1], select [Execute], press EXE

Press ⬆ x, scan the QR code to display a graph.



PRACTICE

◆ Find the number of different real roots in the equation $x^3 - 6x^2 + 2 = 0$.

For the function $y = x^3 - 6x^2 + 2$, we get $y' = 3x^2 - 12x = 3x(x-4)$.

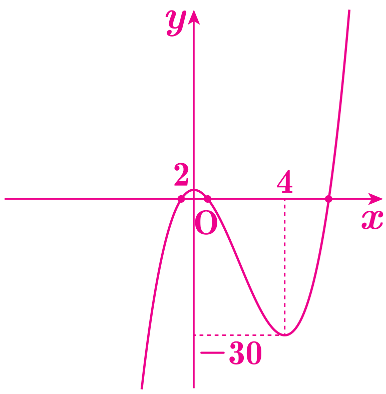
When $y'=0$, we get $x=0, 4$.

The increase/decrease table and graph are shown on the right.

Therefore, the number of real roots is 3.

x	$\cdots$	0	$\cdots$	4	$\cdots$
y'	+	0	−	0	+
y	↗	2	↘	−30	↗

3



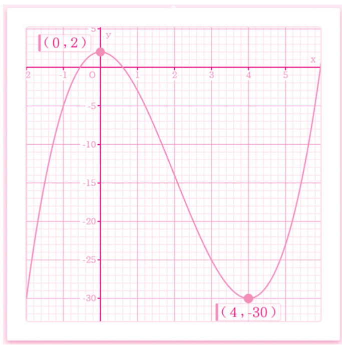
check

Press ⌂, select [Table], press OK, then clear the previous data by pressing ⏻

Press ⋯, select [Define f(x)/g(x)], press OK, select [Define f(x)], press OK, after inputting f(x)$=x^3 - 6x^2 + 2$, press EXE

Press ⋯, select [Table Range], press OK, after inputting [Start:−2, End:6, Step:1], select [Execute], press EXE

Press ⇧ x, scan the QR code to display a graph.



Increase/decrease of functions and applicable graphs (3)

TARGET To understand proofs of and methods to prove inequalities of cubic functions.

STUDY GUIDE

Proving inequalities of cubic functions

We can prove the inequality $f(x) \geq g(x)$ $(x \geq a)$ in the following way, by determining the increases and decreases of the function. Use the property of $f(x) \geq g(x) \Leftrightarrow f(x)-g(x) \geq 0$ to determine the increase and decrease of the function $f(x)-g(x)$ at $x \geq a$ to show the minimum value is ≥ 0.

Proof of $f(x) \geq g(x)$

$F(x)=f(x)-g(x)$ minimum value ≥ 0

EXERCISE

◆ If $x \geq 1$, then prove the inequality $x^3+9x \geq 6x^2$. Also, solve for when the equality sign holds.

[Proof]

Given $f(x)=x^3-6x^2+9x$, we can get

$f'(x)=3x^2-12x+9=3(x-1)(x-3)$.

When $f'(x)=0$, we get $x=1, 3$.

The increase/decrease table and graph are shown on the right.

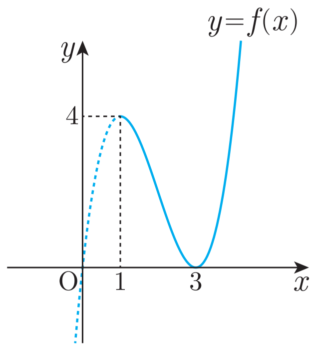
Thus, when $x \geq 1$, then $f(x)$ has a minimum value of 0 at $x=3$.

Therefore, when $x \geq 1$, then $f(x) \geq 0$, such that $x^3-6x^2+9x \geq 0$

Specifically, $x^3+9x \geq 6x^2$.

Also, the equality sign holds when $x=3$.

x	1	…	3	…
$f'(x)$	0	−	0	+
$f(x)$	4	↘	0	↗



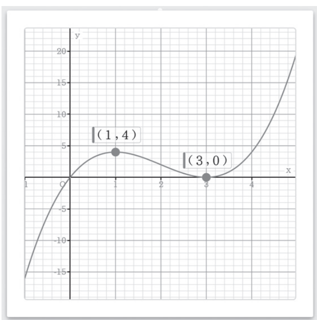
check

Press ⌂, select [Table], press OK, then clear the previous data by pressing ⏻

Press ⋯, select [Define f(x)/g(x)], press OK, select [Define f(x)], press OK, after inputting $f(x)=x^3-6x^2+9x$, press EXE

Press ⋯, select [Table Range], press OK, after inputting [Start:−1, End:5, Step:1], select [Execute], press EXE

Press ⇧ ⓧ, scan the QR code to display a graph.

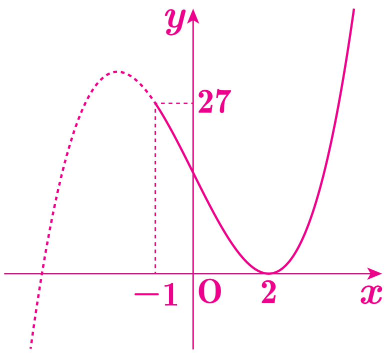


PRACTICE

◆ If $x \geq -1$, then prove the inequality $x^3+16 \geq 12x$. Also, solve for when the equality sign holds.

[Proof]

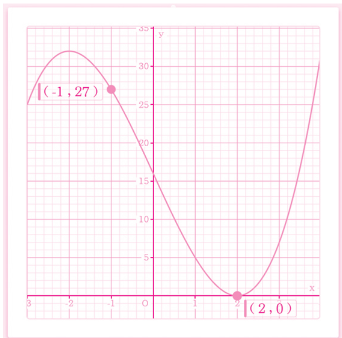
x	-1	$\cdots$	2	$\cdots$
$f'(x)$	$-$	$-$	0	$+$
$f(x)$	27	↘	0	↗



Given $f(x)=x^3+16-12x$, we can get
$f'(x)=3x^2-12=3(x-2)(x+2)$.
When $f'(x)=0$, we get $x=\pm 2$.
The increase/decrease table and graph are shown on the right.
Thus, when $x \geq -1$, then $f(x)$ has a minimum value of 0 at $x=2$.
Therefore, when $x \geq -1$, then $f(x) \geq 0$, such that $x^3+16-12x \geq 0$
Specifically, $x^3+16 \geq 12x$.
Also, the equality sign holds when $x=2$.

check

Press ⌂, select [Table], press OK, then clear the previous data by pressing ⏻
Press ⋯, select [Define f(x)/g(x)], press OK, select [Define f(x)], press OK, after inputting $f(x)=x^3-12x+16$, press EXE
Press ⋯, select [Table Range], press OK, after inputting [Start:−3, End:4, Step:1], select [Execute], press EXE
Press ⇧ ⓧ, scan the QR code to display a graph.



Indefinite integrals

TARGET To understand indefinite integrals.

STUDY GUIDE

Indefinite integrals

For a function $f(x)$, $f(x)$ is differentiated to the function $F(x)$, which is called the **primitive function** of $f(x)$. Specifically, it is $F'(x)=f(x)$.

Ex. $(x^3)' = 3x^2, (x^3+5)' = 3x^2, (x^3-8)' = 3x^2$ such that x^3, x^3+5, x^3-8 are all primitive functions of $3x^2$.

Thus, there are an infinite number of primitive functions for $f(x)$, and the only differences are the constants.
Given that $F(x)$ is 1 of the primitive functions of the function $f(x)$, then $F(x)+C$ (where C is any constant) is written as $\int f(x)dx$, which we call an **indefinite integral** of $f(x)$.
Finding the indefinite integral $\int f(x)dx$ of the function $f(x)$ is called **integration**, and the constant C is called a **constant of integration**.

$$F'(x) = f(x) \Leftrightarrow \int f(x)dx = F(x) + C$$

Ex. $\int 3x^2\,dx = x^3 + C$ (C is the constant of integration)

EXTRA Info.
Integration and differentiation are inverse operations.

$$3x^2 \underset{\text{Differentiation}}{\overset{\text{Integration}}{\rightleftarrows}} x^3 + C$$

EXERCISE

◆ Solve the following problems.

(1) Select all the answers from the following (a) to (e) that are primitive functions of $2x$.

(a) x^2 (b) $2x^2$ (c) x^2-9 (d) 2 (e) $x^2+\frac{1}{2}$

$$(x^2)' = 2x, (2x^2)' = 4x, (x^2-9)' = 2x, (2)' = 0, \left(x^2+\frac{1}{2}\right)' = 2x$$

All functions differentiated to $2x$ are primitive functions of $2x$.

(a), (c), (e)

(2) Use $(x^4)' = 4x^3$ to find $\int 4x^3\,dx$.

Since $F'(x) = f(x) \Leftrightarrow \int f(x)dx = F(x) + C$, we get $\int 4x^3\,dx = x^4 + C$ (C is the constant of integration).

$x^4 + C$ (C is the constant of integration)

(3) Integrate the function $f(x)=x$.

From $\left(\frac{1}{2}x^2\right)' = x$, since $\frac{1}{2}x^2$ is 1 primitive function of x, we get $\int x\,dx = \frac{1}{2}x^2 + C$ (C is the constant of integration).

$\frac{1}{2}x^2 + C$ (C is the constant of integration)

PRACTICE

◆ Solve the following problems.

(1) Select all the answers from the following (a) to (e) that are primitive functions of $5x^4$.

(a) $5x^5$ (b) x^5 (c) $20x^3$ (d) x^5+3 (e) $x^5-\frac{4}{5}$

$$(5x^5)'=25x^4, (x^5)'=5x^4, (20x^3)'=60x^2, (x^5+3)'=5x^4, \left(x^5-\frac{4}{5}\right)'=5x^4$$

All functions differentiated to $5x^4$ are primitive functions of $5x^4$.

(b), (d), (e)

(2) Use $\left(\frac{1}{3}x^3\right)'=x^2$ to find $\int x^2\,dx$.

Since $F'(x)=f(x) \Leftrightarrow \int f(x)dx=F(x)+C$, we get $\int x^2\,dx=\frac{1}{3}x^3+C$ (C is the constant of integration).

$\frac{1}{3}x^3+C$ **(C is the constant of integration)**

(3) Integrate the function $f(x)=x^3$.

From $\left(\frac{1}{4}x^4\right)'=x^3$, since $\frac{1}{4}x^4$ is 1 primitive function of x^3, we get

$\int x^3dx=\frac{1}{4}x^4+C$ (C is the constant of integration).

$\frac{1}{4}x^4+C$ **(C is the constant of integration)**

Properties of indefinite integrals

TARGET To understand the properties of indefinite integrals and how to calculate them.

STUDY GUIDE

How to find indefinite integrals

From $(x)' = 1, \left(\frac{1}{2}x^2\right)' = x, \left(\frac{1}{3}x^3\right)' = x^2, \cdots$, the following equation holds when n is an integer greater than 0 and C is a constant.

$$\int x^n dx = \frac{1}{n+1}x^{n+1} + C$$

EXTRA Info.

When $n=0$, we get $\int 1dx$, but it is usually written without the 1, as $\int dx$. Also, C is called the constant of integration, however it is often not written.

Properties of indefinite integrals

The following equations hold for indefinite integrals of constant factor, sums, and differences of functions.

(1) $\int kf(x)dx = k\int f(x)dx$ (k **is a constant**)

(2) $\int \{f(x)+g(x)\}dx = \int f(x)dx + \int g(x)dx$

(3) $\int \{f(x)-g(x)\}dx = \int f(x)dx - \int g(x)dx$

explanation

Let the primitive functions of the functions $f(x)$ and $g(x)$ be $F(x)$ and $G(x)$ respectively, then we get $F'(x)=f(x)$ and $G'(x)=g(x)$.

(1) Given k is a constant, then we get $\{kF(x)\}'=kF'(x)=kf(x)$.

(2) We get $\{F(x)+G(x)\}'=F'(x)+G'(x)=f(x)+g(x)$.

(3) We get $\{F(x)-G(x)\}'=F'(x)-G'(x)=f(x)-g(x)$.

EXERCISE

1 Find the following indefinite integrals.

(1) $\int x^2\,dx$

$= \dfrac{1}{2+1}x^{2+1} + C = \dfrac{1}{3}x^3 + C$

$\boldsymbol{\dfrac{1}{3}x^3 + C}$

(2) $\int x^3\,dx$

$= \dfrac{1}{3+1}x^{3+1} + C = \dfrac{1}{4}x^4 + C$

$\boldsymbol{\dfrac{1}{4}x^4 + C}$

(3) $\int 1dx$

$= \int x^0\,dx = \dfrac{1}{0+1}x^{0+1} + C = x + C$

$\boldsymbol{x + C}$

2 Solve the following problems.

(1) Find $\int (5x^3 - 7x^2 + 3x - 4)dx$.

$= 5\int x^3\,dx - 7\int x^2\,dx + 3\int x\,dx - 4\int dx$

$= 5 \cdot \dfrac{1}{4}x^4 - 7 \cdot \dfrac{1}{3}x^3 + 3 \cdot \dfrac{1}{2}x^2 - 4 \cdot x + C$

$= \dfrac{5}{4}x^4 - \dfrac{7}{3}x^3 + \dfrac{3}{2}x^2 - 4x + C$

The constant of integration is written as $+C$, not separately for each term, but as 1.

$\boldsymbol{\dfrac{5}{4}x^4 - \dfrac{7}{3}x^3 + \dfrac{3}{2}x^2 - 4x + C}$

(2) Find $\int (12t^2 - 6t + 5)dt$.

$= 12\int t^2\,dt - 6\int t\,dt + 5\int dt$

$= 12 \cdot \dfrac{1}{3}t^3 - 6 \cdot \dfrac{1}{2}t^2 + 5 \cdot t + C$

$= 4t^3 - 3t^2 + 5t + C$

This formula also holds for variables other than x.

$\boldsymbol{4t^3 - 3t^2 + 5t + C}$

(3) Find a function $F(x)$ that satisfies the conditions $F'(x)=8x-3$ and $F(-1)=5$.

$F(x) = \int (8x-3)dx = 8 \cdot \dfrac{1}{2}x^2 - 3 \cdot x + C = 4x^2 - 3x + C$

From $F(-1) = 4 \cdot (-1)^2 - 3 \cdot (-1) + C = 5$, we can get $7 + C = 5, C = -2$.

Therefore, we get $F(x) = 4x^2 - 3x - 2$.

$\boldsymbol{F(x) = 4x^2 - 3x - 2}$

PRACTICE

1 Find the following indefinite integrals.

(1) $\int x^4\,dx$

$= \frac{1}{4+1}x^{4+1} + C = \frac{1}{5}x^5 + C$

$\frac{1}{5}x^5 + C$

(2) $\int x\,dx$

$= \frac{1}{1+1}x^{1+1} + C = \frac{1}{2}x^2 + C$

$\frac{1}{2}x^2 + C$

2 Solve the following problems.

(1) Find $\int (3x^2 + 4x - 1)dx$.

$= 3\int x^2\,dx + 4\int x\,dx - \int dx$

$= 3 \cdot \frac{1}{3}x^3 + 4 \cdot \frac{1}{2}x^2 - 1 \cdot x + C$

$= x^3 + 2x^2 - x + C$

$x^3 + 2x^2 - x + C$

(2) Find $\int \left(t^2 - \frac{1}{3}t - 9\right)dt$.

$= \int t^2\,dt - \frac{1}{3}\int t\,dt - 9\int dt$

$= \frac{1}{3}t^3 - \frac{1}{3} \cdot \frac{1}{2}t^2 - 9 \cdot t + C$

$= \frac{1}{3}t^3 - \frac{1}{6}t^2 - 9t + C$

$\frac{1}{3}t^3 - \frac{1}{6}t^2 - 9t + C$

(3) Find a function $F(x)$ that satisfies the conditions $F'(x)=6x-5$ and $F(2)=3$.

$F(x) = \int (6x-5)dx = 6 \cdot \frac{1}{2}x^2 - 5 \cdot x + C = 3x^2 - 5x + C$

From $F(2) = 3 \cdot 2^2 - 5 \cdot 2 + C = 3$**, we can get** $2 + C = 3, C = 1$**.**

Therefore, we get $F(x) = 3x^2 - 5x + 1$**.**

$F(x) = 3x^2 - 5x + 1$

Properties of definite integrals

TARGET To understand the properties of definite integrals and how to calculate them.

STUDY GUIDE

Definite integrals

Given $F(x)$ is 1 of the primitive functions of the function $f(x)$, then we say that the value of $F(b)-F(a)$ for 2 real numbers a and b is the **definite integral** of $f(x)$ from a to b, which is expressed as follows.

When $F'(x)=f(x)$, $\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$

Here, we say that a and b are respectively the **lower limit** and **upper limit** of the definite integral; and to find this definite integral is called **integrating the function $f(x)$ from a to b**.

Also, since $[F(x)+C]_a^b = \{F(b)+C\} - \{F(a)+C\} = F(b) - F(a)$, the constant of integration C is not needed in the definite integral.

Properties of definite integrals

The following equations hold for definite integrals of constant multiples, sums, and differences of functions.

(1) $\int_a^b kf(x)dx = k\int_a^b f(x)dx$ (**k is a constant**)

(2) $\int_a^b \{f(x)+g(x)\}dx = \int_a^b f(x)dx + \int_a^b g(x)dx$

(3) $\int_a^b \{f(x)-g(x)\}dx = \int_a^b f(x)dx - \int_a^b g(x)dx$

(4) $\int_a^a f(x)dx = 0$

(5) $\int_a^b f(x)dx = -\int_b^a f(x)dx$

(6) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

(Magnitude relationship of a, b, and c does not matter)

EXERCISE

◆ Find the following definite integrals.

(1) $\int_{-1}^{2}(x^2-3x+1)dx$

Use properties (1), (2), and (3). These properties are the same as for the indefinite integral, so find the indefinite integral and then substitute the value of x and do the calculation.

$$=\left[\frac{1}{3}x^3-\frac{3}{2}x^2+x\right]_{-1}^{2}=\left(\frac{1}{3}\cdot 2^3-\frac{3}{2}\cdot 2^2+2\right)-\left\{\frac{1}{3}\cdot(-1)^3-\frac{3}{2}\cdot(-1)^2+(-1)\right\}=\frac{3}{2}$$

$\frac{3}{2}$

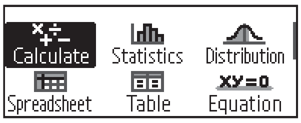
OTHER METHODS This calculation can also be done by substituting the value of x for each term.

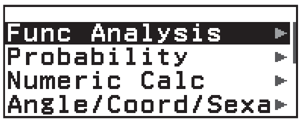
$$=\left[\frac{1}{3}x^3-\frac{3}{2}x^2+x\right]_{-1}^{2}=\frac{1}{3}\cdot\{2^3-(-1)^3\}-\frac{3}{2}\cdot\{2^2-(-1)^2\}+\{2-(-1)\}=\frac{3}{2}$$

check

Press [HOME], select [Calculate], press [OK]

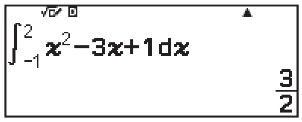
Press [CATALOG], select [Func Analysis], press [OK], select [Integration($\int$)], press [OK]





On the screen that is displayed, you can input the function and the upper limit and lower limit values to find the definite integral.

[x] [■²] [−] [3] [x] [+] [1] [∨] [−] [1] [∧] [2] [EXE]



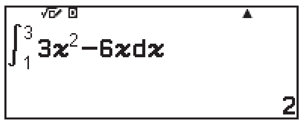
(2) $\int_{1}^{3}(4x^2-3x+2)dx-\int_{1}^{3}(x^2+3x+2)dx$

Since the upper limit and lower limit of the definite integral are equal, we can use property (3) to combine them into 1.

$$=\int_{1}^{3}\{(4x^2-3x+2)-(x^2+3x+2)\}dx=\int_{1}^{3}(3x^2-6x)\,dx=[x^3-3x^2]_1^3=(3^3-3\cdot 3^2)-(1^3-3\cdot 1^2)=2$$

2

[CATALOG] [OK] [∨] [OK]   [3] [x] [■²] [−] [6] [x] [∨] [1] [∧] [3] [EXE]



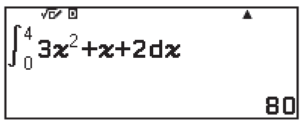
(3) $\int_{0}^{3}(3x^2+x+2)dx+\int_{3}^{4}(3x^2+x+2)dx$

Use property (6) to combine them into 1.

$$=\int_{0}^{4}(3x^2+x+2)dx=\left[x^3+\frac{1}{2}x^2+2x\right]_0^4=\left(4^3+\frac{1}{2}\cdot 4^2+2\cdot 4\right)-\left(0^3+\frac{1}{2}\cdot 0^2+2\cdot 0\right)=80$$

80

[CATALOG] [OK] [∨] [OK]   [3] [x] [■²] [+] [x] [+] [2] [∨] [0] [∧] [4] [EXE]



PRACTICE

◆ Find the following definite integrals.

(1) $\int_{-2}^{3}(2x^2+x-3)dx$

$$=\left[\frac{2}{3}x^3+\frac{1}{2}x^2-3x\right]_{-2}^{3}=\left(\frac{2}{3}\cdot 3^3+\frac{1}{2}\cdot 3^2-3\cdot 3\right)-\left\{\frac{2}{3}\cdot(-2)^3+\frac{1}{2}\cdot(-2)^2-3\cdot(-2)\right\}$$

$$=\frac{65}{6}$$

$\frac{65}{6}$

[MENU] [OK] [∨] [OK] [2] [x] [■²] [+] [x] [−] [3] [∨] [−] [2] [∧] [3] [EXE]

$\int_{-2}^{3} 2x^2+x-3dx$ $\quad \frac{65}{6}$

(2) $\int_{1}^{2}(2x^2-3x+4)dx+\int_{1}^{2}(x^2+3x+1)dx$

$$=\int_{1}^{2}\{(2x^2-3x+4)+(x^2+3x+1)\}dx=\int_{1}^{2}(3x^2+5)dx$$

$$=[x^3+5x]_{1}^{2}=(2^3+5\cdot 2)-(1^3+5\cdot 1)=12$$

12

[MENU] [OK] [∨] [OK] [3] [x] [■²] [+] [5] [∨] [1] [∧] [2] [EXE]

$\int_{1}^{2} 3x^2+5dx$ $\quad 12$

(3) $\int_{0}^{1}(x^2-3x+4)dx+\int_{1}^{3}(x^2-3x+4)dx$

$$=\int_{0}^{3}(x^2-3x+4)dx=\left[\frac{1}{3}x^3-\frac{3}{2}x^2+4x\right]_{0}^{3}$$

$$=\left(\frac{1}{3}\cdot 3^3-\frac{3}{2}\cdot 3^2+4\cdot 3\right)-\left(\frac{1}{3}\cdot 0^3-\frac{3}{2}\cdot 0^2+4\cdot 0\right)=\frac{15}{2}$$

$\frac{15}{2}$

[MENU] [OK] [∨] [OK] [x] [■²] [−] [3] [x] [+] [4] [∨] [0] [∧] [3] [EXE]

$\int_{0}^{3} x^2-3x+4dx$ $\quad \frac{15}{2}$

Definite integrals and differential calculus

TARGET To understand the relation between definite integrals and differential calculus.

STUDY GUIDE

Definite integrals and differential calculus

When a is a constant, the definite integral $\int_a^x f(t)dt$ is a function of the upper limit x. When this is differentiated by x, its derivative becomes $f(x)$.

$$\frac{d}{dx}\int_a^x f(t)dt = f(x) \quad (a \text{ is a constant})$$

explanation

Given $F(t)$ is 1 of the primitive functions of $f(t)$, then $F'(t)=f(t)$.

Therefore, we get $\frac{d}{dx}\int_a^x f(t)dt = \frac{d}{dx}[F(t)]_a^x = \frac{d}{dx}\{F(x)-F(a)\}$.

Since $F(a)$ is a constant, when differentiated it becomes 0, so we get $\frac{d}{dx}\{F(x)-F(a)\} = \frac{d}{dx}F(x) = F'(x) = f(x)$.

Definite integral of even functions and odd functions

As with $f(x)=x^2, f(x)=x^4, \cdots$ we say that a function $f(x)$, such that $f(-x)=f(x)$ always holds, is an **even function**, and the graph of an even function is symmetric with respect to the y axis.

As with $f(x)=x, f(x)=x^3, \cdots$ we say that a function $f(x)$, such that $f(-x)=f(x)$ always holds, is an **odd function**, and the graph of an odd function is symmetric with respect to the origin.

The following equations hold for definite integrals of even functions and odd functions.

$$f(x) \text{ is an even function} \Rightarrow \int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$$

$$f(x) \text{ is an odd function} \Rightarrow \int_{-a}^{a} f(x)dx = 0$$

Ex. (1) $\int_{-3}^{3} x^4\,dx = 2\int_0^3 x^4\,dx = 2\left[\frac{1}{5}x^5\right]_0^3 = 2\left(\frac{1}{5}\cdot 3^5 - \frac{1}{5}\cdot 0^5\right) = \frac{486}{5}$

(2) $\int_{-12}^{12} x^5\,dx = 0$

(3) $\int_{-2}^{2}(x^3-3x^2+5x+1)dx = 2\int_0^2(-3x^2+1)dx = 2[-x^3+x]_0^2 = 2\{(-2^3+2)-(-0^3+0)\} = -12$

EXERCISE

◆ Solve the following problems.

(1) Find the derivative of the function $f(x)=\int_2^x (6t^2-2t)dt$.

$f'(x)=\dfrac{d}{dx}\int_2^x (6t^2-2t)dt=6x^2-2x$

$$\boldsymbol{f'(x)=6x^2-2x}$$

(2) Given a is a constant. Find the values of the constant a and the function $f(x)$ that satisfy $\int_a^x f(t)dt=x^2+3x-4$.

Differentiate both sides of the given equation by x.

From $\dfrac{d}{dx}\int_a^x f(t)dt=\dfrac{d}{dx}(x^2+3x-4)$, we get $f(x)=2x+3$.

Let $x=a$ in the given equation.

Since $\int_a^a f(t)dt=0$, we get $0=a^2+3a-4, a=1,-4$.

$$\boldsymbol{f(x)=2x+3, a=1,-4}$$

(3) Find the function $f(x)$ that satisfies the equation $f(x)=x^2+2\int_0^1 f(t)dt$.

If we let $\int_0^1 f(t)dt=a$ (a is a constant)…(i), then we get $f(x)=x^2+2a$ …(ii).

From (i) and (ii), we get $\int_0^1 (t^2+2a)dt=a$ and $\int_0^1 (t^2+2a)dt=\left[\dfrac{1}{3}t^3+2at\right]_0^1=\dfrac{1}{3}+2a$.

Therefore, since we get $\dfrac{1}{3}+2a=a, a=-\dfrac{1}{3}$, then we get $f(x)=x^2-\dfrac{2}{3}$.

$$\boldsymbol{f(x)=x^2-\frac{2}{3}}$$

PRACTICE

◆ Solve the following problems.

(1) Find the derivative of the function $f(x)=\int_1^x (3t^2+4t)dt$.

$$f'(x)=\frac{d}{dx}\int_1^x (3t^2+4t)dt = 3x^2+4x$$

$$f'(x)=3x^2+4x$$

(2) Given a is a constant. Find the values of the constant a and the function $f(x)$ that satisfies the equation $\int_a^x f(t)dt = x^2-5x+6$.

Differentiate both sides of the given equation by x.

From $\frac{d}{dx}\int_a^x f(t)dt = \frac{d}{dx}(x^2-5x+6)$**, we get** $f(x)=2x-5$.

Let $x=a$ in the given equation.

Since $\int_a^a f(t)dt = 0$**, we get** $0=a^2-5a+6, a=2,3$.

$$f(x)=2x-5, a=2,3$$

(3) Find the function $f(x)$ that satisfies the equation $f(x)=3x^2+\int_{-1}^2 f(t)dt$.

If we let $\int_{-1}^2 f(t)dt = a$ (a **is a constant**)...(i)**, then we get** $f(x)=3x^2+a$ ...(ii).

From (i) **and** (ii)**, we get** $\int_{-1}^2 (3t^2+a)dt = a$ **and**

$\int_{-1}^2 (3t^2+a)dt = [t^3+at]_{-1}^2 = 9+3a$.

Therefore, since we get $9+3a=a, a=-\frac{9}{2}$**, then we get** $f(x)=3x^2-\frac{9}{2}$.

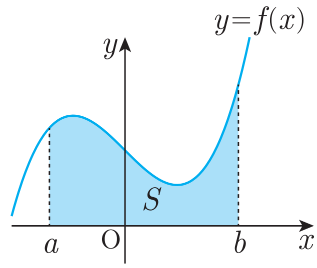
$$f(x)=3x^2-\frac{9}{2}$$

Definite integrals and the area of a shape (1)

TARGET To understand the area of the shape bounded by the curve and the x axis, and the 2 straight lines $x=a$ and $x=b$.

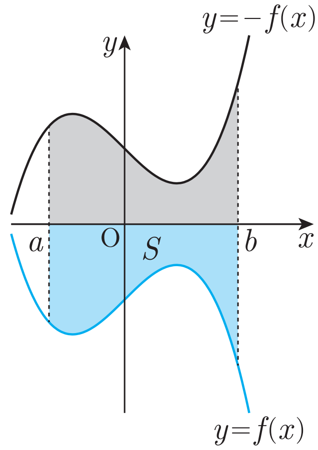
STUDY GUIDE

Definite integrals and the area of a shape



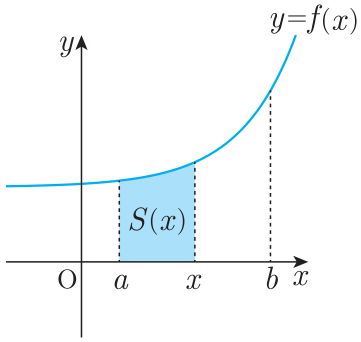
Given $a \leq x \leq b$, then when $f(x) \geq 0$, the area S of the part bounded by the curve $y=f(x)$ and the x axis, and the 2 straight lines $x=a$ and $x=b$, is as follows.

$$S = \int_a^b f(x)dx$$



Also, given $a \leq x \leq b$, then when $f(x) \leq 0$, the area S of the part bounded by the curve $y=f(x)$ and the x axis and the 2 straight lines $x=a$ and $x=b$ is the area of the part bounded by the curve $y=f(x)$ reflected across the x axis, and the x axis, and the 2 straight lines $x=a$ and $x=b$, given that $a \leq x \leq b$, which gives us $S = \int_a^b \{-f(x)\}dx = -\int_a^b f(x)dx$.

explanation



To find the area S from $x=a$ to $x=b$, we consider the function $S(x)$, which expresses the area from $x=a$ to $x=x$.

So $S(a)=0$ and $S(b)=S$.

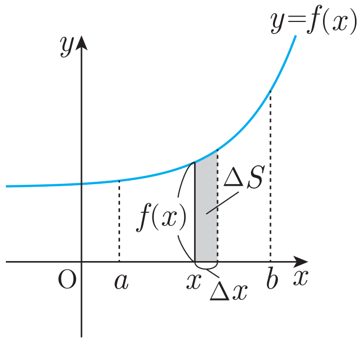
Given ΔS is the area from x to $x+\Delta x$, then ΔS can be approximated by the rectangular areas $f(x)\Delta x$ for very small Δx.

Thus, we get

$$\Delta S = f(x)\Delta x \Rightarrow \frac{\Delta S}{\Delta x} = f(x) \Rightarrow \lim_{\Delta x \to 0} \frac{\Delta S}{\Delta x} = \lim_{\Delta x \to 0} f(x) \Rightarrow \frac{dS}{dx} = f(x) .$$

Therefore, $S(x)$ is 1 of the primitive functions of $f(x)$.

Specifically, since $S(x) = \int f(x)dx$, we get $S = S(b) - S(a) = \int_a^b f(x)dx$.



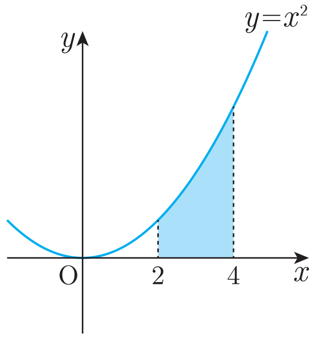
EXERCISE

◆ Find the area S of the part bounded by the following curve and straight lines.

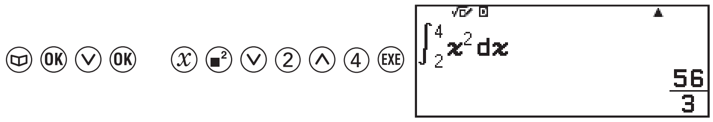
(1) The curve $y=x^2$ and the x axis, and 2 straight lines $x=2$ and $x=4$

For $2 \leq x \leq 4$, we get $y=x^2>0$.

$$S=\int_2^4 x^2\,dx=\left[\frac{1}{3}x^3\right]_2^4=\frac{1}{3}\cdot 4^3-\frac{1}{3}\cdot 2^3=\frac{56}{3}$$



$\dfrac{\mathbf{56}}{\mathbf{3}}$

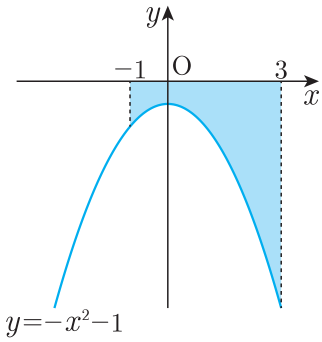


(2) The curve $y=-x^2-1$ and the x axis, and 2 straight lines $x=-1$ and $x=3$

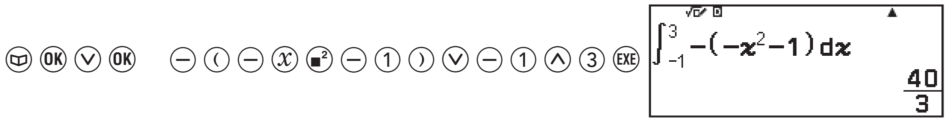
For $-1 \leq x \leq 3$, we get $y=-x^2-1<0$.

$$S=\int_{-1}^{3}\{-(-x^2-1)\}dx=-\int_{-1}^{3}(-x^2-1)dx=-\left[-\frac{1}{3}x^3-x\right]_{-1}^{3}$$

$$=-\left[\left(-\frac{1}{3}\cdot 3^3-3\right)-\left\{-\frac{1}{3}\cdot(-1)^3-(-1)\right\}\right]=\frac{40}{3}$$



$\dfrac{\mathbf{40}}{\mathbf{3}}$

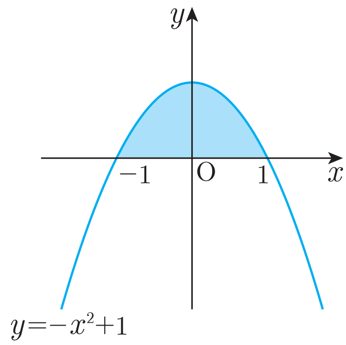


(3) The curve $y=-x^2+1$ and the x axis

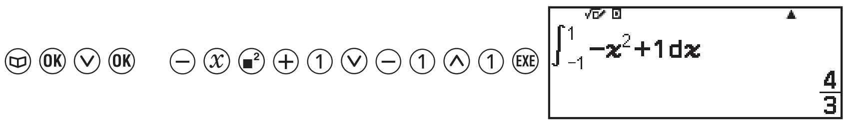
By solving $-x^2+1=0$, we get $x=\pm 1$, so the curve $y=-x^2+1$ crosses the x axis at $x=\pm 1$.

$$S=\int_{-1}^{1}(-x^2+1)dx=\left[-\frac{1}{3}x^3+x\right]_{-1}^{1}$$

$$=\left(-\frac{1}{3}\cdot 1^3+1\right)-\left\{-\frac{1}{3}\cdot(-1)^3+(-1)\right\}=\frac{4}{3}$$



$\dfrac{\mathbf{4}}{\mathbf{3}}$



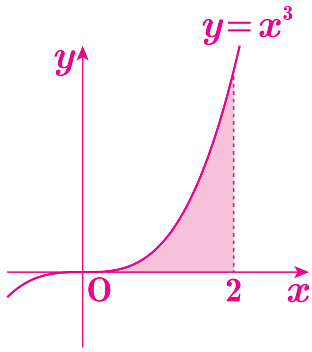
PRACTICE

◆ Find the area S of the part bounded by the following curve and straight lines.

(1) The curve $y=x^3$ and the x axis, and 2 straight lines $x=0$ and $x=2$

For $0 \leq x \leq 2$, we get $y=x^3 \geq 0$.

$$S=\int_0^2 x^3\,dx=\left[\frac{1}{4}x^4\right]_0^2=\frac{1}{4}\cdot 2^4-\frac{1}{4}\cdot 0^4=4$$



4

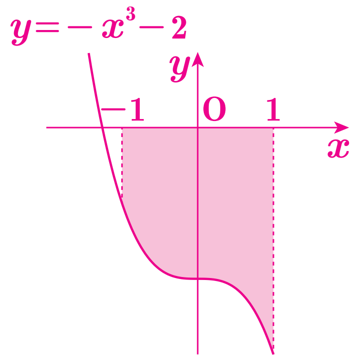
(2) The curve $y=-x^3-2$ and the x axis, and 2 straight lines $x=-1$ and $x=1$

For $-1 \leq x \leq 1$, we get $y=-x^3-2<0$.

$$S=\int_{-1}^{1}\{-(-x^3-2)\}dx=-\int_{-1}^{1}(-x^3-2)dx$$

$$=-\left[-\frac{1}{4}x^4-2x\right]_{-1}^{1}$$

$$=-\left[\left(-\frac{1}{4}\cdot 1^4-2\cdot 1\right)-\left\{-\frac{1}{4}\cdot(-1)^4-2\cdot(-1)\right\}\right]=4$$



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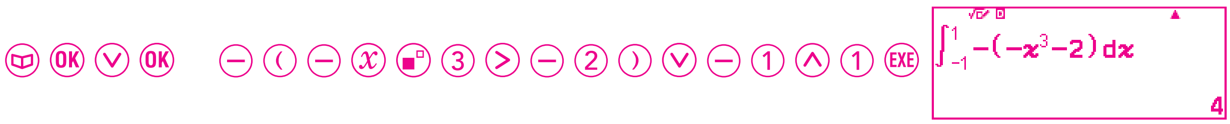


(3) The curve $y=3x^2-12$ and the x axis

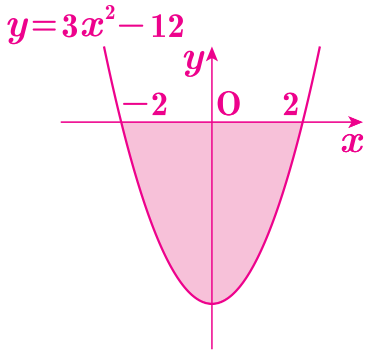
By solving $3x^2-12=0$, we get $x=\pm 2$, so the curve $y=3x^2-12$ crosses the x axis at $x=\pm 2$.

For $-2 \leq x \leq 2$, we get $y=3x^2-12 \leq 0$.

$$S=\int_{-2}^{2}\{-(3x^2-12)\}dx=-\int_{-2}^{2}(3x^2-12)dx$$

$$=-[x^3-12x]_{-2}^{2}=-[(2^3-12\cdot 2)-\{(-2)^3-12\cdot(-2)\}]$$

$$=32$$



32

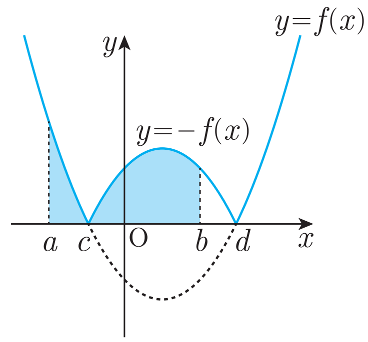
Definite integrals and the area of a shape (2)

TARGET To understand the definite integral of a function that has an absolute value.

STUDY GUIDE

Definite integral of a function that has an absolute value

If the function to be integrated has an absolute value, then we integrate the function after removing the absolute value symbol by splitting the integration interval.



$$|A|=\begin{cases} A & (\text{when } A\geq 0) \\ -A & (\text{when } A\leq 0)\end{cases}$$

As shown in the figure on the right, the function $y=|f(x)|$ is, when $x\leq c$ and $d\leq x$, $y=f(x)$, and when $c\leq x\leq d$, it is $y=-f(x)$.

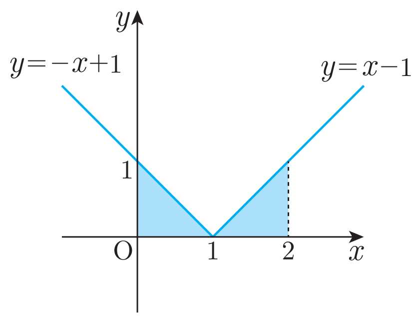
The definite integral from a to b of the function $y=|f(x)|$ is $S=\int_a^b |f(x)|dx=\int_a^c f(x)dx+\int_c^b -f(x)dx$.

EXERCISE

◆ Find the following definite integrals.

(1) $\int_0^2 |x-1|\,dx$

$|x-1|$ is $-x+1$ for $x\leq 1$, and it is $x-1$ for $1\leq x$.



$$=\int_0^1(-x+1)dx+\int_1^2(x-1)dx=\left[-\frac{1}{2}x^2+x\right]_0^1+\left[\frac{1}{2}x^2-x\right]_1^2$$

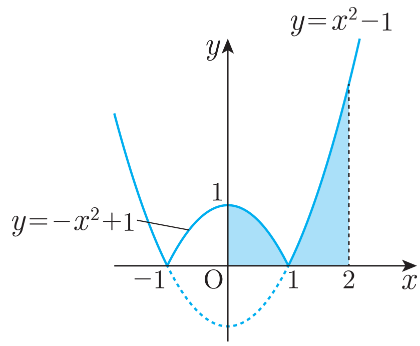
$$=\left(-\frac{1}{2}\cdot 1^2+1\right)+\left\{\frac{1}{2}\cdot 2^2-2-\left(\frac{1}{2}\cdot 1^2-1\right)\right\}=1$$

1

(book) OK ∨ OK (book) ∨ ∨ OK OK x − 1 ∨ 0 ∧ 2 EXE

(2) $\int_0^2 |x^2-1|\,dx$

$|x^2-1|$ is x^2-1 for $x\leq -1$ and $1\leq x$, and it is $-x^2+1$ for $-1\leq x\leq 1$.

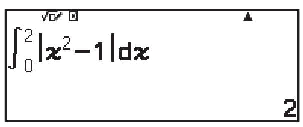


$$=\int_0^1(-x^2+1)dx+\int_1^2(x^2-1)dx=\left[-\frac{1}{3}x^3+x\right]_0^1+\left[\frac{1}{3}x^3-x\right]_1^2$$

$$=\left(-\frac{1}{3}\cdot 1^3+1\right)+\left\{\frac{1}{3}\cdot 2^3-2-\left(\frac{1}{3}\cdot 1^3-1\right)\right\}=2$$

2

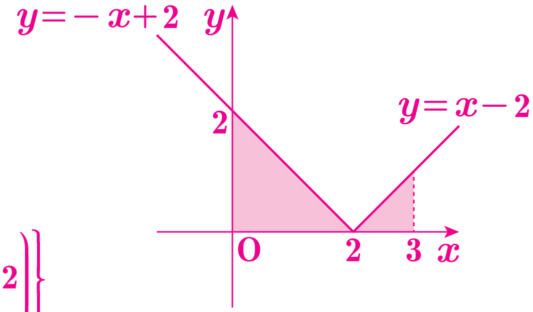
(book) OK ∨ OK (book) ∨ ∨ OK OK x x^2 − 1 ∨ 0 ∧ 2 EXE



◆ Find the following definite integrals.

(1) $\displaystyle\int_0^3 |x-2|\,dx$

$|x-2|$ is $-x+2$ for $x \leq 2$, and it is $x-2$ for $2 \leq x$.



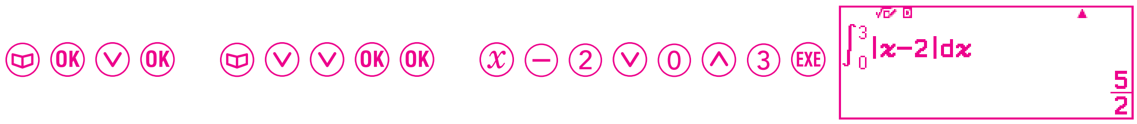
$$= \int_0^2 (-x+2)dx + \int_2^3 (x-2)dx$$

$$= \left[-\frac{1}{2}x^2 + 2x\right]_0^2 + \left[\frac{1}{2}x^2 - 2x\right]_2^3$$

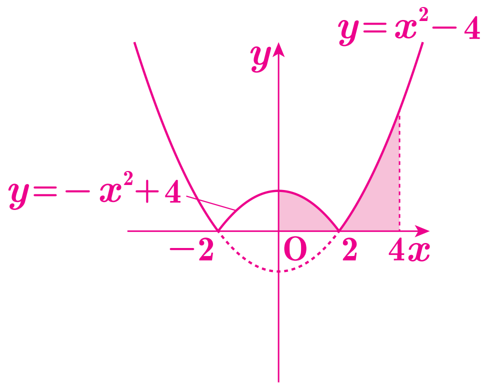
$$= \left(-\frac{1}{2}\cdot 2^2 + 2\cdot 2\right) + \left\{\frac{1}{2}\cdot 3^2 - 2\cdot 3 - \left(\frac{1}{2}\cdot 2^2 - 2\cdot 2\right)\right\}$$

$$= \frac{5}{2}$$

$$\frac{5}{2}$$



(2) $\displaystyle\int_0^4 |x^2-4|\,dx$



$|x^2-4|$ is x^2-4 for $x \leq -2$ and $2 \leq x$, and it is $-x^2+4$ for $-2 \leq x \leq 2$.

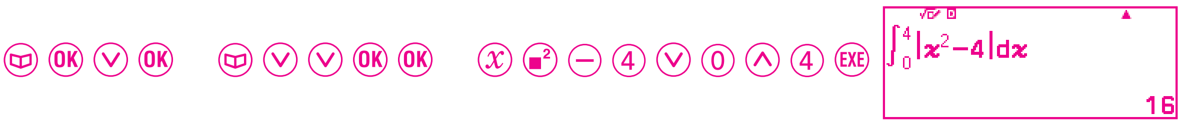
$$= \int_0^2 (-x^2+4)dx + \int_2^4 (x^2-4)dx$$

$$= \left[-\frac{1}{3}x^3 + 4x\right]_0^2 + \left[\frac{1}{3}x^3 - 4x\right]_2^4$$

$$= \left(-\frac{1}{3}\cdot 2^3 + 4\cdot 2\right) + \left\{\frac{1}{3}\cdot 4^3 - 4\cdot 4 - \left(\frac{1}{3}\cdot 2^3 - 4\cdot 2\right)\right\}$$

$$= 16$$

16



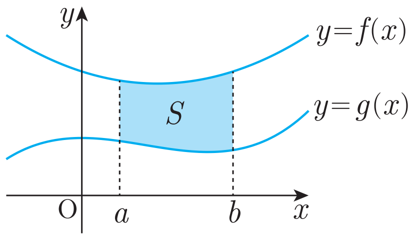
Definite integrals and the area of a shape (3)

TARGET To understand the area between 2 curves.

STUDY GUIDE

Area between 2 curves

Given 2 curves $y=f(x)$ and $y=g(x)$ with an interval of $a \leq x \leq b$, then when $f(x) \geq g(x)$, the area S of the part bounded by these 2 curves and 2 straight lines $x=a$ and $x=b$ is as follows.

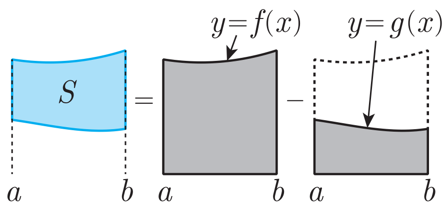


$$S = \int_a^b \{f(x) - g(x)\}dx$$

explanation

Given the interval is $a \leq x \leq b$ when $f(x) \geq g(x) \geq 0$, then we can consider the difference of the areas of the 2 shapes as in the figure on the right.

$$S = \int_a^b f(x)dx - \int_a^b g(x)dx = \int_a^b \{f(x) - g(x)\}dx$$



Also, given the interval is $a \leq x \leq b$, when $f(x)$ or $g(x)$ has a negative value, then the 2 curves are parallel translated in the y axis direction so that both have a positive value.

Given $y=f(x)+c$ and $y=g(x)+c$, we get $S = \int_a^b [f(x) + c - \{g(x) + c\}]dx = \int_a^b \{f(x) - g(x)\}dx$, which can be considered the same as above.

EXERCISE

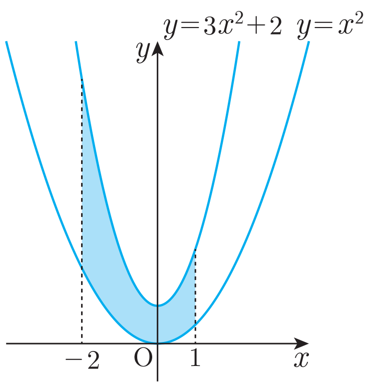
◆ Find the area S of the part bounded by the following curve and straight lines.

(1) The 2 curves $y = 3x^2 + 2$ and $y = x^2$, and the 2 straight lines $x=-2$ and $x=1$

Given $-2 \leq x \leq 1$, from $3x^2 + 2 - x^2 = 2x^2 + 2 > 0$, we get $3x^2 + 2 > x^2$.

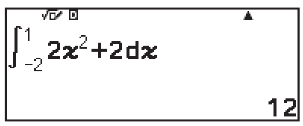
$$S = \int_{-2}^{1}(3x^2 + 2 - x^2)dx = \int_{-2}^{1}(2x^2 + 2)dx = \left[\frac{2}{3}x^3 + 2x\right]_{-2}^{1}$$

$$= \frac{2}{3} \cdot 1^3 + 2 \cdot 1 - \left\{\frac{2}{3} \cdot (-2)^3 + 2 \cdot (-2)\right\} = 12$$



12

[menu] [OK] [∨] [OK] [2] [x] [■²] [+] [2] [∨] [−] [2] [∧] [1] [EXE]



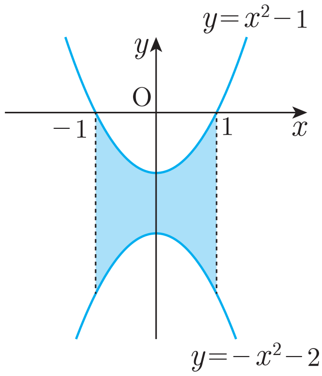
(2) 2 curves $y=x^2-1$ and $y=-x^2-2$, and 2 straight lines $x=-1$ and $x=1$

Given $-1\leq x\leq 1$, from $x^2-1-(-x^2-2)=2x^2+1>0$, we get

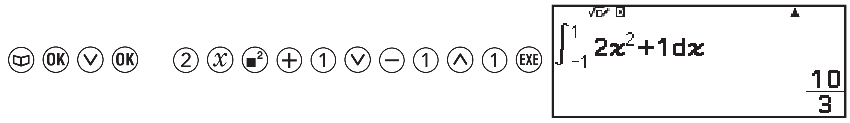
$x^2-1>-x^2-2$.

$$S=\int_{-1}^{1}\{x^2-1-(-x^2-2)\}dx=\int_{-1}^{1}(2x^2+1)dx=\left[\frac{2}{3}x^3+x\right]_{-1}^{1}$$

$$=\frac{2}{3}\cdot 1^3+1-\left\{\frac{2}{3}\cdot(-1)^3+(-1)\right\}=\frac{10}{3}$$



$\dfrac{10}{3}$



(3) 2 curves $y=x^2-1$ and $y=-x^2+1$

The x coordinates of the intersections of the 2 curves are

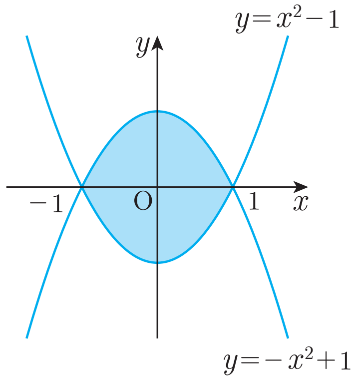
$x^2-1=-x^2+1, 2x^2=2, x=\pm 1$.

Given $-1\leq x\leq 1$, from $x^2-1-(-x^2+1)=2x^2-2\leq 0$, we get

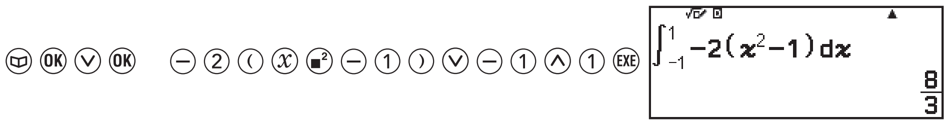
$x^2-1\leq -x^2+1$.

$$S=\int_{-1}^{1}\{-x^2+1-(x^2-1)\}dx=\int_{-1}^{1}\{-2(x^2-1)\}dx=-2\left[\frac{1}{3}x^3-x\right]_{-1}^{1}$$

$$=-2\left[\frac{1}{3}\cdot 1^3-1-\left\{\frac{1}{3}\cdot(-1)^3-(-1)\right\}\right]=\frac{8}{3}$$



$\dfrac{8}{3}$



PRACTICE

◆ Find the area S of the part bounded by the following curve and straight lines.

(1) 2 curves $y=2x^2+3$ and $y=x^2$, and 2 straight lines $x=-1$ and $x=2$

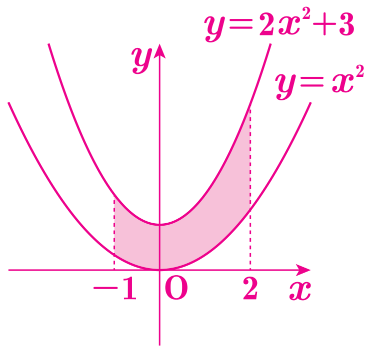
Given $-1\leq x\leq 2$, from $2x^2+3-x^2=x^2+3>0$,

we get $2x^2+3>x^2$.

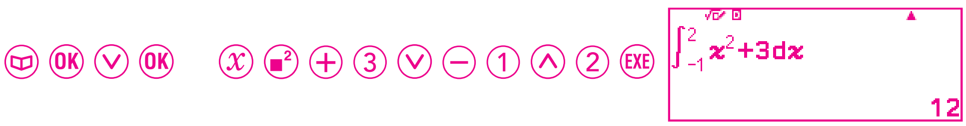
$$S=\int_{-1}^{2}(2x^2+3-x^2)dx=\int_{-1}^{2}(x^2+3)dx$$

$$=\left[\frac{1}{3}x^3+3x\right]_{-1}^{2}=\frac{1}{3}\cdot 2^3+3\cdot 2-\left\{\frac{1}{3}\cdot(-1)^3+3\cdot(-1)\right\}$$

$$=12$$

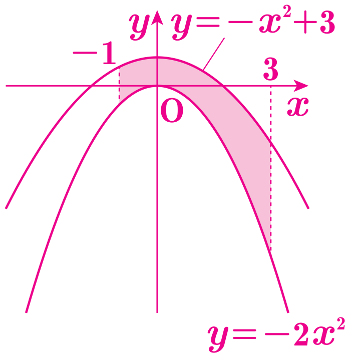


12



(2) 2 curves $y=-2x^2$ and $y=-x^2+3$, and 2 straight lines $x=-1$ and $x=3$

Given $-1 \leq x \leq 3$, from $-2x^2-(-x^2+3)=-x^2-3<0$,
we get $-2x^2<-x^2+3$.

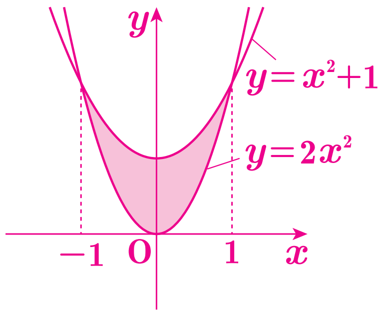


$$S=\int_{-1}^{3}\{-x^2+3-(-2x^2)\}dx=\int_{-1}^{3}(x^2+3)dx$$
$$=\left[\frac{1}{3}x^3+3x\right]_{-1}^{3}=\frac{1}{3}\cdot 3^3+3\cdot 3-\left\{\frac{1}{3}\cdot(-1)^3+3\cdot(-1)\right\}$$
$$=\frac{64}{3}$$

$$\frac{64}{3}$$

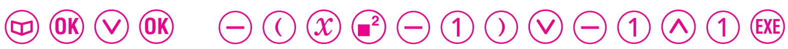
(3) 2 curves $y=2x^2$ and $y=x^2+1$

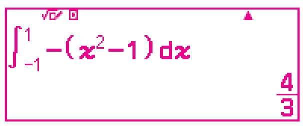
The x coordinates of the intersections of the 2 curves
are $2x^2=x^2+1, x^2=1, x=\pm 1$.
Given $-1 \leq x \leq 1$, from $2x^2-(x^2+1)=x^2-1 \leq 0$,
we get $2x^2 \leq x^2+1$.



$$S=\int_{-1}^{1}(x^2+1-2x^2)dx=\int_{-1}^{1}-(x^2-1)dx$$
$$=-\left[\frac{1}{3}x^3-x\right]_{-1}^{1}=-\left[\frac{1}{3}\cdot 1^3-1-\left\{\frac{1}{3}\cdot(-1)^3-(-1)\right\}\right]$$
$$=\frac{4}{3}$$

$$\frac{4}{3}$$





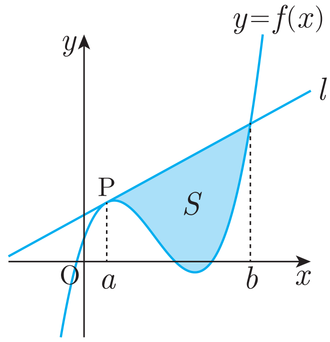
Definite integrals and the area of a shape (4)

TARGET To understand the area of a shape bounded by a curve and the tangent to that curve.

STUDY GUIDE

Area of a shape bounded by a curve and the tangent to that curve

The area of the shape bounded by the curve of the cubic function $y=f(x)$ and a tangent to a point P on its curve can be found by using the following method.



(1) Find the equation of the tangent l to the point P on the curve $y=f(x)$.

(2) Find the x coordinates of the intersections of the tangent l and the curve $y=f(x)$.

(3) Find the area of the bounded area by using the definite integral.

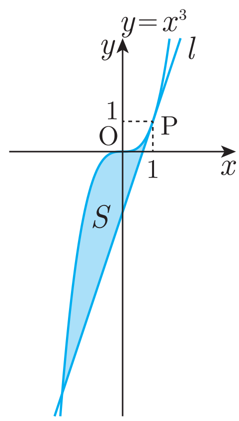
EXTRA Info.

Equation of the tangent to the point $(a, f(a))$ on the graph of the function $y=f(x)$

$$y-f(a)=f'(a)(x-a)$$

EXERCISE

◆ Solve the following problems with regard to the curve $y=x^3$.



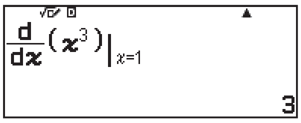
(1) Find the equation for the tangent l to the point P $(1, 1)$ on the curve $y=x^3$.

From $y=x^3$, and since $y'=3x^2$, by substituting $x=1$, we can get $y'=3\cdot 1^2=3$.

The equation of the tangent l is $y-1=3(x-1), y=3x-2$.

$$y=3x-2$$

[⌂] [OK] [OK] [x] [■□] [3] [>] [>] [1] [EXE]



(2) Find the x coordinates of the intersections of the tangent l and the curve $y=x^3$.

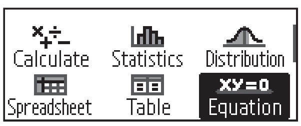
$x^3=3x-2, x^3-3x+2=0, (x-1)^2(x+2)=0, x=-2, 1$

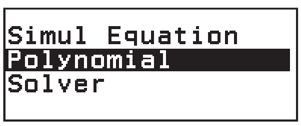
$$x=-2, 1$$

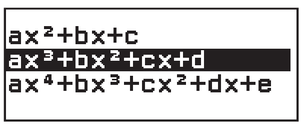
check

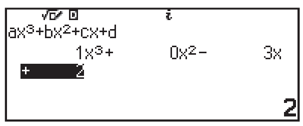
Press [⌂], select [Equation], press [OK], select [Polynomial], press [OK], select [ax^3+bx^2+cx+d], press [OK]

[1] [EXE] [0] [EXE] [(−)] [3] [EXE] [2] [EXE]



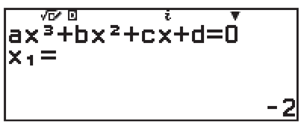


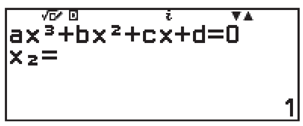




Determines the solution $x=-2$ and 1 for the cubic equation.

▷ [EXE] [EXE]





(3) Find the area S of the shape bounded by the tangent l and the curve $y = x^3$.

Given $-2 \leq x \leq 1$, we can get $x^3 \geq 3x-2$.

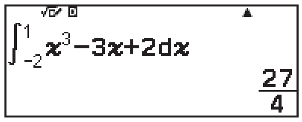
$$S = \int_{-2}^{1} \{x^3 - (3x-2)\}dx = \int_{-2}^{1} (x^3 - 3x + 2)dx = \left[\frac{1}{4}x^4 - \frac{3}{2}x^2 + 2x\right]_{-2}^{1}$$

$$= \frac{1}{4} \cdot 1^4 - \frac{3}{2} \cdot 1^2 + 2 \cdot 1 - \left\{\frac{1}{4} \cdot (-2)^4 - \frac{3}{2} \cdot (-2)^2 + 2 \cdot (-2)\right\} = \frac{27}{4}$$

$\frac{27}{4}$

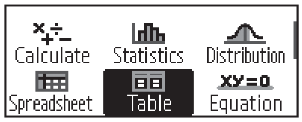
check

[CATALOG] [OK] [▽] [OK] [x] [■] [3] [>] [−] [3] [x] [+] [2] [▽] [−] [2] [△] [1] [EXE]

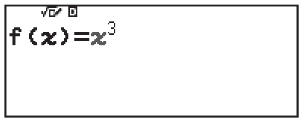


Checking graphs

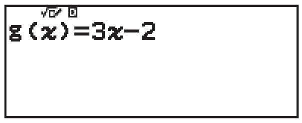
Press [HOME], select [Table], press [OK], then clear the previous data by pressing [AC]



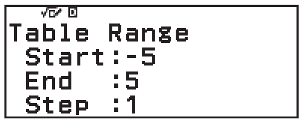
Press [TOOLS], select [Define f(x)/g(x)], press [OK], select [Define f(x)], press [OK], after inputting f(x)= x^3 , press [EXE]



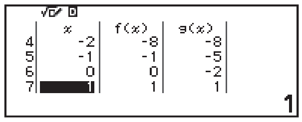
In the same way, input g(x)=3x−2.

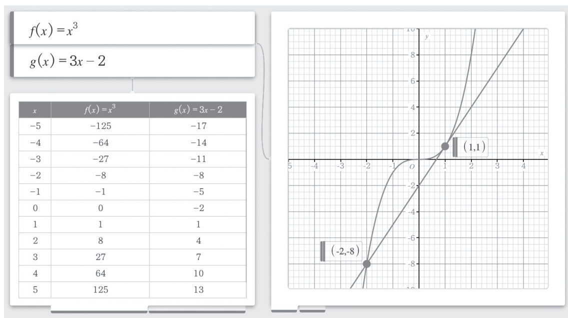


Press [TOOLS], select [Table Range], press [OK], after inputting [Start:−5, End:5, Step:1], select [Execute], press [EXE]



Press [SHIFT] [x], scan the QR code to display a graph.





x	$f(x)=x^3$	$g(x)=3x-2$
−5	−125	−17
−4	−64	−14
−3	−27	−11
−2	−8	−8
−1	−1	−5
0	0	−2
1	1	1
2	8	4
3	27	7
4	64	10
5	125	13

PRACTICE

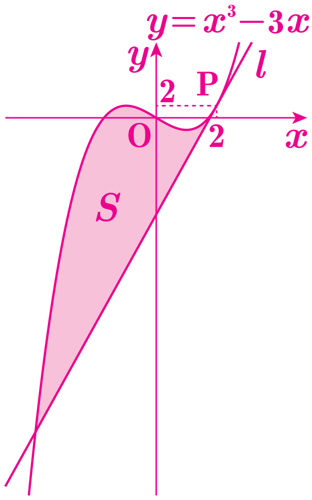
Solve the following problems with regard to the curve $y = x^3 - 3x$.

(1) Find the equation for the tangent l at point P $(2, 2)$ to the curve $y = x^3 - 3x$.

From $y = x^3 - 3x$, and since $y' = 3x^2 - 3$, by substituting $x=2$, we can get $y' = 3 \cdot 2^2 - 3 = 9$.

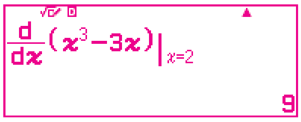
The equation of the tangent l is

$y - 2 = 9(x - 2), y = 9x - 16$.



$y=9x-16$

(📖) (OK) (OK) (x) (■□) (3) (>) (−) (3) (x) (>) (2) (EXE)



(2) Find the x coordinates of the intersections of the tangent l and the curve $y = x^3 - 3x$.

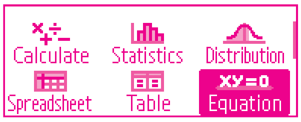
$x^3 - 3x = 9x - 16, x^3 - 12x + 16 = 0, (x - 2)^2(x + 4) = 0, x = 2, -4$

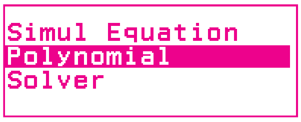
$x=2, -4$

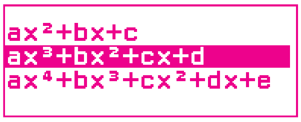
check

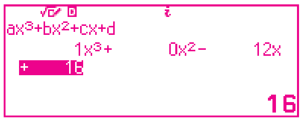
Press (⌂), select [Equation], press (OK), select [Polynomial], press (OK), select [$ax^3 + bx^2 + cx + d$], press (OK)

(1) (EXE) (0) (EXE) (−) (1) (2) (EXE) (1) (6) (EXE)

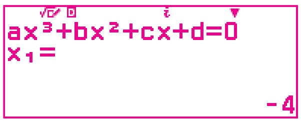


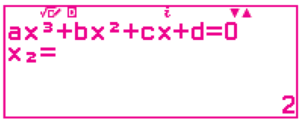






(EXE) (EXE)





(3) Find the area S of the shape bounded by the tangent l and the curve $y=x^3-3x$.

Given $-4 \leq x \leq 2$, we can get $x^3-3x \geq 9x-16$.

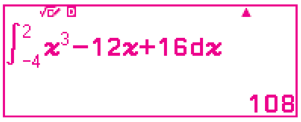
$$S=\int_{-4}^{2}\{x^3-3x-(9x-16)\}dx=\int_{-4}^{2}(x^3-12x+16)dx=\left[\frac{1}{4}x^4-6x^2+16x\right]_{-4}^{2}$$

$$=\frac{1}{4}\cdot 2^4-6\cdot 2^2+16\cdot 2-\left\{\frac{1}{4}\cdot(-4)^4-6\cdot(-4)^2+16\cdot(-4)\right\}=108$$

108

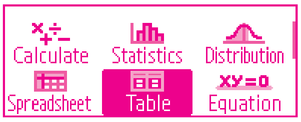
check

(menu) (OK) (∨) (OK) (x) (■□) (3) (>) (−) (1) (2) (x) (+) (1) (6) (∨) (−) (4) (∧) (2) (EXE)

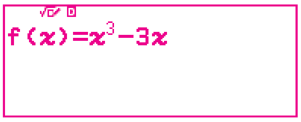


Checking graphs

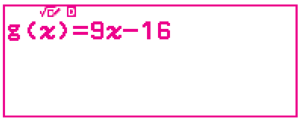
Press (home), select [Table], press (OK), then clear the previous data by pressing (clear)



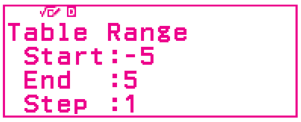
Press (∘∘∘), select [Define f(x)/g(x)], press (OK), select [Define f(x)], press (OK),
after inputting f(x)=x^3-3x, press (EXE)



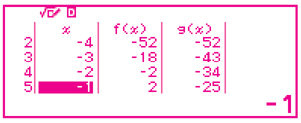
In the same way, input g(x)=9x−16.

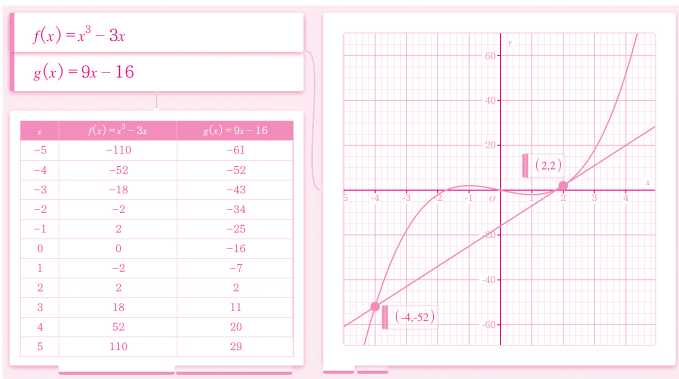


Press (∘∘∘), select [Table Range], press (OK), after inputting [Start:−5, End:5, Step:1], select [Execute], press (EXE)



Press (⇧) (x), scan the QR code to display a graph.





x	$f(x)=x^3-3x$	$g(x)=9x-16$
−5	−110	−61
−4	−52	−52
−3	−18	−43
−2	−2	−34
−1	2	−25
0	0	−16
1	−2	−7
2	2	2
3	18	11
4	52	20
5	110	29

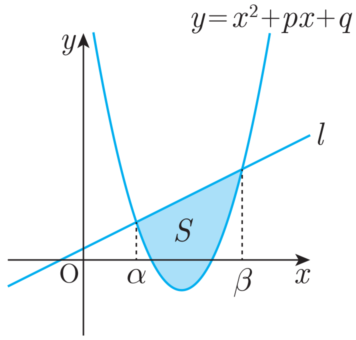
Definite integrals and the area of a shape (5)

TARGET To understand the area of a shape bounded by a parabola and a straight line or by 2 parabolas.

STUDY GUIDE

Area of a shape bounded by a parabola and a straight line

Let the x coordinates of the intersections of the straight line l with the parabola $y = x^2 + px + q$ be α and β $(\alpha<\beta)$. Now, we can find the area S of the shape bounded by the parabola and the straight line l.

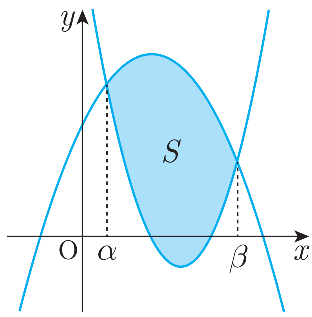


$$S = -\int_{\alpha}^{\beta}(x-\alpha)(x-\beta)dx = \frac{1}{6}(\beta-\alpha)^3$$

When the parabola is $y = ax^2 + bx + c$, we can find the area as follows.

$$S = \frac{|a|}{6}(\beta-\alpha)^3$$

Area of a shape bounded by 2 parabolas



Let the x coordinates of the intersections 2 parabolas be α and $\beta(\alpha<\beta)$, then we can find the area S of the shape bounded by the 2 parabolas by using the above expression.

EXERCISE

◆ Find the area S of the part bounded by the following curve and straight lines.

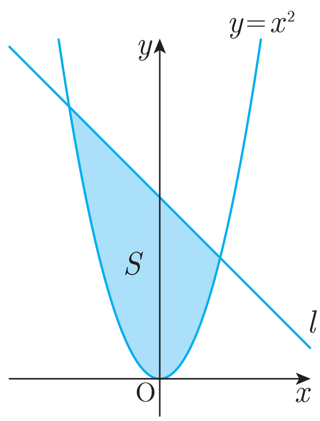
(1) Parabola $y = x^2$ and straight line $y=-x+6$

$x^2 = -x+6, x^2+x-6=0, (x-2)(x+3)=0, x=2, -3$

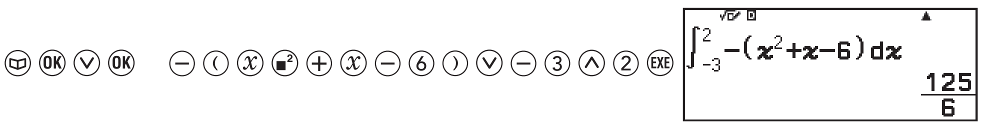
Given $-3 \leq x \leq 2$, we get $x^2 \leq -x+6$.

$$S = \int_{-3}^{2}(-x+6-x^2)dx = \int_{-3}^{2}-(x^2+x-6)dx = -\int_{-3}^{2}(x-2)(x+3)dx$$

$$= \frac{1}{6}\{2-(-3)\}^3 = \frac{125}{6}$$



$\frac{125}{6}$



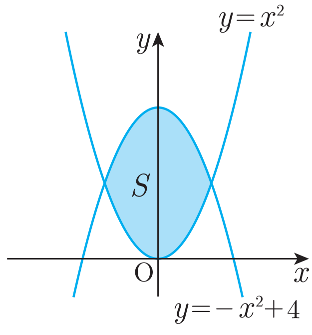
(2) 2 parabolas $y = x^2$ and $y = -x^2 + 4$

$x^2 = -x^2 + 4, x^2 = 2, x = \pm\sqrt{2}$

Given $-\sqrt{2} \leq x \leq \sqrt{2}$, we get $x^2 \leq -x^2 + 4$.

$$S = \int_{-\sqrt{2}}^{\sqrt{2}} (-x^2 + 4 - x^2)dx = \int_{-\sqrt{2}}^{\sqrt{2}} -2(x^2 - 2)dx = -2\int_{-\sqrt{2}}^{\sqrt{2}} (x + \sqrt{2})(x - \sqrt{2})dx$$

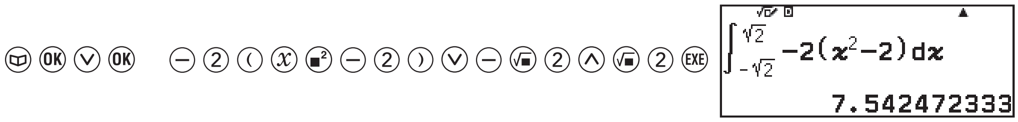
$$= \frac{2}{6}\{\sqrt{2} - (-\sqrt{2})\}^3 = \frac{16\sqrt{2}}{3}$$



$\frac{16\sqrt{2}}{3}$

check

The results of the scientific calculator are shown as a decimal, so confirm that it is equal to the results as an irrational number.



PRACTICE

◆ Find the area S of the part bounded by the following curve and straight lines.

(1) Parabola $y = -x^2$ and straight line $y=2x-8$

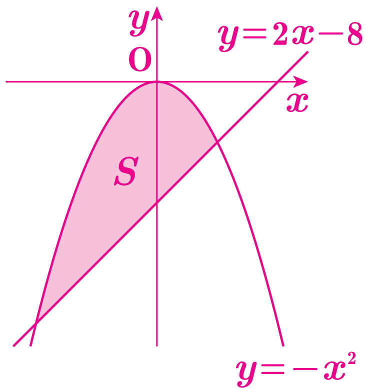
$-x^2 = 2x - 8, x^2 + 2x - 8 = 0, (x - 2)(x + 4) = 0,$

$x = 2, -4$

Given $-4 \leq x \leq 2$, we can get $-x^2 \geq 2x-8$.

$$S = \int_{-4}^{2} \{-x^2 - (2x - 8)\}dx = \int_{-4}^{2} -(x^2 + 2x - 8)dx$$

$$= -\int_{-4}^{2} (x - 2)(x + 4)dx = \frac{1}{6}\{2 - (-4)\}^3 = 36$$



36

(2) 2 parabolas $y = x^2 - 2$ and $y = -x^2 + 6$

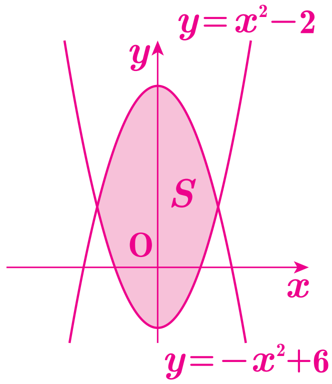
$x^2 - 2 = -x^2 + 6, x^2 = 4, x = \pm 2$

Given $-2 \leq x \leq 2$, we get $x^2 - 2 \leq -x^2 + 6$.

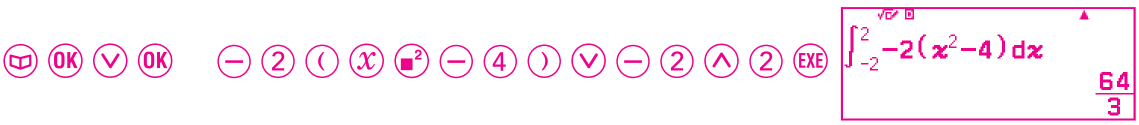
$$S = \int_{-2}^{2} \{-x^2 + 6 - (x^2 - 2)\}dx = \int_{-2}^{2} -2(x^2 - 4)dx$$

$$= -2\int_{-2}^{2} (x^2 - 4)dx = -2\int_{-2}^{2} (x + 2)(x - 2)dx$$

$$= \frac{2}{6}\{2 - (-2)\}^3 = \frac{64}{3}$$



$\frac{64}{3}$



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