14 Probability

CASIO
Essential Materials

Introduction

These teaching materials were created with the hope of conveying to many teachers and students the appeal of scientific calculators.

(1) Change awareness (emphasizing the thinking process) and boost efficiency in learning mathematics

- By reducing the time spent on manual calculations, we can have learning with a focus on the thinking process that is more efficient.
- This reduces the aversion to mathematics caused by complicated calculations, and allows students to experience the joy of thinking, which is the essence of mathematics.

(2) Diversification of learning materials and problem-solving methods

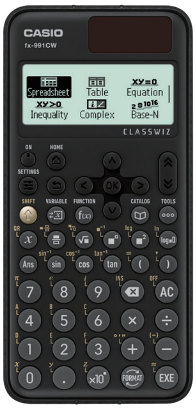
- Making it possible to do difficult calculations manually allows for diversity in learning materials and problem-solving methods.

(3) Promoting understanding of mathematical concepts

- By using the various functions of the scientific calculator in creative ways, students are able to deepen their understanding of mathematical concepts through calculations and discussions from different perspectives than before.
- This allows for exploratory learning through easy trial and error of questions.
- Listing and graphing of numerical values by means of tables allows students to discover laws and to understand visually.

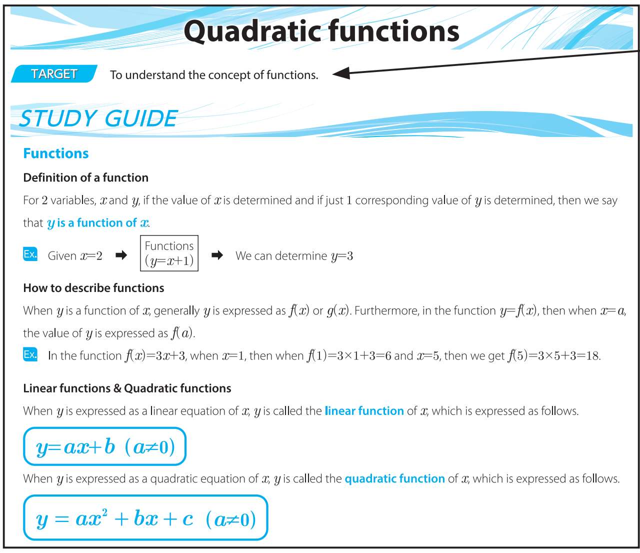
Features of this book

- As well as providing first-time scientific calculator users with opportunities to learn basic scientific calculator functions from the ground up, the book also has material to show people who already use scientific calculators the appeal of scientific calculators described above.
- You can also learn about functions and techniques that are not available on conventional Casio models or other brands of scientific calculators.
- This book covers many units of high school mathematics, allowing students to learn how to use the scientific calculator as they study each topic.
- This book can be used in a variety of situations, from classroom activities to independent study and homework by students.



Better Mathematics Learning with Scientific Calculator

Structure



Quadratic functions

TARGET To understand the concept of functions.

STUDY GUIDE

Functions

Definition of a function

For 2 variables, x and y, if the value of x is determined and if just 1 corresponding value of y is determined, then we say that **y is a function of x.**

Ex. Given $x=2$ ➡ Functions ($y=x+1$) ➡ We can determine $y=3$

How to describe functions

When y is a function of x, generally y is expressed as $f(x)$ or $g(x)$. Furthermore, in the function $y=f(x)$, then when $x=a$, the value of y is expressed as $f(a)$.

Ex. In the function $f(x)=3x+3$, when $x=1$, then when $f(1)=3\times1+3=6$ and $x=5$, then we get $f(5)=3\times5+3=18$.

Linear functions & Quadratic functions

When y is expressed as a linear equation of x, y is called the **linear function** of x, which is expressed as follows.

$\boxed{y=ax+b \ (a\neq0)}$

When y is expressed as a quadratic equation of x, y is called the **quadratic function** of x, which is expressed as follows.

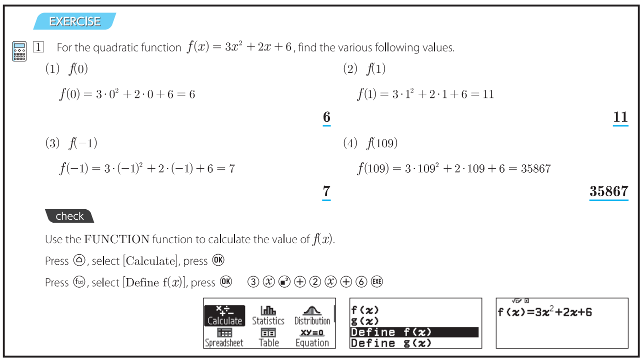
$\boxed{y=ax^2+bx+c \ (a\neq0)}$

TARGET

Students can identify the objective to learn in each section.

STUDY GUIDE

Mathematical theorems and concepts are explained in detail. A scientific calculator is used to check and derive formulas according to the topic.



EXERCISE

1 For the quadratic function $f(x)=3x^2+2x+6$, find the various following values.

(1) $f(0)$

$f(0)=3\cdot0^2+2\cdot0+6=6$ **6**

(2) $f(1)$

$f(1)=3\cdot1^2+2\cdot1+6=11$ **11**

(3) $f(-1)$

$f(-1)=3\cdot(-1)^2+2\cdot(-1)+6=7$ **7**

(4) $f(109)$

$f(109)=3\cdot109^2+2\cdot109+6=35867$ **35867**

check

Use the FUNCTION function to calculate the value of $f(x)$.

Press ⌂, select [Calculate], press OK

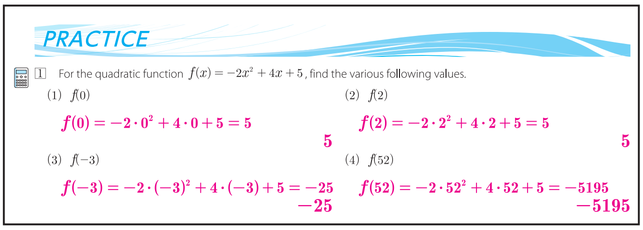
Press f(x), select [Define f(x)], press OK ③ x x² + ② x + ⑥ EXE

EXERCISE

Students learn basic examples based on the explanation in Study Guide.

check

Explains how to use the scientific calculator to solve problems and check answers.



PRACTICE

1 For the quadratic function $f(x)=-2x^2+4x+5$, find the various following values.

(1) $f(0)$

$f(0)=-2\cdot0^2+4\cdot0+5=5$ **5**

(2) $f(2)$

$f(2)=-2\cdot2^2+4\cdot2+5=5$ **5**

(3) $f(-3)$

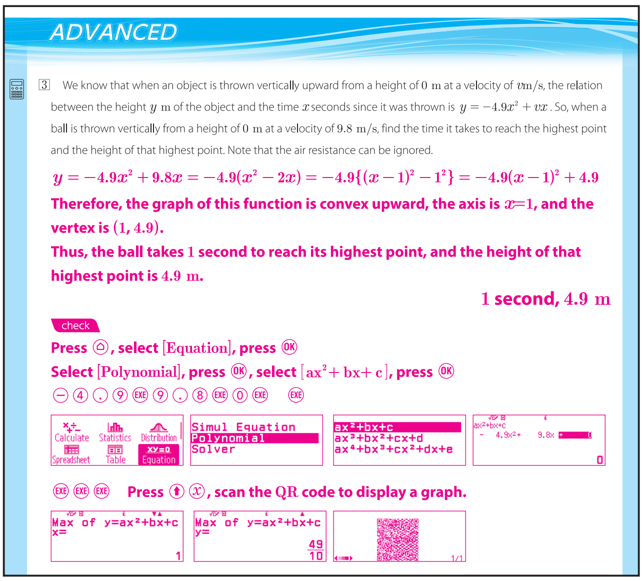
$f(-3)=-2\cdot(-3)^2+4\cdot(-3)+5=-25$ **−25**

(4) $f(52)$

$f(52)=-2\cdot52^2+4\cdot52+5=-5195$ **−5195**

PRACTICE

Students can do practice problems similar to those in EXERCISE. They can also practice using the scientific calculator as they learned to in Check.



ADVANCED

3 We know that when an object is thrown vertically upward from a height of 0 m at a velocity of v m/s, the relation between the height y m of the object and the time x seconds since it was thrown is $y=-4.9x^2+vx$. So, when a ball is thrown vertically from a height of 0 m at a velocity of 9.8 m/s, find the time it takes to reach the highest point and the height of that highest point. Note that the air resistance can be ignored.

$y=-4.9x^2+9.8x=-4.9(x^2-2x)=-4.9\{(x-1)^2-1^2\}=-4.9(x-1)^2+4.9$

Therefore, the graph of this function is convex upward, the axis is $x=1$, and the vertex is (1, 4.9).

Thus, the ball takes 1 second to reach its highest point, and the height of that highest point is 4.9 m.

1 second, 4.9 m

check

Press ⌂, select [Equation], press OK

Select [Polynomial], press OK, select [ax²+bx+c], press OK

⊖ ④ . ⑨ EXE ⑨ . ⑧ EXE ⓪ EXE EXE

EXE EXE EXE **Press ⇧ x, scan the QR code to display a graph.**

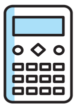
ADVANCED

Practical problems have been included in several topics. Solutions using a scientific calculator are also presented as necessary.

Other marks

Ex.	Simple examples on how to apply equations and theorems
explanation	Formulas and their supplementary explanations
proof	Proofs and checks of mathematical formulas
EXTRA Info.	Knowledge and information on formulas and other supplementary information in other units
OTHER METHODS	Alternative solutions and different verification methods for previously presented problems

Calculator mark



Where to use the scientific calculator

Colors of fonts in the teaching materials

- In STUDY GUIDE, important mathematical terms and formulas are printed in blue.
- In PRACTICE and ADVANCED the answers are printed in red.
  (Separate data is also available without the red parts, so it can be used for exercises.)

Applicable models

The applicable model is fx-991CW.
(Instructions on how to do input are for the fx-991CW, but in many cases similar calculations can be done on other models.)

Related Links

- Information and educational materials relevant to scientific calculators can be viewed on the following site.
  https://edu.casio.com
- The following video can be viewed to learn about the multiple functions of scientific calculators.
  https://www.youtube.com/playlist?list=PLRgxo9AwbIZLurUCZnrbr4cLfZdqY6aZA

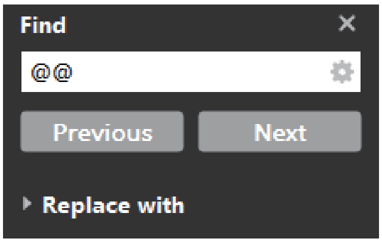
How to use PDF data

About types of data

- Data for all unit editions and data for each unit are available.
- For the above data, the PRACTICE and ADVANCED data without the answers in red is also available.

How to find where the scientific calculator is used

(1) Open a search window in the PDF Viewer.
(2) Type in "@@" as a search term.
(3) You can sequentially check where the calculator marks appear in the data.



How to search for a unit and section

(1) Search for units of data in all unit editions

- The data in all unit editions has a unit table of contents.
- Selecting a unit in the table of contents lets you jump to the first page of that unit.
- There is a bookmark on the first page of each unit, so you can jump from there also.

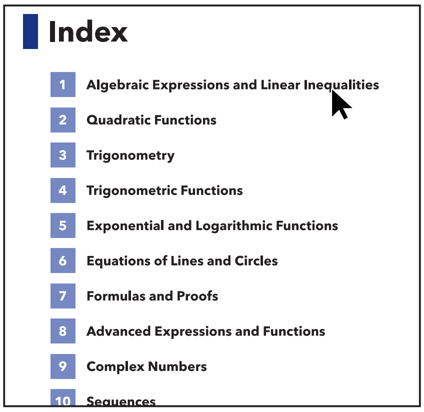
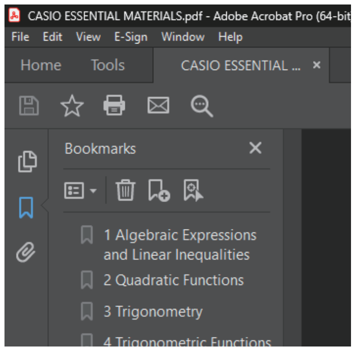


Table of contents of unit



Bookmark of unit

(2) Search for sections

- There are tables of contents for sections on the first page of units.
- Selecting a section in the table of contents takes you to the first page of that section.

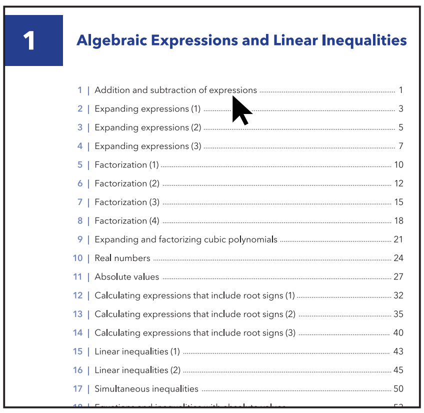


Table of contents of section

Number of cases

TARGET To understand how to find the number of cases by using a tree diagram.

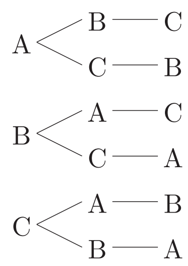
STUDY GUIDE

Tree diagrams

Tree diagrams are a method to count all the possible cases, without repeating any or missing any.

Ex. Count how many possible ways there are to arrange the 3 letters A, B, and C.

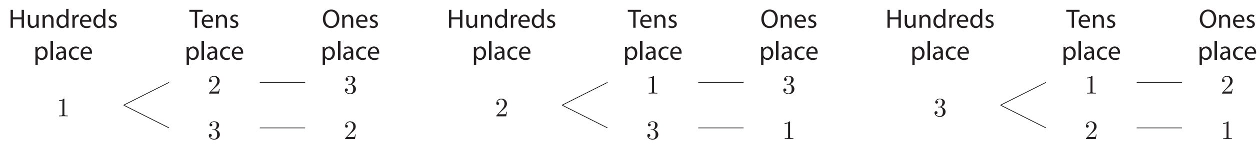
From the tree diagram on the right, there are 6 ways to arrange the 3 letters.



EXERCISE

1 Find how many 3 digit numbers can be made by using the 3 digits, 1, 2, and 3, just 1 time each.

Use a tree diagram to show how to arrange the 3 numbers 1, 2, and 3.

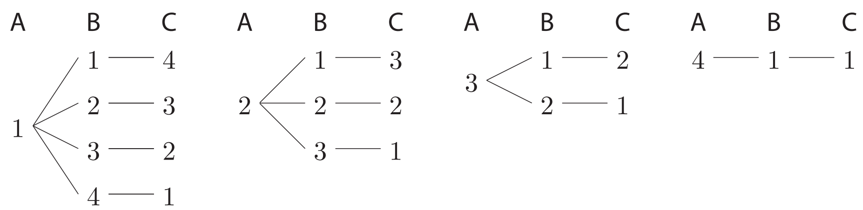


From the tree diagram, there are 6 ways.

6 ways

2 Find the number of cases in which the sum of the rolled dice is 6 when 3 dice, A, B, and C, are rolled 1 time each.

Use a tree diagram to show the outcomes of the 3 dice rolls.

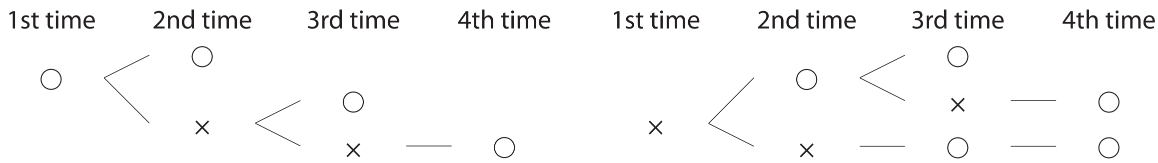


From the tree diagram, there are 10 ways.

10 ways

3 In a game of tossing 1 coin repeatedly, you win by getting heads 2 times. However, the game ends after tossing the coin 4 times or getting heads 2 times. Find the number of cases in which this game is won.

Use a tree diagram to show the results given ○ is heads and × is tails.



From the tree diagram, there are 6 ways.

6 ways

PRACTICE

1 Find the number of cases in which the product of the rolled dice is 6 when 3 dice, A, B, and C, are rolled 1 time each.

Use a tree diagram to show the outcomes of the 3 dice rolls.

A	B	C	A	B	C	A	B	C	A	B	C
1	1	6	2	1	3	3	1	2	6	1	1
	2	3		3	1		2	1			
	3	2									
	6	1									

From the tree diagram, there are 9 ways.

9 ways

2 In a game of tossing 1 coin repeatedly, you win by getting heads 3 times. However, the game ends after tossing the coin 4 times or getting heads 3 times. Find the number of cases in which this game is won.

Use a tree diagram to show the results given ○ is heads and × is tails.

1st time	2nd time	3rd time	4th time
○	○	○	
		×	○
	×	○	○
×	○	○	○

From the tree diagram, there are 4 ways.

4 ways

3 Find how many 3 digit numbers can be made by selecting 3 digits from the 4 digits 1, 2, 3, and 3.

Note that there are 2 3s, and use a tree diagram to show the results.

Hundreds place	Tens place	Ones place	Hundreds place	Tens place	Ones place	Hundreds place	Tens place	Ones place
1	2	3	2	1	3	3	1	2
	3	2		3	1			3
	3	3		3	3		2	1
								3
							3	1
								2

From the tree diagram, there are 12 ways.

12 ways

Law of addition and law of multiplication

TARGET To understand how to find the number of ways that either of 2 events can occur or that both events can occur.

STUDY GUIDE

Law of addition

Assume that there are 2 events, A and B, and that they do not occur simultaneously. When there are m ways for A to happen and n ways for B to happen, the following **law of addition** holds.

Number of ways that A or B occurs $\Rightarrow$ $(m+n)$ ways

(The law of addition also holds when there are 3 or more events.)

explanation

Given A and B do not occur simultaneously $\rightarrow$ by combining the m ways of selecting A and the n ways of selecting B, then the total ways they can occur is $(m+n)$.

Law of multiplication

Assume that there are 2 events, A and B. When there are m ways for A to happen, and for every one of those occurrences there are n ways for B to happen, then the following **law of multiplication** holds.

Number of ways that B occurs for each A $\Rightarrow$ $(m\times n)$ ways

(The law of multiplication also holds when there are 3 or more events.)

explanation

Given A and B occur simultaneously $\rightarrow$ since there are n ways of selecting B for every m way of selecting A, then the total ways they can occur is $(m\times n)$.

How A occurs (m ways)	A_1	,	A_2	, ... ,	A_m
How B occurs (n ways)	$\mathrm{B}_1, \mathrm{B}_2, \ldots\ldots, \mathrm{B}_n$	,	$\mathrm{B}_1, \mathrm{B}_2, \ldots\ldots, \mathrm{B}_n$	, ... ,	$\mathrm{B}_1, \mathrm{B}_2, \ldots\ldots, \mathrm{B}_n$

EXERCISE

1 On the drink menu of a certain restaurant, there are 4 kinds of fruit juice and 5 kinds of carbonated drinks. Find the number of ways that you can select 1 kind of drink from this menu.

You cannot select from the fruit juices and from the carbonated drinks at the same time.

Since there are 4 ways and 5 ways to select one, the law of addition gives us $4+5=9$ (ways)

9 ways

2 Find the number of cases in which the sum of the dice is a multiple of 3 when 2 dice, A and B, are rolled 1 time each.

When the sum of the dice is any of 3, 6, 9, or 12, it is a multiple of 3.

There are 2 ways to get a sum of 3 on the dice, they are $(1, 2)$ and $(2, 1)$.

There are 5 ways to get a sum of 6 on the dice, they are $(1, 5)$, $(2, 4)$, $(3, 3)$, $(4, 2)$, and $(5, 1)$.

There are 4 ways to get a sum of 9 on the dice, they are $(3, 6)$, $(4, 5)$, $(5, 4)$, and $(6, 3)$.

There is 1 way to get a sum of 12 on the dice, it is $(6, 6)$.

Since the sums on the dice of 3, 6, 9, and 12 cannot occur simultaneously, the law of addition gives us $2+5+4+1=12$ (ways)

12 ways

[3] There are 5 ways to go from town A to town B and 4 ways to go from town B to town C. In this case, find the total number of ways to go from town A through town B to town C.

There are 5 ways to go from town A to town B, and for each of those ways there are 4 ways to go from town B to town C.

Therefore, from the law of multiplication, we get $5\times4=20$ (ways)

20 ways

[4] Mr. D has 6 types of shirts and 5 types of trousers. Find the number of ways that he can select 1 type of shirt and 1 type of trousers to wear.

There are 6 ways to select a shirt, and for each of those there are 5 ways to select trousers.

Therefore, from the law of multiplication, we get $6\times5=30$ (ways)

30 ways

[5] Find the number of positive divisors of 144.

Factorize 144 into its prime factors to get $144=2^4\times3^2$

Then, the 5 divisors of 2^4 are $1, 2, 2^2, 2^3, 2^4$, and the 3 divisors of 3^2 are $1, 3, 3^2$.

We can get all the positive divisors of 144 by multiplying each of the 5 divisors of 2^4 by each of the 3 divisors of 3^2.

Therefore, from the law of multiplication, we get $5\times3=15$ (divisors)

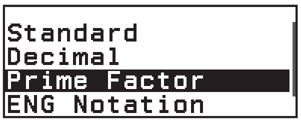
15 divisors

check

On the scientific calculator, use the FORMAT function to factorize 144 into prime factors.

Press ⌂, select [Calculate], press (OK)

① ④ ④ (EXE), press (FORMAT), select [Prime Factor], press (OK)



PRACTICE

[1] In a given class, there are 16 students that ride a bicycle to school and 19 students that ride a bus to school. Find the number of ways that you can select 1 typical student from among them.

You cannot select 1 student that rides a bicycle to school and 1 student that rides a bus to school at the same time.

Since there are 16 ways and 19 ways to select one, the law of addition gives us $16+19=35$ (ways)

35 ways

[2] Find the number of ways in which the sum of the dice is greater than or equal to 9 when 2 dice, large and small, are rolled 1 time each.

When the sum of the dice is any of 9, 10, 11, or 12, it is greater than or equal to 9.

There are 4 ways to get a sum of 9 on the dice, they are (3, 6), (4, 5), (5, 4), and (6, 3).

There are 3 ways to get a sum of 10 on the dice, they are (4, 6), (5, 5), and (6, 4).

There are 2 ways to get a sum of 11 on the dice, they are (5, 6) and (6, 5).

There is 1 way to get a sum of 12 on the dice, it is (6, 6).

Since the sums on the dice of 9, 10, 11, and 12 cannot occur simultaneously, the law of addition gives us $4+3+2+1=10$ (ways)

10 ways

3 On the menu of a certain restaurant, there are 6 types of drinks, 8 types of main dishes, and 5 types of desserts. Find the number of ways that you can select 1 type of each item.

There are 6 ways to select a drink, 8 ways to select a main dish, and 5 ways to select a dessert.

Therefore, from the law of multiplication, we get $6\times8\times5=240$ (ways)

240 ways

4 Find the number of outcomes that 3 dice, A, B, and C, can be rolled 1 time each.

There are 6 outcomes for A, 6 outcomes for B, and 6 outcomes for C.

Therefore, from the law of multiplication, we get $6\times6\times6=216$ (ways)

216 ways

5 Find the number of positive divisors of 360.

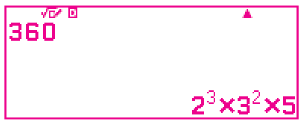
Factorize 360 into its prime factors to get $360=2^3\times3^2\times5$

Then, the 4 divisors of 2^3 are $1, 2, 2^2, 2^3$, and the 3 divisors of 3^2 are $1, 3, 3^2$, and the 2 divisors of 5 are 1 and 5.

Therefore, from the law of multiplication, we get $4\times3\times2=24$ (divisors)

24

③ ⑥ ⓪ EXE, press FORMAT, select [Prime Factor], press OK



Permutations

TARGET To understand the total number of ways that several things can be arranged in 1 ordered sequence.

STUDY GUIDE

Permutations

When several things are arranged in 1 ordered sequence, each sequence is called a **permutation**. If you take r things from n different things and arrange them in 1 row, we call that a **permutation of r things taken from n things**, and that total number is shown as ${}_n\mathrm{P}_r$. However, $r \leq n$. You can find the total number of permutations of r things taken from n things as follows.

$$ {}_n\mathrm{P}_r = \underbrace{n(n-1)(n-2)\cdots\cdots(n-r+1)}_{\text{Product of } r \text{ things}} $$

Ex. The total number of permutations of 5 things taken from 8 things is

${}_8\mathrm{P}_5 = 8\cdot(8-1)\cdot(8-2)\cdot(8-3)\cdot(8-4) = 8\cdot 7\cdot 6\cdot 5\cdot 4 = 6720$.

check

Use the scientific calculator to calculate the total number of permutations of 5 things taken from 8 things.

Press ⌂, select $[\text{Calculate}]$, press OK

⑧, press ⊟, select $[\text{Probability}]$, press OK, select $[\text{Permutation(P)}]$, press OK, ⑤ EXE

explanation

Consider taking 1 thing at a time in order from n things.

The 1st thing There are n ways to take 1 thing from n things.

The 2nd thing There are $(n-1)$ ways to take 1 thing from the remaining $(n-1)$ things.

The 3rd thing There are $(n-2)$ ways to take 1 thing from the remaining $(n-2)$ things.

⋮

The rth thing Since $(r-1)$ things have already been taken from the n things, the remaining number of things is $n-(r-1)=(n-r+1)$ things.

There are $(n-r+1)$ ways to take 1 thing from here.

From the law of multiplication, there are $n\times(n-1)\times(n-2)\times\cdots\cdots\times(n-r+1)$ ways to take n things from r things.

Specifically, the total number of permutations of taking all n things from n different things is

${}_n\mathrm{P}_n = n(n-1)(n-2)\cdots\cdots 3\cdot 2\cdot 1$

The product of all natural numbers from 1 to n, such as the right-hand side of the above equation, is called the **factorial of n** and is expressed as $n!$. Also defined as $0!=1$.

$$ n! = n(n-1)(n-2)\cdots\cdots 3\cdot 2\cdot 1 $$

Ex. $6!=6\cdot5\cdot4\cdot3\cdot2\cdot1=720$

check

Use the scientific calculator to calculate the factorial of 6.

Press [HOME], select [Calculate], press [OK]

[6], press [CATALOG], select [Probability], press [OK], select [Factorial(!)], press [OK], [EXE]

By using the factorial sign, which we can transform as ${}_9\mathrm{P}_5=9\cdot8\cdot7\cdot6\cdot5=9\cdot8\cdot7\cdot6\cdot5\cdot\dfrac{4\cdot3\cdot2\cdot1}{4\cdot3\cdot2\cdot1}=\dfrac{9!}{4!}=\dfrac{9!}{(9-5)!}$, we can express the formula for finding the total number of permutations as follows.

$$\boxed{{}_n\mathrm{P}_r=\frac{n!}{(n-r)!}}$$

When $r=0$ in the above equation, it becomes ${}_n\mathrm{P}_0=\dfrac{n!}{(n-0)!}=\dfrac{n!}{n!}=1$, which we define as ${}_n\mathrm{P}_0=1$.

EXERCISE

1 Solve the following problems.

(1) Find the value of ${}_6\mathrm{P}_4$.

${}_6\mathrm{P}_4=6\cdot5\cdot4\cdot3=360$

360

[6] [CATALOG] [∨] [OK] [∨] [∨] [OK] [4] [EXE]

(2) Find the value of ${}_{10}\mathrm{P}_3$.

${}_{10}\mathrm{P}_3=10\cdot9\cdot8=720$

720

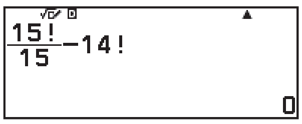
[1] [0] [CATALOG] [∨] [OK] [∨] [∨] [OK] [3] [EXE]

(3) When $\dfrac{15!}{15}=A!$, find the value of A.

From $\dfrac{15!}{15}=\dfrac{15\cdot14!}{15}=14!$, we get $A=14$

14

[1] [5] [CATALOG] [∨] [OK] [∨] [OK] [FRACTION] [1] [5] [>] [−] [1] [4] [CATALOG] [∨] [OK] [∨] [OK] [EXE]



2 Find how many ways there are to arrange the 7 numbers $1, 2, 3, 4, 5, 6$, and 7 in 1 row, as shown below.

(1) Select and arrange 4 numbers from the 7 numbers.

Since it is a permutation of 4 things taken from 7 things, there are ${}_7P_4 = 7 \cdot 6 \cdot 5 \cdot 4 = 840$ (ways)

840 ways

7 ∨ OK ∨ ∨ OK 4 EXE

(2) Arrange all 7 of the numbers.

Since it is a permutation of all 7 things, there are $7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040$ (ways)

5040 ways

7 ∨ OK ∨ OK EXE

(3) Arrange all 7 numbers into 7-digit even numbers.

There are 3 ways to be an even number, the ones place must be 2, 4, or 6, then the other places are 6 permutations, which excludes the 1 even number in the ones place.

Therefore, from the law of multiplication, we get $3 \times 6! = 3 \times 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 2160$ (ways)

2160 ways

3 × 6 ∨ OK ∨ OK EXE

PRACTICE

1 Solve the following problems.

(1) Find the value of ${}_9P_3$.

$${}_9P_3 = 9 \cdot 8 \cdot 7 = 504$$

504

9 ∨ OK ∨ ∨ OK 3 EXE

(2) Find the value of ${}_7P_5$.

$${}_7P_5 = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 = 2520$$

2520

7 ∨ OK ∨ ∨ OK 5 EXE

(3) When $35 \times 34! = A!$, find the value of A.

From $35 \times 34! = 35 \times (34 \cdot 33 \cdot 32 \cdots\cdots 3 \cdot 2 \cdot 1) = 35!$**, we get** $A = 35$

35

2 Find how many ways there are to arrange the 8 numbers 1, 2, 3, 4, 5, 6, 7, and 8 in 1 row, as shown below.

(1) Select and arrange 4 numbers from the 8 numbers.

Since it is a permutation of 4 things taken from 8 things, there are

$_8P_4 = 8 \cdot 7 \cdot 6 \cdot 5 = 1680$ **(ways)**

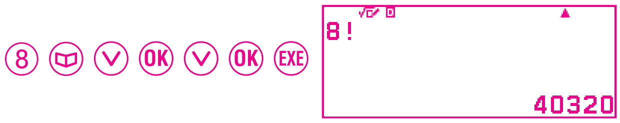
1680 ways

(2) Arrange all 8 of the numbers.

Since it is a permutation of all 8 things, there are

$8! = 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 40320$ **(ways)**

40320 ways



(3) Arrange all 8 numbers into 8-digit odd numbers.

There are 4 ways to be an odd number, the ones place must be 1, 3, 5 or 7, then the other places are 7 permutations, which excludes the 1 odd number in the ones place.

Therefore, from the law of multiplication, we get

$4 \times 7! = 4 \times 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 20160$ **(ways)**

20160 ways

Various permutations

TARGET To understand how to calculate the total number of permutations in which certain things are arranged next to each other or at both ends.

STUDY GUIDE

Various permutations

Consider the following arrangements in which certain things are arranged either next to each other or at both ends, and use the law of multiplication to solve them.

Arrangement of certain things to be next to each other

> **Consider arranging things specified as adjacent as 1 thing.**

Ex. When arranging 6 letters, A, B, C, D, E, and F horizontally in 1 row, consider how many arrangements there are in which A and B are next to each other.

Since A and B are always adjacent, AB is regarded as 1 letter, so together with the other letters, C, D, E, and F, we can consider this as arrangements of 5 letters.

There are ${}_5\mathrm{P}_5$ ways to arrange them.

In each of these arrangements, there are ${}_2\mathrm{P}_2$ ways to arrange just A and B.

Therefore, from the law of multiplication, we can find a total of ${}_5\mathrm{P}_5\times{}_2\mathrm{P}_2=5!\times2!=240$ (ways)

check

Use the scientific calculator to calculate the factorial of $5!\times2!$.

5 📖 ⋁ OK ⋁ OK × 2 📖 ⋁ OK ⋁ OK EXE

Concept of arrangement of certain things to be on both ends

> **Consider arranging things at both ends, and then arrange the remaining things between them.**

Ex. When arranging 6 letters, A, B, C, D, E, and F horizontally in 1 row, consider how many arrangements there are in which A and B are on both ends.

An arrangement with A and B at both ends is a permutation of 2 things taken from 2 things, so there are ${}_2\mathrm{P}_2$ ways.

In each of these arrangements, there are ${}_4\mathrm{P}_4$ ways to arrange the other letters C, D, E, and F.

Therefore, from the law of multiplication, we can find a total of ${}_2\mathrm{P}_2\times{}_4\mathrm{P}_4=2!\times4!=48$ (ways)

check

Use the scientific calculator to calculate the factorial of $2!\times4!$.

2 📖 ⋁ OK ⋁ OK × 4 📖 ⋁ OK ⋁ OK EXE

EXERCISE

1. Given 8 students, A, B, C, D, E, F, G, and H, who line up horizontally in 1 row, find how many ways they can line up, as shown below.

(1) Students C and D line up next to each other

Since C and D alway line up next to each other, regard them as one group.

There are ${}_7P_7$ ways for the other 7 students to line up.

In each of these lines, there are ${}_2P_2$ ways for just C and D to line up.

Therefore, from the law of multiplication, we can find a total of ${}_7P_7 \times {}_2P_2 = 7! \times 2! = 10080$ (ways)

10080 ways

(2) Students E and F line up on both ends

A line with E and F at both ends is a permutation of selecting 2 people from 2 people, so there are ${}_2P_2$ ways.

In each of these lines, there are ${}_6P_6$ ways to line up the other students.

Therefore, from the law of multiplication, we can find a total of ${}_2P_2 \times {}_6P_6 = 2! \times 6! = 1440$ (ways)

1440 ways

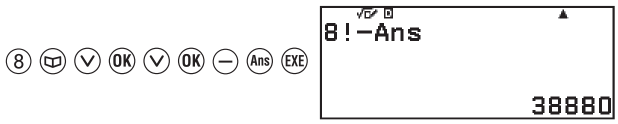
(3) Students G and H do not line up on the ends

The set of a line in which students G and H are at both ends and the set of a line in which students G and H are not at both ends have a complementary relation.

Therefore, we can subtract the total number of lines in which students G and H are at both ends, ${}_2P_2 \times {}_6P_6$, from the total number of lines in which 8 students line up horizontally in 1 row ${}_8P_8$.

Thus, we can find a total of ${}_8P_8 - {}_2P_2 \times {}_6P_6 = 8! - 1440 = 38880$ (ways)

38880 ways



2 Given 3 soccer players and 4 basketball players line up horizontally in 1 row, find how many ways they can line up, as shown below.

(1) The 3 soccer players line up in a row

Since the 3 soccer players alway line up in a row, regard them as one group.

There are ${}_5P_5$ ways for the other 5 athletes to line up.

In each of these lines, there are ${}_3P_3$ ways for just the 3 soccer players to line up.

Therefore, from the law of multiplication, we can find a total of ${}_5P_5 \times {}_3P_3 = 5! \times 3! = 720$ (ways)

720 ways

(2) Basketball players line up on both ends

A line with 2 people at both ends is a permutation of selecting 2 people from 4 people, so there are ${}_4P_2$ ways.

In each of these lines, there are ${}_5P_5$ ways for the remaining 5 athletes to line up.

Therefore, from the law of multiplication, we can find a total of ${}_4P_2 \times {}_5P_5 = \dfrac{4!}{(4-2)!} \times 5! = 1440$ (ways)

1440 ways

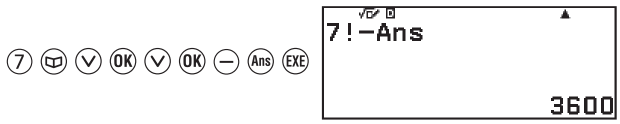
(3) Basketball players do not line up on the ends

The set of a line in which basketball players are at both ends and the set of a line in which basketball players are not at both ends have a complementary relation.

Therefore, we can subtract the total number of lines in which basketball players are at both ends, ${}_4P_2 \times {}_5P_5$, from the total number of lines in which 7 athletes line up horizontally in 1 row ${}_7P_7$.

Thus, we can find a total of ${}_7P_7 - {}_4P_2 \times {}_5P_5 = 7! - 1440 = 3600$ (ways)

3600 ways



PRACTICE

◆ Given 4 1st-year students and 4 2nd year students are put in a running order for a relay race, find how may running orders there are, as shown below.

(1) 4 1st-year students run consecutively

Regard the 4 1st-year students as a group.
There are ${}_5P_5$ ways to arrange the other 5 students in the running order.
In each of these running orders, there are ${}_4P_4$ ways to arrange the running order for the 4 1st-year students.
Therefore, from the law of multiplication, we can find a total of
${}_5P_5 \times {}_4P_4 = 5! \times 4! = 2880$ **(ways)**

2880 ways

5 ⊞ ∨ OK ∨ OK × 4 ⊞ ∨ OK ∨ OK EXE

5!×4!
2880

(2) 2nd-year students run in both the first and last positions

A running order with 2nd-year students running in the first and last positions is a permutation of selecting 2 people from 4 2nd-year students, so there are ${}_4P_2$ ways.
In each of these running orders, there are ${}_6P_6$ ways to arrange the running order for the remaining 6 people, which excludes the 2 people running in first and last positions.
Therefore, from the law of multiplication, we can find a total of

$${}_4P_2 \times {}_6P_6 = \frac{4!}{(4-2)!} \times 6! = 8640 \text{ (ways)}$$

8640 ways

4 ⊞ ∨ OK ∨ ∨ OK 2 × 6 ⊞ ∨ OK ∨ OK EXE

4P2×6!
8640

(3) 2nd-year students do not run in the first or last positions

The set of a running order in which the first and last positions are both 2nd-year students and the set of a running order in which the first and last position are both not 2nd-year students have a complementary relation.
Therefore, we can subtract the total number of running orders in which the first and last positions are 2nd-year students, ${}_4P_2 \times {}_6P_6$, from the total number of running orders of 8 people ${}_8P_8$.
Therefore, we can find a total of ${}_8P_8 - {}_4P_2 \times {}_6P_6 = 8! - 8640 = 31680$ **(ways)**

31680 ways

8 ⊞ ∨ OK ∨ OK − Ans EXE

8!−Ans
31680

Circular permutations

TARGET To understand the permutations of things arranged in a circle.

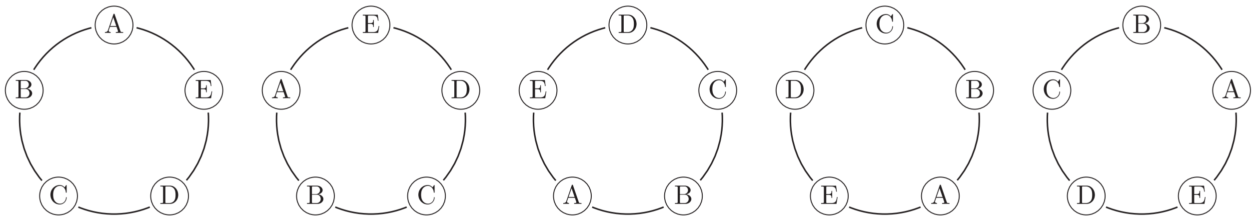
STUDY GUIDE

Circular permutations

A permutation in which n different things are arranged in a circle is called a **circular permutation**. In a circular permutation, we consider arrangements that are the same when rotated to be the same arrangement. You can find the total number of circular permutations of n different things as follows.

$$\frac{{}_{n}\mathrm{P}_{n}}{n}=\frac{n!}{n}=(n-1)!$$

Ex. Consider the total number of circular arrangements of 5 letters, A, B, C, D, and E.



The 5 arrangements shown in the figure above are all 1 arrangement because they coincide when rotated.
The number of permutations in which A, B, C, D, and E are arranged in 1 line is ${}_{5}\mathrm{P}_{5}$ ways, and when we make it circular, each of the 5 ways is the same.

Therefore, the total number of circular permutations is $\frac{{}_{5}\mathrm{P}_{5}}{5}=\frac{5!}{5}=4!$ (ways).

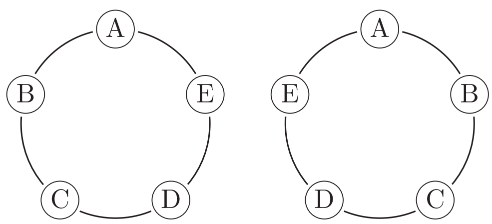
From this, we can understand that a permutation of n things arranged in a circle can be considered by fixing **1 thing and then arranging the remaining $(n-1)$ things**.
Specifically, it is equal to a permutation of $(n-1)$ things, so 5 things have permutations of $(5-1)=4$ (things), so there are ${}_{4}\mathrm{P}_{4}=4!$ (ways).

Necklace permutations

A permutation in which there are n different things, like beads on a necklace, is called a **necklace permutation**. In a necklace permutation, we consider arrangements that are the same when flipped to be the same arrangement. You can find the total number of necklace permutations of n different things as follows.

$$\frac{(n-1)!}{2}$$

Ex. Consider the total number of ways of making a necklace by stringing together 5 different beads, A, B, C, D, and E.



The 2 arrangements shown in the figure on the right are considered the same because they coincide when the necklace is flipped.
A circular permutation in which 5 different beads are arranged in a circle has $(5-1)!$ arrangements, 2 of which coincide when flipped.
Therefore, when making a necklace by stringing together 5 different beads, there are a total of $\frac{(5-1)!}{2}$ ways to make it.

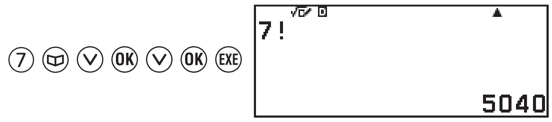
EXERCISE

1 Given 4 uppercase letters, A, B, C, and D, and 4 lowercase letters, e, f, g, and h, arranged in a circle, find how many ways they can be arranged, as shown below.

(1) Arrange the 8 letters freely.

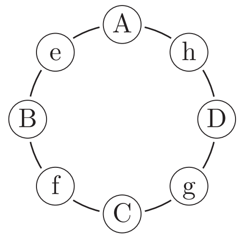
For circular permutations of 8 things, there are $(8-1)!=7!=5040$ (ways)

5040 ways



(2) Alternate the uppercase and lowercase letters.

Arrange 4 uppercase letters in a circle, and arrange the lowercase letters between them.



The total number of arrangements of 4 uppercase letters is a circular permutation of 4 things, so there are $(4-1)!=3!$ (ways)

The total number of arrangements of 4 lowercase letters is a permutation of 4 lowercase letters, considering e is the first, as in the figure on the right.

Therefore, we get ${}_4P_4$ (ways)

Thus, from the law of multiplication, we can arrange them in a total of $3!\times{}_4P_4=3!\times4!=144$ (ways)

144 ways

(3) Arrange the 4 lowercase letters in a row.

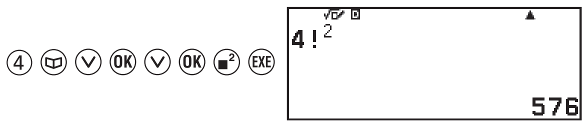
Regard the 4 lowercase letters as a group, and consider the circular permutations of the 5 letters.

This circular permutation has $(5-1)!=4!$ (ways)

In each of these arrangements, there are ${}_4P_4$ ways for the 4 lowercase letters to be arranged.

Therefore, from the law of multiplication, we can find a total of $4!\times{}_4P_4=4!\times4!=576$ (ways)

576 ways



2 Given 2 students wearing white shoes and 8 students wearing black shoes are to be seated at a round table, find how many ways they can be seated, as shown below.

(1) The 2 students wearing white shoes face each other.

Consider the 2 people wearing white shoes are seated facing each other and the 8 people wearing black shoes are seated between them.

The 2 people wearing white shoes are seated in a circular permutation of 2 people, so there are $(2-1)!=1!$ (ways)

If we consider the seating arrangement of the 8 people in black shoes to start with whoever is seated beside 1 of the people in white shoes, then it becomes a permutation of 8 people, so there are ${}_8P_8$ ways

Therefore, from the law of multiplication, we can find a total of $1!\times{}_8P_8=1\times8!=40320$ (ways)

40320 ways

8 ∨ OK ∨ OK EXE

(2) The 2 students wearing white shoes are beside each other.

Regard the 2 people in white shoes as a group, and consider the circular permutations of the 9 people.

This circular permutation has $(9-1)!=8!$ (ways)

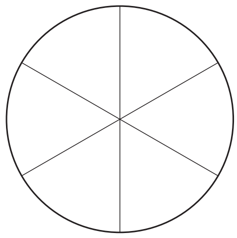
In each of these seating arrangements, there are ${}_2P_2$ ways for just the 2 people in white shoes to be seated.

Therefore, from the law of multiplication, we can find a total of $8!\times{}_2P_2=8!\times2!=80640$ (ways)

80640 ways

8 ∨ OK ∨ OK × 2 ∨ OK ∨ OK EXE

3 Given a circle divided into 6 equal parts, each of which is painted using all of 6 different colors, find how many ways they can be painted.



We can consider this a circular permutation of 6 colors arranged in a circle.

Therefore, there are $(6-1)!=5!=120$ (ways)

120 ways

5 ∨ OK ∨ OK EXE

4 Given a necklace made by stringing 7 different colored beads, find how many ways it could be made.

We can consider this a necklace permutation of 7 different colors.

Therefore, we get $\frac{(7-1)!}{2}=\frac{6!}{2}=360$ (ways)

360 ways

6 ∨ OK ∨ OK 2 EXE

PRACTICE

1 Given 3 students wearing red shirts and 3 students wearing blue shirts are to be seated at a round table, find how many ways they can be seated, as shown below.

(1) Seat the 6 people freely.

For circular permutations of 6 people, there are $(6-1)!=5!=120$ (ways)

120 ways

5 ◫ ∨ OK ∨ OK EXE

5!
120

(2) Alternate the seats of the 3 students in red shirts with the 3 students in blue shirts.

Consider the 3 people in red shirts are seated in a circle, and the 3 people in blue shirts are seated between them.

The 3 people wearing red shirts are seated in a circular permutation of 3 people, so there are $(3-1)!=2!$ (ways)

If we consider starting beside 1 person in a red shirt, then it becomes a permutation of 3 people in blue shirts, so there are ${}_3P_3$ ways

Therefore, from the law of multiplication, we can find a total of $2!\times{}_3P_3=2!\times3!=12$ (ways)

12 ways

2 ◫ ∨ OK ∨ OK × 3 ◫ ∨ OK ∨ OK EXE

2!×3!
12

(3) Seat the 3 people in blue shirts in a row.

Regard the 3 people in blue shirts as a group, and consider the circular permutations of the 4 people.

This circular permutation has $(4-1)!=3!$ (ways)

In each of these arrangements, there are ${}_3P_3$ ways for just the 3 people in blue shirts to be seated.

Therefore, from the law of multiplication, we can find a total of $3!\times{}_3P_3=3!\times3!=36$ (ways)

36 ways

3 ◫ ∨ OK ∨ OK $\blacksquare^2$ EXE

$3!^2$
36

2 Given 2 adults and 4 children are arranged in a circle, find how many ways they can line up, as shown below.

(1) The 2 adults face each other.

Consider the 2 adults are facing each other and the children are arranged between them.

The 2 adults are in a circular permutation of 2 people, so there are $(2-1)!=1!$ (ways)

If we consider the seating arrangement of the 4 children to start beside 1 of the adults, then it becomes a permutation of 4 people, so there are ${}_4P_4$ ways.

Therefore, from the law of multiplication, we can find a total of

$1!\times{}_4P_4=1\times4!=24$ (ways)

24 ways

4 ∨ OK ∨ OK EXE

(2) The 2 adults are beside each other.

Regard the 2 adults as a group, and consider the circular permutations of the 5 people.

This circular permutation has $(5-1)!=4!$ (ways)

In each of these arrangements, there are ${}_2P_2$ ways for just the 2 adults to be arranged.

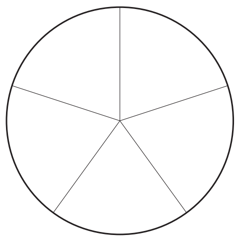
Therefore, from the law of multiplication, we can find a total of

$4!\times{}_2P_2=4!\times2!=48$ (ways)

48 ways

4 ∨ OK ∨ OK × 2 ∨ OK ∨ OK EXE

3 Given a circle divided into 5 equal parts, each of which is painted using each of all 5 different colors, find how many ways they can be painted.



We can consider this a circular permutation of 5 colors arranged in a circle.

Therefore, there are $(5-1)!=4!=24$ (ways)

24 ways

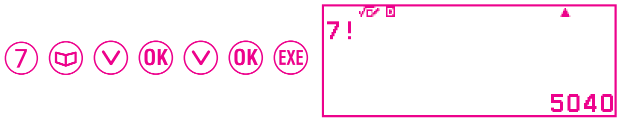
4 ∨ OK ∨ OK EXE

[4] Given 1 ball each with the letters A, B, C, D, E, F, G, and H written on it, solve the following problems.

(1) How many ways are there to arrange all the balls in a circle?

For circular permutations of 8 different things, there are $(8-1)!=7!=5040$ (ways)

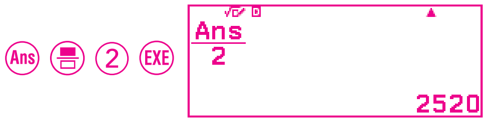
5040 ways



(2) How many ways are there to string the balls in a bracelet?

For necklace permutations of 8 different things, there are $\dfrac{(8-1)!}{2}=\dfrac{7!}{2}=2520$ (ways)

2520 ways



Repeated permutations

TARGET To understand permutations in which the same thing can be used repeatedly.

STUDY GUIDE

Repeated permutations

If you take r things from n different things, of which you can use the same thing several times, and arrange them in a permutation of 1 row, we call that a **repeated permutation of r things taken from n things**, and we find that total number as follows.

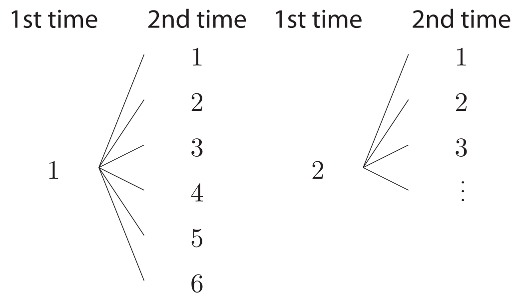
$$\underbrace{n \times n \times n \times \cdots\cdots \times n}_{\text{Product of } r \text{ things}} = n^r$$

Ex. Consider the total number of outcomes for 1 die that is rolled 4 times.

Given a die is rolled 2 times, there are 6 ways, 1 to 6, for the 1st outcome, and for each of these the 2nd outcome has 6 ways.

When a 3rd successive roll is made, there are 6 possible outcomes for the 3rd roll for each of the outcomes of the 2nd.

When a 4th roll is made, there are 6 possible outcomes for the 4th roll for each of the outcomes of the 3rd roll.



Therefore, according to the law of multiplication, the total number of outcomes is $6 \times 6 \times 6 \times 6 = 6^4$ (ways).

EXERCISE

◆ Find how many whole numbers can be made as shown below by using the 5 digits, 1, 2, 3, 4, and 5. However, you can use the same number multiple times.

(1) 6-digit whole number

Since it is a repeated permutation of 6 things taken from 5 things, there are $5^6 = 15625$ (ways)

15625 ways

⑤ ■□ ⑥ EXE

(2) 4-digit even number

To be an even number, the ones place must be 2 or 4, so there are 2 ways to select a number for the ones place.

For any of these, the selection of a number for the remaining tens, hundreds, and thousands places is a repeated permutation of 3 things taken from 5 things, so there are 5^3 ways.

Therefore, from the law of multiplication, we find a total of $2 \times 5^3 = 250$ (ways)

250 ways

② ⊗ ⑤ ■□ ③ EXE

(3) Include 0, to use 6 numbers $0, 1, 2, 3, 4$, and 5, to make 5-digit odd numbers

There are 5 ways, which excludes 0, to select a number for the ten thousands place.

There are 3 ways, $1, 3$, and 5, to select a number for the ones place.

The selection of a number for the remaining places is a repeated permutation of 3 things taken from 6 things, so there are 6^3 ways.

Therefore, from the law of multiplication, we find a total of $5\times3\times6^3=3240$ (ways)

3240 ways

⑤ ⊗ ③ ⊗ ⑥ (x■) ③ (EXE)

5×3×6³
3240

PRACTICE

◆ Find how many whole numbers can be made as shown below by using the 4 digits, $1, 2, 3$, and 4. However, you can use the same number multiple times.

(1) 5-digit whole number

Since it is a repeated permutation of 5 things taken from 4 things, there are $4^5=1024$ (ways)

1024 ways

④ (x■) ⑤ (EXE)

4⁵
1024

(2) 4-digit odd number

To be an odd number, the ones place must be 1 or 3, so there are 2 ways to select a number for the ones place.

For any of these, the selection of a number for the remaining tens, hundreds, and thousands places is a repeated permutation of 3 things taken from 4 things, so there are 4^3 ways.

Therefore, from the law of multiplication, we find a total of $2\times4^3=128$ (ways)

128 ways

② ⊗ ④ (x■) ③ (EXE)

2×4³
128

(3) Include 0, to use 5 numbers $0, 1, 2, 3$, and 4, to make 6-digit even numbers

There are 4 ways, which excludes 0, to select a number for the hundred thousands place.

There are 3 ways, $0, 2$, and 4, to select a number for the ones place.

The selection of a number for the remaining places is a repeated permutation of 4 things taken from 5 things, so there are 5^4 ways.

Therefore, from the law of multiplication, we find a total of $4\times3\times5^4=7500$ (ways)

7500 ways

④ ⊗ ③ ⊗ ⑤ (x■) ④ (EXE)

4×3×5⁴
7500

Combinations

TARGET To understand the total number of ways that several things can be taken from somewhere without considering the order.

STUDY GUIDE

Combinations

Combinations and their totals

We say a **combination** is 1 set of several things taken out without considering the order in which they are arranged. If you take r things from n different things and combine them in 1 set, we call that a **combination of r things taken from n things**, and that total number is shown as ${}_nC_r$. However, $r \leq n$. You can find the total number of these as follows.

$$ {}_nC_r = \frac{{}_nP_r}{r!} = \frac{\overbrace{n(n-1)(n-2)\cdots\cdots(n-r+1)}^{\text{Product of } r \text{ things}}}{\underbrace{r(r-1)(r-2)\cdots\cdots 1}_{\text{Product of } r \text{ things}}} $$

Ex. Consider the total of (${}_5C_3$) combinations possible by taking 3 things from 5 different letters A, B, C, D, and E.

1 combination of $\{A, B, C\}$ can be arranged as ABC, ACB, BAC, BCA, CAB, and CBA, so for a permutation of 3 things taken from 3 things, there are $3!$ ways.

For all of the ${}_5C_3$ combinations, each has $3!$ arrangements, which is equivalent to a total number of ${}_5P_3$ permutations, such that ${}_5C_3 \times 3! = {}_5P_3$

Therefore, there are ${}_5C_3 = \dfrac{{}_5P_3}{3!} = \dfrac{5 \cdot 4 \cdot 3}{3 \cdot 2 \cdot 1} = 10$ (ways).

Properties of combinations

If we define a combination as 3 things taken from 7 things, we also simultaneously define the combination of things that are not taken, which is $7-3=4$ (things). That is to say, the total number of combinations of taking 3 things from 7 is equivalent to the total number of combinations of taking 4 things from 7 things. From this, we can generally derive the following relation.

$$ \boxed{{}_nC_r = {}_nC_{n-r}} $$

This relational expression is useful for long calculations, such as ${}_9C_7 = \dfrac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3}{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 36$.

$$ {}_9C_7 = {}_9C_2 = \frac{9 \cdot 8}{2 \cdot 1} = 36 $$

When $r=0$ in this relational expression, we get ${}_nC_0 = {}_nC_n$. Since ${}_nC_n = 1$, we can determine that ${}_nC_0 = 1$.

By using the permutation formula ${}_nP_r = \dfrac{n!}{(n-r)!}$ in ${}_nC_r = \dfrac{{}_nP_r}{r!}$, then when $0<r<n$, we can also express it as ${}_nC_r = \dfrac{n!}{r!(n-r)!}$.

Also, since ${}_nC_n = 1$ and ${}_nC_0 = 1$, if we set $0! = 1$, then this formula also holds for $r=0$ or $r=n$.

proof

On the scientific calculator, use the Table function to check the relational expression of combinations for $n=7$.

Press (HOME), select [Table], press (OK), then clear the previous data by pressing (AC)

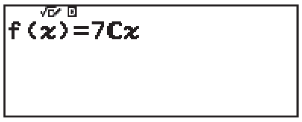
Press (TOOLS), select [Define f(x)/g(x)],

press (OK), select [Define f(x)],

press (OK), input $f(x)={}_7C_x$.

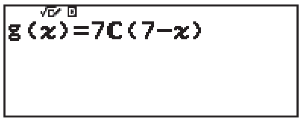
⑦, press (CATALOG), select [Probability], press (OK),

select [Combination(C)], press (OK), (x) (EXE)



In the same way, input $g(x)={}_7C_{(7-x)}$.

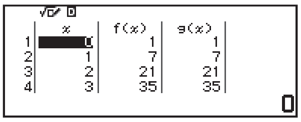
⑦ (CATALOG) (∨) (OK) (∨) (∨) (∨) (OK) (( ) ⑦ (−) (x) ( )) (EXE)

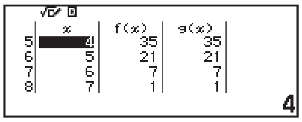


Press (TOOLS), select [Table Range], press (OK)

After inputting [Start:0, End:7, Step:1], select [Execute], press (EXE)

From the table, you can confirm the relational expression of the combination.





EXERCISE

1 Find the following values.

(1) ${}_9C_3$

$${}_9C_3=\frac{9\cdot 8\cdot 7}{3\cdot 2\cdot 1}=84$$

84

check

Use the scientific calculator to calculate the total number of combinations of 3 things taken from 9 things.

Press (HOME), select [Calculate], press (OK)

⑨ (CATALOG) (∨) (OK) (∨) (∨) (∨) (OK) ③ (EXE)

(2) ${}_{13}C_{11}$

$${}_{13}C_{11}={}_{13}C_2=\frac{13\cdot 12}{2\cdot 1}=78$$

78

① ③ (CATALOG) (∨) (OK) (∨) (∨) (∨) (OK) ① ① (EXE)

2 Given 8 letters, A, B, C, D, E, F, G, and H, find how many ways there are to select combinations of 4 letters.

Since it is a combination of 4 things taken from 8 things, there are ${}_8C_4=\frac{8\cdot 7\cdot 6\cdot 5}{4\cdot 3\cdot 2\cdot 1}=70$ (ways)

70 ways

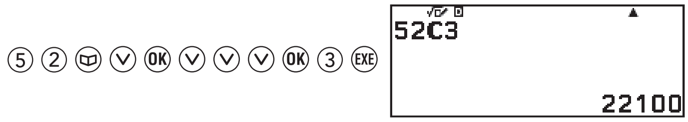
⑧ (CATALOG) (∨) (OK) (∨) (∨) (∨) (OK) ④ (EXE)

3 There are a total of 52 cards in a deck, numbered from 1 to 13, 1 card each, in all 4 suits, diamonds, spades, hearts, and clubs. In this case, solve the following problems.

(1) How many combinations of 3 cards can be selected from all 52 cards?

Since it is a combination of 3 things taken from 52 things, there are ${}_{52}C_3=\dfrac{52\cdot 51\cdot 50}{3\cdot 2\cdot 1}=22100$ (ways)

22100 ways



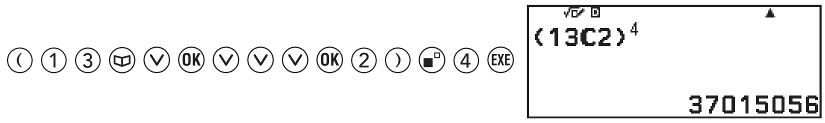
(2) How many combinations of 2 cards from each of 4 suits, for a total of 8 cards, can be selected?

Since it is a combination of 2 cards taken from 13 diamonds, there are ${}_{13}C_2$ ways.

In the same way, 2 cards are selected from each of the spades, hearts, and clubs, for ${}_{13}C_2$ ways.

Therefore, from the law of multiplication, we get ${}_{13}C_2\times{}_{13}C_2\times{}_{13}C_2\times{}_{13}C_2=\left(\dfrac{13\cdot 12}{2\cdot 1}\right)^4=37015056$ (ways)

37015056 ways

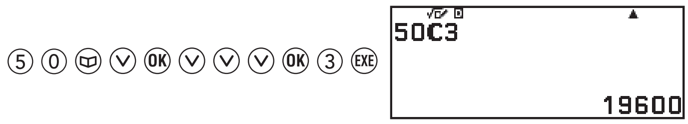


(3) How many combinations of 5 cards, including 2 specified cards, can be selected from all 52 cards?

Since the 2 specified cards are already selected, the other 3 cards are selected from the remaining 50 cards.

Therefore, there are ${}_{50}C_3=\dfrac{50\cdot 49\cdot 48}{3\cdot 2\cdot 1}=19600$ (ways)

19600 ways



PRACTICE

1 Find the following values.

(1) ${}_6C_3$

$${}_6C_3=\frac{6\cdot5\cdot4}{3\cdot2\cdot1}=20$$

20

(2) ${}_{14}C_{10}$

$${}_{14}C_{10}={}_{14}C_4=\frac{14\cdot13\cdot12\cdot11}{4\cdot3\cdot2\cdot1}=1001$$

1001

2 How many combinations are there of 7 cards selected from 9 cards, on each of which is written 1 number, 1 through 9.

Since it is a combination of 7 things taken from 9 things,

there are ${}_9C_7={}_9C_2=\frac{9\cdot8}{2\cdot1}=36$ **(ways)**

36 ways

3 There are 5 different kinds of fish dishes and 6 different kinds of meat dishes. In this case, solve the following problems.

(1) How many combinations of 4 different types of dishes can be selected from all 11 types of fish dishes and meat dishes combined?

Since it is a combination of 4 things taken from 11 things,

there are $_{11}C_4=\dfrac{11\cdot10\cdot9\cdot8}{4\cdot3\cdot2\cdot1}=330$ **(ways)**

330 ways

1 1 ∨ OK ∨ ∨ ∨ OK 4 EXE

11C4
330

(2) How many combinations of 2 different types of fish dishes and 2 different types of meat dishes, for a total of 4 types, can be selected?

There are $_5C_2$ **ways to select combinations of 2 different types from the fish dishes.**

There are $_6C_2$ **ways to select combinations of 2 different types from the meat dishes.**

Therefore, from the law of multiplication, we get $_5C_2\times{}_6C_2=\dfrac{5\cdot4}{2\cdot1}\times\dfrac{6\cdot5}{2\cdot1}=150$ **(ways)**

150 ways

5 ∨ OK ∨ ∨ ∨ OK 2 × 6 ∨ OK ∨ ∨ ∨ OK 2 EXE

5C2×6C2
150

(3) How many combinations of 6 different types, including 2 specified types of dishes, can be selected from all 11 types of fish and meat dishes?

Since the 2 specified types of dishes are already selected, the other 4 types are selected from the remaining 9 types.

Therefore, there are $_9C_4=\dfrac{9\cdot8\cdot7\cdot6}{4\cdot3\cdot2\cdot1}=126$ **(ways)**

126 ways

9 ∨ OK ∨ ∨ ∨ OK 4 EXE

9C4
126

Total number of groups

TARGET To understand the total number of ways that several things can be grouped.

STUDY GUIDE

Grouping

The method to find the total number of ways to group several things varies according to whether there are distinctions between the groups.

Groups with distinctions

Select the ones that go into the 1st group, then select the ones that go into the 2nd group from the rest,... and so on. The combination in the last group is determined automatically.

Groups without distinctions

Use $\dfrac{\text{(Total number when groups have distinctions)}}{\text{(Total number of ways to distinguish groups)}}$ to find the answer.

Ex. Consider the total number of ways that 6 students, A, B, C, D, E, and F, can be divided as follows.

(1) Divide them into 3 rooms, P, Q, and R, with 2 people in each.

For the selection of 2 people taken from 6 people, to go into the first room P, there are ${}_6C_2$ ways.

For the selection of 2 people taken from the remaining 4 people, to go into the next room Q, there are ${}_4C_2$ ways.

For the selection of 2 people taken from the remaining 2 people, to go into the last room R, all the remaining people go in automatically, so there is 1 way.

Therefore, from the law of multiplication, we get ${}_6C_2 \times {}_4C_2 \times 1 = 90$ (ways)

(2) Divide them into 3 rooms, with 2 people in each.

There is no distinction between the 3 rooms when dividing them this way.

In (1), for example, for the 3 groups AB, CD, and EF, A and B might go into P, or Q, or R, so it is a permutation arranging 3 things, so there are $3!$ ways.

If there is no distinction between rooms, then the $3!$ ways to divide them, $\{AB, CD, \text{and } EF\}$, are all the same.

Therefore, when there is no distinction between rooms, we can find the total number by dividing the total number in (1) by $3!$.

P	Q	R
A,B	C,D	E,F
	E,F	C,D
C,D	A,B	E,F
	E,F	A,B
E,F	A,B	C,D
	C,D	A,B

$\dfrac{90}{3!} = \dfrac{90}{6} = 15$ (ways)

EXERCISE

◆ Given 9 people divided as follows, find the total number of ways to divide them.

(1) Divide them into 3 rooms, A, B, and C, with 3 people in each.

For the selection of 3 people taken from 9 people, to go into room A, there are ${}_9\mathrm{C}_3$ ways.

For the selection of 3 people taken from the remaining 6 people, to go into room B, there are ${}_6\mathrm{C}_3$ ways.

For the selection of 3 people taken from the remaining 3 people, to go into room C, all the remaining people go in automatically, so there is 1 way.

Therefore, from the law of multiplication, we get ${}_9\mathrm{C}_3\times{}_6\mathrm{C}_3\times1=\dfrac{9\cdot8\cdot7}{3\cdot2\cdot1}\times\dfrac{6\cdot5\cdot4}{3\cdot2\cdot1}=1680$ (ways)

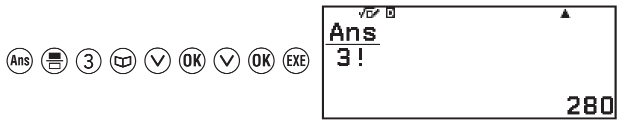
1680 ways

(2) Divide them into 3 rooms, with 3 people in each.

By eliminating the distinction of A, B, and C in (1), we get $3!$ ways repeated.

Therefore, the total number of ways to divide them is $\dfrac{1680}{3!}=\dfrac{1680}{3\cdot2\cdot1}=280$ (ways)

280 ways



(3) Divide them into 3 rooms, with 5 people, 2 people, and 2 people in them.

For the selection of 5 people taken from 9 people, to go into the 5-person room, there are ${}_9\mathrm{C}_5$ ways.

For the selection of the remaining 4 people to be divided into the 2 undistinguished 2-person rooms, there are $\dfrac{{}_4\mathrm{C}_2}{2!}$ ways.

Therefore, from the law of multiplication, we get ${}_9\mathrm{C}_5\times\dfrac{{}_4\mathrm{C}_2}{2!}=\dfrac{9\cdot8\cdot7\cdot6\cdot5}{5\cdot4\cdot3\cdot2\cdot1}\times\dfrac{\frac{4\cdot3}{2\cdot1}}{2\cdot1}=378$ (ways)

378 ways

PRACTICE

◆ Given 8 people divided as follows, find the total number of ways to divide them.

(1) Divide them into 4 rooms, A, B, C, and D, with 2 people in each.

For the selection of 2 people taken from 8 people, to go into room A, there are ${}_8C_2$ ways.
For the selection of 2 people taken from the remaining 6 people, to go into room B, there are ${}_6C_2$ ways.
For the selection of 2 people taken from the remaining 4 people, to go into room C, there are ${}_4C_2$ ways.
For the selection of 2 people taken from the remaining 2 people, to go into room D, all the remaining people go in automatically, so there is 1 way.
Therefore, from the law of multiplication, we get

$${}_8C_2\times{}_6C_2\times{}_4C_2\times1=\frac{8\cdot7}{2\cdot1}\times\frac{6\cdot5}{2\cdot1}\times\frac{4\cdot3}{2\cdot1}=2520 \text{ (ways)}$$

2520 ways

8 📖 ∨ OK ∨ ∨ ∨ OK 2 × 6 📖 ∨ OK ∨ ∨ ∨ OK 2 × 4 📖 ∨ OK ∨ ∨ ∨ OK 2 EXE

8C2×6C2×4C2
2520

(2) Divide them into 4 rooms, with 2 people in each.

By eliminating the distinction of A, B, C, and D in (1), we get $4!$ ways repeated.
Therefore, the total number of ways to divide them is

$$\frac{2520}{4!}=\frac{2520}{4\cdot3\cdot2\cdot1}=105 \text{ (ways)}$$

105 ways

Ans ⊟ 4 📖 ∨ OK ∨ OK EXE

Ans / 4 !
105

(3) Divide them into 4 rooms, with 3 people, 3 people, 1 person, and 1 person in them.

For the selection of 3 people and 3 people from 8 people to be put into 2 undistinguished 3-person rooms, there are $\frac{{}_8C_3\times{}_5C_3}{2!}$ ways.
For the selection of the remaining 2 people to be divided into the 2 undistinguished 1-person rooms, there are $\frac{{}_2C_1}{2!}$ ways.
Therefore, from the law of multiplication, we get

$$\frac{{}_8C_3\times{}_5C_3}{2!}\times\frac{{}_2C_1}{2!}=\frac{\frac{8\cdot7\cdot6}{3\cdot2\cdot1}\times\frac{5\cdot4\cdot3}{3\cdot2\cdot1}}{2\cdot1}\times\frac{2}{2\cdot1}=280 \text{ (ways)}$$

280 ways

⊟ 8 📖 ∨ OK ∨ ∨ ∨ OK 3 × 5 📖 ∨ OK ∨ ∨ ∨ OK 3 ∨ 2 📖 ∨ OK ∨ OK EXE

8C3×5C3 / 2 !
280

Permutations containing similar things

TARGET To understand the total number of permutations that contain similar things.

STUDY GUIDE

Permutations containing similar things

Given n number of objects, of which some of them are the same type of thing, a things, b things.... r things, we can find the total number of permutations arranged in 1 row as follows.

$$\frac{n!}{a!b!\cdots\cdots r!} \quad \textbf{However, } a+b+\cdots\cdots+r=n$$

Ex. Consider arranging 9 balls, comprising 4 red balls, 3 white balls, and 2 black balls, in 1 row.

Start by considering

(Total number of ways to divide them when there are no distinctions)$=\dfrac{\text{(Total number when there are distinctions)}}{\text{(Total number of ways to distinguish)}}$.

You can find the total number of ways by dividing $9!$, for the arrangement of all 9 distinguished balls, by the product of $4!$ to distinguish the red balls, $3!$ to distinguish the white balls, and $2!$ to distinguish the black balls.

That is to say, there are $\dfrac{9!}{4!3!2!}=1260$ (ways).

check

Use the scientific calculator to calculate the total number of permutations that contain same things.

9 ∨ OK ∨ OK

4 ∨ OK ∨ OK × 3 ∨ OK ∨ OK × 2 ∨ OK ∨ OK EXE

```
 9!
4!×3!×2!
        1260
```

OTHER METHODS

Consider 9 locations in which to place all the balls.

For the selection of 4 locations taken from 9 locations, to place the 4 red balls, there are ${}_9C_4$ ways.

For the selection of 3 locations taken from the remaining 5 locations, to place the 3 white balls, there are ${}_5C_3$ ways.

The locations to place the 2 black balls are determined automatically.

Therefore, from the law of multiplication, we get ${}_9C_4\times{}_5C_3\times1=1260$ (ways) to arrange 9 things in 1 row

EXERCISE

1 Find the total number of ways to arrange 5 blue balls, 4 yellow balls, and 3 green balls in 1 row.

This is a permutation of 12 things, which includes 5 things, 4 things, and 3 things that are similar, so there are

$\dfrac{12!}{5!4!3!}=27720$ (ways)

27720 ways

2 Find all of the ways to make whole numbers when making 6-digit whole numbers by using all 6 digits, 2, 2, 3, 3, 4, and 4.

This is a permutation of 6 things, which includes 2 each of 3 types of similar things, so there are $\dfrac{6!}{2!2!2!}=90$ (ways)

90 ways

3 Find how many ways there are to arrange the 8 letters x, x, y, y, y, z, z, and z in 1 row, as shown below.

(1) Arrange the 8 letters freely.

This is a permutation of 8 things, which includes 2 things, 3 things, and 3 things that are similar, so there are

$\dfrac{8!}{2!3!3!}=560$ (ways)

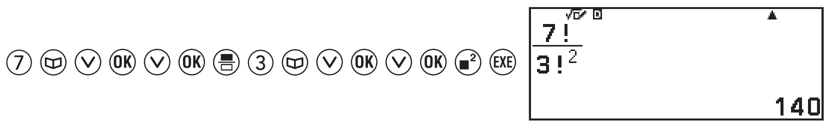
560 ways

(2) Arrange them so the 2 x's are consecutive.

Regard the 2 x's as 1 set, and consider it a permutation of 7 things, including 1 set for x, 3 y's, and 3 z's.

$\dfrac{7!}{1!3!3!}=140$ (ways)

140 ways



PRACTICE

1 Find the total number of ways to arrange 4 red balls and 3 white balls in 1 row of 7 balls.

This is a permutation of 7 things, which includes 4 things and 3 things that are similar, so there are $\dfrac{7!}{4!3!}=35$ **(ways)**

35 ways

2 Find all of the ways to make whole numbers when making 8-digit whole numbers by using all 8 digits, 4, 4, 5, 5, 6, 6, 6, and 6.

This is a permutation of 8 things, which includes 2 things, 2 things, and 4 things that are similar, so there are $\dfrac{8!}{2!2!4!}=420$ **(ways)**

420 ways

3 Given 9 letters, A, A, B, B, B, C, C, C, C arranged in 1 row, find how many ways they can be arranged, as shown below.

(1) Arrange the 9 letters freely.

This is a permutation of 9 things, which includes 2 things, 3 things, and 4 things that are similar, so there are $\dfrac{9!}{2!3!4!}=1260$ **(ways)**

1260 ways

(2) Arrange the 3 B's consecutively.

Regard the 3 B's as 1 set, and consider it a permutation of 7 things, including 1 set for B, 2 A's, and 4 C's.

$\dfrac{7!}{1!2!4!}=105$ **(ways)**

105 ways

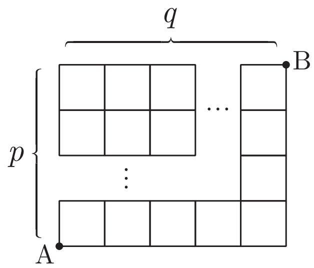
Shortest path

TARGET To understand how to find the total number of shortest routes.

STUDY GUIDE

Shortest path

You can find the total number of shortest paths (shortest routes) that go from A to B through the $p\times q$ grid shown in the figure on the right, by considering them to be permutations containing similar things, as follows.



$$\boxed{\frac{(p+q)!}{p!q!}}$$

Ex. There are paths through the 2×3 grid as in figure 1 on the right. Consider the total number of shortest routes from A to B that go along these paths.

The shortest way to go from A to B is to go 2 cells up and 3 cells to the right.

Here, going 1 cell upward is denoted by ↑, and going 1 cell to the right is denoted by →.

For example, the progress shown in figure 2 on the right is expressed as "→↑→↑→".

That is to say, we can consider the total number of shortest routes as the total number of 2 ↑ and 3 → permutations.

This is a permutations containing similar things, so the total number we find is

$\dfrac{(2+3)!}{2!3!}=10$ (ways).

Figure 1

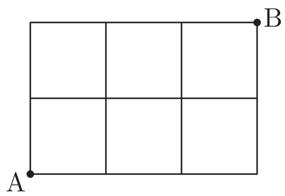
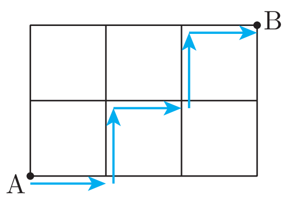


Figure 2

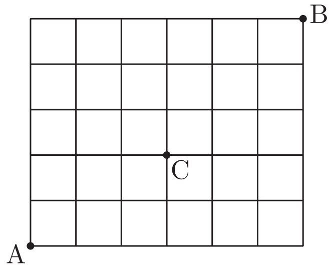


check

Use the scientific calculator to calculate the total number of shortest routes.

EXERCISE

◆ There are paths through the 5×6 grid in the figure on the right. In this case, find how many shortest routes go through as shown below.



(1) To go from A to B.

Going 1 cell upward is denoted by ↑, and going 1 cell to the right is denoted by →.

The total number of shortest routes is the same as the total number of 5 ↑ and 6 → permutations.

Therefore, we find the total number is $\dfrac{(5+6)!}{5!6!}=462$ (ways)

462 ways

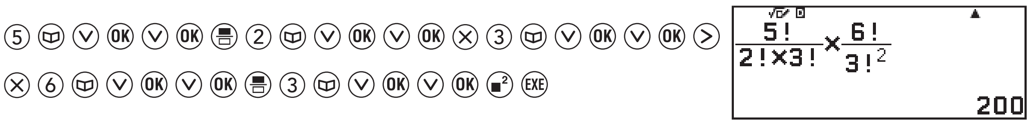
(2) To go from A, through C to B.

The total number of shortest routes from A to C is the same as the total number of 2 ↑ and 3 → permutations, which is $\dfrac{(2+3)!}{2!3!}$ (ways)

The total number of shortest routes from C to B is the same as the total number of 3 ↑ and 3 → permutations, which is $\dfrac{(3+3)!}{3!3!}$ (ways)

Therefore, from the law of multiplication, we find a total of $\dfrac{(2+3)!}{2!3!} \times \dfrac{(3+3)!}{3!3!}=200$ (ways)

200 ways



(3) To go from A to B, without passing through C.

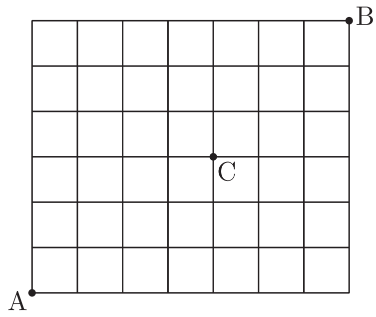
To find the answer, subtract the number found in (2) from the number found in (1).

Therefore, we get $462-200=262$ (ways)

262 ways

PRACTICE

◆ There are paths through the 6×7 grid in the figure on the right. In this case, find how many shortest routes go through as shown below.



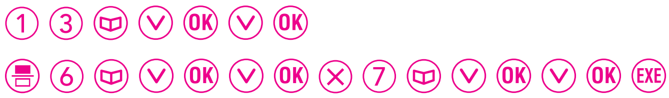
(1) To go from A to B.

Going 1 cell upward is denoted by ↑, and going 1 cell to the right is denoted by →.

The total number of shortest routes is the same as the total number of 6 ↑ and 7 → permutations.

Therefore, we find the total number is $\dfrac{(6+7)!}{6!7!}=1716$ **(ways)**

1716 ways



(2) To go from A, through C to B.

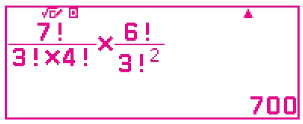
The total number of shortest routes from A to C is the same as the total number of 3 ↑ and 4 → permutations, which is $\dfrac{(3+4)!}{3!4!}$ **(ways)**

The total number of shortest routes from C to B is the same as the total number of 3 ↑ and 3 → permutations, which is $\dfrac{(3+3)!}{3!3!}$ **(ways)**

Therefore, from the law of multiplication, we find a total of

$\dfrac{(3+4)!}{3!4!}\times\dfrac{(3+3)!}{3!3!}=700$ **(ways)**

700 ways



(3) To go from A to B, without passing through C.

To find the answer, subtract the number found in (2) from the number found in (1).

Therefore, we get $1716-700=1016$ **(ways)**

1016 ways

Repeated combinations

TARGET To understand how to find the total number of combinations it is possible to create while allowing repetition.

STUDY GUIDE

Repeated combinations

Combinations that take r things from n types of things, while allowing repetition, are considered as follows.

Consider permutations arranging r ○ and $(n-1)$ partitions | .

Ex. Consider a combination that allows repetition where you take 7 things, from 3 types of fruit: bananas, oranges, and pineapples.

Consider an arrangement of 7 ○ and 2 partitions | , which would be as shown in the following table of fruit combinations.

Combinations of fruit			Arrangement of ○ and |
Banana	Orange	Pineapple	
2	1	4	○○ | ○ | ○○○○
2	2	3	○○ | ○○ | ○○○
0	2	5	| ○○ | ○○○○○
4	0	3	○○○○ | | ○○○
⋮			

2 bananas	|	1 orange	|	4 pineapples
○○	|	○	|	○○○○

Therefore, the total number of fruit combinations is equal to the total number of permutations of an arrangement of

7 ○ and 2 | , which is $\dfrac{(7+2)!}{7!2!}=36$ (ways).

check

Use the scientific calculator to calculate the total number of combinations of 7 things taken from 3 types of things, while allowing repetition.

9 [catalog] ∨ OK ∨ OK

[fraction] 7 [catalog] ∨ OK ∨ OK × 2 [catalog] ∨ OK ∨ OK EXE

```
 9!
7!×2!
                36
```

EXERCISE

1 Find how many combinations there are when you select 12 sheets of paper from the 4 colors of red, blue, white, and yellow. However, you can also not select any of those colors of paper.

The total number you find is equivalent to a permutation arranging 12 ○ and 3 | .

Therefore, we get $\dfrac{(12+3)!}{12!3!}=455$ (ways)

455 ways

1 5 [catalog] ∨ OK ∨ OK

[fraction] 1 2 [catalog] ∨ OK ∨ OK × 3 [catalog] ∨ OK ∨ OK EXE

```
  15!
12!×3!
               455
```

2 Given 15 notebooks to be distributed to 5 students, find how many ways they can be distributed. However, at least 1 notebook is to be distributed to each person.

Assuming that 1 notebook is distributed to each of the 5 people, consider how to distribute the remaining $15-5=10$ (notebooks).

The total number you find is equivalent to a permutation arranging 10 ○ and 4 ｜.

Therefore, we get $\dfrac{(10+4)!}{10!4!}=1001$ (ways)

1001 ways

3 Solve the following problems with regards to a combination of x, y, and z that satisfies $x+y+z=20$.

(1) How many combinations are there where x, y, and z are non-negative whole numbers?

Consider a repeated combination, which allows repetition, where you take 20 letters, from 3 types of letters: x, y, and z.

The total number you find is equivalent to a permutation arranging 20 ○ and 2 ｜.

Therefore, we get $\dfrac{(20+2)!}{20!2!}=231$ (ways)

231 ways

(2) How many combinations of are there where x, y, and z are natural numbers?

Assuming that x, y, and z are distributed 1 at a time, consider the repeated combinations of the remaining $20-3=17$ (letters).

The total number you find is equivalent to a permutation arranging 17 ○ and 2 ｜.

Therefore, we get $\dfrac{(17+2)!}{17!2!}=171$ (ways)

171 ways

PRACTICE

1 Given 4 letters, A, B, C, and D, from which, allowing repetition, you take 25, find how many ways are there to combine them. However, you can also not select any letter.

The total number you find is equivalent to a permutation arranging 25 ○ and 3 | .

Therefore, we get $\dfrac{(25+3)!}{25!3!}=3276$ **(ways)**

3276 ways

(2) (8) (📖) (∨) (OK) (∨) (OK)
(☰) (2) (5) (📖) (∨) (OK) (∨) (OK) (×) (3) (📖) (∨) (OK) (∨) (OK) (EXE)

$\dfrac{28!}{25!\times3!}$ 3276

2 Given 16 pieces of cake to be distributed to 6 students, find how many ways they can be distributed. However, at least 1 piece is to be distributed to each person.

Assuming that 1 piece of cake is distributed to each of the 6 people, consider how to distribute the remaining $16-6=10$ (pieces).

The total number you find is equivalent to a permutation arranging 10 ○ and 5 | .

Therefore, we get $\dfrac{(10+5)!}{10!5!}=3003$ **(ways)**

3003 ways

(1) (5) (📖) (∨) (OK) (∨) (OK)
(☰) (1) (0) (📖) (∨) (OK) (∨) (OK) (×) (5) (📖) (∨) (OK) (∨) (OK) (EXE)

$\dfrac{15!}{10!\times5!}$ 3003

3 Solve the following problems with regards to a combination of x, y, and z that satisfies $x+y+z=35$.

(1) How many combinations are there where x, y, and z are non-negative whole numbers?

Consider a repeated combination, which allows repetition, where you take 35 letters, from 3 types of letters: x, y, and z.

The total number you find is equivalent to a permutation arranging 35 ○ and 2 | .

Therefore, we get $\dfrac{(35+2)!}{35!2!}=666$ **(ways)**

666 ways

(3) (7) (📖) (∨) (OK) (∨) (OK)
(☰) (3) (5) (📖) (∨) (OK) (∨) (OK) (×) (2) (📖) (∨) (OK) (∨) (OK) (EXE)

$\dfrac{37!}{35!\times2!}$ 666

(2) How many combinations of are there where x, y, and z are natural numbers?

Assuming that x, y, and z are distributed 1 at a time, consider the repeated combinations of the remaining $35-3=32$ (letters).

The total number you find is equivalent to a permutation arranging 32 ○ and 2 | .

Therefore, we get $\dfrac{(32+2)!}{32!2!}=561$ **(ways)**

561 ways

(3) (4) (📖) (∨) (OK) (∨) (OK)
(☰) (3) (2) (📖) (∨) (OK) (∨) (OK) (×) (2) (📖) (∨) (OK) (∨) (OK) (EXE)

$\dfrac{34!}{32!\times2!}$ 561

Numbers of sets and elements

TARGET To understand how to find the number and elements of $A \cup B$ and $\bar{A}$.

STUDY GUIDE

Number of elements in a set

If the number of elements in the set A is finite, then $n(A)$ represents the number of elements. In particular, in the case of the empty set $\varnothing$, since it has not elements $n(\varnothing)=0$.

Ex. If $A=\{2, 4, 6, 8, 10, \text{ or } 12\}$, then $n(A)=6$(elements)

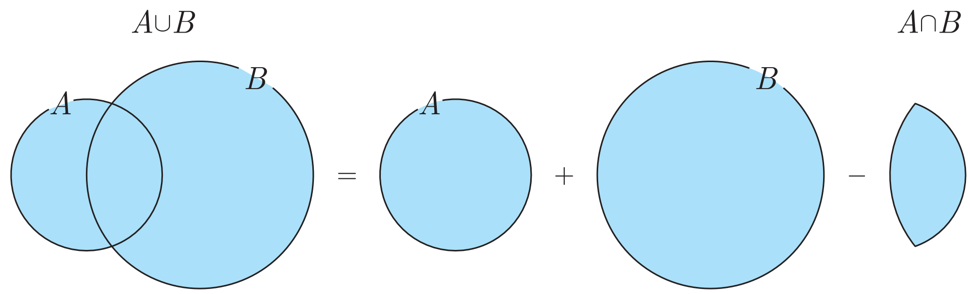
The following relations hold for unions of sets and complementary sets.

(i) $n(A \cup B)=n(A)+n(B)-n(A \cap B)$

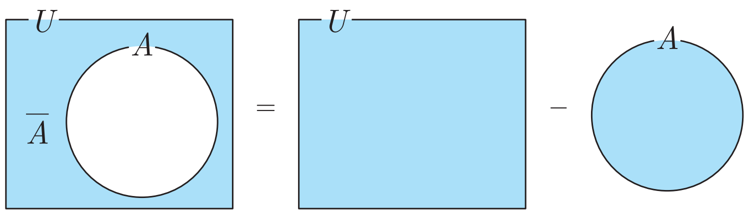
Particularly when $A \cap B=\varnothing$, $n(A \cup B)=n(A)+n(B)$

(ii) $n(\overline{A}) = n(U) - n(A)$

(i)



(ii)



EXERCISE

◆ Let U be a set of natural numbers less than or equal to 30, let A be the set of multiples of 2, let B be the set of multiples of 3, and let C be the set of multiples of 7. In this case, solve the following problems.

(1) Find $n(U)$, $n(A)$, $n(B)$, and $n(C)$.

From the natural numbers less than or equal to 30 being $1, 2, 3, \ldots, 30$, we get $n(U)=30$

From $A=\{2\times1, 2\times2, 2\times3, \ldots, 2\times15\}$, we get $n(A)=15$

From $B=\{3\times1, 3\times2, 3\times3, \ldots, 3\times10\}$, we get $n(B)=10$

From $C=\{7\times1, 7\times2, 7\times3, \text{ and } 7\times4\}$, we get $n(C)=4$

$\mathbf{n(U)=30,\ n(A)=15,\ n(B)=10,\ n(C)=4}$

(2) Find the number of multiples of 2 and 3.

The set of multiples of 2 and multiples of 3 is expressed as $A \cap B$. It is the set of natural numbers less than or equal to 30 that are multiples of 6.

From $A \cap B = \{6\times1, 6\times2, 6\times3, \ldots, 6\times5\}$, we get $n(A \cap B) = 5$ (elements)

5 elements

(3) Find the number of multiples of 2 or 3.

The set of multiples of 2 or multiples of 3 is expressed as $A \cup B$.

$n(A \cup B) = n(A) + n(B) - n(A \cap B) = 15 + 10 - 5 = 20$ (elements)

20 elements

(4) Find the number of elements that are indivisible by 7.

The set of numbers that are indivisible by 7 is expressed as $\overline{C}$.

$n(\overline{C}) = n(U) - n(C) = 30 - 4 = 26$ (elements)

26 elements

PRACTICE

1 Let U be a set of natural numbers less than or equal to 90, let A be the set of multiples of 4, let B be the set of multiples of 5, and let C be the set of multiples of 11. In this case, solve the following problems.

(1) Find $n(U)$, $n(A)$, $n(B)$, and $n(C)$.

From the natural numbers less than or equal to 90 being 1, 2, 3, ..., 90, we get $n(U) = 90$

From $A = \{4\times1, 4\times2, 4\times3, \ldots, 4\times22\}$, we get $n(A) = 22$

From $B = \{5\times1, 5\times2, 5\times3, \ldots, 5\times18\}$, we get $n(B) = 18$

From $C = \{11\times1, 11\times2, 11\times3, \ldots, 11\times8\}$, we get $n(C) = 8$

$n(U) = 90$, $n(A) = 22$, $n(B) = 18$, $n(C) = 8$

(2) Find the number of multiples of 4 and 5.

The set of multiples of 4 and multiples of 5 is expressed as $A \cap B$. It is the set of natural numbers less than or equal to 90 that are multiples of 20.

From $A \cap B = \{20\times1, 20\times2, 20\times3$, and $20\times4\}$, we get $n(A \cap B) = 4$ (elements)

4 elements

(3) Find the number of multiples of 4 or 5.

The set of multiples of 4 or multiples of 5 is expressed as $A \cup B$.

$A \cup B = n(A) + n(B) - n(A \cap B) = 22 + 18 - 4 = 36$ (elements)

36 elements

(4) Find the number of elements that are indivisible by 11.

The set of numbers that are indivisible by 11 is expressed as $\overline{C}$.

$n(\overline{C}) = n(U) - n(C) = 90 - 8 = 82$ (elements)

82 elements

2 Market research was done for 2 models of product, A and B, and the responses of 1000 people were collected. The results were totaled, and product A was purchased by 459, product B was purchased by 472 people, and both of the 2 models were purchased by 236 people. In this case, solve the following problems.

(1) Find the number of people who purchased product A or product B.

The set of all people who responded are represented by U, the set of people who purchased product A are represented by A, and the set of people who purchased product B are represented by B.

The set of people who purchased both of the 2 models is represented by $A \cap B$, and the set of people who purchased product A or product B are represented by $A \cup B$.

$n(A)=459$ (people), $n(B)=472$ (people), and $n(A \cap B)=236$ (people), therefore

$A \cup B = n(A)+n(B)-n(A \cap B)=459+472-236=695$ (people)

695 people

(2) Find the number of people who purchased neither of the 2 models of product.

The set of people who purchased neither of the 2 models of product is $\overline{A} \cap \overline{B}$.

From De Morgan's laws, we get $\overline{A \cup B} = \overline{A} \cap \overline{B}$, so

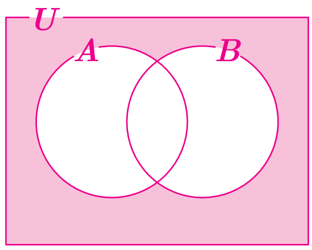
$n(\overline{A} \cap \overline{B}) = n(\overline{A \cup B}) = n(U) - n(A \cup B) = 1000-695=305$ (people)

305 people

EXTRA Info.

De Morgan's laws hold for subsets A and B of the universal set U, as shown below.

$\overline{A \cup B} = \overline{A} \cap \overline{B}$, $\overline{A \cap B} = \overline{A} \cup \overline{B}$



Events and probability

TARGET To understand the basics of probability.

STUDY GUIDE

Events and probability

Trials and events

An experiment or observation that can be repeated under the same conditions, such as rolling dice, is called a **trial**. The result of a trial is called an **event**, which is often expressed using a set. Also, an event that always happens in a trial is called a **sure event** for that trial, and it is expressed by U. Furthermore, an event that cannot be further divided is called a **root event**.

Ex. The sure events in a trial rolling 1 die is $U=\{1, 2, 3, 4, 5, \text{and } 6\}$.

Probability

Given 1 trial, in which any root event is expected to occur equally, where N is the number of all possible events and a is the number of events A occurring, then we say $\dfrac{a}{N}$ is the **probability** of the event A, which we express as $P(A)$.

$$P(A)=\frac{\text{(Number of cases of event } A \text{ occurring)}}{\text{(Number of all cases that could occur)}}=\frac{a}{N}$$

When we consider multiple combinations, such as of coins or dice, to calculate their probability, we need to distinguish them to find the number of cases.

EXERCISE

$\boxed{1}$ Given 1 die is rolled, find the probability of an outcome of a 3 or less.

There are a total of 6 possible outcomes.

Furthermore, there are 3 ways, 1, 2, and 3, to roll a 3 or less.

Therefore, we find a probability of $\dfrac{3}{6}=\dfrac{1}{2}$

$\dfrac{1}{2}$

check

In Math Box on the scientific calculator, you can use the Dice Roll or Coin Toss simulation to check probability (statistical probability).

Simulate rolling 1 die 250 times.

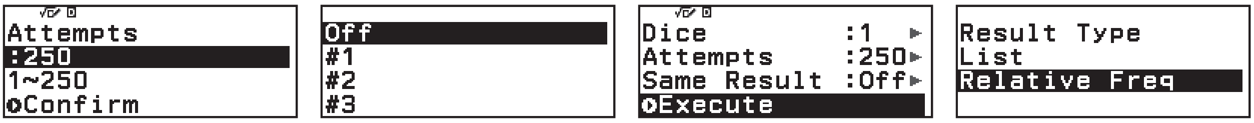
Press ⌂, select [Math Box], press OK, select [Dice Roll], press OK

Select [Dice], press OK, select [1 Die], press OK

Select [Attempts], press OK, after inputting 250 (number of attempts), select [Confirm], press OK

Select [Same Result], press OK, select [Off], press OK

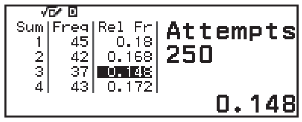
Select [Execute], press EXE, select [Relative Freq], press OK



Read the values in the table for when the sum of the die was 1, 2, and 3 , then

add them to get $0.18+0.168+0.148=0.496$, and you can confirm that the

probability is approximately $0.5=\frac{1}{2}$.



(When Same Result is Off, the results of each trial are different.)

2 Given 2 coins being tossed at the same time, find the probability of 1 being heads and 1 being tails.

Consider the combinations of heads and tails of the 2 coins.

There are a total of 4 combinations: (Heads, Heads), (Heads, Tails), (Tails, Heads), and (Tails, Tails).

Furthermore, there are 2 ways in which 1 is heads and 1 is tails, as shown above.

Therefore, we find a probability of $\frac{2}{4}=\frac{1}{2}$

$\frac{1}{2}$

check

On the scientific calculator, simulate tossing 2 coins 250 times.

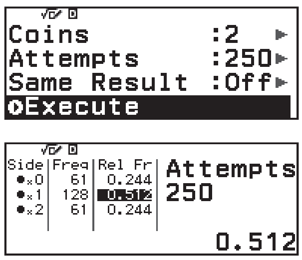
Press ⌂, select [Math Box], press OK, select [Coin Toss], press OK

Select [Coins], press OK, select [2 Coins], press OK

Select [Attempts], press OK, after inputting 250, select [Confirm], press OK

Select [Same Result], press OK, select [Off], press OK

Select [Execute], press EXE, select [Relative Freq], press OK



From the table, the value of heads coming up 1 time [●×1] is about 0.51, and you can confirm that the probability is approximately $\frac{1}{2}$.

[3] Given 2 dice are rolled at the same time, find the probability of their sum being 5.

Consider the combinations of the outcome of the 2 dice rolls.

From the law of multiplication, we get $6\times6=36$ (ways) for all of the combinations.

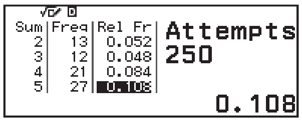
Furthermore, there are 4 ways to get a sum of 5, they are $(1, 4)$, $(2, 3)$, $(3, 2)$, and $(4, 1)$.

Therefore, we find a probability of $\frac{4}{36}=\frac{1}{9}$ **$\frac{1}{9}$**

check

From Dice Roll, select 2 Dice, and simulate rolling 2 dice 250 times.

After the trial, select [Relative Freq], press (OK), select [Sum], press (OK)



From the table, value of the sum of the rolled dice being 5 is about 0.11,

and you can confirm that the probability is approximately $\frac{1}{9}$.

PRACTICE

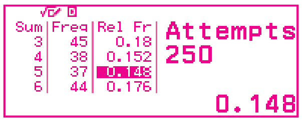
[1] Given 1 die is rolled, find the probability of an outcome of a 5 or greater.

There are a total of 6 possible outcomes.

Furthermore, there are 2 ways, 5, and 6, to roll a 5 or greater.

Therefore, we find a probability of $\frac{2}{6}=\frac{1}{3}$ **$\frac{1}{3}$**

check



Press (⌂), select [Math Box], press (OK), select [Dice Roll], press (OK), select [Dice], press (OK), select [1 Die], press (OK)

Select [Attempts], press (OK), after inputting 250, select [Confirm], press (OK)

Select [Same Result], press (OK), select [Off], press (OK), select [Execute], press (EXE), select [Relative Freq], press (OK)

Read the values in the table for when the sum of the rolled die was 5 or 6, then add them.

[2] Given 2 coins being tossed at the same time, find the probability of the 2 coins both being heads.

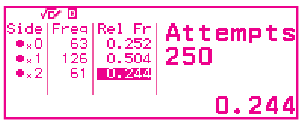
Consider the combinations of heads and tails of the 2 coins.

There are a total of 4 combinations: (Heads, Heads), (Heads, Tails), (Tails, Heads), and (Tails, Tails).

Furthermore, there is 1 way in which 2 are heads, as shown above.

Therefore, we find a probability of $\frac{1}{4}$ **$\frac{1}{4}$**

check



Press (⌂), select [Math Box], press (OK), select [Coin Toss], press (OK), select [Coins], press (OK), select [2 Coins], press (OK)

Select [Attempts], press (OK), after inputting 250, select [Confirm], press (OK)

Select [Same Result], press (OK), select [Off], press (OK), select [Execute], press (EXE), select [Relative Freq], press (OK)

3 Given 2 dice are rolled at the same time, find the following probabilities.

(1) Probability that the sum of the rolled dice is 9 or greater

Consider the combinations of the outcome of the 2 dice rolls.

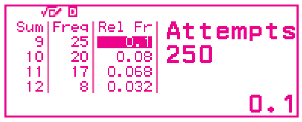
From the law of multiplication, we get $6\times6=36$ (ways) for all of the combinations.

Furthermore, there are 10 ways, (3, 6), (4, 5), (4, 6), (5, 4), (5, 5), (5, 6), (6, 3), (6, 4), (6, 5), (6, 6) to roll a 9 or greater.

Therefore, we find a probability of $\dfrac{10}{36}=\dfrac{5}{18}$

$\dfrac{5}{18}$

check



Press ⌂, select [Math Box], press (OK), select [Dice Roll], press (OK), select [Dice], press (OK), select [2 Dice], press (OK)

Select [Attempts], press (OK), after inputting 250, select [Confirm], press (OK)

Select [Same Result], press (OK), select [Off], press (OK), select [Execute], press (EXE)

Select [Relative Freq], press (OK), select [Sum], press (OK)

Read the values in the table for when the sum of the rolled dice was 9 or greater, then add them.

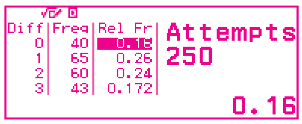
(2) Probability of the same numbers coming up

There are 6 ways for the same numbers to come up, they are (1, 1), (2, 2), (3, 3), (4, 4), (5, 5) , and (6, 6).

Therefore, we find a probability of $\dfrac{6}{36}=\dfrac{1}{6}$

$\dfrac{1}{6}$

check



In the screen showing the results for (1), press ⮌, select [Relative Freq], press (OK), select [Difference], press (OK)

The same number coming up on 2 dice together is equal to a difference of 0.

Probabilities of various events

TARGET To understand how to find probabilities by using the concepts of permutations and combinations.

STUDY GUIDE

Probabilities of various events

You can use concepts such as permutations and combinations to find probabilities.

EXTRA Info.

$${}_n\mathrm{P}_r = n(n-1)(n-2)\cdots\cdots(n-r+1)$$

$$n! = n(n-1)(n-2)\cdots\cdots 3\cdot 2\cdot 1$$

$$(n+1)! = (n+1)n!$$

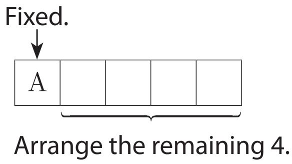
$${}_n\mathrm{C}_r = \frac{{}_n\mathrm{P}_r}{r!} \qquad {}_n\mathrm{C}_r = {}_n\mathrm{C}_{n-r}$$

Probability by using permutations

Ex. Given 5 cards, each with 1 letter A, B, C, D, or E written on them, being arranged in 1 row from the left at random, find the probability that the card on the left end is A.

An arrangement of 5 cards is a permutation of 5 things taken from 5 things, so there are ${}_5\mathrm{P}_5$ ways.

An arrangement with A on the left end is a permutation in which A is fixed on the left end and 4 things are taken from the remaining 4 things, so there are ${}_4\mathrm{P}_4$ ways.



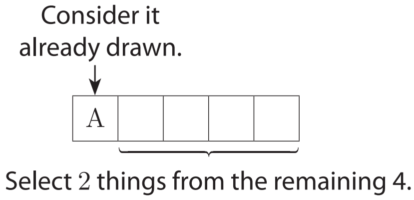
Therefore, we find a probability of $\dfrac{{}_4\mathrm{P}_4}{{}_5\mathrm{P}_5} = \dfrac{4!}{5!} = \dfrac{4!}{5\cdot 4!} = \dfrac{1}{5}$.

Probability by using combinations

Ex. Given 5 cards, each with 1 letter A, B, C, D, or E written on them, from which 3 cards are drawn at random, find the probability that 1 card is A.

The selection of 3 cards taken from 5 cards is a combination of 3 things taken from 5 things, so there are ${}_5\mathrm{C}_3$ ways.

A selection that includes A in the 3 cards is a combination of 2 things taken from the remaining 4 things, as A has already been taken, so there are ${}_4\mathrm{C}_2$ ways.



Therefore, we find a probability of $\dfrac{{}_4\mathrm{C}_2}{{}_5\mathrm{C}_3} = \dfrac{\frac{4\cdot 3}{2\cdot 1}}{\frac{5\cdot 4\cdot 3}{3\cdot 2\cdot 1}} = \dfrac{4\cdot 3}{2\cdot 1}\cdot\dfrac{3\cdot 2\cdot 1}{5\cdot 4\cdot 3} = \dfrac{3}{5}$.

EXERCISE

1 Given a relay team of 6 athletes, A, B, C, D, E, and F, if the running order is decided by drawing lots, find the probability that A will be the first runner and that F will be the last runner.

The 6 runners are a permutation of 6 runners, so there are ${}_6\mathrm{P}_6$ ways.

Making A the first runner and F the last runner gives us a permutation of the remaining 4 people, so there are ${}_4\mathrm{P}_4$ ways.

Therefore, we find a probability of $\dfrac{{}_4\mathrm{P}_4}{{}_6\mathrm{P}_6}=\dfrac{4!}{6!}=\dfrac{4!}{6\cdot5\cdot4!}=\dfrac{1}{30}$

$$\mathbf{\frac{1}{30}}$$

check

Press ⌂, select [Calculate], press OK

4 ∨ OK ∨ OK 6 ∨ OK ∨ OK EXE

2 There is a bag holding 7 red balls and 5 white balls. Given 3 balls are taken from the bag at the same time, find the probability that the balls taken out are 2 red balls and 1 white ball.

The red balls and white balls add up to 12 balls, from which 3 balls are taken, so there are ${}_{12}\mathrm{C}_3$ ways.

From the law of multiplication, we find there are ${}_7\mathrm{C}_2\times{}_5\mathrm{C}_1$ (ways) to take 2 red balls from 7 red balls and to take 1 white ball from 5 white balls

Therefore, we find a probability of $\dfrac{{}_7\mathrm{C}_2\times{}_5\mathrm{C}_1}{{}_{12}\mathrm{C}_3}=\dfrac{\dfrac{7\cdot6}{2\cdot1}\cdot5}{\dfrac{12\cdot11\cdot10}{3\cdot2\cdot1}}=\dfrac{7\cdot6}{2\cdot1}\cdot5\cdot\dfrac{3\cdot2\cdot1}{12\cdot11\cdot10}=\dfrac{21}{44}$

$$\mathbf{\frac{21}{44}}$$

7 ∨ OK ∨ ∨ ∨ OK 2 × 5 ∨ OK ∨ ∨ ∨ OK 1
∨ 1 2 ∨ OK ∨ ∨ ∨ OK 3 EXE

PRACTICE

1 Given 5 children and 2 adults are arranged in 1 line, find the following probabilities.

(1) Probability that the 2 adults are beside each other

To arrange 7 people, including children and adults, in 1 row is a permutation of 7 people, so there are ${}_7P_7$ ways.

Treat the 2 adults as a group, so it is an arrangement of 6 things, so there are ${}_6P_6$ ways.

In each of these arrangements, there are ${}_2P_2$ ways for the 2 adults to be arranged.

Therefore, we find a probability of $\dfrac{{}_6P_6 \times {}_2P_2}{{}_7P_7} = \dfrac{6!2!}{7!} = \dfrac{6!\cdot 2}{7\cdot 6!} = \dfrac{2}{7}$

$$\frac{2}{7}$$

6 ∨ OK ∨ OK × 2 ∨ OK ∨ OK ∨ 7 ∨ OK ∨ OK EXE

$\dfrac{6!\times 2!}{7!}$ $\dfrac{2}{7}$

(2) Probability that the 2 adults will be on both ends

An arrangement with the adults on both ends has ${}_2P_2$ ways.

In each of these arrangements, there are ${}_5P_5$ ways for the 5 children to be arranged.

Therefore, we find a probability of $\dfrac{{}_2P_2 \times {}_5P_5}{{}_7P_7} = \dfrac{2!5!}{7!} = \dfrac{2\cdot 5!}{7\cdot 6\cdot 5!} = \dfrac{1}{21}$

$$\frac{1}{21}$$

2 ∨ OK ∨ OK × 5 ∨ OK ∨ OK ∨ 7 ∨ OK ∨ OK EXE

$\dfrac{2!\times 5!}{7!}$ $\dfrac{1}{21}$

2 There is a bag holding 8 red balls and 6 white balls. Given 4 balls are taken from the bag at the same time, find the probability that the balls taken out are 2 red balls and 2 white ball.

The red balls and white balls add up to 14 balls, from which 4 balls are taken, so there are ${}_{14}C_4$ ways.

From the law of multiplication, we find there are ${}_8C_2 \times {}_6C_2$ (ways) to take 2 red balls from 8 red balls and to take 2 white balls from 6 white balls

Therefore, we find a probability of

$$\frac{{}_8C_2 \times {}_6C_2}{{}_{14}C_4} = \frac{\dfrac{8\cdot 7}{2\cdot 1}\cdot\dfrac{6\cdot 5}{2\cdot 1}}{\dfrac{14\cdot 13\cdot 12\cdot 11}{4\cdot 3\cdot 2\cdot 1}} = \frac{8\cdot 7}{2\cdot 1}\cdot\frac{6\cdot 5}{2\cdot 1}\cdot\frac{4\cdot 3\cdot 2\cdot 1}{14\cdot 13\cdot 12\cdot 11} = \frac{60}{143}$$

$$\frac{60}{143}$$

8 ∨ OK ∨ ∨ ∨ OK 2 × 6 ∨ OK ∨ ∨ ∨ OK 2 ∨ 1 4 ∨ OK ∨ ∨ ∨ OK 4 EXE

$\dfrac{8C2\times 6C2}{14C4}$ $\dfrac{60}{143}$

Basic properties of probability (1)

TARGET To understand about product events, sum events, and exclusive events.

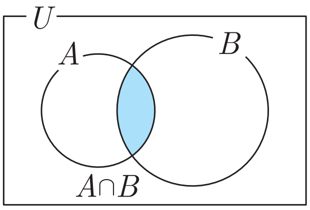
STUDY GUIDE

Product event, sum event, exclusive event

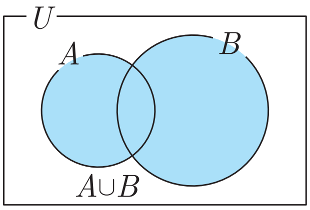
A sure event U that includes 2 events, A and B, has the following events.

- An event in which both A and B occur is called a **product event**, and is expressed as $\boldsymbol{A \cap B}$.
- An event in which at least one of A or B occurs is called a **sum event**, and is expressed as $\boldsymbol{A \cup B}$.
- In an event in which A and B do not occur at the same time, we say that A and B are mutually **exclusive events**, which is expressed as $\boldsymbol{A \cap B = \varnothing}$.

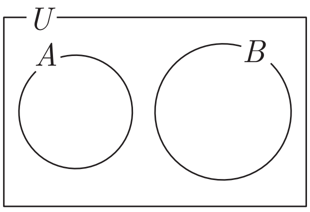
  Events indicated by the empty set $\varnothing$ are called an **empty event**.



Product event



Sum event



Exclusive event

EXERCISE

$\boxed{1}$ Given 1 die is rolled, then an outcome of 4 or greater is the A event, and an outcome of an even number is the B event. In this case, show the following events as sets.

(1) $A \cap B$

Each event is represented as a set: $A=\{4, 5, 6\}$ and $B=\{2, 4, 6\}$

The elements common to A and B are the $A \cap B$ elements, so $A \cap B=\{4, 6\}$

$$\boldsymbol{A \cap B=\{4, 6\}}$$

(2) $A \cup B$

All of the elements included in A and B are all the elements $A \cup B$, so $A \cup B=\{2, 4, 5, 6\}$

$$\boldsymbol{A \cup B=\{2, 4, 5, 6\}}$$

2 Given 1 die is rolled, then an outcome of an even number is the A event, an outcome of an odd number is the B event, an outcome of 3 or less is the C event, and an outcome of 4 or greater is the D event. In this case, solve the following problems.

(1) For all sets, show the product events for each of 2 events.

Each event is represented as a set: $A=\{2, 4, 6\}$, $B=\{1, 3, 5\}$, $C=\{1, 2, 3\}$, $D=\{4, 5, 6\}$

A and B have no common elements, so $A\cap B=\varnothing$

A and C have 2 as a common element, so $A\cap C=\{2\}$

A and D have 4 and 6 as common elements, so $A\cap D=\{4, 6\}$

B and C have 1 and 3 as common elements, so $B\cap C=\{1, 3\}$

B and D have 5 as a common element, so $B\cap D=\{5\}$

C and D have no common elements, so $C\cap D=\varnothing$

$A\cap B=\varnothing$, $A\cap C=\{2\}$, $A\cap D=\{4, 6\}$, $B\cap C=\{1, 3\}$, $B\cap D=\{5\}$, $C\cap D=\varnothing$

(2) Describe all the combinations of these events that are mutually exclusive events.

Combinations that become empty events when put in a product event are exclusive events, so A and B, as well as C and D are exclusive events.

A and B as well as C and D

PRACTICE

1 Given 1 card being drawn from 15 cards, on each of which is written 1 number, 1 through 15, then an outcome of a multiple of 2 is the A event and an outcome of a multiple of 3 is the B event. In this case, show the following events as sets.

(1) $A\cap B$

Each event is represented as a set: $A=\{2, 4, 6, 8, 10, 12, 14\}$, $B=\{3, 6, 9, 12, 15\}$

The elements common to A and B are the $A\cap B$ elements, so $A\cap B=\{6, 12\}$

$A\cap B=\{6, 12\}$

(2) $A\cup B$

All of the elements included in A and B are all the elements $A\cup B$, so $A\cup B=\{2, 3, 4, 6, 8, 9, 10, 12, 14, 15\}$

$A\cup B=\{2, 3, 4, 6, 8, 9, 10, 12, 14, 15\}$

2 Given that 1 card is drawn from 1 deck of 52 cards, excluding jokers, of the following A - D events, find all the combinations that are mutually exclusive events. Note that the picture cards appear as jacks, queens, and kings.

A…Picture card is drawn. B…Ace of hearts is drawn.

C…A queen is drawn. D…A number of 10 or less is drawn.

Simply find the combinations that cannot occur at the same time, which are A and B, A and D, B and C, as well as C and D.

A and B, A and D, B and C, as well as C and D

Basic properties of probability (2)

TARGET To understand how to find the probability of the occurrence of at least 1 of 2 events.

STUDY GUIDE

Basic properties of probability

When a sure event U includes 2 events, A and B, then the probability of those events occurring is $P(U)$, $P(A)$, or $P(B)$. It has the following properties in this case.

(i) **For any event A, $0 \leq P(A) \leq 1$**

(ii) **The probability of a sure event U is $P(U)=1$**

The probability of an empty event $\varnothing$ is $P(\varnothing)=0$

(iii) $P(A \cup B)=P(A)+P(B)-P(A \cap B)$

(iv) **When A and B are mutually exclusive events, then $P(A \cap B)=P(\varnothing)=0$, so $P(A \cup B)=P(A)+P(B)$**

In particular, (iv) is called the **addition theorem** of probability.

EXERCISE

1 Given 1 die is rolled, find the probability of the outcome being a multiple of 3 or an odd number.

Given outcomes that are a multiple of 3 are A events, and outcomes that are odd are B events, then the product events $A \cap B$ are outcomes that are multiples of 3 and odd events. Each event is represented as a set: $A=\{3, 6\}$, $B=\{1, 3, 5\}$, $A \cap B=\{3\}$

Therefore, $P(A)=\frac{2}{6}=\frac{1}{3}, P(B)=\frac{3}{6}=\frac{1}{2}, P(A \cap B)=\frac{1}{6}$

Thus, the probability of a sum event $A \cup B$, whose outcome is a multiple of 3 or an odd number, is

$$P(A \cup B)=P(A)+P(B)-P(A \cap B)=\frac{1}{3}+\frac{1}{2}-\frac{1}{6}=\frac{2}{3}$$

$\frac{2}{3}$

2 There is a bag holding 7 red balls and 5 blue balls. Given 2 balls are taken from the bag at the same time, find the probability that 2 the balls taken out are the same color.

An event in which the 2 balls that are taken out are the same color is a sum event $A \cup B$ of event A, in which the 2 balls are red, and event B, in which the 2 balls are blue.

The red balls and blue balls add up to 12 balls, from which 2 balls are taken, so there are ${}_{12}C_2$ ways.

2 red balls can be taken out in ${}_7C_2$ ways. 2 blue balls can be taken out in ${}_5C_2$ ways.

From the addition theorem of probability, the probability of the sum event $A \cup B$ of mutually exclusive events A and B is

$$P(A \cup B)=P(A)+P(B)=\frac{{}_7C_2}{{}_{12}C_2}+\frac{{}_5C_2}{{}_{12}C_2}=\frac{\frac{7 \cdot 6}{2 \cdot 1}}{\frac{12 \cdot 11}{2 \cdot 1}}+\frac{\frac{5 \cdot 4}{2 \cdot 1}}{\frac{12 \cdot 11}{2 \cdot 1}}=\frac{7 \cdot 6}{12 \cdot 11}+\frac{5 \cdot 4}{12 \cdot 11}=\frac{31}{66}$$

$\frac{31}{66}$

PRACTICE

1 There are 25 cards, on each of which is written 1 number, from 1 to 25. Find the following probabilities when cards are drawn.

(1) Probability of the outcome being odd or a multiple of 5 when 1 card is drawn

The sure event is that a number 1 to 25 is drawn, and since $U=\{1, 2, \cdots, 25\}$, we have $n(U)=25$

Outcomes that are odd are A events and outcomes that are a multiple of 5 are B events.

$A=\{1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25\}$, so $n(A)=13$

$B=\{5, 10, 15, 20, 25\}$, so $n(B)=5$

$A\cap B=\{5, 15, 25\}$, so $n(A\cap B)=3$

The probability we want is $A\cup B$, so we get

$$P(A\cup B)=P(A)+P(B)-P(A\cap B)=\frac{13}{25}+\frac{5}{25}-\frac{3}{25}=\frac{3}{5}$$

$$\frac{3}{5}$$

(2) Probability of both being odd or both being even when 2 cards are drawn

Event A, where both are even numbers, and event B, where both are odd numbers, are mutually exclusive events.

The probability we want is $A\cup B$, so from the addition theorem of probability, we get $P(A\cup B)=P(A)+P(B)$

The selection is to draw 2 cards from 25 cards, so there are ${}_{25}C_2$ ways.

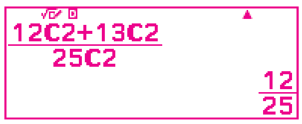
Then, it is to draw 2 even cards from 12 even cards, so there are ${}_{12}C_2$ ways.

Then, it is to draw 2 odd cards from 13 odd cards, so there are ${}_{13}C_2$ ways.

Therefore, the probability of both being odd or both being even when 2 cards are drawn is $\dfrac{{}_{12}C_2}{{}_{25}C_2}+\dfrac{{}_{13}C_2}{{}_{25}C_2}=\dfrac{\frac{12\cdot 11}{2\cdot 1}}{\frac{25\cdot 24}{2\cdot 1}}+\dfrac{\frac{13\cdot 12}{2\cdot 1}}{\frac{25\cdot 24}{2\cdot 1}}=\dfrac{12\cdot 11}{25\cdot 24}+\dfrac{13\cdot 12}{25\cdot 24}=\dfrac{12}{25}$

$$\frac{12}{25}$$

1 2 ∨ OK ∨ ∨ ∨ OK 2
+ 1 3 ∨ OK ∨ ∨ ∨ OK 2
∨ 2 5 ∨ OK ∨ ∨ ∨ OK 2 EXE



[2] There is a bag holding 9 red balls and 7 blue balls. Given 3 balls are taken from the bag at the same time, find the probability that 3 of the balls taken out are the same color.

An event in which the 3 balls that are taken out are the same color is a sum event $A \cup B$ of event A, in which the 3 balls are red, and event B, in which the 3 balls are blue.

The red balls and blue balls add up to 16 balls, from which 3 balls are taken, so there are ${}_{16}\mathrm{C}_3$ ways.

3 red balls can be taken out in ${}_{9}\mathrm{C}_3$ ways.

2 blue balls can be taken out in ${}_{7}\mathrm{C}_3$ ways.

From the addition theorem of probability, the probability of the sum event $A \cup B$ of mutually exclusive events A and B is $P(A \cup B) = P(A) + P(B)$

$$= \frac{{}_{9}\mathrm{C}_3}{{}_{16}\mathrm{C}_3} + \frac{{}_{7}\mathrm{C}_3}{{}_{16}\mathrm{C}_3} = \frac{\frac{9 \cdot 8 \cdot 7}{3 \cdot 2 \cdot 1}}{\frac{16 \cdot 15 \cdot 14}{3 \cdot 2 \cdot 1}} + \frac{\frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1}}{\frac{16 \cdot 15 \cdot 14}{3 \cdot 2 \cdot 1}} = \frac{9 \cdot 8 \cdot 7}{16 \cdot 15 \cdot 14} + \frac{7 \cdot 6 \cdot 5}{16 \cdot 15 \cdot 14} = \frac{17}{80}$$

$$\frac{17}{80}$$

⊜ 9 📖 ∨ OK ∨ ∨ ∨ OK 3 + 7 📖 ∨ OK ∨ ∨ ∨ OK 3 ∨ 1 6 📖 ∨ OK ∨ ∨ ∨ OK 3 EXE

$\frac{9C3+7C3}{16C3}$
$\frac{17}{80}$

Basic properties of probability (3)

TARGET To understand the probability that some event will not happen.

STUDY GUIDE

Complementary events

For an event A, which is in the sure event U, an event in which "A does not occur" is called a **complementary event** of A, and is expressed as $\overline{A}$. The complementary event $\overline{A}$ has nothing in common with event A, so they are mutually exclusive. Therefore, this gives us $A\cap\overline{A}=\varnothing$. Furthermore, since A and $\overline{A}$ are sure events, we have $A\cup\overline{A}=U$.

For $P(A\cup\overline{A})=P(U)$, then from $P(U)=1$ and $P(A\cap\overline{A})=0$, we get $P(A)+P(\overline{A})=1$.

From this, we can find the probability that "A does not occur ($\overline{A}$)" by using the following relation.

$$\boxed{P(\overline{A})=1-P(A)}$$

EXERCISE

1 There is a bag holding 7 red balls and 6 white balls. Given 3 balls are taken from the bag at the same time, find the following probabilities.

(1) Probability of all 3 being red balls

The red balls and white balls add up to 13 balls, from which 3 balls are taken, so there are ${}_{13}C_3$ ways.

Of these, the number of cases where all 3 are red balls, in which 3 red balls are taken from the 7 red balls, is ${}_{7}C_3$ ways.

Therefore, we find a probability of $\dfrac{{}_{7}C_{3}}{{}_{13}C_{3}}=\dfrac{\dfrac{7\cdot6\cdot5}{3\cdot2\cdot1}}{\dfrac{13\cdot12\cdot11}{3\cdot2\cdot1}}=\dfrac{7\cdot6\cdot5}{13\cdot12\cdot11}=\dfrac{35}{286}$

$\dfrac{\mathbf{35}}{\mathbf{286}}$

[MENU] [7] [CATALOG] [▽] [OK] [▽] [▽] [▽] [OK] [3] [▽] [1] [3] [CATALOG] [▽] [OK] [▽] [▽] [▽] [OK] [3] [EXE]

```
7C3
13C3
                35
               286
```

(2) Probability of at least 1 being a white ball

An event, such that "at least 1 ball is white" is the complementary event $\overline{A}$ of A, which is "all 3 balls are red".

Therefore, we find a probability of $P(\overline{A})=1-P(A)=1-\dfrac{35}{286}=\dfrac{251}{286}$

$\dfrac{\mathbf{251}}{\mathbf{286}}$

[1] [−] [Ans] [EXE]

```
1−Ans
               251
               286
```

[2] Find the following probabilities when 6 people simultaneously draw 1 lot each from 30 lots, which includes 3 winning lots.

(1) Probability of all 6 people losing

The selection is to draw 6 lots from 30 lots, so there are ${}_{30}C_6$ ways.

The number of cases in which all 6 people lose is a selection of 6 lots from 27 losing lots, so there are ${}_{27}C_6$ ways.

Therefore, we find a probability of $\dfrac{{}_{27}C_6}{{}_{30}C_6} = \dfrac{\dfrac{27\cdot 26\cdot 25\cdot 24\cdot 23\cdot 22}{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}}{\dfrac{30\cdot 29\cdot 28\cdot 27\cdot 26\cdot 25}{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}} = \dfrac{27\cdot 26\cdot 25\cdot 24\cdot 23\cdot 22}{30\cdot 29\cdot 28\cdot 27\cdot 26\cdot 25} = \dfrac{506}{1015}$

$\dfrac{506}{1015}$

⊟ 2 7 ⊞ ∨ OK ∨ ∨ ∨ OK 6 ∨ 3 0 ⊞ ∨ OK ∨ ∨ ∨ OK 6 EXE

27C6
30C6
506/1015

(2) Probability of at least 1 person winning

An event, such that "at least 1 person wins" is the complementary event $\overline{A}$ of A, which is "all 6 people lose".

Therefore, we find a probability of $P(\overline{A}) = 1 - P(A) = 1 - \dfrac{506}{1015} = \dfrac{509}{1015}$

$\dfrac{509}{1015}$

1 ⊖ Ans EXE

1−Ans
509/1015

PRACTICE

[1] There is a bag holding 10 red balls and 10 white balls. Given 4 balls are taken from the bag at the same time, find the following probabilities.

(1) Probability of all 4 being red balls

The red balls and white balls add up to 20 balls, from which 4 balls are taken, so there are ${}_{20}C_4$ ways.

Of these, the number of cases where all 4 are red balls, in which 4 red balls are taken from the 10 red balls, is ${}_{10}C_4$ ways.

Therefore, we find a probability of $\dfrac{{}_{10}C_4}{{}_{20}C_4} = \dfrac{\dfrac{10\cdot 9\cdot 8\cdot 7}{4\cdot 3\cdot 2\cdot 1}}{\dfrac{20\cdot 19\cdot 18\cdot 17}{4\cdot 3\cdot 2\cdot 1}} = \dfrac{10\cdot 9\cdot 8\cdot 7}{20\cdot 19\cdot 18\cdot 17} = \dfrac{14}{323}$

$\dfrac{14}{323}$

⊟ 1 0 ⊞ ∨ OK ∨ ∨ ∨ OK 4
∨ 2 0 ⊞ ∨ OK ∨ ∨ ∨ OK 4 EXE

10C4
20C4
14/323

(2) Probability of at least 1 being a white ball

An event, such that "at least 1 ball is white" is the complementary event $\overline{A}$ of A, which is "all 4 balls are red".

Therefore, we find a probability of $P(\overline{A}) = 1 - P(A) = 1 - \dfrac{14}{323} = \dfrac{309}{323}$

$\dfrac{309}{323}$

1 ⊖ Ans EXE

1−Ans
309/323

2 There are 40 lots, which includes 5 winning lots. Find the following probabilities when 4 people simultaneously draw 1 lot each from them.

(1) Probability of all 4 people losing

The selection is to draw 4 lots from 40 lots, so there are ${}_{40}\mathrm{C}_4$ ways.

The number of cases in which all 4 people lose is a selection of 4 lots from 35 losing lots, so there are ${}_{35}\mathrm{C}_4$ ways.

Therefore, we find a probability of $\dfrac{{}_{35}\mathrm{C}_4}{{}_{40}\mathrm{C}_4} = \dfrac{\dfrac{35\cdot34\cdot33\cdot32}{4\cdot3\cdot2\cdot1}}{\dfrac{40\cdot39\cdot38\cdot37}{4\cdot3\cdot2\cdot1}} = \dfrac{35\cdot34\cdot33\cdot32}{40\cdot39\cdot38\cdot37} = \dfrac{5236}{9139}$

$$\frac{5236}{9139}$$

3 5 ∨ OK ∨ ∨ ∨ OK 4
∨ 4 0 ∨ OK ∨ ∨ ∨ OK 4 EXE

35C4 / 40C4 = 5236/9139

(2) Probability of at least 1 person winning

An event, such that "at least 1 person wins" is the complementary event $\overline{A}$ of A, which is "all 4 people lose".

Therefore, we find a probability of $P(\overline{A}) = 1 - P(A) = 1 - \dfrac{5236}{9139} = \dfrac{3903}{9139}$

$$\frac{3903}{9139}$$

1 − Ans EXE

1−Ans = 3903/9139

Probability of independent trials

TARGET To understand the probability when there are multiple trials that do not affect each other.

STUDY GUIDE

Probability of independent trials

2 trials are said to be **independent** if their respective results do not affect each other. Given 2 trials T_1 and T_2 are independent, such that the probability of event A occurring in T_1 is $P(A)$, and the probability of event B occurring in T_2 is $P(B)$, then we can find the probability p of A and B occurring as follows.

$$\boxed{p = P(A) \times P(B)}$$

The same also holds when there are 3 or more independent trials.

EXERCISE

1 Given 1 die is rolled 2 times consecutively, find the probability that the outcomes of the 2 rolls are both 1.

The outcome of the 1st trial and the outcome of the 2nd trial are independent because they do not affect each other.

Given A and B are the events in which the 1st and 2nd outcomes are 1 events, respectively, A and B, then $P(A)=\frac{1}{6}$ and $P(B)=\frac{1}{6}$.

Therefore, we find a probability of $P(A)\times P(B)=\frac{1}{6}\times\frac{1}{6}=\frac{1}{36}$

$\frac{1}{36}$

2 Given 1 coin is tossed 3 times consecutively, find the probability that the 1st time is heads, the 2nd time is tails, and the 3rd time is heads.

The 1st, 2nd, and 3rd tosses are independent because they do not affect each other.

Given A, B, and C are the events in which the 1st time is heads, the 2nd time is tails, and the 3rd time is heads, respectively, then $P(A)=\frac{1}{2}$, $P(B)=\frac{1}{2}$, and $P(C)=\frac{1}{2}$.

Therefore, we find a probability of $P(A)\times P(B)\times P(C)=\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}=\frac{1}{8}$

$\frac{1}{8}$

3 Do a trial in which 1 card is drawn from 1 deck of 52 cards, excluding jokers, and next 1 die is rolled. In this case, find the probability that the card will be a heart and the rolled die will be an odd number.

The drawing 1 card trial and the rolling 1 die trial are independent because they do not affect each other.

Given A and B are the events in which the suit is hearts and the dice roll is odd, respectively, then $P(A)=\frac{13}{52}=\frac{1}{4}$ and $P(B)=\frac{3}{6}=\frac{1}{2}$.

Therefore, we find a probability of $P(A)\times P(B)=\frac{1}{4}\times\frac{1}{2}=\frac{1}{8}$

$\frac{1}{8}$

4 There is a bag holding 6 red balls and 4 white balls. 1 ball is taken out of the bag, returned to the bag, and then another 1 is taken out. In this case, find the following probabilities.

(1) Probability of the 1st time being a red ball and the 2nd time being a white ball

The taking out 1 ball trial and then taking out 1 more ball trial are independent because they do not affect each other. Given A and B are the events in which the 1st is a red ball and the 2nd is a white ball, respectively, then

$P(A)=\frac{6}{10}=\frac{3}{5}$ and $P(B)=\frac{4}{10}=\frac{2}{5}$.

Therefore, we find a probability of $P(A)\times P(B)=\frac{3}{5}\times\frac{2}{5}=\frac{6}{25}$

$\mathbf{\frac{6}{25}}$

(2) Probability of the same color ball being taken out 2 times

There are 2 ways for the event of the same color ball being taken out 2 times: the event of red balls being taken out 2 times (i) and the event of white balls being taken out 2 times (ii).

The probability of (i), the 1st time being a red ball and the 2nd time being a red ball, is $\frac{6}{10}\times\frac{6}{10}=\frac{9}{25}$

The probability of (ii), the 1st time being a white ball and the 2nd time being a white ball, is $\frac{4}{10}\times\frac{4}{10}=\frac{4}{25}$

Since (i) and (ii) cannot occur at the same time and are exclusive events, we find the probability is $\frac{9}{25}+\frac{4}{25}=\frac{13}{25}$

$\mathbf{\frac{13}{25}}$

PRACTICE

1 Given 1 die is rolled 3 times consecutively, find the probability that the outcomes of the 3 rolls are all multiples of 3.

The 1st, 2nd, and 3rd rolls are independent because they do not affect each other. Given A, B, and C are the events in which the 1st time, 2nd time, and 3rd time are multiples of 3, respectively, then

$P(A)=\frac{2}{6}=\frac{1}{3}$, $P(B)=\frac{1}{3}$, and $P(C)=\frac{1}{3}$.

Therefore, we find a probability of $P(A)\times P(B)\times P(C)=\frac{1}{3}\times\frac{1}{3}\times\frac{1}{3}=\frac{1}{27}$

$\mathbf{\frac{1}{27}}$

2 Given 1 coin is tossed 3 times consecutively, find the probability that the outcome is 2 heads and 1 tails.

The 1st, 2nd, and 3rd tosses are independent because they do not affect each other. There are 3 ways for the event where the outcome is 2 heads and 1 tails: (1st, 2nd, 3rd)$=$(heads, heads, tails), (heads, tails, heads), (tails, heads, heads).

The probability for all of them is $\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}=\frac{1}{8}$

Therefore, since the 3 events are exclusive events, we find a probability of $\frac{1}{8}+\frac{1}{8}+\frac{1}{8}=\frac{3}{8}$

$\mathbf{\frac{3}{8}}$

$\boxed{3}$ Do a trial in which 1 card is drawn from 1 deck of 52 cards, excluding jokers, and next 1 die is rolled. In this case, find the probability of the card being a diamond and the rolled die being 2 or less.

The drawing 1 card trial and the rolling 1 die trial are independent because they do not affect each other.

Given A and B are the events in which the suit is diamonds and the rolled die is 2 or less, respectively, then $P(A)=\frac{13}{52}=\frac{1}{4}$ and $P(B)=\frac{2}{6}=\frac{1}{3}$.

Therefore, we find a probability of $P(A)\times P(B)=\frac{1}{4}\times\frac{1}{3}=\frac{1}{12}$

$$\frac{1}{12}$$

$\boxed{4}$ Bag A contains 5 red balls and 3 white balls, and bag B contains 2 red balls and 4 white balls. Given 1 ball is taken from each bag A and B, find the probability of the same color ball being taken out.

There are 2 ways for the event of the same color being taken from A and B: the event of red balls being take from A and B (i) and the event of white balls being taken from A and B (ii).

The probability of (i), a red ball from A and a red ball from B, is $\frac{5}{8}\times\frac{2}{6}=\frac{5}{24}$

The probability of (ii), a white ball from A and a white ball from B, is $\frac{3}{8}\times\frac{4}{6}=\frac{1}{4}$

Since (i) and (ii) cannot occur at the same time and are exclusive events, we find the probability is $\frac{5}{24}+\frac{1}{4}=\frac{11}{24}$

$$\frac{11}{24}$$

Probability of repeated trials (1)

TARGET To understand the probability when independent trials are repeated.

STUDY GUIDE

Repeated trials

When the same trial is repeated under the same conditions, and each trial is independent, then that series of trials is called a **repeated trial**. Given the probability p of event A occurring at the 1st trial and repeating the trial n times, then we can find the probability that A occurs r times as follows.

$$\boxed{{}_nC_r\, p^r(1-p)^{n-r}}$$

Ex. Given 1 die is rolled 5 times consecutively, consider the probability of 1 being rolled 2 times.

By marking ○ each time the outcome is 1, and marking × each time the outcome is not 1, then the ways that 1 can appear 2 times are shown in the table on the right.

For this number of cases, consider that 2 ○ and 3 × line up. That is to say, the total number of combinations of 2 ○ out of 5 attempts is ${}_5C_2$ ways.

1st time	2nd time	3rd time	4th time	5th time
○	○	×	×	×
○	×	○	×	×
○	×	×	○	×
○	×	×	×	○
⋮	⋮	⋮	⋮	⋮

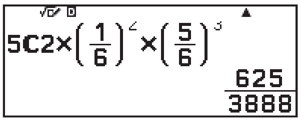
Then, given 1 die is rolled 1 time, the probability that the outcome is 1 is $\frac{1}{6}$, and the probability that the outcome is not 1 is a complementary event, so it is $1-\frac{1}{6}$.

Therefore, the probability of the 1st line in the table is $\frac{1}{6}\times\frac{1}{6}\times\left(1-\frac{1}{6}\right)\times\left(1-\frac{1}{6}\right)\times\left(1-\frac{1}{6}\right)=\left(\frac{1}{6}\right)^2\left(1-\frac{1}{6}\right)^3$.

Similarly, the probability of the other ${}_5C_2$ ways are all the same.

All the cases of the ${}_5C_2$ ways are exclusive, so we simply find the sum of these probabilities.

Thus, we find a probability of ${}_5C_2\left(\frac{1}{6}\right)^2\left(1-\frac{1}{6}\right)^3=\frac{5\cdot 4}{2\cdot 1}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^3=\frac{625}{3888}$.



EXERCISE

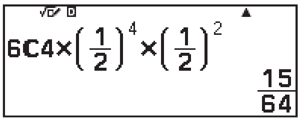
1. Given 1 coin is tossed 6 times consecutively, find the probability that the outcome is 4 heads.

Given 1 coin is tossed 1 time, the probability that the outcome is heads is $\frac{1}{2}$.

Therefore, we find a probability of ${}_6C_4\left(\frac{1}{2}\right)^4\left(1-\frac{1}{2}\right)^2={}_6C_2\left(\frac{1}{2}\right)^4\left(1-\frac{1}{2}\right)^2=\frac{6\cdot 5}{2\cdot 1}\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^2=\frac{15}{64}$

$\frac{15}{64}$

⑥ [⊟] [∨] [OK] [∨] [∨] [∨] [OK] ④
[×] [(] ① [÷] ② [>] [)] [x■] ④ [>] [×] [(] ① [÷] ② [>] [)] [x²] [EXE]



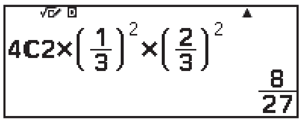
2 Given 1 die is rolled 4 times consecutively, find the probability of the outcome being 5 or greater 2 times.

Given 1 die is rolled 1 time, the probability of an outcome of 5 or greater is $\frac{2}{6}=\frac{1}{3}$.

Therefore, we find a probability of ${}_4\mathrm{C}_2\left(\frac{1}{3}\right)^2\left(1-\frac{1}{3}\right)^2=\frac{4\cdot 3}{2\cdot 1}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^2=\frac{8}{27}$

$$\frac{8}{27}$$

4 [CATALOG] [∨] [OK] [∨] [∨] [∨] [OK] 2
[×] [(] 1 [fraction] 3 [>] [)] [x²] [×] [(] 2 [fraction] 3 [>] [)] [x²] [EXE]



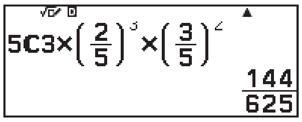
3 There is a bag holding 6 red balls and 4 white balls. 1 ball is taken out of the bag, the color is confirmed, and then it is returned to the bag. Given this trial is repeated 5 times, find the probability that the outcome is 3 white balls.

Given 1 ball is taken from the bag, the probability that the outcome is a white ball is $\frac{4}{10}=\frac{2}{5}$.

Therefore, we find a probability of ${}_5\mathrm{C}_3\left(\frac{2}{5}\right)^3\left(1-\frac{2}{5}\right)^2={}_5\mathrm{C}_2\left(\frac{2}{5}\right)^3\left(1-\frac{2}{5}\right)^2=\frac{5\cdot 4}{2\cdot 1}\left(\frac{2}{5}\right)^3\left(\frac{3}{5}\right)^2=\frac{144}{625}$

$$\frac{144}{625}$$

5 [CATALOG] [∨] [OK] [∨] [∨] [∨] [OK] 3
[×] [(] 2 [fraction] 5 [>] [)] [x^□] 3 [>] [×] [(] 3 [fraction] 5 [>] [)] [x²] [EXE]



PRACTICE

1 Given 1 coin is tossed 5 times consecutively, find the probability that the outcome is 3 heads.

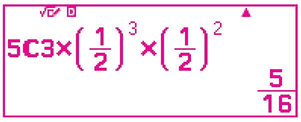
Given 1 coin is tossed 1 time, the probability that the outcome is heads is $\frac{1}{2}$.

Therefore, we find a probability of

$$ {}_5C_3\left(\frac{1}{2}\right)^3\left(1-\frac{1}{2}\right)^2 = {}_5C_2\left(\frac{1}{2}\right)^3\left(1-\frac{1}{2}\right)^2 = \frac{5\cdot 4}{2\cdot 1}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^2 = \frac{5}{16} $$

$$ \frac{5}{16} $$

5 (book) ∨ OK ∨ ∨ ∨ OK 3
× (1 (fraction) 2 >) $\blacksquare^{\square}$ 3 > × (1 (fraction) 2 >) $\blacksquare^2$ EXE



2 Given 1 die is rolled 6 times consecutively, find the probability of the outcome being 3 or greater 4 times.

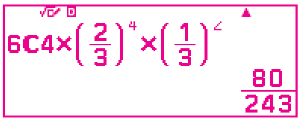
Given 1 die is rolled 1 time, the probability of an outcome of 3 or greater is $\frac{4}{6}=\frac{2}{3}$.

Therefore, we find a probability of

$$ {}_6C_4\left(\frac{2}{3}\right)^4\left(1-\frac{2}{3}\right)^2 = {}_6C_2\left(\frac{2}{3}\right)^4\left(1-\frac{2}{3}\right)^2 = \frac{6\cdot 5}{2\cdot 1}\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right)^2 = \frac{80}{243} $$

$$ \frac{80}{243} $$

6 (book) ∨ OK ∨ ∨ ∨ OK 4
× (2 (fraction) 3 >) $\blacksquare^{\square}$ 4 > × (1 (fraction) 3 >) $\blacksquare^2$ EXE



3 There is a bag holding 8 red balls and 12 white balls. 1 ball is taken out of the bag, the color is confirmed and then it is returned to the bag. Given this trial is repeated 7 times, find the probability that the outcome is 4 white balls.

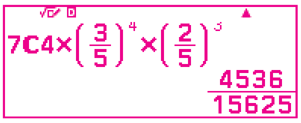
Given 1 ball is taken from the bag, the probability that the outcome is a white ball is $\frac{12}{20}=\frac{3}{5}$.

Therefore, we find a probability of

$$ {}_7C_4\left(\frac{3}{5}\right)^4\left(1-\frac{3}{5}\right)^3 = {}_7C_3\left(\frac{3}{5}\right)^4\left(1-\frac{3}{5}\right)^3 = \frac{7\cdot 6\cdot 5}{3\cdot 2\cdot 1}\left(\frac{3}{5}\right)^4\left(\frac{2}{5}\right)^3 = \frac{4536}{15625} $$

$$ \frac{4536}{15625} $$

7 (book) ∨ OK ∨ ∨ ∨ OK 4
× (3 (fraction) 5 >) $\blacksquare^{\square}$ 4 > × (2 (fraction) 5 >) $\blacksquare^{\square}$ 3 EXE



Probability of repeated trials (2)

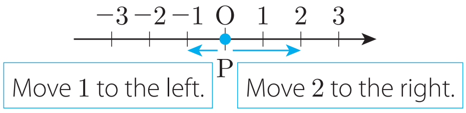
TARGET To understand about problems of using repeated trials.

STUDY GUIDE

Using repeated trials

Consider problems in which points are moved or scores are given according to the results of a tossed coin or die, such that x is the number of heads or tails that appears or a specified number on the die appears, and that a graph of the points or a total of the scores are shown by x.

Ex. There is a point P at the origin of a number line. Toss a coin and move the point 2 to the right for heads, and move it 1 to the left for tails. Consider the probability of whether the point P returns to the origin after the coin is tossed 6 times.

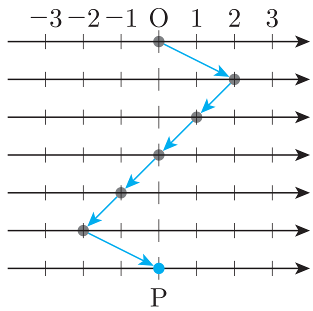


We can formulate the following equation for x by assuming that the point P returns to the origin O position if heads comes up x times when a coin is tossed 6 times.

$2\times x+(-1)\times(6-x)=0$

By solving this, we get $x=2$

Therefore, we can see that the point P returns to the origin if heads appears 2 times within 6 repeated trials.



Point P returns to the origin if heads appears 2 times and tails 4 times within 6 repeated trials.

Also, for 1 coin toss, the probability of heads is $\dfrac{1}{2}$.

Thus, we find a probability of ${}_6\mathrm{C}_2\left(\dfrac{1}{2}\right)^2\left(1-\dfrac{1}{2}\right)^4=\dfrac{6\cdot 5}{2\cdot 1}\left(\dfrac{1}{2}\right)^2\left(\dfrac{1}{2}\right)^4=\dfrac{15}{64}$.

proof

On the scientific calculator, use the $\mathrm{Math\ Box}$ function to check the probability statistically.

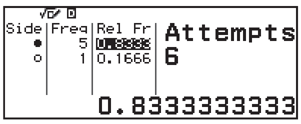
Press ⌂, select $[\mathrm{Math\ Box}]$, press OK, select $[\mathrm{Coin\ Toss}]$, press OK

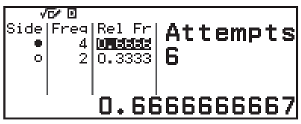
Simulate tossing 1 coin 6 times.

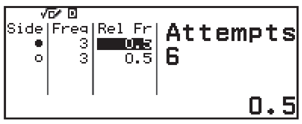
The number of times heads ● and tails ○ appear when a coin is tossed 6 times are displayed respectively.

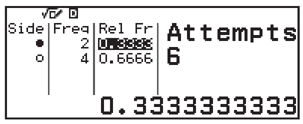
In the results display screen, press ⮌ ⮌ to repeat a similar simulation and to count the number of times for each combination.

If 64 trials of tossing a coin 6 times are done, then theoretically 2 heads ● and 4 tails ○ should appear 15 times.









Fill in the table below based on the results of the 64 simulations. However, you could also do this trial with the cooperation of several people.

Counts of tails ● and heads ○	$0, 6$	$1, 5$	$2, 4$	$3, 3$	$4, 2$	$5, 1$	$6, 0$
Theoretical value for each time (count)	1	6	15	20	15	6	1
Trial results from scientific calculator (count)							

EXERCISE

◆ There is a point P at the origin of a number line. Roll 1 die and move the point 3 to the right for odd numbers or 1 to the left for even numbers. In this case, find the following probabilities.

(1) Given the die is rolled 12 times, the probability of the point P returning to the origin

If the die is rolled 12 times, and we let the number of odd-numbered outcomes be x, then the number of even-numbered outcomes is $(12-x)$.

Since the position of the point P is the origin in this case, then $3x+(-1)(12-x)=0$, which we can solve as $x=3$

Therefore, the point P is at the origin if odd-numbered outcomes appear 3 times within 12 repeated trials.

Also, for 1 roll of the die, the probability of an odd-numbered outcome is $\frac{1}{2}$.

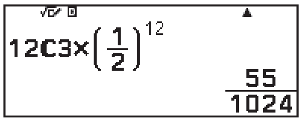
Thus, we find a probability of ${}_{12}\mathrm{C}_{3}\left(\frac{1}{2}\right)^{3}\left(1-\frac{1}{2}\right)^{9}=\frac{12\cdot11\cdot10}{3\cdot2\cdot1}\left(\frac{1}{2}\right)^{3}\left(\frac{1}{2}\right)^{9}=\frac{55}{1024}$

$$\boldsymbol{\frac{55}{1024}}$$

check

Press ⌂, select [Calculate], press OK

① ② ⓣ ∨ OK ∨ ∨ ∨ OK ③ ⊗ (① ▤ ② >) ■ ① ② EXE



(2) Given the die is rolled 10 times, the probability of the point P being at a position of $+6$

If the die is rolled 10 times, and we let the number of odd-numbered outcomes be y, then the number of even-numbered outcomes is $(10-y)$.

Since the position of the point P is $+6$ in this case, then $3y+(-1)(10-y)=6$, which we can solve as $y=4$

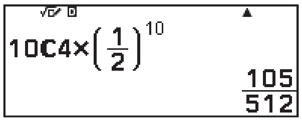
Therefore, the point P is at a $+6$ position if odd-numbered outcomes appear 4 times within 10 repeated trials.

Also, for 1 roll of the die, the probability of an odd-numbered outcome is $\frac{1}{2}$.

Thus, we find a probability of ${}_{10}\mathrm{C}_{4}\left(\frac{1}{2}\right)^{4}\left(1-\frac{1}{2}\right)^{6}=\frac{10\cdot9\cdot8\cdot7}{4\cdot3\cdot2\cdot1}\left(\frac{1}{2}\right)^{4}\left(\frac{1}{2}\right)^{6}=\frac{105}{512}$

$$\boldsymbol{\frac{105}{512}}$$

① ⓪ ⓣ ∨ OK ∨ ∨ ∨ OK ④ ⊗ (① ▤ ② >) ■ ① ⓪ EXE



PRACTICE

◆ Play a game in which you roll 1 die and outcomes of 5 or greater get 4 points, and any other outcome gets 2 points. In this case, find the following probabilities.

(1) Given the die is rolled 6 times, the probability of the score being exactly 20 points

If the die is rolled 6 times, and we let the number of outcomes of 5 or greater be x, then the number of other outcomes is $(6-x)$.

Since the total score is 20 points in this case, then $4x+2(6-x)=20$, which we can solve as $x=4$

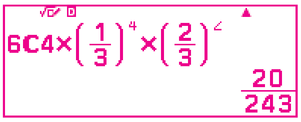
Also, for 1 roll of the die, the probability of an outcome of 5 or greater is $\frac{2}{6}=\frac{1}{3}$.

Therefore, we find a probability of

$${}_6C_4\left(\frac{1}{3}\right)^4\left(1-\frac{1}{3}\right)^2={}_6C_2\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^2=\frac{6\cdot5}{2\cdot1}\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^2=\frac{20}{243}$$

$$\frac{20}{243}$$

6 [MENU] ∨ OK ∨ ∨ ∨ OK 4
× (1 [FRACTION] 3 >) [x^□] 4 > × (2 [FRACTION] 3 >) [x²] EXE



(2) Given the die is rolled 5 times, the probability of the score being exactly 16 points

If the die is rolled 5 times, and we let the number of outcomes of 5 or greater be y, then the number of even-numbered outcomes is $(5-y)$.

Since the total score is 16 points in this case, then $4y+2(5-y)=16$, which we can solve as $y=3$

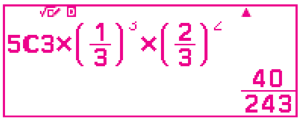
Also, for 1 roll of the die, the probability of an outcome of 5 or greater is $\frac{1}{3}$.

Therefore, we find a probability of

$${}_5C_3\left(\frac{1}{3}\right)^3\left(1-\frac{1}{3}\right)^2={}_5C_2\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^2=\frac{5\cdot4}{2\cdot1}\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^2=\frac{40}{243}$$

$$\frac{40}{243}$$

5 [MENU] ∨ OK ∨ ∨ ∨ OK 3
× (1 [FRACTION] 3 >) [x^□] 3 > × (2 [FRACTION] 3 >) [x²] EXE



Conditional probability

TARGET To understand probabilities with conditions.

STUDY GUIDE

Conditional probability

For 2 events, A and B, the probability of B occurring when A occurs is called the **conditional probability** of B occurring when A occurs, and is expressed as $P_A(B)$. $P_A(B)$ is the probability that event B will occur when event A occurs, which we can find as follows.

$$P_A(B)=\frac{\text{(Number of cases of events } A \text{ and } B \text{ occurring together)}}{\text{(Number of cases of event } A \text{ occurring)}}=\frac{n(A\cap B)}{n(A)}$$

Let the number of cases in the sure event U be $n(U)$, then by using $P(A)=\frac{n(A)}{n(U)}$ and $P(A\cap B)=\frac{n(A\cap B)}{n(U)}$, we get $P_A(B)=\frac{n(A\cap B)}{n(A)}=\frac{P(A\cap B)\times n(U)}{P(A)\times n(U)}=\frac{P(A\cap B)}{P(A)}$, so we can express the conditional probability as follows.

$$P_A(B)=\frac{P(A\cap B)}{P(A)}$$

By removing the denominator of this equation, we get the following formula.

$$P(A\cap B)=P(A)P_A(B)$$

This relational expression is used to find the probability of A and B occurring together, $P(A\cap B)$, which we call the **multiplication theorem of probability**.

Ex. In a bag are 4 red balls, and on each is written a number, $1, 2, 3,$ or 4; there are also 5 white balls, and on each is written a number, $5, 6, 7, 8,$ or 9. If you take 1 ball from this bag, find the conditional probability of the ball being red and having an even number written on it.

The sure event U has 9 ways to take out a ball, so $n(U)=9$.

Given A is the event in which a red ball is taken out, and B is the event in which it has an odd number, then $n(A)=4$ and $n(A\cap B)=2$.

Therefore, $P(A)=\frac{n(A)}{n(U)}=\frac{4}{9}$ and $P(A\cap B)=\frac{n(A\cap B)}{n(U)}=\frac{2}{9}$

Thus, we find a probability of $P_A(B)=\frac{P(A\cap B)}{P(A)}=\frac{\frac{2}{9}}{\frac{4}{9}}=\frac{1}{2}$.

EXERCISE

1 From a bag holding 5 red balls and 6 white balls, take out 1 ball 2 times. Find the conditional probability of the 2nd ball taken out being red when the 1st ball taken out was red. However, the balls taken are not returned to the bag.

Let the 1st ball taken out being a red ball be event A, and let the 2nd ball taken out being a red ball be event B.

Since there are 5 balls, we get $n(A)=5$.

Since the 1st time the ball is taken from 5 red balls, and the 2nd time the ball is taken from the remaining 4 red balls, we get $n(A\cap B)=5\cdot4$.

Therefore, $P(A)=\frac{5}{11}, P(A\cap B)=\frac{5\cdot4}{11\cdot10}=\frac{2}{11}$

Thus, we find a probability of $P_A(B)=\frac{\frac{2}{11}}{\frac{5}{11}}=\frac{2}{5}$

$\frac{2}{5}$

2 There are 20 lots, of which 5 are winning lots. From these, first the older brother draws 1 lot, then the younger brother draws 1 lot. Find the conditional probability of the younger brother winning after the older brother has lost. However, the lots that are drawn are not returned.

The older brother losing is event A and the younger brother winning is event B.

Since there are 20 lots, of which 5 are winning lots, we get $n(A)=20-5=15$.

Since there are 15 losing lots and 5 winning lots, we get $n(A\cap B)=15\cdot5$.

Therefore, $P(A)=\frac{15}{20}=\frac{3}{4}, P(A\cap B)=\frac{15\cdot5}{20\cdot19}=\frac{15}{76}$

Thus, we find a probability of $P_A(B)=\frac{\frac{15}{76}}{\frac{3}{4}}=\frac{5}{19}$

$\frac{5}{19}$

PRACTICE

◆ From a bag holding 3 red balls and 4 white balls, take out 1 ball 2 times. Find the conditional probability of the 2nd ball taken out being red when the 1st ball taken out was red. However, the balls taken are not returned to the bag.

Let the 1st ball taken out being a red ball be event A, and let the 2nd ball taken out being a red ball be event B.

Since there are 3 balls, we get $n(A)=3$.

Since the 1st time the ball is taken from 3 red balls, and the 2nd time the ball is taken from the remaining 2 red balls, we get $n(A\cap B)=3\cdot2$.

Therefore, $P(A)=\frac{3}{7}, P(A\cap B)=\frac{3\cdot2}{7\cdot6}=\frac{1}{7}$

Thus, we find a probability of $P_A(B)=\frac{\frac{1}{7}}{\frac{3}{7}}=\frac{1}{3}$

$\frac{1}{3}$

Calculating probability

TARGET To understand how to find probabilities by using the addition theorem and multiplication theorem.

STUDY GUIDE

Addition theorem and multiplication theorem

We can use the addition theorem and multiplication theorem to find probabilities in applied problems of probability.

Addition theorem for probability

When event A and B are mutually exclusive, the following formula holds.

$$\boxed{P(A \cup B) = P(A) + P(B)}$$

Multiplication theorem of probability

When event A and B occur together, the following formula holds.

$$\boxed{P(A \cap B) = P(A)P_A(B)}$$

EXERCISE

$\boxed{1}$ There is a bag holding 4 red balls and 5 white balls. From this bag, A takes out 1 ball, and then B takes out 1. Find the probability of B taking out a red ball in this case. However, the balls taken are not returned to the bag.

There are the following 2 ways in which B takes out a red ball.

A takes out a red ball, and then B takes out a red ball. $\ldots(\mathrm{i})$

A takes out a white ball, and then B takes out a red ball. $\ldots(\mathrm{ii})$

From the multiplication theorem, the probability of (i) is $\dfrac{4}{9} \times \dfrac{3}{8} = \dfrac{1}{6}$

From the multiplication theorem, the probability of (ii) is $\dfrac{5}{9} \times \dfrac{4}{8} = \dfrac{5}{18}$

Since (i) and (ii) are exclusive, from the addition theorem, we find the probability is $\dfrac{1}{6} + \dfrac{5}{18} = \dfrac{4}{9}$

$$\mathbf{\frac{4}{9}}$$

[2] There are 30 lots, of which 6 are winning lots. In this case, find the following probabilities. However, the lots that are drawn are not returned.

(1) Given 2 people A and B draw lots in this order, the probability of B winning

There are the following 2 ways in which B wins.

A wins and then B also wins. $\ldots(\mathrm{i})$

A loses and then B wins. $\ldots(\mathrm{ii})$

From the multiplication theorem, the probability of (i) is $\dfrac{6}{30}\times\dfrac{5}{29}=\dfrac{1}{29}$

From the multiplication theorem, the probability of (ii) is $\dfrac{24}{30}\times\dfrac{6}{29}=\dfrac{24}{145}$

Since (i) and (ii) are exclusive, from the addition theorem, we find the probability is $\dfrac{1}{29}+\dfrac{24}{145}=\dfrac{1}{5}$

$$\mathbf{\frac{1}{5}}$$

(2) Given 3 people A, B, and C draw lots in this order, the probability that at least A and C win

There are the following 2 ways in which at least A and C win.

All 3 people win. $\ldots(\mathrm{i})$

A wins, B loses, and then C wins. $\ldots(\mathrm{ii})$

From the multiplication theorem, the probability of (i) is $\dfrac{6}{30}\times\dfrac{5}{29}\times\dfrac{4}{28}=\dfrac{1}{203}$

From the multiplication theorem, the probability of (ii) is $\dfrac{6}{30}\times\dfrac{24}{29}\times\dfrac{5}{28}=\dfrac{6}{203}$

Since (i) and (ii) are exclusive, from the addition theorem, we find the probability is $\dfrac{1}{203}+\dfrac{6}{203}=\dfrac{1}{29}$

$$\mathbf{\frac{1}{29}}$$

⊟ 6 × 5 × 4 + 6 × 2 4 × 5 ∨ 3 0 × 2 9 × 2 8 EXE

$$\frac{6\times5\times4+6\times24\times5}{30\times29\times28} \qquad \frac{1}{29}$$

PRACTICE

1 There is a bag holding 12 red balls and 10 white balls. From this bag, A takes out 1 ball, and then B takes out 1. Find the probability of B taking out a red ball in this case. However, the balls taken are not returned to the bag.

There are the following 2 ways in which B takes out a red ball.

A takes out a red ball, and then B takes out a red ball. …(i)

A takes out a white ball, and then B takes out a red ball. …(ii)

From the multiplication theorem, the probability of (i) is $\frac{12}{22}\times\frac{11}{21}=\frac{2}{7}$

From the multiplication theorem, the probability of (ii) is $\frac{10}{22}\times\frac{12}{21}=\frac{20}{77}$

Since (i) and (ii) are exclusive, from the addition theorem, we find the probability is

$\frac{2}{7}+\frac{20}{77}=\frac{6}{11}$

$\frac{6}{11}$

2 There are 25 lots, of which 5 are winning lots. In this case, find the following probabilities. However, the lots that are drawn are not returned.

(1) Given 2 people A and B draw lots in this order, the probability of B winning

There are the following 2 ways in which B wins.

A wins and then B also wins. …(i)

A loses and then B wins. …(ii)

From the multiplication theorem, the probability of (i) is $\frac{5}{25}\times\frac{4}{24}=\frac{1}{30}$

From the multiplication theorem, the probability of (ii) is $\frac{20}{25}\times\frac{5}{24}=\frac{1}{6}$

Since (i) and (ii) are exclusive, from the addition theorem, we find the probability is $\frac{1}{30}+\frac{1}{6}=\frac{1}{5}$

$\frac{1}{5}$

(2) Given 3 people A, B, and C, draw lots in this order, the probability that at least B and C win

There are the following 2 ways in which at least B and C win.

All 3 people win. …(i)

A loses, B wins, and then C wins. … (ii)

From the multiplication theorem, the probability of (i) is $\frac{5}{25}\times\frac{4}{24}\times\frac{3}{23}=\frac{1}{230}$

From the multiplication theorem, the probability of (ii) is $\frac{20}{25}\times\frac{5}{24}\times\frac{4}{23}=\frac{2}{69}$

Since (i) and (ii) are exclusive, from the addition theorem, we find the probability is $\frac{1}{230}+\frac{2}{69}=\frac{1}{30}$

$\frac{1}{30}$

5 × 4 × 3 + 2 0 × 5 × 4 ▽ 2 5 × 2 4 × 2 3 EXE

$\frac{5\times4\times3+20\times5\times4}{25\times24\times23}$

$\frac{1}{30}$

Expected value

TARGET To understand how to evaluate the value of a trial according to its probability.

STUDY GUIDE

Expected value

When a numerical value, such as "prize money", is obtained by a trial, such as "drawing lots", the average of the obtained values is called the **expected value**.

If the possible values of a quantity X are x_1, x_2, ……, x_n, and the probability p of obtaining these values is p_1, p_2, ……, p_n $(p_1+p_2+\ldots\ldots+p_n=1)$, then the expected value E can be found by using the following formula.

$$E = x_1p_1 + x_2p_2 + \cdots\cdots + x_np_n$$

EXERCISE

1 There is a lottery as shown in the table on the right.

In this case, solve the following problems.

	1st prize	2nd prize	3rd prize	Loser	Total
Prize money ($)	3000	1000	400	0	
Count (lots)	2	8	10	80	100

(1) Find the probabilities of drawing 1st, 2nd, or 3rd prize, or losing, when you draw only 1 time.

The are a total of 100 lots.

1st prize has 2 lots, so the probability is $\frac{2}{100}=\frac{1}{50}$

2nd prize has 8 lots, so the probability is $\frac{8}{100}=\frac{2}{25}$

3rd prize has 10 lots, so the probability is $\frac{10}{100}=\frac{1}{10}$

Losing has 80 lots, so the probability is $\frac{80}{100}=\frac{4}{5}$

1st… $\frac{1}{50}$, 2nd… $\frac{2}{25}$, 3rd… $\frac{1}{10}$, loser… $\frac{4}{5}$

(2) Find the expected value of the prize money for drawing 1 lot.

Since the expected value is the sum of the products of each prize and its probability, we get

$$3000\times\frac{1}{50}+1000\times\frac{2}{25}+400\times\frac{1}{10}+0\times\frac{4}{5}=180(\$)$$

$180

2 Given 1 coin being tossed 5 times consecutively, solve the following problems.

(1) Find the probabilities for each number of times the outcome is heads.

Given the coin is tossed 1 time, the probability that the outcome is heads is $\frac{1}{2}$.

The outcome can be heads 0 times, 1 time, 2 times, 3 times, 4 times, or 5 times.

Given $k(k=0, 1, 2, 3, 4,$ and $5)$ is the number of times that heads can appear, then the probability that heads appears just those numbers of times can be expressed as ${}_5\mathrm{C}_k\left(\frac{1}{2}\right)^k\left(1-\frac{1}{2}\right)^{5-k}={}_5\mathrm{C}_k\left(\frac{1}{2}\right)^5$.

Therefore, the probability of 0 heads is ${}_5\mathrm{C}_0\left(\frac{1}{2}\right)^5=\left(\frac{1}{2}\right)^5=\frac{1}{32}$, the probability of 1 head is ${}_5\mathrm{C}_1\left(\frac{1}{2}\right)^5=5\cdot\left(\frac{1}{2}\right)^5=\frac{5}{32}$,

the probability of 2 heads is ${}_5\mathrm{C}_2\left(\frac{1}{2}\right)^5=\frac{5\cdot 4}{2\cdot 1}\cdot\left(\frac{1}{2}\right)^5=\frac{5}{16}$, the probability of 3 heads is ${}_5\mathrm{C}_3\left(\frac{1}{2}\right)^5={}_5\mathrm{C}_2\left(\frac{1}{2}\right)^5=\frac{5}{16}$,

the probability of 4 heads is ${}_5\mathrm{C}_4\left(\frac{1}{2}\right)^5={}_5\mathrm{C}_1\left(\frac{1}{2}\right)^5=\frac{5}{32}$, the probability of 5 heads is ${}_5\mathrm{C}_5\left(\frac{1}{2}\right)^5=\left(\frac{1}{2}\right)^5=\frac{1}{32}$

0 times … $\frac{1}{32}$, 1 time … $\frac{5}{32}$, 2 times … $\frac{5}{16}$, 3 times … $\frac{5}{16}$, 4 times … $\frac{5}{32}$, 5 times … $\frac{1}{32}$

(2) Find the expected value of the number of times heads appears.

Since the expected value is the sum of the products of each number of times and its probability, we get

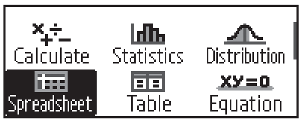
$0\times\frac{1}{32}+1\times\frac{5}{32}+2\times\frac{5}{16}+3\times\frac{5}{16}+4\times\frac{5}{32}+5\times\frac{1}{32}=\frac{5}{2}$ (times)

$\frac{5}{2}$ times

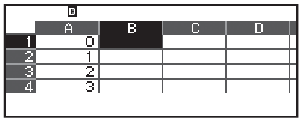
check

Since the expected value to be found is the sum of the values of x=0, 1, 2, 3, 4, and 5 in $x \cdot {}_5\mathrm{C}_x\left(\frac{1}{2}\right)^5$, we can confirm it by using the Spreadsheet function on the scientific calculator, as shown below.

Press ⌂, select [Spreadsheet], press OK, then clear the previous data by pressing ⏻

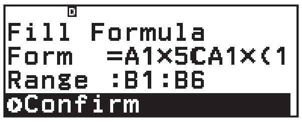


After inputting [A1:0, A2:1, A3:2, A4:3, A5:4, and A6:5] respectively, press EXE, move to [B1].



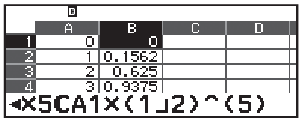
Press ⋯, select [Fill Formula], press OK

After inputting [Form=A1× ${}_5\mathrm{C}_{A1}$ × $\left(\frac{1}{2}\right)^{(5)}$], press EXE (*)

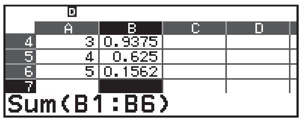


After inputting [Range:B1:B6], press EXE, select [Confirm], press OK (*)

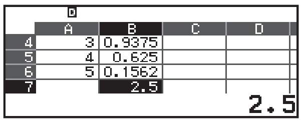
When the sheet is displayed, move to [B7].



After inputting [Sum(B1:B6)], press EXE (*)



The calculated expected value $2.5 = \frac{5}{2}$ is displayed in [B7].



EXTRA Info.

How to enter letters A to F in the scientific calculator

[A]=⇧ ④, [B]=⇧ ⑤, [C]=⇧ ⑥, [D]=⇧ ①, [E]=⇧ ②, [F]=⇧ ③

*In this question, [A] and [B] are used because calculations done in column B are based on values input in column A.

How to input operation symbols specific to Spreadsheet

On the cell in which to do a calculation, press ⊡, select [Spreadsheet], press OK, select one of the 8 selections below to use, then press OK

[Grab, =, $, :, Min, Max, Mean, Sum]

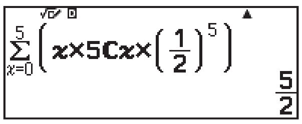
* In this question, [Sum] is used to calculate the total of a selected range specified by [:].

OTHER METHODS

Press ⌂, select [Calculate], press OK

Press ⊡, select [Func Analysis], press OK, select [Summation(Σ)], press OK

𝑥 × 5 ⊡ ∨ OK ∨ ∨ ∨ OK 𝑥 × (1 ⊟ 2 >) ■ 5 ∨ 0 ∧ 5 EXE



Details about the symbol Σ (sigma), which is used in the separate solution, are explained in the separate volume "Sequences".

PRACTICE

1 There is a lottery as shown in the table on the right. In this case, solve the following problems.

	1st prize	2st prize	3st prize	Loser	Total
Prize money ($)	10000	1000	100	0	
Count (lots)	1	10	100	889	1000

(1) Find the probabilities of drawing 1st, 2nd, or 3rd prize, or losing, when you draw only 1 time.

The are a total of 1000 lots.

1st prize has 1 lot, so the probability is $\frac{1}{1000}$

2nd prize has 10 lots, so the probability is $\frac{10}{1000}=\frac{1}{100}$

3rd prize has 100 lots, so the probability is $\frac{100}{1000}=\frac{1}{10}$

Losing has 889 lots, so the probability is $\frac{889}{1000}$

1st… $\frac{1}{1000}$**, 2nd…** $\frac{1}{100}$**, 3rd…** $\frac{1}{10}$**, loser…** $\frac{889}{1000}$

(2) Find the expected value of the prize money for drawing 1 lot.

Since the expected value is the sum of the products of each prize and its probability, we get $10000\times\frac{1}{1000}+1000\times\frac{1}{100}+100\times\frac{1}{10}+0\times\frac{889}{1000}=30(\$)$

$30

2 Given 1 die being rolled 3 times consecutively, solve the following problems.

(1) Find the probabilities for each number of times the outcome is 1.

Given a die is rolled 1 time, the probability of an outcome of 1 is $\frac{1}{6}$.

The outcome can be 1 0 times, 1 time, 2 times, or 3 times.

Given k(k=0, 1, 2, and 3) is the number of times that 1 can appear, then the probability that 1 appears just those numbers of times can be expressed as

$${}_3C_k\left(\frac{1}{6}\right)^k\left(1-\frac{1}{6}\right)^{3-k}={}_3C_k\frac{5^{3-k}}{6^3}.$$

Therefore, the probability of 0 1s is ${}_3C_0\frac{5^3}{6^3}=\frac{5^3}{6^3}=\frac{125}{216}$

the probability of 1 1 is ${}_3C_1\frac{5^2}{6^3}=3\cdot\frac{5^2}{6^3}=\frac{25}{72}$

the probability of 2 1s is ${}_3C_2\frac{5^1}{6^3}={}_3C_1\frac{5}{6^3}=3\cdot\frac{5}{6^3}=\frac{5}{72}$

the probability of 3 1s is ${}_3C_3\frac{5^0}{6^3}=\frac{1}{6^3}=\frac{1}{216}$

0 times … $\frac{125}{216}$, 1 time … $\frac{25}{72}$, 2 times … $\frac{5}{72}$, 3 times … $\frac{1}{216}$

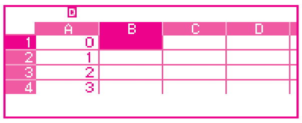
(2) Find the expected value of the number of times the outcome is 1.

Since the expected value is the sum of the products of each number of times and its probability, we get $0\times\frac{125}{216}+1\times\frac{25}{72}+2\times\frac{5}{72}+3\times\frac{1}{216}=\frac{1}{2}$ **(times)**

$\frac{1}{2}$ times

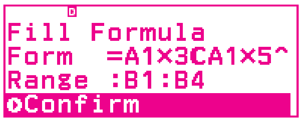
check

Press ⌂, select [Spreadsheet], press OK, then clear the previous data by pressing ⏻

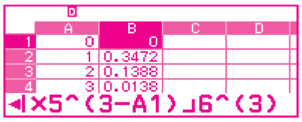


After inputting [A1:0, A2:1, A3:2, and A4:3] respectively, press EXE, move to [B1].

Press ○○○, select [Fill Formula], press OK

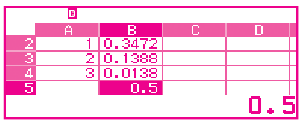


After inputting $\left[\text{Form}=A1\times{}_3C_{A1}\times\frac{5^{(3-A1)}}{6^{(3)}}\right]$**, press EXE**



After inputting [Range:B1:B4], press EXE, select [Confirm], press OK

When the sheet is displayed, move to [B5].



After inputting [Sum(B1:B4)], press EXE

The calculated expected value $0.5=\frac{1}{2}$ is displayed in [B5].

CASIO Essential Materials

Publisher: CASIO Institute for Educational Development

Date of Publication: 2023/12/22 (1st edition)

https://edu.casio.com