13 Sets

CASIO
Essential Materials

Introduction

These teaching materials were created with the hope of conveying to many teachers and students the appeal of scientific calculators.

(1) Change awareness (emphasizing the thinking process) and boost efficiency in learning mathematics

- By reducing the time spent on manual calculations, we can have learning with a focus on the thinking process that is more efficient.
- This reduces the aversion to mathematics caused by complicated calculations, and allows students to experience the joy of thinking, which is the essence of mathematics.

(2) Diversification of learning materials and problem-solving methods

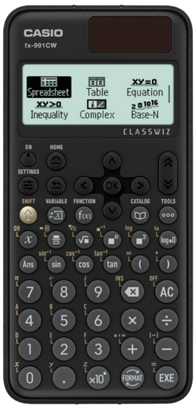
- Making it possible to do difficult calculations manually allows for diversity in learning materials and problem-solving methods.

(3) Promoting understanding of mathematical concepts

- By using the various functions of the scientific calculator in creative ways, students are able to deepen their understanding of mathematical concepts through calculations and discussions from different perspectives than before.
- This allows for exploratory learning through easy trial and error of questions.
- Listing and graphing of numerical values by means of tables allows students to discover laws and to understand visually.

Features of this book

- As well as providing first-time scientific calculator users with opportunities to learn basic scientific calculator functions from the ground up, the book also has material to show people who already use scientific calculators the appeal of scientific calculators described above.
- You can also learn about functions and techniques that are not available on conventional Casio models or other brands of scientific calculators.
- This book covers many units of high school mathematics, allowing students to learn how to use the scientific calculator as they study each topic.
- This book can be used in a variety of situations, from classroom activities to independent study and homework by students.



Better Mathematics Learning
with Scientific Calculator

Structure

Quadratic functions

TARGET To understand the concept of functions.

STUDY GUIDE

Functions

Definition of a function

For 2 variables, x and y, if the value of x is determined and if just 1 corresponding value of y is determined, then we say that **y is a function of x.**

Ex. Given $x=2$ ➡ Functions ($y=x+1$) ➡ We can determine $y=3$

How to describe functions

When y is a function of x, generally y is expressed as $f(x)$ or $g(x)$. Furthermore, in the function $y=f(x)$, then when $x=a$, the value of y is expressed as $f(a)$.

Ex. In the function $f(x)=3x+3$, when $x=1$, then when $f(1)=3\times1+3=6$ and $x=5$, then we get $f(5)=3\times5+3=18$.

Linear functions & Quadratic functions

When y is expressed as a linear equation of x, y is called the **linear function** of x, which is expressed as follows.

$$y=ax+b \quad (a\neq0)$$

When y is expressed as a quadratic equation of x, y is called the **quadratic function** of x, which is expressed as follows.

$$y=ax^2+bx+c \quad (a\neq0)$$

TARGET

Students can identify the objective to learn in each section.

STUDY GUIDE

Mathematical theorems and concepts are explained in detail. A scientific calculator is used to check and derive formulas according to the topic.

EXERCISE

1 For the quadratic function $f(x)=3x^2+2x+6$, find the various following values.

(1) $f(0)$

$f(0)=3\cdot0^2+2\cdot0+6=6$

6

(2) $f(1)$

$f(1)=3\cdot1^2+2\cdot1+6=11$

11

(3) $f(-1)$

$f(-1)=3\cdot(-1)^2+2\cdot(-1)+6=7$

7

(4) $f(109)$

$f(109)=3\cdot109^2+2\cdot109+6=35867$

35867

check

Use the FUNCTION function to calculate the value of $f(x)$.

Press ⌂, select [Calculate], press OK

Press f(x), select [Define f(x)], press OK ③ ⓧ ⓧ² ⊕ ② ⓧ ⊕ ⑥ EXE

Calculate Statistics Distribution Spreadsheet Table Equation | f(x) g(x) Define f(x) Define g(x) | f(x)=3x²+2x+6

EXERCISE

Students learn basic examples based on the explanation in Study Guide.

check

Explains how to use the scientific calculator to solve problems and check answers.

PRACTICE

1 For the quadratic function $f(x)=-2x^2+4x+5$, find the various following values.

(1) $f(0)$

$f(0)=-2\cdot0^2+4\cdot0+5=5$

5

(2) $f(2)$

$f(2)=-2\cdot2^2+4\cdot2+5=5$

5

(3) $f(-3)$

$f(-3)=-2\cdot(-3)^2+4\cdot(-3)+5=-25$

−25

(4) $f(52)$

$f(52)=-2\cdot52^2+4\cdot52+5=-5195$

−5195

PRACTICE

Students can do practice problems similar to those in EXERCISE. They can also practice using the scientific calculator as they learned to in Check.

ADVANCED

3 We know that when an object is thrown vertically upward from a height of 0 m at a velocity of v m/s, the relation between the height y m of the object and the time x seconds since it was thrown is $y=-4.9x^2+vx$. So, when a ball is thrown vertically from a height of 0 m at a velocity of 9.8 m/s, find the time it takes to reach the highest point and the height of that highest point. Note that the air resistance can be ignored.

$y=-4.9x^2+9.8x=-4.9(x^2-2x)=-4.9\{(x-1)^2-1^2\}=-4.9(x-1)^2+4.9$

Therefore, the graph of this function is convex upward, the axis is $x=1$, and the vertex is (1, 4.9).

Thus, the ball takes 1 second to reach its highest point, and the height of that highest point is 4.9 m.

1 second, 4.9 m

check

Press ⌂, select [Equation], press OK

Select [Polynomial], press OK, select [ax²+bx+c], press OK

⊖ ④ ⊙ ⑨ EXE ⑨ ⊙ ⑧ EXE ⓪ EXE EXE

Calculate Statistics Distribution Spreadsheet Table Equation | Simul Equation Polynomial Solver | ax²+bx+c ax³+bx²+cx+d ax⁴+bx³+cx²+dx+e | ax²+bx+c −4.9x²+ 9.8x 0

EXE EXE EXE **Press ⇧ ⓧ, scan the QR code to display a graph.**

Max of y=ax²+bx+c x= 1 | Max of y=ax²+bx+c y= 49/10 | 1/1

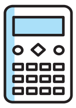
ADVANCED

Practical problems have been included in several topics. Solutions using a scientific calculator are also presented as necessary.

Other marks

Ex.	Simple examples on how to apply equations and theorems
explanation	Formulas and their supplementary explanations
proof	Proofs and checks of mathematical formulas
EXTRA Info.	Knowledge and information on formulas and other supplementary information in other units
OTHER METHODS	Alternative solutions and different verification methods for previously presented problems

Calculator mark



Where to use the scientific calculator

Colors of fonts in the teaching materials

- In STUDY GUIDE, important mathematical terms and formulas are printed in blue.
- In PRACTICE and ADVANCED the answers are printed in red.
  (Separate data is also available without the red parts, so it can be used for exercises.)

Applicable models

The applicable model is fx-991CW.
(Instructions on how to do input are for the fx-991CW, but in many cases similar calculations can be done on other models.)

Related Links

- Information and educational materials relevant to scientific calculators can be viewed on the following site.
  https://edu.casio.com
- The following video can be viewed to learn about the multiple functions of scientific calculators.
  https://www.youtube.com/playlist?list=PLRgxo9AwbIZLurUCZnrbr4cLfZdqY6aZA

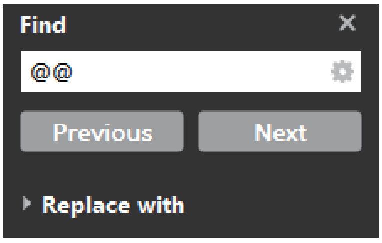
How to use PDF data

About types of data

- Data for all unit editions and data for each unit are available.
- For the above data, the PRACTICE and ADVANCED data without the answers in red is also available.

How to find where the scientific calculator is used

(1) Open a search window in the PDF Viewer.
(2) Type in "@@" as a search term.
(3) You can sequentially check where the calculator marks appear in the data.



How to search for a unit and section

(1) Search for units of data in all unit editions

- The data in all unit editions has a unit table of contents.
- Selecting a unit in the table of contents lets you jump to the first page of that unit.
- There is a bookmark on the first page of each unit, so you can jump from there also.

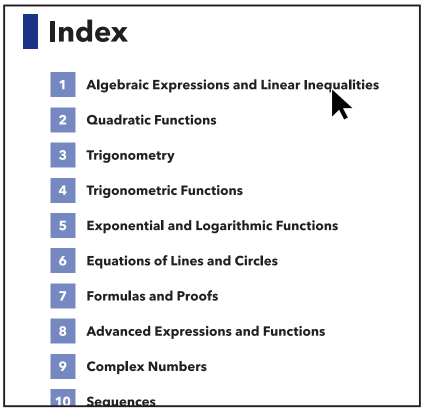
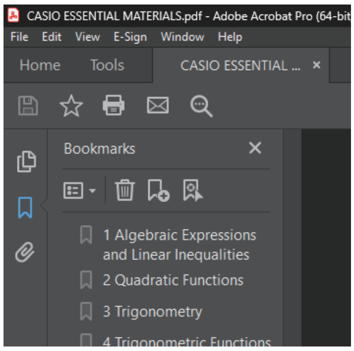


Table of contents of unit



Bookmark of unit

(2) Search for sections

- There are tables of contents for sections on the first page of units.
- Selecting a section in the table of contents takes you to the first page of that section.

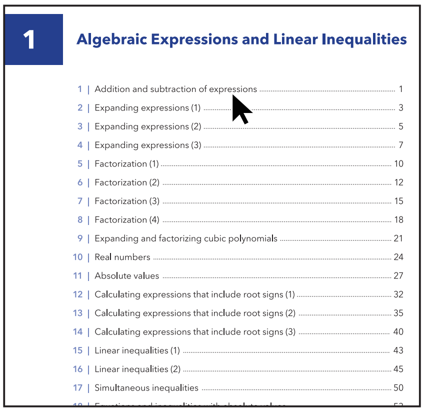


Table of contents of section

Sets and elements

TARGET To understand how to describe various sets.

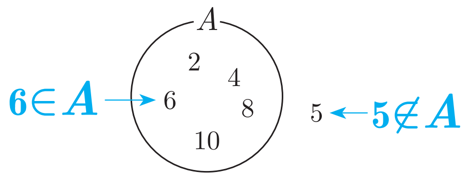
STUDY GUIDE

How to describe sets

A set and its elements

A **set** is a collection of things whose range is clearly defined, such as "positive even numbers less than or equal to 10", and each object belonging to the set is an **element** of the set.
When x is an element of the set A, we denote it as $x \in A$, and read it as x belongs to A. Conversely, if y is not an element of the set A, we denote it as $y \notin A$.

$A = \{x \mid x$ are positive even numbers less than or equal to $10\}$



How to describe sets

There are two ways to represent a set: (a) where all the elements are listed in curly brackets { }, and (b) where the conditions of the elements are written.

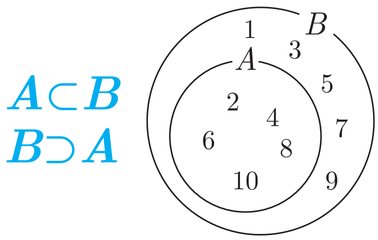
(a) $A = \{2, 4, 6, 8, 10\}$

(b) $A = \{x \mid x$ are positive even numbers less than or equal to $10\}$

Subsets

When all elements in set A are elements of set B, then we say that **A is a subset of B**, which is denoted as $A \subset B$ or $B \supset A$.
When $A \subset B$, if $x \in A$ then $x \in B$.
When $A \subset B$ and $B \subset A$, then sets A and B are equal, denoted as $A = B$.

$A = \{x \mid x$ are positive even numbers less than or equal to $10\}$
$B = \{y \mid y$ are positive numbers less than or equal to $10\}$



EXERCISE

◆ Given A is the set of all positive divisors of 18, solve the following problems.

(1) Use symbols to describe whether 6 and 15 are elements of A.

Since 6 is a positive divisor of 18, then 6 is an element of the set A.

Since 15 is not a positive divisor of 18, then 15 is not an element of the set A.

$6 \in A, 15 \notin A$

(2) Use the method of listing all the elements to describe set A.

The positive divisors of 18 are 1, 2, 3, 6, 9, and 18.

$A=\{1,2,3,6,9,18\}$

(3) Given B is the set of positive numbers from 1 to 20, use symbols to describe the relation of A and B.

$A=\{1, 2, 3, 6, 9, 18\}$

$B=\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20\}$

Because all elements in set A are elements of set B, set A is a subset of set B.

$A \subset B$ or $B \supset A$

PRACTICE

◆ Given A is the set of all positive divisors of 24, solve the following problems.

(1) Use symbols to describe whether 9 and 12 are elements of A.

Since 9 is not a positive divisor of 24, then 9 is not an element of the set A.

Since 12 is a positive divisor of 24, then 12 is an element of the set A.

$9 \notin A, 12 \in A$

(2) Use the method of listing all the elements to describe set A.

The positive divisors of 24 are 1, 2, 3, 4, 6, 8, 12, and 24.

$A=\{1,2,3,4,6,8,12,24\}$

(3) Given B is the set of positive divisors of 8, use symbols to describe the relation of A and B.

$A=\{1, 2, 3, 4, 6, 8, 12, 24\}$

$B=\{1, 2, 4, 8\}$

Because all elements in set B are elements of set A, then set B is a subset of set A.

$B \subset A$ or $A \supset B$

Intersections, unions, and complements

TARGET To understand the relations between sets.

STUDY GUIDE

Intersections and unions

Intersections

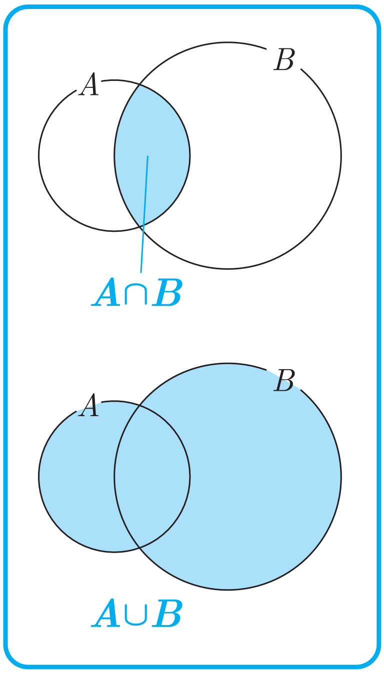
The set that contains all elements that belong to both the 2 sets A and B is called the **intersection** of A and B, and it is denoted as $\boldsymbol{A \cap B}$.

Union of sets

The set that contains all the elements that belong to either of the 2 sets A or B is called the **union** of A and B, and it is denoted as $\boldsymbol{A \cup B}$.

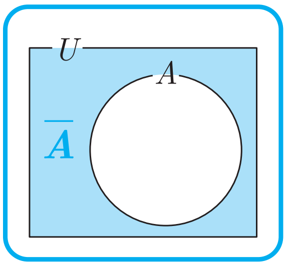
Empty set

A set that contains no elements is called an **empty set**, and it is denoted as $\varnothing$.
An empty set is considered to be a subset of any set. That is to say, $\varnothing \subset \boldsymbol{A}$ always holds for set A.



Complementary sets

We usually consider a set to be a subset of a universal set U, the content of which is decided in advance. In this case, the set U is called the **universal set**. Also, the set of elements in the universal set U that do not belong to set A are the **complementary set** of A and is denoted as $\overline{A}$.



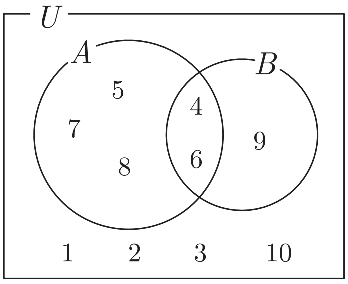
EXERCISE

◆ Let $U=\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ be the universal set. Solve the following problems for sets A and B below.

$A=\{4, 5, 6, 7, 8\}$ $B=\{4, 6, 9\}$

(1) Find $A\cap B$ for sets A and B.

As per the diagram on the right, the elements that belong to both A and B are 4 and 6.



$A\cap B=\{4,6\}$

(2) Find $A\cup B$ for sets A and B.

As per the diagram on the right, the elements that belong to at least one of A or B are 4, 5, 6, 7, 8 and 9.

$A\cup B=\{4,5,6,7,8,9\}$

(3) Find the complement of set A.

As per the diagram on the right, the elements in U that do not belong to A are 1, 2, 3, 9, and 10.

$\overline{A} = \{1, 2, 3, 9, 10\}$

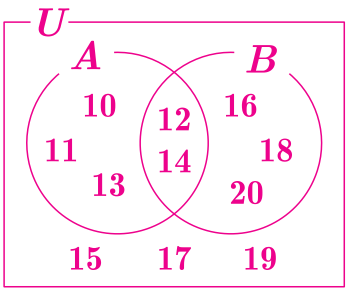
PRACTICE

◆ Let $U=\{10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20\}$ be the universal set. Solve the following problems for sets A and B below.

$A=\{10, 11, 12, 13, 14\}$ $B=\{12, 14, 16, 18, 20\}$

(1) Find $A\cap B$ for sets A and B.

As per the diagram on the right, the elements that belong to both A and B are 12 and 14.



$A\cap B=\{12,14\}$

(2) Find $A\cup B$ for sets A and B.

As per the diagram in (1), the elements that belong to at least one of A or B are 10, 11, 12, 13, 14, 16, 18 and 20.

$A\cup B=\{10,11,12,13,14,16,18,20\}$

(3) Find the complement of set A.

As per the diagram in (1), the elements in U that do not belong to A are 15, 16, 17, 18, 19 and 20.

$\overline{A} = \{15, 16, 17, 18, 19, 20\}$

Intersections and unions of inequalities

TARGET To understand intersections and unions for inequalities described as sets.

STUDY GUIDE

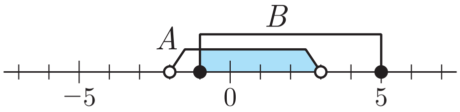
Inequalities described as sets

Use all real numbers as the universal set to consider the intersection and union of inequalities described as sets.

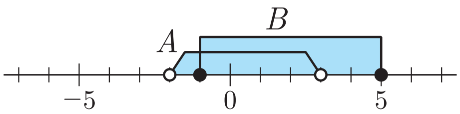
Ex. Given all real numbers as the universal set, then A and B are its subsets as shown below. Provided that x is a real number.

$A=\{x|-2<x<3\}$ $B=\{x|-1\leq x\leq 5\}$

At this point, consider how $A\cap B$ and $A\cup B$ appear on the number line.



$A\cap B=\{x|-1\leq x<3\}$

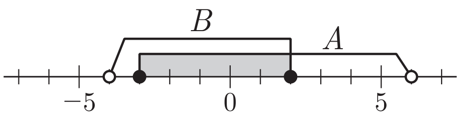


$A\cup B=\{x|-2<x\leq 5\}$

EXERCISE

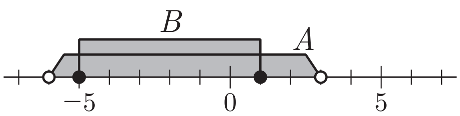
◆ Given all real numbers as the universal set, then solve the following problems for sets A and B. Provided that x is a real number.

(1) Find $A\cap B$ for $A=\{x|-3\leq x<6\}$ and $B=\{x|-4<x\leq 2\}$.



$A\cap B=\{x|-3\leq x\leq 2\}$

(2) Find $A\cup B$ for $A=\{x|-6<x<3\}$ and $B=\{x|-5\leq x\leq 1\}$.

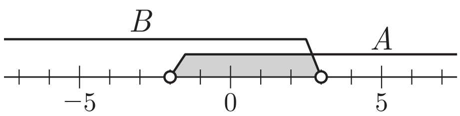


$A\cup B=\{x|-6<x<3\}$

(3) Find $A\cap B$ for $A=\{x|3x-1>-7\}$ and $B=\{x|2x+7>6x-5\}$.

From $3x-1>-7$, we get $x>-2$.

From $2x+7>6x-5$, we get $x<3$.

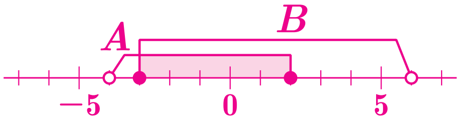


$A\cap B=\{x|-2<x<3\}$

PRACTICE

◆ Given all real numbers as the universal set, then solve the following problems for sets A and B. Provided that x is a real number.

(1) Find $A\cap B$ for $A=\{x|-4<x\le 2\}$ and $B=\{x|-3\le x<6\}$.

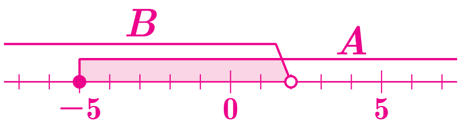


$A\cap B=\{x|-3\le x\le 2\}$

(2) Find $A\cap B$ for $A=\{x|2x-7\le 4x+3\}$ and $B=\{x|5x+1>7x-3\}$.

From $2x-7\le 4x+3$, we get $x\ge -5$.

From $5x+1>7x-3$, we get $x<2$.

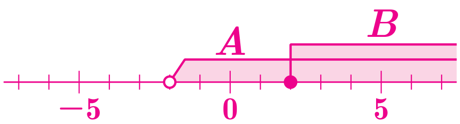


$A\cap B=\{x|-5\le x<2\}$

(3) Find $A\cup B$ for $A=\{x|3x+8>x+4\}$ and $B=\{x|3x-5\le 6x-11\}$.

From $3x+8>x+4$, we get $x>-2$.

From $3x-5\le 6x-11$, we get $x\ge 2$.



$A\cup B=\{x|x>-2\}$

De Morgan's laws

TARGET To understand De Morgan's laws.

STUDY GUIDE

De Morgan's laws

De Morgan's laws hold for subsets A and B of the universal set U, as shown below.

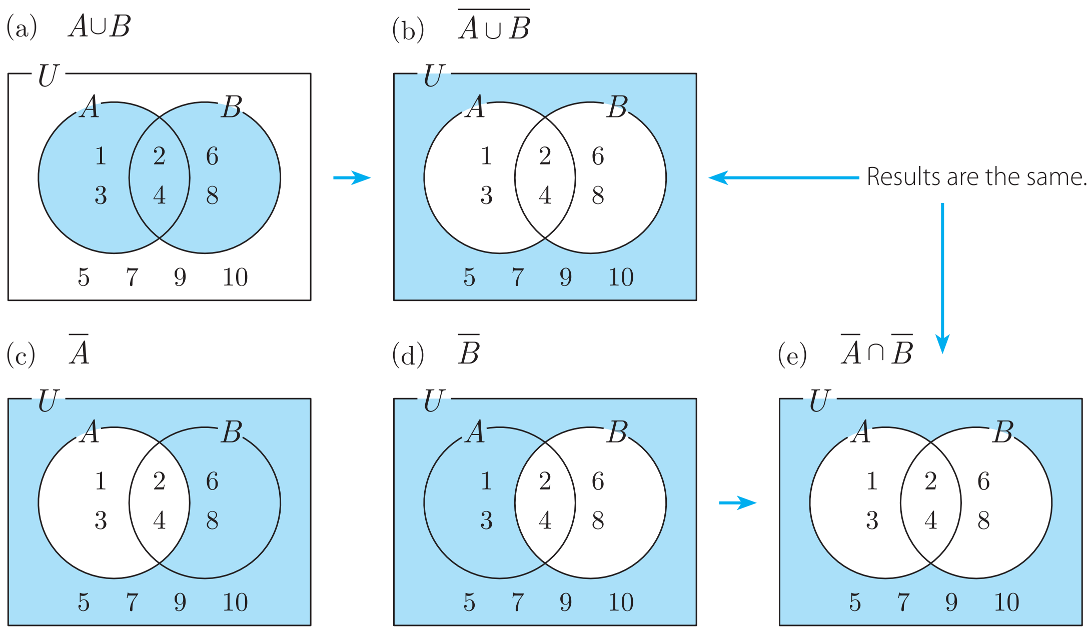
$$\overline{A \cup B} = \overline{A} \cap \overline{B},\ \overline{A \cap B} = \overline{A} \cup \overline{B}$$

explanation

Let U=$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ be the universal set, and let A=$\{1, 2, 3, 4\}$ and B=$\{2, 4, 6, 8\}$ be the subsets.

Such that A∪B=$\{1, 2, 3, 4, 6, 8\}\ldots$(a), we get $\overline{A \cup B} = \{5, 7, 9, 10\}\ldots$(b).

Furthermore, for $\overline{A} = \{5, 6, 7, 8, 9, 10\}\ldots$(c), $\overline{B} = \{1, 3, 5, 7, 9, 10\}\ldots$(d), we get $\overline{A} \cap \overline{B} = \{5, 7, 9, 10\}\ldots$(e).



Therefore, $\overline{A \cup B} = \overline{A} \cap \overline{B}$ holds.

The other law can be confirmed in the same way.

EXERCISE

◆ Given $U=\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ as the universal set, solve the following problems for sets A and B below.

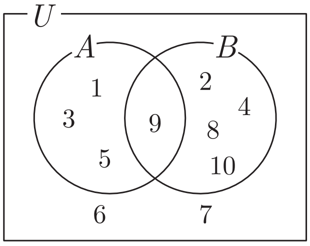
$A=\{1, 3, 5, 9\}$ $B=\{2, 4, 8, 9, 10\}$

(1) Find $\overline{A} \cap \overline{B}$ by using De Morgan's laws.

From De Morgan's laws, $\overline{A} \cap \overline{B} = \overline{A \cup B}$

From $A \cup B=\{1, 2, 3, 4, 5, 8, 9, 10\}$, we get $\overline{A \cup B} = \{6,7\}$

Therefore, $\overline{A} \cap \overline{B} = \{6,7\}$



$\overline{A} \cap \overline{B} = \{6,7\}$

(2) Find $\overline{A} \cup \overline{B}$ by using De Morgan's laws.

From De Morgan's laws, $\overline{A} \cup \overline{B} = \overline{A \cap B}$

From $A \cap B=\{9\}$, we get $\overline{A \cap B} = \{1,2,3,4,5,6,7,8,10\}$

Therefore, $\overline{A} \cup \overline{B} = \{1,2,3,4,5,6,7,8,10\}$

$\overline{A} \cup \overline{B} = \{1,2,3,4,5,6,7,8,10\}$

PRACTICE

◆ Given $U=\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ as the universal set, solve the following problems for sets A and B below.

$A=\{1, 4, 7, 8\}$ $B=\{2, 4, 6, 8, 9\}$

(1) Find $\overline{A} \cap \overline{B}$ by using De Morgan's laws.

From De Morgan's laws, $\overline{A} \cap \overline{B} = \overline{A \cup B}$

From $A \cup B=\{1, 2, 4, 6, 7, 8, 9\}$**, we get** $\overline{A \cup B} = \{3,5\}$

Therefore, $\overline{A} \cap \overline{B} = \{3,5\}$

$\overline{A} \cap \overline{B} = \{3,5\}$

(2) Find $\overline{A} \cup \overline{B}$ by using De Morgan's laws.

From De Morgan's laws, $\overline{A} \cup \overline{B} = \overline{A \cap B}$

From $A \cap B=\{4, 8\}$**, we get** $\overline{A \cap B} = \{1,2,3,5,6,7,9\}$

Therefore, $\overline{A} \cup \overline{B} = \{1,2,3,5,6,7,9\}$

$\overline{A} \cup \overline{B} = \{1,2,3,5,6,7,9\}$

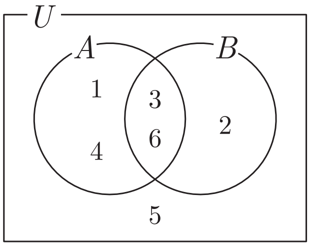
The problem of finding various sets

TARGET To consider sets by using intersections, unions, and complements.

STUDY GUIDE

Ex. Let $U=\{1, 2, 3, 4, 5, 6\}$ be the universal set. Consider the following (1) to (4) for sets A and B below.

$A=\{1, 3, 4, 6\}$ $B=\{2, 3, 6\}$



(1) $A\cap B$

The elements that belong to both A and B are 3 and 6.

Therefore, we get $A\cap B=\{3, 6\}$.

(2) $\overline{A}$

The elements in U that do not belong to A are 2 and 5.

Therefore, we get $\overline{A}=\{2, 5\}$.

(3) $\overline{A}\cap B$

The element that belongs to both $\overline{A}=\{2, 5\}$ and B is 2.

Therefore, we get $\overline{A}\cap B=\{2\}$.

(4) $A\cup \overline{B}$

The elements in U that do not belong to B are 1, 4, and 5, so we get $\overline{B}=\{1, 4, 5\}$

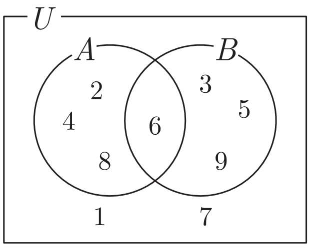
The elements that belong to at least one of A or $\overline{B}$ are 1, 3, 4, 5, and 6.

Therefore, we get $A\cup \overline{B}=\{1, 3, 4, 5, 6\}$.

EXERCISE

◆ Given $U=\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ as the universal set, solve the following problems for sets A and B below.

$A=\{2, 4, 6, 8\}$ $B=\{3, 5, 6, 9\}$



(1) Find $A\cap B$

The element that belongs to both A and B is 6.

Therefore, we get $A\cap B=\{6\}$.

$\boldsymbol{A\cap B=\{6\}}$

(2) Find $\overline{A}$.

The elements in U that do not belong to A are 1, 3, 5, 7, and 9.

Therefore, we get $\overline{A}=\{1, 3, 5, 7, 9\}$.

$\boldsymbol{\overline{A}=\{1,3,5,7,9\}}$

(3) Find $\overline{A}\cap B$.

The elements that belong to both $\overline{A}=\{1, 3, 5, 7, 9\}$ and B are 3, 5, and 9.

Therefore, we get $\overline{A}\cap B=\{3, 5, 9\}$.

$\boldsymbol{\overline{A}\cap B=\{3,5,9\}}$

PRACTICE

◆ Given $U=\{x|x$ is a positive number less than or equal to $11\}$ as the universal set, solve the following problems for sets A, B, and C below.

$A=\{1, 3, 6, 9, 11\}$ $B=\{2, 4, 6, 8, 10\}$ $C=\{3, 5, 7, 10, 11\}$

(1) Find $B\cup C$.

The elements that belong to at least one of B or C are 2, 3, 4, 5, 6, 7, 8, 10, and 11.

Therefore, we get $B\cup C=\{2, 3, 4, 5, 6, 7, 8, 10, 11\}$.

$B\cup C=\{2,3,4,5,6,7,8,10,11\}$

(2) Find $\overline{A}$.

The elements in U that do not belong to A are 2, 4, 5, 7, 8, and 10.

Therefore, we get $\overline{A}=\{2,4,5,7,8,10\}$.

$\overline{A}=\{2,4,5,7,8,10\}$

(3) Find $\overline{A}\cap C$.

The elements that belong to both $\overline{A}=\{2,4,5,7,8,10\}$ and C are 5, 7, and 10.

Therefore, we get $\overline{A}\cap C=\{5,7,10\}$.

$\overline{A}\cap C=\{5,7,10\}$

(4) Find $\overline{B}\cap\overline{C}$.

From De Morgan's laws, we get $\overline{B}\cap\overline{C}=\overline{B\cup C}$.

The elements in $\overline{B\cup C}$ do not belong to $B\cup C$.

From (1) we get $\overline{B\cup C}=\{1, 9\}$

Therefore, $\overline{B}\cap\overline{C}=\overline{B\cup C}=\{1,9\}$

$\overline{B}\cap\overline{C}=\{1,9\}$

Propositions: True/false and counterexamples

TARGET To understand the concept of true and false propositions.

STUDY GUIDE

Propositions are true/false

Proposition

A **proposition** is a statement or formula that is clearly determined to be correct or incorrect.

Propositions are true/false

When a proposition is correct, it is said to be **true**, and when it is incorrect, it is said to be **false**.

Conditions

Conditions are used to determine whether a proposition is true or false by assigning values to variables in a statement or formula.

Assumptions and conclusions

Propositions generally use 2 conditions, p and q, which are often expressed as "If p then q". In this case, p is the **assumption** and q is the **conclusion**. Propositions are expressed as follows.

Proposition "If p then q" is denoted as $p \Longrightarrow q$.

Relation between propositions and sets

In the proposition "$p \Longrightarrow q$", a set that satisfies the condition p is P, and a set that satisfies the condition q is Q, such that **if the proposition "$p \Longrightarrow q$" is true, then we have $P \subset Q$**. Conversely, if $P \subset Q$, then the proposition "$p \Longrightarrow q$" is true.

Counterexample

To show that the proposition "$p \Longrightarrow q$" is false, we only need 1 example that "satisfies p but does not satisfy q". Such examples are called **counterexamples**.

EXERCISE

◆ Determine whether the next proposition is true or false. And, if it is false, give a counterexample.

(1) $x=3 \Longrightarrow x^2=9$

Squaring both sides of $x=3$ gives $x^2=9$, so this is true.

True

(2) $x^2=9 \Longrightarrow x=3$

Solving $x^2=9$ gives us $x=3$ or $x=-3$. From which, when $x=-3$ we also get $x^2=9$, so this is false.

False, with a counterexample of $x=-3$

(3) $x<y \Longrightarrow ax<ay$ (where a is a positive real number)

Since $a>0$, multiplying both sides of $x<y$ by a gives us $ax<ay$, so this is true.

True

PRACTICE

◆ Determine whether the next proposition is true or false. And, if it is false, give a counterexample.

(1) $x=-5 \Longrightarrow x^2=25$

Squaring both sides of $x=-5$ gives $x^2=25$, so this is true.

True

(2) $x^2=81 \Longrightarrow x=9$

Solving $x^2=81$ gives us $x=9$ or $x=-9$.

From which, when $x=-9$ we also get $x^2=81$, so this is false.

False, with a counterexample of $x=-9$

(3) $x=y \Longrightarrow x+a=y+a$ (where a is a real number)

Adding a to both sides of $x=y$ gives $x+a=y+a$, so this is true.

True

(4) If it is a triangle, then all 3 interior angles are equal.

For example, all 3 interior angles of a right triangle cannot be equal, so this is false.

False, with a counterexample of a right triangle

(5) If it is a rhombus, all 4 sides are equal.

The definition of a rhombus is that all 4 sides are equal, so this is true.

True

Necessary conditions and sufficient conditions

TARGET To understand necessary conditions, sufficient conditions, and necessary and sufficient conditions.

STUDY GUIDE

Necessary conditions and sufficient conditions

Necessary conditions and sufficient conditions

For 2 conditions, p and q, when the proposition "$p \Longrightarrow q$" is true, p is said to be a **sufficient condition** for q, and q is said to be a **necessary condition** for p.

$p \Longrightarrow q$

Sufficient condition **Necessary condition**

Necessary and sufficient conditions

For 2 conditions, p and q, when propositions "$p \Longrightarrow q$" and "$q \Longrightarrow p$" are both true, p is said to be a **necessary and sufficient condition** for q and is denoted as $p \Longleftrightarrow q$. Also, p and q are said to be **equivalent**.

EXERCISE

◆ Given that x and y are real numbers. For the following 2 conditions, p and q, determine which is most appropriate, whether p is a "necessary condition", a "sufficient condition", or a "necessary and sufficient condition" for q.

(1) p : $xy>0$ q : $x>0$ and $y>0$

"$xy>0 \Longrightarrow x>0$ and $y>0$" is false because of the counterexamples $x=-1$ and $y=-1$.

"$x>0$ and $y>0 \Longrightarrow xy>0$" are true.

Therefore, p is a necessary condition for q.

This is a necessary condition

(2) p : $x=0$ and $y=0$ q : $x+y=0$

"$x=0$ and $y=0 \Longrightarrow x+y=0$" are true.

"$x+y=0 \Longrightarrow x=0$ and $y=0$" is false because of the counterexamples $x=-1$ and $y=1$.

Therefore, p is a sufficient condition for q.

This is a sufficient condition

(3) p : $|x|<2$ q : $-2<x<2$

"$|x|<2 \Longrightarrow -2<x<2$" is true.

"$-2<x<2 \Longrightarrow |x|<2$" is also true.

Therefore, p is a necessary and sufficient condition for q.

This is a necessary and sufficient condition

PRACTICE

◆ Given that x and y are real numbers. For the following 2 conditions, p and q, determine which is most appropriate, whether p is a "necessary condition", a "sufficient condition", or a "necessary and sufficient condition" for q.

(1) p : $x-y>0$ q : $x>y$

"$x-y>0 \Longrightarrow x>y$" is true.

"$x>y \Longrightarrow x-y>0$" is also true.

Therefore, p is a necessary and sufficient condition for q.

This is a necessary and sufficient condition

(2) p : $x=3$ and $y=5$ q : $xy=15$

"$x=3$ and $y=5 \Longrightarrow xy=15$" is true.

"$xy=15 \Longrightarrow x=3$ and $y=5$" is false because of the counterexamples $x=5$ and $y=3$.

Therefore, p is a sufficient condition for q.

This is a sufficient condition

(3) p : $(x+5)^2=0$ q : $x=-5$

"$(x+5)^2=0 \Longrightarrow x=-5$" is true.

"$x=-5 \Longrightarrow (x+5)^2=0$" is also true.

Therefore, p is a necessary and sufficient condition for q.

This is a necessary and sufficient condition

(4) p : quadrangle ABCD is a parallelogram q : quadrangle ABCD is a rectangle

"Quadrangle ABCD is a parallelogram $\Longrightarrow$ Quadrangle ABCD is a rectangle" is false because some parallelograms exist whose 4 corners are not all 90°.

"Quadrangle ABCD is a rectangle $\Longrightarrow$ Quadrangle ABCD is a parallelogram" is true.

Therefore, p is a necessary condition for q.

This is a necessary condition

(5) p : quadrangle ABCD is a square q : quadrangle ABCD is a rhombus

"Quadrangle ABCD is a square $\Longrightarrow$ Quadrangle ABCD is a rhombus" is true.

"Quadrangle ABCD is a rhombus $\Longrightarrow$ Quadrangle ABCD is a square" is false because some rhombuses exist whose 4 corners are not all 90°.

Therefore, p is a sufficient condition for q.

This is a sufficient condition

Negation of conditions

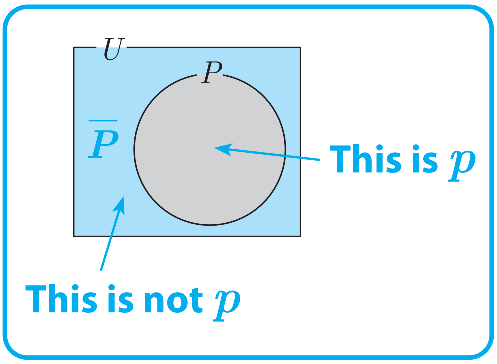
TARGET To understand the negation of conditions.

STUDY GUIDE

Negation of conditions

For condition p, the condition "not p" is the **negation** of p, and it is denoted as $\overline{p}$.

Assuming that set P satisfies condition p, set $\overline{P}$ satisfies $\overline{p}$, and the universal set is U, then the relations of P, $\overline{P}$, and U are shown in the diagram on the right.



EXERCISE

◆ Determine the negations of the conditions below.

(1) Given x is a real number, and $x>5$

Then "Not $x>5$", gives us "$x\leq5$".

$x\leq5$

(2) Given x is a whole number, and x is an even number

Then "x is not an even number", gives us "x is an odd number".

x is an odd number

(3) Given x is a real number, and x is a positive number

Then "x is a negative number" is incorrect because it does not include 0.

x is a negative number or 0

PRACTICE

◆ Determine the negations of the conditions below.

(1) Given x is a real number, and $x \geq -8$

Then "Not $x \geq -8$", gives us "$x < -8$".

$x < -8$

(2) Given x is a real number, and $x < 4$

Then "Not $x < 4$", gives us "$x \geq 4$".

$x \geq 4$

(3) Given x is a real number, and x is a rational number

Then "x is not a rational number", gives us "x is an irrational number".

x is an irrational number

(4) Given x is a whole number, and x is a negative number

Then "x is a positive number" is incorrect because it does not include 0.

x is a positive number or 0

Negations of "And" and "Or"

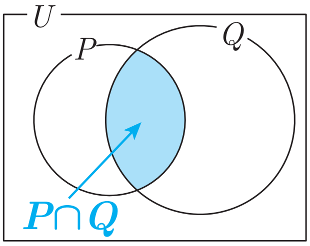
TARGET To understand the negation of "And" and "Or".

STUDY GUIDE

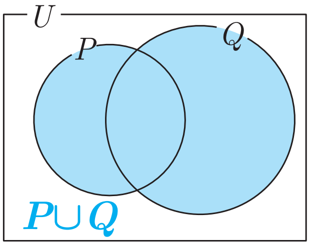
"And", "Or", and their negations

Given a universal set U, and a set P that satisfies condition p, and set Q that satisfies condition q, then sets that satisfy conditions "p and q", "p or q", and "not p" are as follows.

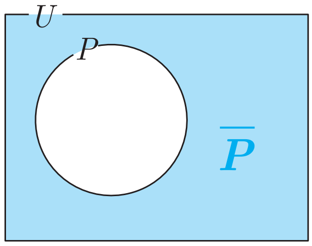
p and q



p or q



This is not p



For the negation of the condition "p and q" and the negation of "p or q", De Morgan's laws give us the following.

$$\overline{p \text{ and } q} \iff \overline{p} \text{ or } \overline{q}$$

$$\overline{p \text{ or } q} \iff \overline{p} \text{ and } \overline{q}$$

EXERCISE

◆ Given that x and y are real numbers. Determine the negations of the conditions below.

(1) $x<0$ and $y\leq 0$

The negation of $x<0$ is $x\geq 0$ and the negation of $y\leq 0$ is $y>0$, so the negation of "$x<0$ and $y\leq 0$" is "$x\geq 0$ or $y>0$".

$x\geq 0$ or $y>0$

(2) $x>2$ or $x<-1$

The negation of $x>2$ is $x\leq 2$ and the negation of $x<-1$ is $x\geq -1$, so the negation of "$x>2$ or $x<-1$" is "$x\leq 2$ and $x\geq -1$".

In other words, "$-1\leq x\leq 2$".

$-1\leq x\leq 2$

(3) At least 1 of x or y is 0

In other words, since "$x=0$ or $y=0$", the negation is "$x\neq 0$ and $y\neq 0$".

That is, "both x and y are not 0".

Both x and y are not 0

PRACTICE

◆ Given that x and y are real numbers. Determine the negations of the conditions below.

(1) $x \geq 3$ and $y = -4$

The negation of $x \geq 3$ is $x < 3$ and the negation of $y = -4$ is $y \neq -4$, so the negation of "$x \geq 3$ and $y = -4$" is "$x < 3$ or $y \neq -4$".

$x < 3$ or $y \neq -4$

(2) $x > -2$ or $x < -7$

The negation of $x > -2$ is $x \leq -2$ and the negation of $x < -7$ is $x \geq -7$, so the negation of "$x > -2$ or $x < -7$" is "$x \leq -2$ and $x \geq -7$".
In other words, "$-7 \leq x \leq -2$".

$-7 \leq x \leq -2$

(3) At least 1 of x or y is -3

In other words, since "$x = -3$ or $y = -3$", the negation is "$x \neq -3$ and $y \neq -3$".
That is, "both x and y are not -3".

Both x and y are not -3

(4) Both x and y are negative numbers

In other words, since "$x < 0$ and $y < 0$", the negation is "$x \geq 0$ or $y \geq 0$".
That is, "at least 1 of x or y is a positive number or 0".

At least 1 of x or y is a positive number or 0

Converse, inverse, and contraposition

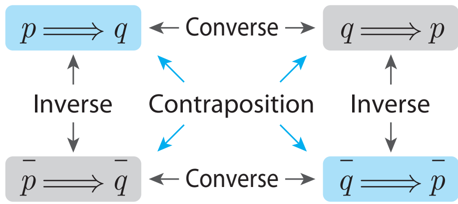
TARGET To understand converse, inverse, and contraposition.

STUDY GUIDE

Converse, inverse, and contraposition

For the proposition "$p \Longrightarrow q$",

we say that "$q \Longrightarrow p$" is the **converse**, "$\overline{p} \Longrightarrow \overline{q}$" is the **inverse**, and "$\overline{q} \Longrightarrow \overline{p}$" is the **contraposition**.



The following holds regarding the truth of propositions and their converses, inverses, and contrapositions.

(a) Even if the original proposition is true, the converse and inverse are not always true.

(b) The original proposition and its contraposition always have the same truth value.

explanation Regarding (b) above

Given a universal set U, and a set P that satisfies condition p, and set Q that satisfies condition q.

The proposition "$p \Longrightarrow q$" is true → $P \subset Q$ → $\overline{Q} \subset \overline{P}$ → The proposition "$\overline{q} \Longrightarrow \overline{p}$" is true.

EXERCISE

◆ Determine the converse, inverse, and contraposition of the following propositions. Also, determine whether they are true or false.

"$x=y \Longrightarrow x^2 = y^2$"

The converse is "$x^2 = y^2 \Longrightarrow x=y$".

The inverse is "$x \neq y \Longrightarrow x^2 \neq y^2$".

The contraposition is "$x^2 \neq y^2 \Longrightarrow x \neq y$".

Consider whether the proposition's converse, inverse, and contraposition are true or false.

The converse is false because $x = -y$ is the counterexample.

The inverse is false because $x=-y$ is the counterexample.

Since the original proposition is true, the contraposition is true.

The converse "$x^2 = y^2 \Longrightarrow x=y$" is false

The inverse "$x \neq y \Longrightarrow x^2 \neq y^2$" is false

The contraposition "$x^2 \neq y^2 \Longrightarrow x \neq y$" is true

PRACTICE

◆ Determine the converse, inverse, and contraposition of the propositions below. Also, determine whether they are true or false.

(1) $x=3 \Longrightarrow x^2=9$

The converse "$x^2=9 \Longrightarrow x=3$" is false because $x=-3$ is a counterexample.
The inverse "$x\neq 3 \Longrightarrow x^2\neq 9$" is also false because $x=-3$ is a counterexample.
The contraposition "$x^2\neq 9 \Longrightarrow x\neq 3$" is true because the original proposition is true.

The converse "$x^2=9 \Longrightarrow x=3$" is false
The inverse "$x\neq 3 \Longrightarrow x^2\neq 9$" is false
The contraposition "$x^2\neq 9 \Longrightarrow x\neq 3$" is true

(2) For 2 triangles, if the lengths of each of their 3 pairs of sides are equal, then they are congruent.

The converse "for 2 triangles, if they are congruent, then the lengths of each of their 3 pairs of sides are equal" is true.
The inverse "for 2 triangles, if the lengths of each of their 3 pairs of sides are not equal, then they are not congruent" is true.
The contraposition "for 2 triangles, if they are not congruent, then the lengths of each of their 3 pairs of sides are not equal" is true.

The converse "for 2 triangles, if they are congruent, then the lengths of each of their 3 pairs of sides are equal" is true
The inverse "for 2 triangles, if the lengths of each of their 3 pairs of sides are not equal, then they are not congruent" is true
The contraposition "for 2 triangles, if they are not congruent, then the lengths of each of their 3 pairs of sides are not equal" is true

Proofs using contraposition

TARGET To understand about using contraposition for proofs.

STUDY GUIDE

Proofs using contraposition

When proving a proposition, it may be difficult to prove it as is. In such situations, 1 way to prove the proposition is to use its contraposition to prove it.

Because "a proposition and its contraposition must have the same truth value", proving that the contraposition of a proposition is true, also proves that the original proposition holds.

EXERCISE

◆ For an integer x, if x^2 is odd, then x is odd. This is proved in the following way. Fill in the blanks [a] to [e] with the appropriate term or expression.

[Proof]

The contraposition of this proposition is "If an integer x is not odd, then x^2 is not [a]."

From that we can get, "If the integer x is even, then x^2 is [b].", so it suffices to prove this.

Suppose that x is an even number, then we can represent x by using some integer n, such that

$x=$[c]

This means that

$x^2=($[c]$)^2=$[d]$=2\cdot$[e]

Since [e] is an integer, then x^2 is even.

Therefore, since the contraposition is proved to be true, the original proposition also holds true.

[a]**...odd,** [b]**...even,** [c]**...$2n$,** [d]**...$4n^2$,** [e]**...$2n^2$**

PRACTICE

1 For real numbers a and b, if $a+b$ is an irrational number, then at least one of a or b is an irrational number. This is proved in the following way. Fill in the blanks [a] to [d] with the appropriate term or expression.

[Proof]

The contraposition of this proposition is "If real numbers a and b are both not irrational numbers, then $a+b$ is not [a]."

From that we can get, "If the real numbers a and b are both rational numbers, then $a+b$ is [b].", so it suffices to prove this.

Suppose that a and b are rational numbers,

then a can be written as $\frac{n}{m}$ for some integers $m(\neq 0)$ and n, and b can be written as $\frac{q}{p}$ for some integers $p(\neq 0)$ and q.

This means that

$$a+b=\frac{n}{m}+\frac{q}{p}=\frac{[\ \mathrm{c}\ ]}{[\ \mathrm{d}\ ]}$$

Since [c] and [d] are both integers, then $a+b$ is a rational number.

Therefore, since the contraposition is proved to be true, the original proposition also holds true.

[a]…Irrational number, [b]…Rational number, [c]…$np+mq$, and [d]…mp

2 For integers x and y, prove that if xy is an odd number, then at least one of x or y is an odd number.

[Proof]

The contraposition of this proposition is "If integers x and y are both not odd numbers, then xy is not an odd number."

From that we can get, "If the integers x and y are even, then xy is even.", so it suffices to prove this.

Suppose that x and y are even numbers,

then we can represent x and y by using some integers m and n, such that

$x=2m$, $y=2n$.

This means that

$xy=2m\times 2n=4mn=2(2mn)$.

Since $2mn$ is an integer, then xy is even.

Therefore, since the contraposition is proved to be true, the original proposition also holds true.

Proofs using contradictions

TARGET To understand about using proof by contradiction.

STUDY GUIDE

Proofs using contradictions

For example, to share 4 balls among 3 people, we know that 1 person is certain to receive 2 or more balls. This is because, assuming that 1 person cannot be given 2 or more balls (only 1 ball can be given), then at least 4 people are needed, which creates a contradiction because there are only 3 people. This method of using a contradiction of things is called **proof by contradiction**.

> **Proof by contradiction**
> **A method for proving a proposition by making an assumption for which the proposition does not hold such that it leads to a contradiction.**

First, negate the conclusion to lead to a contradiction of an assumption. Generally, when a proposition cannot be proved as it is, it is good to use "proof by using contraposition" or "proof by contradiction".

EXERCISE

◆ The fact that $\sqrt{2}$ is an irrational number is used to prove that $\sqrt{2}+5$ is an irrational number in the following way. Fill in the blanks [a] to [b] with the appropriate term.

[Proof]

Assume that $\sqrt{2}+5$ is a rational number p.
That is, $\sqrt{2}+5=p$.
From this, we get $\sqrt{2}=p-5$.
Since $p-5$ is [a], then $\sqrt{2}$ is also [a].
This contradicts that $\sqrt{2}$ is [b].
Therefore, $\sqrt{2}+5$ is an irrational number.

[a]**…Rational number** [b]**…Irrational number**

PRACTICE

1 Prove that $\sqrt{3}$ is an irrational number in the following way. Fill in the blanks [a] to [e] with the appropriate expression. Note that, for an integer n, if n^2 is a multiple of 3, then it is good to use a multiple of 3 for n also.

[Proof]

Assume that $\sqrt{3}$ is not an irrational number, in other words it is a rational number.

This can be expressed as $\sqrt{3} = \dfrac{n}{m}$ by using positive integers m and n (where m and n have no common denominator other than 1).

By squaring both sides of this equation, we get $3m^2 = n^2$ $\quad\ldots\ldots(\mathrm{i})$

Here, $3m^2$ is a multiple of [a], so n^2 is a multiple of [a] also. Therefore, since n is also a multiple of [a], we can show n=[a]k, where k is a positive integer. Assigning this value to (i), we get

$3m^2 = (3k)^2 =$ [b],

such that, m^2 =[c], where [c] is a multiple of [d], so m^2 is a multiple of [d] also.

Therefore, m is a multiple of [d] also.

It follows that m and n are both multiples of [e], which contradicts that they have no common denominator except 1.

Accordingly, $\sqrt{3}$ is not a rational number. Therefore, $\sqrt{3}$ is an irrational number.

[a]…3,[b]…$9k^2$,[c]…$3k^2$,[d]…3,[e]…3

2 Use the fact that $\sqrt{5}$ is an irrational number to prove that $2\sqrt{5}-1$ is an irrational number.

[Proof]

Assume that $2\sqrt{5}-1$ is a rational number p. That is, $2\sqrt{5}-1=p$.

From this, we get $2\sqrt{5}=p+1$, $\sqrt{5}=\dfrac{p+1}{2}$

Since $\dfrac{p+1}{2}$ is a rational number, then $\sqrt{5}$ is also a rational number.

This contradicts that $\sqrt{5}$ is an irrational number.

Therefore, $2\sqrt{5}-1$ is an irrational number.

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