11 Vectors

CASIO
Essential Materials

Introduction

These teaching materials were created with the hope of conveying to many teachers and students the appeal of scientific calculators.

(1) Change awareness (emphasizing the thinking process) and boost efficiency in learning mathematics

- By reducing the time spent on manual calculations, we can have learning with a focus on the thinking process that is more efficient.
- This reduces the aversion to mathematics caused by complicated calculations, and allows students to experience the joy of thinking, which is the essence of mathematics.

(2) Diversification of learning materials and problem-solving methods

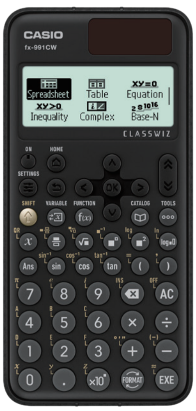
- Making it possible to do difficult calculations manually allows for diversity in learning materials and problem-solving methods.

(3) Promoting understanding of mathematical concepts

- By using the various functions of the scientific calculator in creative ways, students are able to deepen their understanding of mathematical concepts through calculations and discussions from different perspectives than before.
- This allows for exploratory learning through easy trial and error of questions.
- Listing and graphing of numerical values by means of tables allows students to discover laws and to understand visually.

Features of this book

- As well as providing first-time scientific calculator users with opportunities to learn basic scientific calculator functions from the ground up, the book also has material to show people who already use scientific calculators the appeal of scientific calculators described above.
- You can also learn about functions and techniques that are not available on conventional Casio models or other brands of scientific calculators.
- This book covers many units of high school mathematics, allowing students to learn how to use the scientific calculator as they study each topic.
- This book can be used in a variety of situations, from classroom activities to independent study and homework by students.



Better Mathematics Learning with Scientific Calculator

Structure

Quadratic functions

TARGET To understand the concept of functions.

STUDY GUIDE

Functions

Definition of a function

For 2 variables, x and y, if the value of x is determined and if just 1 corresponding value of y is determined, then we say that **y is a function of x.**

Ex. Given $x=2$ ➡ Functions ($y=x+1$) ➡ We can determine $y=3$

How to describe functions

When y is a function of x, generally y is expressed as $f(x)$ or $g(x)$. Furthermore, in the function $y=f(x)$, then when $x=a$, the value of y is expressed as $f(a)$.

Ex. In the function $f(x)=3x+3$, when $x=1$, then when $f(1)=3\times1+3=6$ and $x=5$, then we get $f(5)=3\times5+3=18$.

Linear functions & Quadratic functions

When y is expressed as a linear equation of x, y is called the **linear function** of x, which is expressed as follows.

$$\boxed{y=ax+b \quad (a\neq0)}$$

When y is expressed as a quadratic equation of x, y is called the **quadratic function** of x, which is expressed as follows.

$$\boxed{y=ax^2+bx+c \quad (a\neq0)}$$

TARGET

Students can identify the objective to learn in each section.

STUDY GUIDE

Mathematical theorems and concepts are explained in detail. A scientific calculator is used to check and derive formulas according to the topic.

EXERCISE

1 For the quadratic function $f(x)=3x^2+2x+6$, find the various following values.

(1) $f(0)$

$f(0)=3\cdot0^2+2\cdot0+6=6$ **6**

(2) $f(1)$

$f(1)=3\cdot1^2+2\cdot1+6=11$ **11**

(3) $f(-1)$

$f(-1)=3\cdot(-1)^2+2\cdot(-1)+6=7$ **7**

(4) $f(109)$

$f(109)=3\cdot109^2+2\cdot109+6=35867$ **35867**

check

Use the FUNCTION function to calculate the value of $f(x)$.

Press ⓞ, select [Calculate], press OK

Press f(x), select [Define f(x)], press OK ③ ⓧ ⓧ² ⊕ ② ⓧ ⊕ ⑥ EXE

Calculate Statistics Distribution Spreadsheet Table Equation | f(x) g(x) Define f(x) Define g(x) | f(x)=3x²+2x+6

EXERCISE

Students learn basic examples based on the explanation in Study Guide.

check

Explains how to use the scientific calculator to solve problems and check answers.

PRACTICE

1 For the quadratic function $f(x)=-2x^2+4x+5$, find the various following values.

(1) $f(0)$

$f(0)=-2\cdot0^2+4\cdot0+5=5$ **5**

(2) $f(2)$

$f(2)=-2\cdot2^2+4\cdot2+5=5$ **5**

(3) $f(-3)$

$f(-3)=-2\cdot(-3)^2+4\cdot(-3)+5=-25$ **-25**

(4) $f(52)$

$f(52)=-2\cdot52^2+4\cdot52+5=-5195$ **-5195**

PRACTICE

Students can do practice problems similar to those in EXERCISE. They can also practice using the scientific calculator as they learned to in Check.

ADVANCED

3 We know that when an object is thrown vertically upward from a height of 0 m at a velocity of v m/s, the relation between the height y m of the object and the time x seconds since it was thrown is $y=-4.9x^2+vx$. So, when a ball is thrown vertically from a height of 0 m at a velocity of 9.8 m/s, find the time it takes to reach the highest point and the height of that highest point. Note that the air resistance can be ignored.

$y=-4.9x^2+9.8x=-4.9(x^2-2x)=-4.9\{(x-1)^2-1^2\}=-4.9(x-1)^2+4.9$

Therefore, the graph of this function is convex upward, the axis is $x=1$, and the vertex is $(1, 4.9)$.

Thus, the ball takes 1 second to reach its highest point, and the height of that highest point is 4.9 m.

1 second, 4.9 m

check

Press ⓞ, select [Equation], press OK

Select [Polynomial], press OK, select [ax²+bx+c], press OK

⊖ ④ ⊙ ⑨ EXE ⑨ ⊙ ⑧ EXE ⓪ EXE EXE

Calculate Statistics Distribution Spreadsheet Table Equation | Simul Equation Polynomial Solver | ax²+bx+c ax³+bx²+cx+d ax⁴+bx³+cx²+dx+e | ax²+bx+c −4.9x²+ 9.8x 0

EXE EXE EXE **Press ⇧ ⓧ, scan the QR code to display a graph.**

Max of y=ax²+bx+c x= 1 | Max of y=ax²+bx+c y= 49/10 | 1/1

ADVANCED

Practical problems have been included in several topics. Solutions using a scientific calculator are also presented as necessary.

Other marks

Ex.	Simple examples on how to apply equations and theorems
explanation	Formulas and their supplementary explanations
proof	Proofs and checks of mathematical formulas
EXTRA Info.	Knowledge and information on formulas and other supplementary information in other units
OTHER METHODS	Alternative solutions and different verification methods for previously presented problems

Calculator mark

Where to use the scientific calculator

Colors of fonts in the teaching materials

- In STUDY GUIDE, important mathematical terms and formulas are printed in blue.
- In PRACTICE and ADVANCED the answers are printed in red.
  (Separate data is also available without the red parts, so it can be used for exercises.)

Applicable models

The applicable model is fx-991CW.
(Instructions on how to do input are for the fx-991CW, but in many cases similar calculations can be done on other models.)

Related Links

- Information and educational materials relevant to scientific calculators can be viewed on the following site.
  https://edu.casio.com
- The following video can be viewed to learn about the multiple functions of scientific calculators.
  https://www.youtube.com/playlist?list=PLRgxo9AwbIZLurUCZnrbr4cLfZdqY6aZA

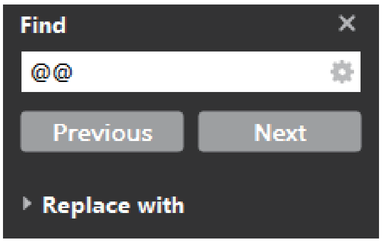
How to use PDF data

About types of data

- Data for all unit editions and data for each unit are available.
- For the above data, the PRACTICE and ADVANCED data without the answers in red is also available.

How to find where the scientific calculator is used

(1) Open a search window in the PDF Viewer.
(2) Type in "@@" as a search term.
(3) You can sequentially check where the calculator marks appear in the data.



How to search for a unit and section

(1) Search for units of data in all unit editions

- The data in all unit editions has a unit table of contents.
- Selecting a unit in the table of contents lets you jump to the first page of that unit.
- There is a bookmark on the first page of each unit, so you can jump from there also.

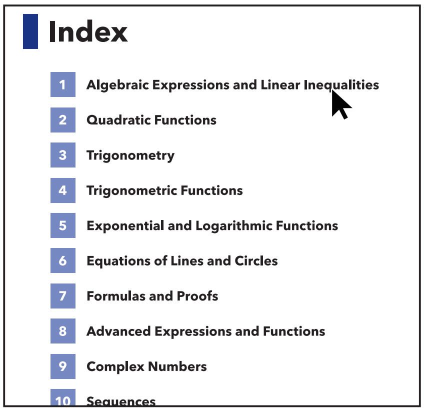
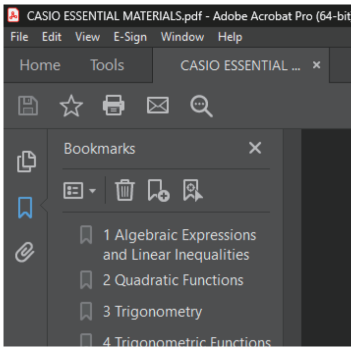


Table of contents of unit



Bookmark of unit

(2) Search for sections

- There are tables of contents for sections on the first page of units.
- Selecting a section in the table of contents takes you to the first page of that section.

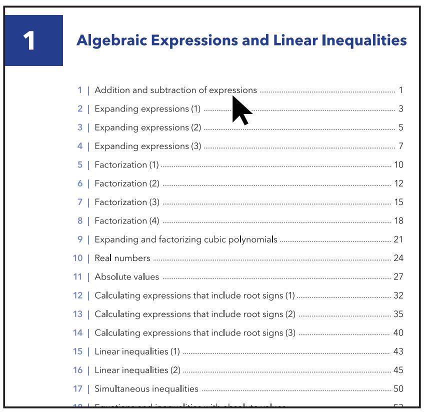


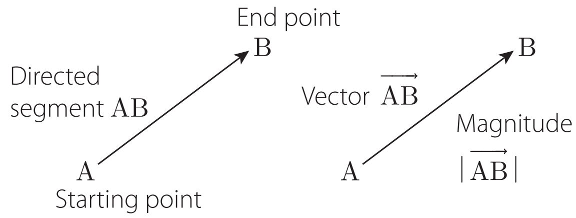
Table of contents of section

Vectors

TARGET To understand the meaning of and how to express vectors.

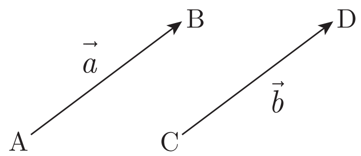
STUDY GUIDE

Directed segments and vectors



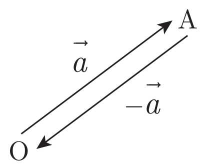
A **vector** is a quantity that has both direction and magnitude. Furthermore, a line segment with a specified direction, as shown in the diagram on the right, is called a **directed segment**. For the directed segment AB, A is called the **starting point**, B is called the **end point**, and the length of the segment AB is the **magnitude** of the directed segment AB. The position of the directed segment does not matter with a vector. A vector represented by a directed segment AB is written as $\overrightarrow{\mathrm{AB}}$, and the magnitude of the vector $\overrightarrow{\mathrm{AB}}$ is written as $|\overrightarrow{\mathrm{AB}}|$. A vector with a magnitude of 1 is called a **unit vector**.

Vectorial equality



When 2 vectors $\vec{a}$ and $\vec{b}$ have the same direction and equal magnitude, then we say $\vec{a}$ and $\vec{b}$ are **equal**, which is denoted as $\vec{a}=\vec{b}$.

Inverse vectors and null vectors



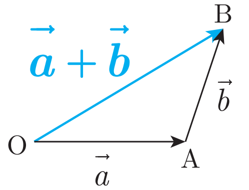
A vector whose magnitude is equal to, but whose direction is opposite to vector $\vec{a}$, is called the **inverse vector** of $\vec{a}$, which is denoted as $-\vec{a}$. When $\vec{a}=\overrightarrow{\mathrm{OA}}$, then $-\vec{a}=\overrightarrow{\mathrm{AO}}$ and $-\overrightarrow{\mathrm{OA}}=\overrightarrow{\mathrm{AO}}$.

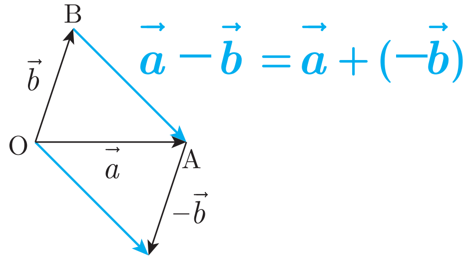
A vector $\overrightarrow{\mathrm{AB}}$ whose starting point A and end point B coincide with each other is regarded as a vector $\overrightarrow{\mathrm{AA}}$ having a magnitude of 0, and we call it a **null vector**, which is denoted as $\vec{0}$. Such that $\overrightarrow{\mathrm{AA}}=\vec{0}$.

Adding and subtracting vectors

Consider 2 vectors $\vec{a}=\overrightarrow{\mathrm{OA}}$ and $\vec{b}=\overrightarrow{\mathrm{AB}}$. We say that $\overrightarrow{\mathrm{OB}}$ is the **sum** of $\vec{a}$ and $\vec{b}$, which is denoted as $\vec{a}+\vec{b}$.
Specifically, $\overrightarrow{\mathrm{OA}}+\overrightarrow{\mathrm{AB}}=\overrightarrow{\mathrm{OB}}$.
Consider 2 vectors $\vec{a}=\overrightarrow{\mathrm{OA}}$ and $\vec{b}=\overrightarrow{\mathrm{OB}}$. We say that $\overrightarrow{\mathrm{BA}}$ is the **difference** between $\vec{a}$ and $\vec{b}$, which is denoted as $\vec{a}-\vec{b}$. Specifically, $\overrightarrow{\mathrm{OA}}-\overrightarrow{\mathrm{OB}}=\overrightarrow{\mathrm{BA}}$ and $\vec{a}-\vec{b}=\vec{a}+(-\vec{b})$.

Adding $\vec{a}+\vec{b}$ **Subtracting** $\vec{a}-\vec{b}=\vec{a}+(-\vec{b})$





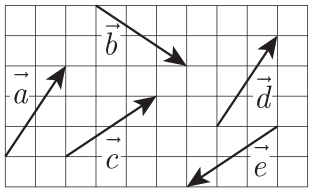
EXERCISE

1 Solve the following problems using the diagram on the right.

(1) Which are equal vectors? Use the equal sign to show them.

Find the vectors with the same direction and equal length.

$\vec{a} = \vec{d}$



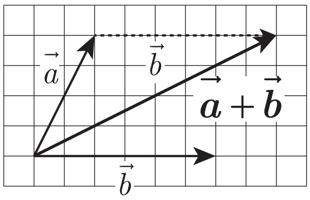
(2) Which are inverse vectors? Use the equal sign to show them.

Find the vectors with the opposite direction and equal length.

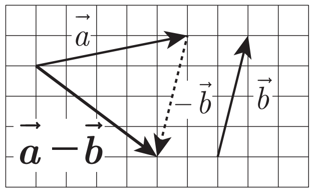
$\vec{c} = -\vec{e}$ or $-\vec{c} = \vec{e}$

2 Draw the following vectors.

(1) $\vec{a} + \vec{b}$

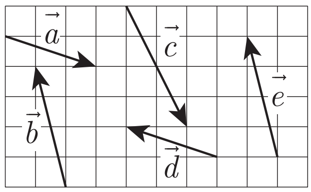


(2) $\vec{a} - \vec{b}$



PRACTICE

1 Solve the following problems using the diagram on the right.



(1) Which are equal vectors? Use the equal sign to show them.

Find the vectors with the same direction and equal length.

$\vec{b} = \vec{e}$

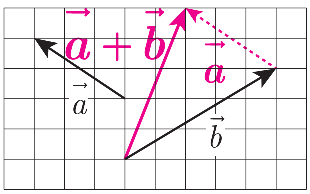
(2) Which are inverse vectors? Use the equal sign to show them.

Find the vectors with the opposite direction and equal length.

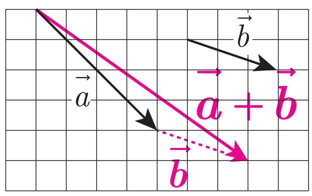
$\vec{a} = -\vec{d}$ **or** $-\vec{a} = \vec{d}$

2 Draw the following vectors.

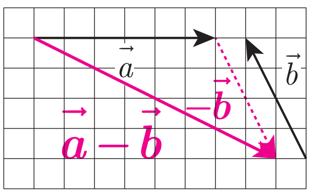
(1) $\vec{a} + \vec{b}$



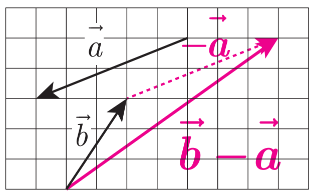
(2) $\vec{a} + \vec{b}$



(3) $\vec{a} - \vec{b}$



(4) $\vec{b} - \vec{a}$



Vector operations

TARGET To understand the operational rules and real multiples vectors.

STUDY GUIDE

Commutative law and associative law

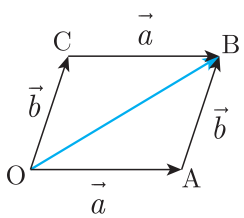
The following laws hold for adding vectors.

Commutative law $\vec{a}+\vec{b}=\vec{b}+\vec{a}$

Associative law $(\vec{a}+\vec{b})+\vec{c}=\vec{a}+(\vec{b}+\vec{c})$

The commutative law and associative law hold for adding vectors in the same way as for integer expressions. This can be explained as follows.

explanation [Commutative law]



For the parallelogram OABC, let $\overrightarrow{OA}=\vec{a},\overrightarrow{OC}=\vec{b}$.

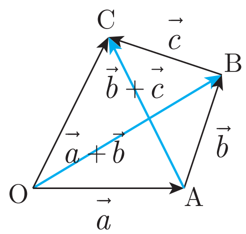
From $\overrightarrow{CB}=\vec{a},\overrightarrow{AB}=\vec{b}$, we get

$\overrightarrow{OB}=\overrightarrow{OA}+\overrightarrow{AB}=\vec{a}+\vec{b}$

$\overrightarrow{OB}=\overrightarrow{OC}+\overrightarrow{CB}=\vec{b}+\vec{a}$.

Therefore, we get $\vec{a}+\vec{b}=\vec{b}+\vec{a}$.

[Associative law]



For the quadrangle OABC, let $\overrightarrow{OA}=\vec{a},\overrightarrow{AB}=\vec{b},\overrightarrow{BC}=\vec{c}$.

From $\overrightarrow{OB}=\overrightarrow{OA}+\overrightarrow{AB}=\vec{a}+\vec{b}$ and

$\overrightarrow{AC}=\overrightarrow{AB}+\overrightarrow{BC}=\vec{b}+\vec{c}$, we get

$\overrightarrow{OC}=\overrightarrow{OB}+\overrightarrow{BC}=(\vec{a}+\vec{b})+\vec{c}$

$\overrightarrow{OC}=\overrightarrow{OA}+\overrightarrow{AC}=\vec{a}+(\vec{b}+\vec{c})$.

Therefore, we get $(\vec{a}+\vec{b})+\vec{c}=\vec{a}+(\vec{b}+\vec{c})$.

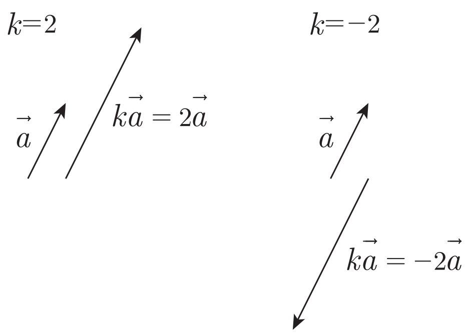
Real multiples

For a vector $\vec{a}$, which is not $\vec{0}$, and a real number k, we can determine $k\vec{a}$, the multiple of k of $\vec{a}$, as follows.

(a) If $k>0$ and the direction is the same as $\vec{a}$, then the vector has a magnitude that is a multiple of k.

(b) If $k<0$ and the direction is the opposite of $\vec{a}$, then the vector has magnitude that is a multiple of $|k|$.

(c) If $k=0$, then we get $\vec{0}$.

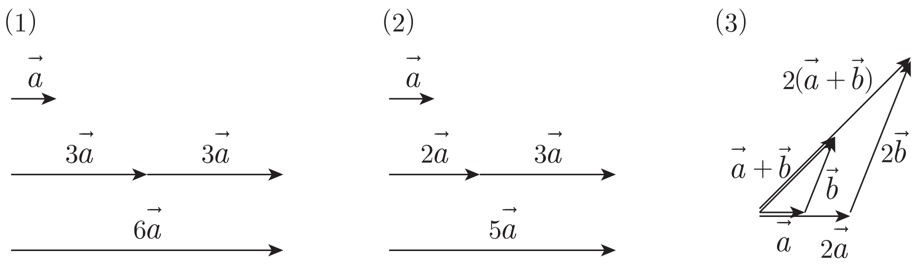


The following 3 laws hold when adding real multiples of vectors, given that k and l are real numbers.

(1) $k(l\vec{a})=(kl)\vec{a}$ (2) $(k+l)\vec{a}=k\vec{a}+l\vec{a}$ (3) $k(\vec{a}+\vec{b})=k\vec{a}+k\vec{b}$

explanation

From the following diagrams, we understand that the laws (1) to (3) hold for real multiples of vectors when k=2 and l=3.

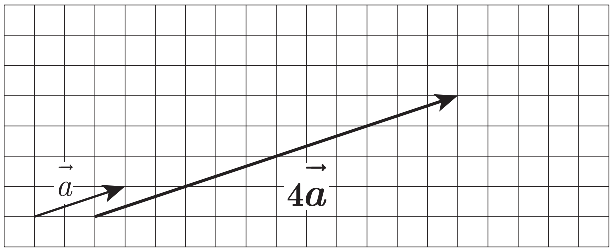


By applying the laws and properties we have learned so far, we can calculate the sums, differences, and real multiples of vectors the same as we calculate integer expressions.

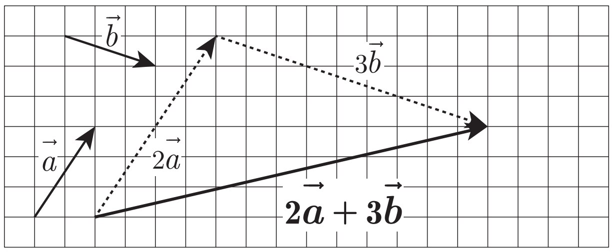
EXERCISE

1 Draw the following vectors.

(1) $4\vec{a}$



(2) $2\vec{a}+3\vec{b}$



2 Simplify the following expressions.

(1) $3\vec{a}+2\vec{b}-5\vec{a}+3\vec{b}=3\vec{a}-5\vec{a}+2\vec{b}+3\vec{b}$

$=(3-5)\vec{a}+(2+3)\vec{b}$

$=-2\vec{a}+5\vec{b}$

$\boldsymbol{-2\vec{a}+5\vec{b}}$

(2) $-8\vec{a}-7\vec{b}+10\vec{a}+\vec{b}=-8\vec{a}+10\vec{a}-7\vec{b}+\vec{b}$

$=(-8+10)\vec{a}+(-7+1)\vec{b}$

$=2\vec{a}-6\vec{b}$

$\boldsymbol{2\vec{a}-6\vec{b}}$

3 Show $\vec{a}$ and $\vec{b}$ such that $\vec{x}$ satisfies the following equations.

(1) $4\vec{x}-3\vec{a}+\vec{b}=2\vec{x}+5\vec{a}+9\vec{b}$

$4\vec{x}-2\vec{x}=5\vec{a}+3\vec{a}+9\vec{b}-\vec{b}$

$2\vec{x}=8\vec{a}+8\vec{b}$

$\vec{x}=4\vec{a}+4\vec{b}$

$\boldsymbol{\vec{x}=4\vec{a}+4\vec{b}}$

(2) $2\vec{x}+3\vec{a}-10\vec{b}=\vec{0}$

$2\vec{x}=-3\vec{a}+10\vec{b}$

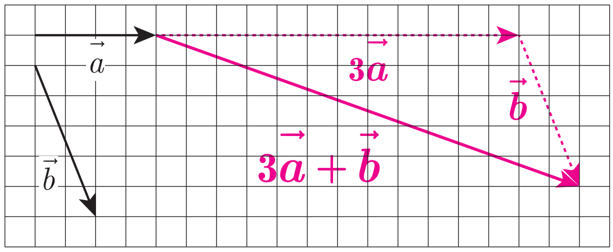
$\vec{x}=-\frac{3}{2}\vec{a}+5\vec{b}$

$\boldsymbol{\vec{x}=-\frac{3}{2}\vec{a}+5\vec{b}}$

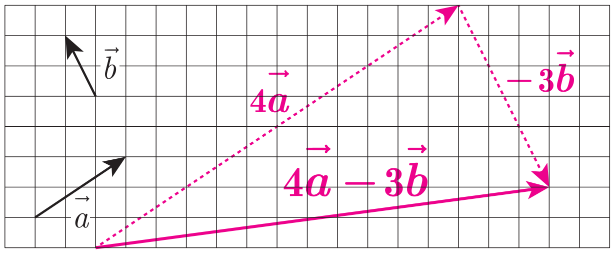
PRACTICE

1 Draw the following vectors.

(1) $3\vec{a}+\vec{b}$



(2) $4\vec{a}-3\vec{b}$



2 Simplify the following expressions.

(1) $-6\vec{a}+11\vec{b}-3\vec{a}-2\vec{b}$

$$=-6\vec{a}-3\vec{a}+11\vec{b}-2\vec{b}$$
$$=(-6-3)\vec{a}+(11-2)\vec{b}$$
$$=-9\vec{a}+9\vec{b}$$

$$-9\vec{a}+9\vec{b}$$

(2) $\frac{1}{3}(9\vec{a}-7\vec{b})-(-\vec{a}-2\vec{b})$

$$=3\vec{a}-\frac{7}{3}\vec{b}+\vec{a}+2\vec{b}$$
$$=3\vec{a}+\vec{a}-\frac{7}{3}\vec{b}+2\vec{b}$$
$$=(3+1)\vec{a}+\left(-\frac{7}{3}+2\right)\vec{b}$$
$$=4\vec{a}-\frac{1}{3}\vec{b}$$

$$4\vec{a}-\frac{1}{3}\vec{b}$$

3 Show $\vec{a}$ and $\vec{b}$ such that $\vec{x}$ satisfies the following equations.

(1) $2\vec{x}+8\vec{a}-4\vec{b}=\vec{0}$

$$2\vec{x}=-8\vec{a}+4\vec{b}$$
$$\vec{x}=-4\vec{a}+2\vec{b}$$

$$\vec{x}=-4\vec{a}+2\vec{b}$$

(2) $\vec{x}+5\vec{a}-2\vec{b}=3\vec{x}+11\vec{a}+2\vec{b}$

$$\vec{x}-3\vec{x}=11\vec{a}+2\vec{b}-5\vec{a}+2\vec{b}$$
$$-2\vec{x}=6\vec{a}+4\vec{b}$$
$$\vec{x}=-3\vec{a}-2\vec{b}$$

$$\vec{x}=-3\vec{a}-2\vec{b}$$

Components of vectors

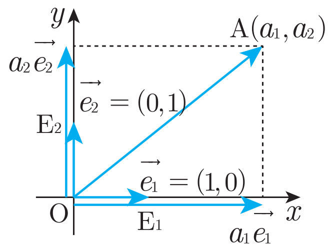
TARGET To understand how to calculate and draw vectors on a coordinate plane.

STUDY GUIDE

Component expression of vectors

On a plane with a coordinate system whose origin is O, we have 2 points $\mathrm{E}_1\,(1,0)$ and $\mathrm{E}_2(0,1)$ on the x axis and y axis. For this, there are 2 vectors $\vec{e_1}=\overrightarrow{\mathrm{OE}_1}, \vec{e_2}=\overrightarrow{\mathrm{OE}_2}$, called **fundamental vectors**. If we take an arbitrary point A, then its coordinates are $(a_1$ and $a_2)$, which are expressed as $\overrightarrow{\mathrm{OA}}=a_1\vec{e_1}+a_2\vec{e_2}$.

These a_1 and a_2 are respectively called the **x component** and **y component** of $\overrightarrow{\mathrm{OA}}$, which is expressed as $\overrightarrow{\mathrm{OA}}=(a_1,a_2)$. The method for showing this is called the **component expression** of $\overrightarrow{\mathrm{OA}}$.



Operations using components

Given $\vec{a}=(a_1,a_2), \vec{b}=(b_1,b_2)$. We can express sums, differences, equality, and real multiples as shown below.

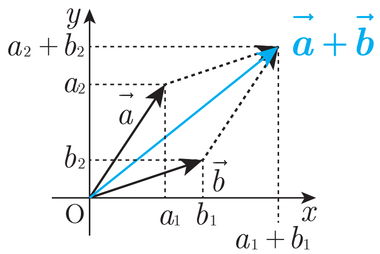
Adding $\quad \vec{a}+\vec{b}=(a_1,a_2)+(b_1,b_2)=(a_1+b_1,a_2+b_2)$

Subtracting $\quad \vec{a}-\vec{b}=(a_1,a_2)-(b_1,b_2)=(a_1-b_1,a_2-b_2)$

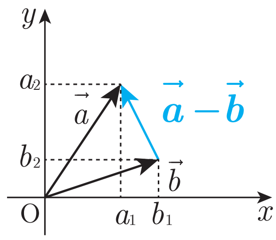
Equality $\quad \vec{a}=\vec{b} \iff (a_1,a_2)=(b_1,b_2) \iff a_1=b_1, a_2=b_2$

Real multiples $\quad k\vec{a}=k(a_1,a_2)=(ka_1,ka_2)$ $\quad$ **(k is a real number)**

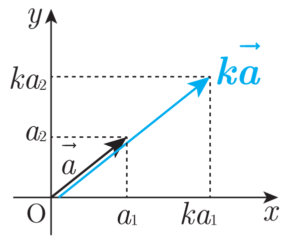
Adding



Subtracting



Real multiples



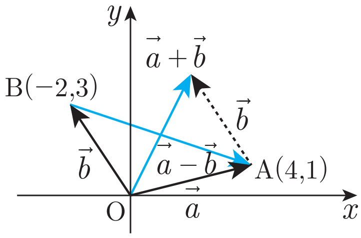
Components and magnitudes

Given $\vec{a}=(a_1,a_2)$. The following shows the magnitude of the vector whose components have been expressed.

$$|\vec{a}|=\sqrt{a_1{}^2+a_2{}^2}$$

EXERCISE

◆ On a coordinate plane with an origin O, take 2 points A$(4, 1)$ and B$(-2, 3)$.
Given $\vec{a} = \overrightarrow{\mathrm{OA}}, \vec{b} = \overrightarrow{\mathrm{OB}}$, solve the following problems.



(1) Express the components of $\vec{a}$ and $\vec{b}$.

From A $(4, 1)$ and B $(-2, 3)$, we get $\vec{a} = \overrightarrow{\mathrm{OA}} = (4,1), \vec{b} = \overrightarrow{\mathrm{OB}} = (-2,3)$.

$$\vec{a} = (4,1), \vec{b} = (-2,3)$$

(2) Express the components of $\vec{a}+\vec{b}, \vec{a}-\vec{b}$.

$\vec{a}+\vec{b} = (4-2, 1+3) = (2,4), \vec{a}-\vec{b} = (4+2, 1-3) = (6,-2)$

$$\vec{a}+\vec{b} = (2,4), \vec{a}-\vec{b} = (6,-2)$$

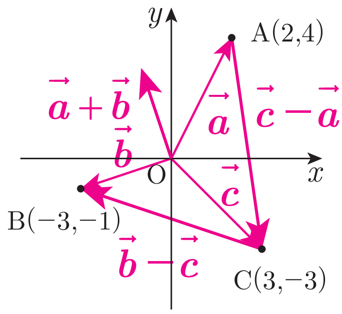
(3) Find the magnitudes of $\vec{a}$ and $\vec{b}$.

$|\vec{a}| = \sqrt{4^2+1^2} = \sqrt{17}, |\vec{b}| = \sqrt{(-2)^2+3^2} = \sqrt{13}$

$$|\vec{a}| = \sqrt{17}, |\vec{b}| = \sqrt{13}$$

PRACTICE

◆ On a coordinate plane with an origin O, take 3 points A $(2, 4)$, B $(-3, -1)$, and C $(3, -3)$. Given $\vec{a} = \overrightarrow{\mathrm{OA}}, \vec{b} = \overrightarrow{\mathrm{OB}}, \vec{c} = \overrightarrow{\mathrm{OC}}$, solve the following problems.



(1) Express the components of $\vec{a}$, $\vec{b}$, and $\vec{c}$.

From A $(2, 4)$, B $(-3, -1)$, C $(3, -3)$, we get
$\vec{a} = \overrightarrow{\mathrm{OA}} = (2,4), \vec{b} = \overrightarrow{\mathrm{OB}} = (-3,-1), \vec{c} = \overrightarrow{\mathrm{OC}} = (3,-3).$

$$\vec{a} = (2,4), \vec{b} = (-3,-1), \vec{c} = (3,-3)$$

(2) Express the components of $\vec{a}+\vec{b}, \vec{b}-\vec{c}, \vec{c}-\vec{a}$.

$\vec{a}+\vec{b} = (2-3, 4-1) = (-1,3)$
$\vec{b}-\vec{c} = (-3-3, -1+3) = (-6,2)$
$\vec{c}-\vec{a} = (3-2, -3-4) = (1,-7)$

$$\vec{a}+\vec{b} = (-1,3), \vec{b}-\vec{c} = (-6,2), \vec{c}-\vec{a} = (1,-7)$$

(3) Find the magnitudes of $\vec{a}$, $\vec{b}$, and $\vec{c}$.

$|\vec{a}| = \sqrt{2^2+4^2} = 2\sqrt{5}$
$|\vec{b}| = \sqrt{(-3)^2+(-1)^2} = \sqrt{10}$
$|\vec{c}| = \sqrt{3^2+(-3)^2} = 3\sqrt{2}$

$$|\vec{a}| = 2\sqrt{5}, |\vec{b}| = \sqrt{10}, |\vec{c}| = 3\sqrt{2}$$

Vector operations with components

TARGET To understand how to calculate vectors on a coordinate plane.

STUDY GUIDE

Operations using components

Given $\vec{a}=(a_1,a_2), \vec{b}=(b_1,b_2)$. Use the following relations to calculate using components of vectors.

Adding $\vec{a}+\vec{b}=(a_1,a_2)+(b_1,b_2)=(a_1+b_1,a_2+b_2)$

Subtracting $\vec{a}-\vec{b}=(a_1,a_2)-(b_1,b_2)=(a_1-b_1,a_2-b_2)$

Real multiples $k\vec{a}=k(a_1,a_2)=(ka_1,ka_2)$ (k **is a real number**)

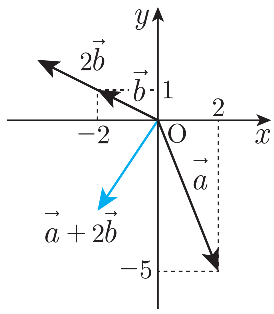
EXERCISE

◆ Given $\vec{a}=(2,-5), \vec{b}=(-2,1)$, solve the following problems.

(1) Express the components of $\vec{a}+2\vec{b}$.

$$\begin{aligned}\vec{a}+2\vec{b}&=(2,-5)+2(-2,1)\\&=(2-4,-5+2)\\&=(-2,-3)\end{aligned}$$

$\vec{a}+2\vec{b}=(-2,-3)$



(2) Find the magnitude of $\vec{a}+2\vec{b}$.

$$\begin{aligned}|\vec{a}+2\vec{b}|&=\sqrt{(-2)^2+(-3)^2}\\&=\sqrt{13}\end{aligned}$$

$|\vec{a}+2\vec{b}|=\sqrt{13}$

PRACTICE

◆ Given $\vec{a}=(1,4), \vec{b}=(-3,2)$, solve the following problems.

(1) Express the components of $2\vec{a}-\vec{b}$.

$$\begin{aligned}2\vec{a}-\vec{b}&=2(1,4)-(-3,2)\\&=(2+3,8-2)\\&=(5,6)\end{aligned}$$

$2\vec{a}-\vec{b}=(5,6)$

(2) Find the magnitude of $2\vec{a}-\vec{b}$.

$$\begin{aligned}|2\vec{a}-\vec{b}|&=\sqrt{5^2+6^2}\\&=\sqrt{61}\end{aligned}$$

$|2\vec{a}-\vec{b}|=\sqrt{61}$

EXTRA Info.

On the scientific calculator, use the Vector function to check basic operations.

In this section, we study elementary ways of using Vector, the vector operation function on the scientific calculator by calculating problems involving the components of vectors.

Vector: With this function, you can systematically do a variety of vector operations by recording vector components in the variable memory of the scientific calculator.

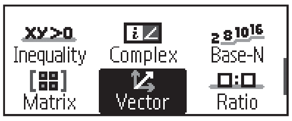
The purpose of this chapter is to approach the academic reasoning diversely by combining basic mathematic studies with vector operations using the Vector function.

EXERCISE

◆ Given $\vec{a}=(3,-4)$ and $\vec{b}=(-5,12)$, solve the following questions by using the Vector function of the scientific calculator.

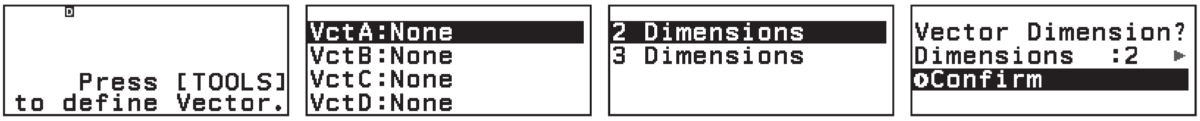
(*)Register the components of $\vec{a}$ and $\vec{b}$ respectively in the scientific calculator.

Press [HOME], select [Vector], press [OK]



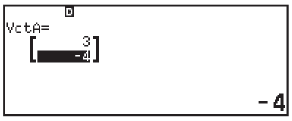
Register $\vec{a}=(3,-4)$.

Press [TOOLS], select [VctA], press [OK], select [Dimensions], press [OK], select [2 Dimensions], press [OK], select [Confirm], press [OK]



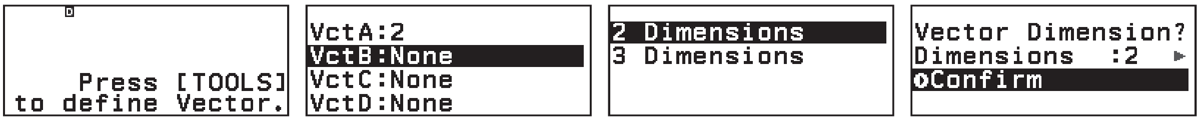
In the displayed screen, input the x component and the y component respectively.

③ [EXE] ⊖ ④ [EXE] [EXE]



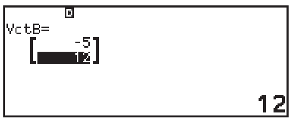
Register $\vec{b}=(-5,12)$.

Press [TOOLS], select [VctB], press [OK], select [Dimensions], press [OK], select [2 Dimensions], press [OK], select [Confirm], press [OK]



In the displayed screen, input the x component and the y component respectively.

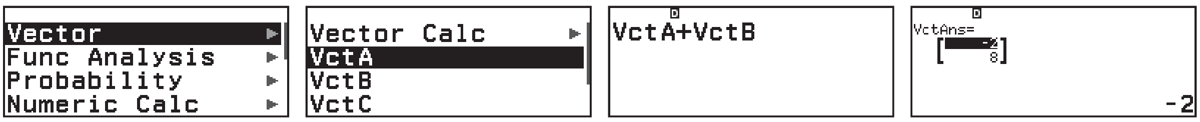
⊖ ⑤ [EXE] ① ② [EXE] [EXE]



(1) Express the components of $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$.

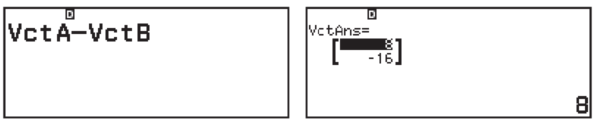
Calculate the components of $\vec{a}+\vec{b}$.

Press [CATALOG], select [Vector], press [OK], select [VctA], press [OK], ⊕, [CATALOG], select [Vector], press [OK], select [VctB], press [OK], [EXE]



Calculate the components of $\vec{a}-\vec{b}$.

[AC] [CATALOG] [OK] ⌄ [OK] ⊖ [CATALOG] [OK] ⌄ ⌄ [OK] [EXE]

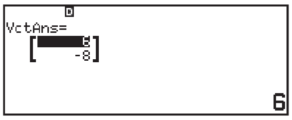


$$\vec{a}+\vec{b}=(-2,8),\ \vec{a}-\vec{b}=(8,-16)$$

(2) Express the components of $2\vec{a}$ and $-\vec{b}$.

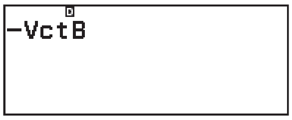
Calculate the components of $2\vec{a}$.

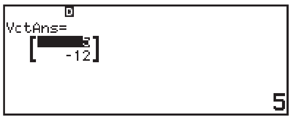
AC 2 ▣ OK ∨ OK EXE



Calculate the components of $-\vec{b}$.

AC ⊖ ▣ OK ∨ ∨ OK EXE



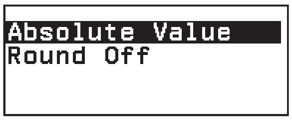


$$2\vec{a} = (6, -8), -\vec{b} = (5, -12)$$

(3) Find the magnitudes of $\vec{a}$ and $\vec{b}$.

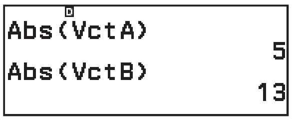
Calculate the value of $|\vec{a}|$.

Press AC, ▣, select [Numeric Calc], press OK, select [Absolute Value], press OK, ▣ OK ∨ OK ⟩ EXE



Calculate the value of $|\vec{b}|$.

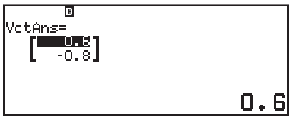
▣ ∨ ∨ ∨ OK OK ▣ OK ∨ ∨ OK ⟩ EXE



$$|\vec{a}| = 5, |\vec{b}| = 13$$

(4) Express the components of the unit vector of $\vec{a}$.

Press AC, ▣, select [Vector], press OK, select [Vector Calc], press OK, select [Unit Vector], press OK, ▣ OK ∨ OK ⟩ EXE



$$\left(\frac{3}{5}, -\frac{4}{5}\right)$$

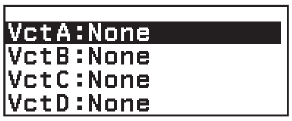
PRACTICE

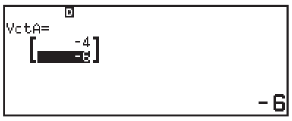
◆ Given $\vec{a}=(-4,-6)$ and $\vec{b}=(8,18)$ solve the following questions by using the Vector function of the scientific calculator.

(*)Register the components of $\vec{a}$ and $\vec{b}$ respectively in the scientific calculator.

Register $\vec{a}=(-4,-6)$.

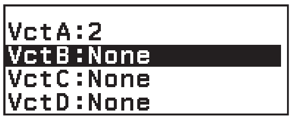
[⋯] [OK] [OK] [OK] [OK]   [(−)] [4] [EXE] [(−)] [6] [EXE] [EXE]

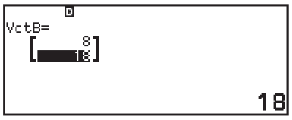




Register $\vec{b}=(8,18)$.

[⋯] [∨] [OK] [OK] [OK] [OK]   [8] [EXE] [1] [8] [EXE] [EXE]

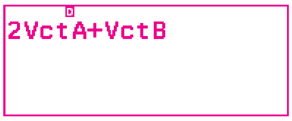




(1) Express the components of $2\vec{a}+\vec{b}$ and $\frac{1}{2}\vec{b}-\vec{a}$.

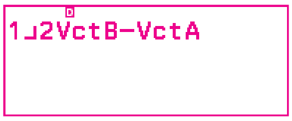
Calculate the components of $2\vec{a}+\vec{b}$.

[2] [📖] [OK] [∨] [OK] [+] [📖] [OK] [∨] [∨] [OK] [EXE]



Calculate the components of $\frac{1}{2}\vec{b}-\vec{a}$.

[AC]   [1] [⊟] [2] [📖] [OK] [∨] [∨] [OK] [−] [📖] [OK] [∨] [OK] [EXE]

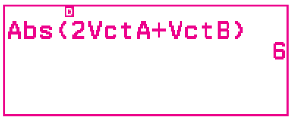


$$2\vec{a}+\vec{b}=(0,6),\ \frac{1}{2}\vec{b}-\vec{a}=(8,15)$$

(2) Find the magnitudes of $2\vec{a}+\vec{b}$ and $\frac{1}{2}\vec{b}-\vec{a}$.

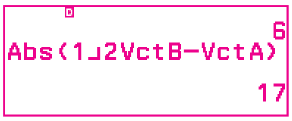
Calculate the value of $|2\vec{a}+\vec{b}|$.

[AC]   [📖] [∨] [∨] [∨] [OK] [OK] [2] [📖] [OK] [∨] [OK] [+] [📖] [OK] [∨] [∨] [OK] [)] [EXE]



Calculate the value of $\left|\frac{1}{2}\vec{b}-\vec{a}\right|$.

[📖] [∨] [∨] [∨] [OK] [OK] [1] [⊟] [2] [📖] [OK] [∨] [∨] [OK] [−] [📖] [OK] [∨] [OK] [)] [EXE]



$$|2\vec{a}+\vec{b}|=6,\ \left|\frac{1}{2}\vec{b}-\vec{a}\right|=17$$

Deformation of vectors

TARGET To understand how to deform vectors by considering the sums and differences of vectors.

STUDY GUIDE

Deformation of vectors

Use the following relations, to deform vectors.

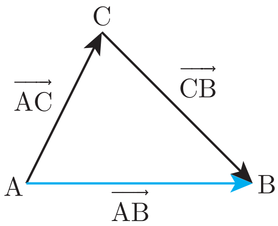
Adding $\overrightarrow{AC}+\overrightarrow{CB}=\overrightarrow{AB}$ **(Combined)**

Subtracting $\overrightarrow{AB}=\overrightarrow{CB}-\overrightarrow{CA}$ **(Divided)**

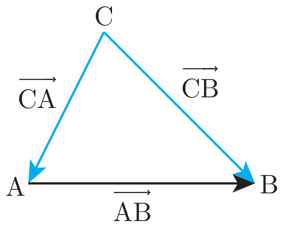
Negative direction $\overrightarrow{AB}=-\overrightarrow{BA}$ **(Starting point and end point are switched)**

Null vector $\overrightarrow{AA}=\vec{0}$ **(Starting point and end point are the same)**

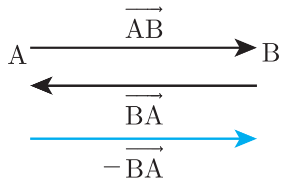
Adding



Subtracting

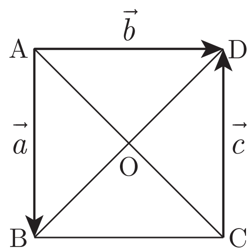


Negative direction



EXERCISE

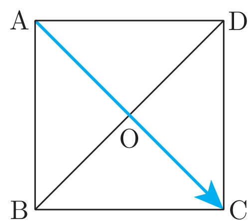
◆ In the diagram on the right, the quadrangle ABCD is a square with a point O at the intersection of the diagonals AC and BD. Given $\overrightarrow{AB}=\vec{a}, \overrightarrow{AD}=\vec{b}, \overrightarrow{CD}=\vec{c}$, solve the following problems.



(1) What vector is formed by $\vec{a}+\vec{b}$? Draw it on the right.

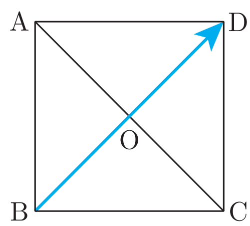
$\overrightarrow{BC}=\overrightarrow{AD}=\vec{b}$

$\vec{a}+\vec{b}=\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$



(2) Show $\overrightarrow{BD}$ in the diagram on the right as $\vec{a},\vec{b}$.

$\overrightarrow{BD}=\overrightarrow{BA}+\overrightarrow{AD}=-\overrightarrow{AB}+\overrightarrow{AD}=-\vec{a}+\vec{b}=\vec{b}-\vec{a}$



$\vec{b}-\vec{a}$

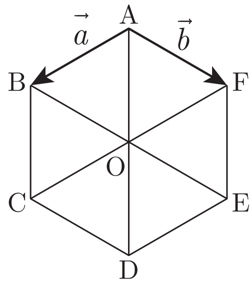
(3) What vector is formed by $\vec{a}+\vec{c}$?

From $\overrightarrow{CD}=\overrightarrow{BA}$, we get $\vec{a}+\vec{c}=\overrightarrow{AB}+\overrightarrow{CD}=\overrightarrow{AB}+\overrightarrow{BA}=\overrightarrow{AA}=\vec{0}$.

Null vectors

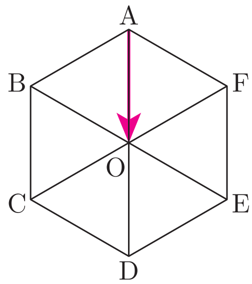
PRACTICE

◆ In the diagram on the right, the hexagon ABCDEF has a point O at the intersection of the diagonals. Given $\overrightarrow{\mathrm{AB}}=\vec{a}, \overrightarrow{\mathrm{AF}}=\vec{b}$, solve the following problems.



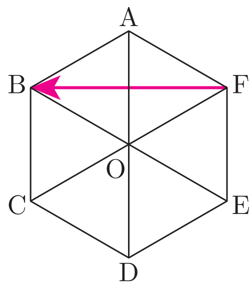
(1) What vector is formed by $\vec{a}+\vec{b}$? Draw it on the right.

$$\overrightarrow{\mathbf{BO}}=\overrightarrow{\mathbf{AF}}=\vec{b}$$
$$\vec{a}+\vec{b}=\overrightarrow{\mathbf{AB}}+\overrightarrow{\mathbf{BO}}=\overrightarrow{\mathbf{AO}}$$



(2) What vector is formed by $\vec{a}-\vec{b}$? Draw it on the right.

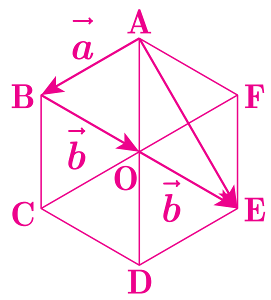
$$\overrightarrow{\mathbf{FA}}=-\overrightarrow{\mathbf{AF}}=-\vec{b}$$
$$\vec{a}-\vec{b}=\overrightarrow{\mathbf{AB}}+\overrightarrow{\mathbf{FA}}=\overrightarrow{\mathbf{FA}}+\overrightarrow{\mathbf{AB}}=\overrightarrow{\mathbf{FB}}$$



(3) Show $\overrightarrow{\mathrm{AE}}$ in the diagram as $\vec{a}, \vec{b}$.

$$\overrightarrow{\mathbf{AE}}=\overrightarrow{\mathbf{AB}}+\overrightarrow{\mathbf{BO}}+\overrightarrow{\mathbf{OE}}=\vec{a}+\vec{b}+\vec{b}=\vec{a}+2\vec{b}$$

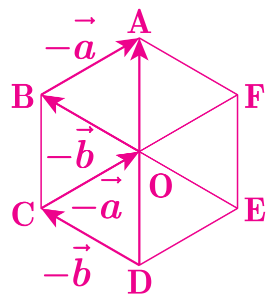
$\vec{a}+2\vec{b}$



(4) Show $\overrightarrow{\mathrm{DA}}$ in the diagram as $\vec{a}, \vec{b}$.

$$\overrightarrow{\mathbf{DA}}=\overrightarrow{\mathbf{DC}}+\overrightarrow{\mathbf{CO}}+\overrightarrow{\mathbf{OB}}+\overrightarrow{\mathbf{BA}}=-\vec{b}-\vec{a}-\vec{b}-\vec{a}=-2\vec{a}-2\vec{b}$$

$-2\vec{a}-2\vec{b}$



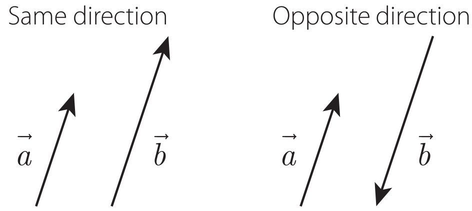
Parallel vector conditions

TARGET To understand parallel vectors.

STUDY GUIDE

Parallel vectors

When 2 vectors, $\vec{a}$ and $\vec{b}$, that are not $\vec{0}$, have the same directions or opposite directions, we say that $\vec{a}$ and $\vec{b}$ are **parallel**, and we write it as $\vec{a} \,//\, \vec{b}$. From the definition of parallel vectors and the definition of real multiples, we can say the following.



When $\vec{a} \neq \vec{0}, \vec{b} \neq \vec{0}$,
we get $\vec{a} \,//\, \vec{b} \iff \vec{a} = k\vec{b}$, in which there exists the real number k.

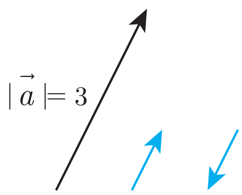
Component expression of parallel vectors

When $\vec{a} \neq \vec{0}, \vec{b} \neq \vec{0}, \vec{a} = (a_1, a_2), \vec{b} = (b_1, b_2)$, we can say the following about parallel vectors.

We get $\vec{a} \,//\, \vec{b} \iff (a_1, a_2) = k(b_1, b_2)$, in which there exists the real number k.

EXERCISE

1 When $|\vec{a}| = 3$, use $\vec{a}$ to show a vector with magnitude 1 that is parallel to $\vec{a}$.



We find 2 vectors, a vector with the same direction $\frac{1}{3}\vec{a}$ and a vector with the opposite direction $-\frac{1}{3}\vec{a}$.

$\frac{1}{3}\vec{a}, -\frac{1}{3}\vec{a}$

2 Determine a value for x such that the 2 vectors $\vec{a} = (x+1, 2x-3)$ and $\vec{b} = (2, 3)$ are parallel.

From $\vec{a} \,//\, \vec{b}$, we get $\vec{a} = k\vec{b}$, in which there exists the real number k.

From $(x+1, 2x-3) = k(2, 3)$, we get $\begin{cases} x+1 = 2k \\ 2x-3 = 3k \end{cases}$.

By solving these simultaneous equations, we get x=9 and k=5.

x=9

PRACTICE

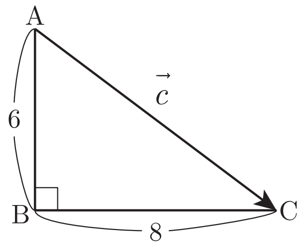
1 When $|\vec{a}| = 2$, use $\vec{a}$ to show a vector with magnitude 5 that is parallel to $\vec{a}$.

A vector with magnitude 1 that is parallel to $\vec{a}$ is $\frac{1}{2}\vec{a}$ and $-\frac{1}{2}\vec{a}$.

Therefore, the vectors we find are $\frac{5}{2}\vec{a}$ and $-\frac{5}{2}\vec{a}$.

$$\frac{5}{2}\vec{a}, -\frac{5}{2}\vec{a}$$

2 In the diagram on the right, AB=6, BC=8, and ∠B=90° in the right triangle ABC, such that $\overrightarrow{AC} = \vec{c}$. Now, use $\vec{c}$ to show the direction is the same as $\vec{c}$ and the magnitude of the vector has a magnitude of 1.



From the Pythagorean theorem, we get $|\vec{c}| = \mathrm{AC} = \sqrt{6^2 + 8^2} = 10$.

Therefore, the vector we find is $\frac{1}{|\vec{c}|}\vec{c} = \frac{1}{10}\vec{c}$.

$$\frac{1}{10}\vec{c}$$

3 Determine a value for x such that the 2 vectors $\vec{a} = (-3x+1, 2x-2)$ and $\vec{b} = (2, -1)$ are parallel.

From $\vec{a} \,/\!/\, \vec{b}$, we get $\vec{a} = k\vec{b}$, in which there exists the real number k.

From $(-3x+1, 2x-2) = k(2, -1)$, we get $\begin{cases} -3x+1 = 2k \\ 2x-2 = -k \end{cases}$.

By solving these simultaneous equations, we get x=3 and k=-4.

$$x=3$$

4 Determine a value for x such that the 2 vectors $\vec{a} = (2x, 8)$ and $\vec{b} = (4, x)$ are parallel.

From $\vec{a} \,/\!/\, \vec{b}$, we get $\vec{a} = k\vec{b}$, in which there exists the real number k.

From $(2x, 8) = k(4, x)$, we get $\begin{cases} 2x = 4k \\ 8 = kx \end{cases}$.

From $2x=4k$, we get $k = \frac{1}{2}x$, for which we substitute $8=kx$.

$8 = \frac{1}{2}x^2, x^2 = 16, x = \pm 4$

$$x=\pm 4$$

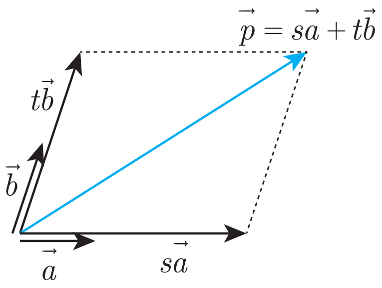
Vector resolution

TARGET To understand how to resolve vectors.

STUDY GUIDE

Vector resolution

When we have 2 vectors $\vec{a} \neq \vec{0}, \vec{b} \neq \vec{0}, \vec{a} \not\parallel \vec{b}$ on a plane, then an arbitrary vector $\vec{p}$ is expressed in a way using real numbers s and t as follows.



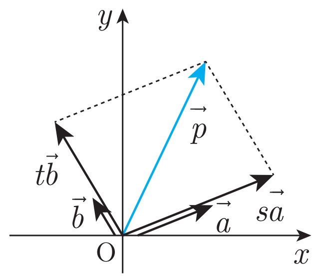
$$\vec{p} = s\vec{a} + t\vec{b}$$

And, $s\vec{a} + t\vec{b} = s'\vec{a} + t'\vec{b} \iff s = s', t = t'$

Specifically, $s\vec{a} + t\vec{b} = \vec{0} \iff s = t = 0$

Component expression for vector resolution

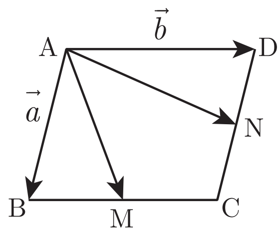
When we have 2 vectors $\vec{a} = (a_1, a_2), \vec{b} = (b_1, b_2)$, such that $\vec{a} \neq \vec{0}, \vec{b} \neq \vec{0}, \vec{a} \not\parallel \vec{b}$, then an arbitrary vector $\vec{p} = (x, y)$ is expressed only 1 way using real numbers s and t as follows.



$$(x, y) = s(a_1, a_2) + t(b_1, b_2)$$

EXERCISE

1 In the diagram on the right, the parallelogram ABCD has midpoints M and N on sides BC and CD; and when $\overrightarrow{AB} = \vec{a}, \overrightarrow{AD} = \vec{b}$, we can use $\vec{a}$ and $\vec{b}$ to express $\overrightarrow{AM}, \overrightarrow{AN}$.



$$\overrightarrow{AM} = \overrightarrow{AB} + \overrightarrow{BM} = \overrightarrow{AB} + \frac{1}{2}\overrightarrow{AD} = \vec{a} + \frac{1}{2}\vec{b}$$

$$\overrightarrow{AN} = \overrightarrow{AD} + \overrightarrow{DN} = \overrightarrow{AD} + \frac{1}{2}\overrightarrow{AB} = \frac{1}{2}\vec{a} + \vec{b}$$

$$\overrightarrow{AM} = \vec{a} + \frac{1}{2}\vec{b}, \overrightarrow{AN} = \frac{1}{2}\vec{a} + \vec{b}$$

2 Given $\vec{a} = (4, -3), \vec{b} = (1, 1)$. Use $\vec{a}$ and $\vec{b}$ to show $\vec{p} = (10, -11)$.

Let $\vec{p} = s\vec{a} + t\vec{b}$ (s and t being real numbers).

From $(10, -11) = s(4, -3) + t(1, 1)$, we get $\begin{cases} 10 = 4s + t \\ -11 = -3s + t \end{cases}$.

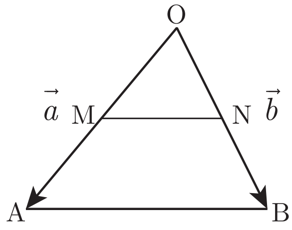
By solving these simultaneous equations, we get s=3 and t=−2.

Therefore, we get $\vec{p} = 3\vec{a} - 2\vec{b}$.

$$\vec{p} = 3\vec{a} - 2\vec{b}$$

PRACTICE

1 In the diagram on the right, $\triangle OAB$ has sides OA and OB with midpoints M and N, such that $\overrightarrow{OA} = \vec{a}, \overrightarrow{OB} = \vec{b}$. Now, use $\vec{a}$ and $\vec{b}$ to show the following vectors.



(1) $\overrightarrow{AN}$

$$\overrightarrow{AN} = \overrightarrow{AO} + \overrightarrow{ON} = -\overrightarrow{OA} + \frac{1}{2}\overrightarrow{OB} = -\vec{a} + \frac{1}{2}\vec{b}$$

$$-\vec{a} + \frac{1}{2}\vec{b}$$

(2) $\overrightarrow{NM}$

$$\overrightarrow{NM} = \overrightarrow{NO} + \overrightarrow{OM} = -\frac{1}{2}\overrightarrow{OB} + \frac{1}{2}\overrightarrow{OA} = -\frac{1}{2}\vec{b} + \frac{1}{2}\vec{a} = \frac{1}{2}\vec{a} - \frac{1}{2}\vec{b}$$

$$\frac{1}{2}\vec{a} - \frac{1}{2}\vec{b}$$

2 Solve the following problems.

(1) Given $\vec{a} = (1, -2), \vec{b} = (2, -1)$. Use $\vec{a}$ and $\vec{b}$ to show $\vec{p} = (-5, -2)$.

Let $\vec{p} = s\vec{a} + t\vec{b}$ (s and t being real numbers).

From $(-5, -2) = s(1, -2) + t(2, -1)$, we get $\begin{cases} -5 = s + 2t \\ -2 = -2s - t \end{cases}$.

By solving these simultaneous equations, we get $s=3$ and $t=-4$.
Therefore, we get $\vec{p} = 3\vec{a} - 4\vec{b}$.

$$\vec{p} = 3\vec{a} - 4\vec{b}$$

(2) Given $\vec{a} = (5, 2), \vec{b} = (-2, 4)$. Use $\vec{a}$ and $\vec{b}$ to show $\vec{p} = (1, 2)$.

Let $\vec{p} = s\vec{a} + t\vec{b}$ (s and t being real numbers).

From $(1, 2) = s(5, 2) + t(-2, 4)$, we get $\begin{cases} 1 = 5s - 2t \\ 2 = 2s + 4t \end{cases}$.

By solving these simultaneous equations, we get $s = \frac{1}{3}, t = \frac{1}{3}$.

Therefore, we get $\vec{p} = \frac{1}{3}\vec{a} + \frac{1}{3}\vec{b}$.

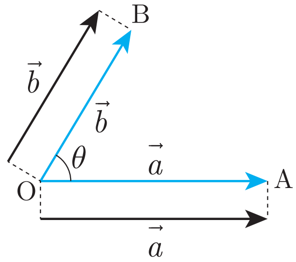
$$\vec{p} = \frac{1}{3}\vec{a} + \frac{1}{3}\vec{b}$$

Inner products of vectors

TARGET To understand the inner product of vectors.

STUDY GUIDE

Inner products of vectors



Decide on 1 point O, then, given $\vec{a}=\overrightarrow{\mathrm{OA}}, \vec{b}=\overrightarrow{\mathrm{OB}}$, when taking the points A and B, the angle formed by $\overrightarrow{\mathrm{OA}}$ and $\overrightarrow{\mathrm{OB}}$ is an angle θ, such that $0° \leq \theta \leq 180°$, which we call **the angle formed by vectors $\vec{a}, \vec{b}$**.

When given an angle, θ, formed by 2 vectors $\vec{a}$ and $\vec{b}$, which are not $\vec{0}$, the product of $|\vec{a}||\vec{b}|\cos\theta$ is called the **inner product** of $\vec{a}$ and $\vec{b}$, and expressed as $\vec{a}\cdot\vec{b}$. When $\vec{a}=\vec{0}$ or $\vec{b}=\vec{0}$, we can determine that the inner product of $\vec{a}$ and $\vec{b}$ is $\vec{a}\cdot\vec{b}=0$.

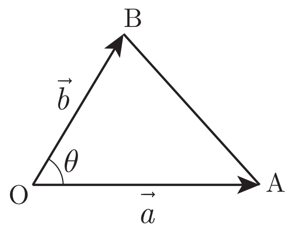
$$\boxed{\vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|\cos\theta}$$

Component expression of the inner products of vectors

When the components of 2 vectors $\vec{a}$ and $\vec{b}$ are expressed as $\vec{a}=(a_1,a_2), \vec{b}=(b_1,b_2)$, the inner product of $\vec{a}$ and $\vec{b}$ can be expressed by using the components as follows.

$$\boxed{\vec{a}\cdot\vec{b}=a_1b_1+a_2b_2}$$

explanation



Given $\vec{a}=\overrightarrow{\mathrm{OA}}, \vec{b}=\overrightarrow{\mathrm{OB}}$, we get $\angle\mathrm{AOB}=\theta$.

From the cosine formula, $\mathrm{AB}^2=\mathrm{OA}^2+\mathrm{OB}^2-2\cdot\mathrm{OA}\cdot\mathrm{OB}\cdot\cos\theta$ …we derive (i).

Now, from $\vec{a}=(a_1,a_2), \vec{b}=(b_1,b_2)$, we get

$\mathrm{AB}^2=|\vec{b}-\vec{a}|^2=(b_1-a_1)^2+(b_2-a_2)^2, \mathrm{OA}^2=|\overrightarrow{\mathrm{OA}}|^2=|\vec{a}|^2=a_1^2+a_2^2, \mathrm{OB}^2=|\overrightarrow{\mathrm{OB}}|^2=|\vec{b}|^2=b_1^2+b_2^2$.

And, from $\mathrm{OA}\cdot\mathrm{OB}\cdot\cos\theta=\vec{a}\cdot\vec{b}$, we can substitute this for (i), to get

$(b_1-a_1)^2+(b_2-a_2)^2=a_1^2+a_2^2+b_1^2+b_2^2-2\vec{a}\cdot\vec{b}$.

Rearranging this leads us to $\vec{a}\cdot\vec{b}=a_1b_1+a_2b_2$.

EXERCISE

1 Let the angle formed by $\vec{a}$ and $\vec{b}$, which are $|\vec{a}|=2, |\vec{b}|=5$, be θ. Find the values of the inner product of $\vec{a}\cdot\vec{b}$ when the magnitude of θ is as follows.

(1) $30°$

$\vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|\cos 30°=2\cdot 5\cdot\frac{\sqrt{3}}{2}=5\sqrt{3}$

$5\sqrt{3}$

(2) $90°$

$\vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|\cos 90°=2\cdot 5\cdot 0=0$

0

(3) $135°$

$\vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|\cos 135°=2\cdot 5\cdot\left(-\frac{\sqrt{2}}{2}\right)=-5\sqrt{2}$

$-5\sqrt{2}$

[2] When $\vec{a}=(2,-6), \vec{b}=(7,3)$, find the value of $\vec{a}\cdot\vec{b}$.

$\vec{a}\cdot\vec{b}=2\cdot 7+(-6)\cdot 3=-4$

-4

PRACTICE

[1] Solve the following problems.

(1) Find the value of the inner product of $\vec{a}\cdot\vec{b}$ when $|\vec{a}|=\sqrt{3}, |\vec{b}|=2\sqrt{3}$ and the angle formed by $\vec{a}$ and $\vec{b}$ is $60°$.

$$\vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|\cos 60^\circ=\sqrt{3}\cdot 2\sqrt{3}\cdot\frac{1}{2}=3$$

3

(2) Find the value of the inner product of $\vec{a}\cdot\vec{b}$ when $|\vec{a}|=3, |\vec{b}|=2$ and the angle formed by $\vec{a}$ and $\vec{b}$ is $150°$.

$$\vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|\cos 150^\circ=3\cdot 2\cdot\left(-\frac{\sqrt{3}}{2}\right)=-3\sqrt{3}$$

$-3\sqrt{3}$

[2] Solve the following problems.

(1) Find the value of $\vec{a}\cdot\vec{b}$ when $\vec{a}=(-3,7), \vec{b}=(4,2)$.

$$\vec{a}\cdot\vec{b}=-3\cdot 4+7\cdot 2=2$$

2

(2) Find the value of $\vec{a}\cdot\vec{b}$ when $\vec{a}=(6,-5), \vec{b}=(-4,3)$.

$$\vec{a}\cdot\vec{b}=6\cdot(-4)+(-5)\cdot 3=-39$$

-39

Calculating angles formed by vectors

TARGET To understand how to use the inner product of vectors to find the angle formed by vectors.

STUDY GUIDE

How to find the angle formed by vectors

When the angle formed by 2 vectors $\vec{a}$ and $\vec{b}$ is θ, the definition of the inner product leads us to the following formula.

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$$

EXERCISE

◆ Find the angle θ $(0°\leq\theta\leq180°)$ formed by $\vec{a}$ and $\vec{b}$ when $|\vec{a}|=3, |\vec{b}|=4, \vec{a}\cdot\vec{b}=6\sqrt{3}$.

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{6\sqrt{3}}{3\cdot 4} = \frac{\sqrt{3}}{2}$$

Since $0°\leq\theta\leq180°$, we get $\theta=30°$. **$30°$**

PRACTICE

1 Find the angle θ $(0°\leq\theta\leq180°)$ formed by $\vec{a}$ and $\vec{b}$ when $|\vec{a}|=4, |\vec{b}|=5, \vec{a}\cdot\vec{b}=-10\sqrt{2}$.

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{-10\sqrt{2}}{4\cdot 5} = -\frac{\sqrt{2}}{2}$$

Since $0°\leq\theta\leq180°$, we get $\theta=135°$.

$135°$

2 Solve the following problems with regards to $\vec{a}=(\sqrt{3},1), \vec{b}=(-3,\sqrt{3})$.

(1) Find the value of $\vec{a}\cdot\vec{b}$.

$$\vec{a}\cdot\vec{b} = \sqrt{3}\cdot(-3)+1\cdot\sqrt{3} = -2\sqrt{3}$$

$-2\sqrt{3}$

(2) Find the angle θ $(0°\leq\theta\leq180°)$ formed by $\vec{a}$ and $\vec{b}$.

$$|\vec{a}|=\sqrt{(\sqrt{3})^2+1^2}=2, |\vec{b}|=\sqrt{(-3)^2+(\sqrt{3})^2}=2\sqrt{3}, \vec{a}\cdot\vec{b}=-2\sqrt{3}$$

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{-2\sqrt{3}}{2\cdot 2\sqrt{3}} = -\frac{1}{2}$$

Since $0°\leq\theta\leq180°$, we get $\theta=120°$.

$120°$

EXTRA Info.

On the scientific calculator, use the Vector function to do calculations related to the inner products and angles formed by vectors.

EXERCISE

◆ Solve the following problems with regards to $\vec{a}=(-1, 3)$ and $\vec{b}=(1, 2)$.

(1) Find the value of $\vec{a}\cdot\vec{b}$.

$\vec{a}\cdot\vec{b}=-1\cdot 1+3\cdot 2=5$ **5**

(2) Find the angle θ $(0°\leq\theta\leq 180°)$ formed by $\vec{a}$ and $\vec{b}$.

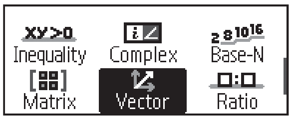
$$|\vec{a}|=\sqrt{(-1)^2+3^2}=\sqrt{10}, |\vec{b}|=\sqrt{1^2+2^2}=\sqrt{5}, \vec{a}\cdot\vec{b}=5$$

$$\cos\theta=\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}=\frac{5}{\sqrt{10}\cdot\sqrt{5}}=\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{2}$$

Since $0°\leq\theta\leq 180°$, we get $\theta=45°$. **45°**

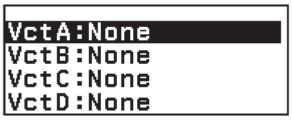
check

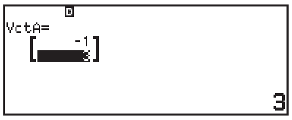
Press (HOME), select [Vector], press (OK)



Register $\vec{a}=(-1, 3)$.

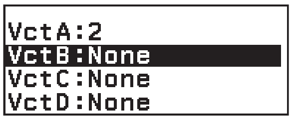
(TOOLS) (OK) (OK) (OK) (OK) (−) (1) (EXE) (3) (EXE) (EXE)

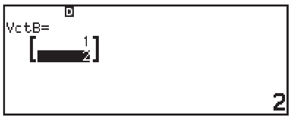




Register $\vec{b}=(1, 2)$.

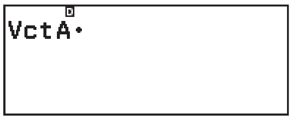
(TOOLS) (∨) (OK) (OK) (OK) (OK) (1) (EXE) (2) (EXE) (EXE)

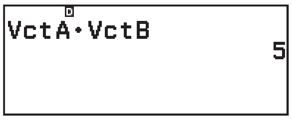




(1) Calculate the value of $\vec{a}\cdot\vec{b}$.

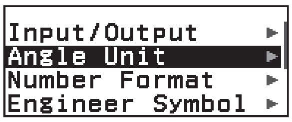
Press (CATALOG) (OK) (∨) (OK), (CATALOG), select [Vector], press (OK), select [Vector Calc], press (OK), select [Dot Product], press (OK), (CATALOG) (OK) (∨) (∨) (OK) (EXE)





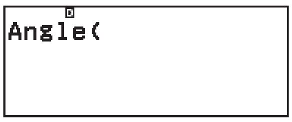
(2) Set the angle display to Degree.

Press (AC) (SETTINGS), select [Calc Settings], press (OK), select [Angle Unit], press (OK), select [Degree], press (OK), (AC)



Calculate the angle θ formed by $\vec{a}$ and $\vec{b}$.

Press (CATALOG), select [Vector], press (OK), select [Vector Calc], press (OK), select [Angle], press (OK), (CATALOG) (OK) (∨) (OK) (SHIFT) ( )) (CATALOG) (OK) (∨) (∨) (OK) ( )) (EXE)



PRACTICE

1 Solve the following problems with regards to $\vec{a}=(0,-5), \vec{b}=(-3,-\sqrt{3})$.

(1) Find the value of $\vec{a}\cdot\vec{b}$.

$$\vec{a}\cdot\vec{b}=0\cdot(-3)-5\cdot(-\sqrt{3})=5\sqrt{3}$$

$5\sqrt{3}$

(2) Find the angle θ $(0°\leq\theta\leq180°)$ formed by $\vec{a}$ and $\vec{b}$.

$$|\vec{a}|=\sqrt{0^2+(-5)^2}=5, |\vec{b}|=\sqrt{(-3)^2+(-\sqrt{3})^2}=2\sqrt{3}, \vec{a}\cdot\vec{b}=5\sqrt{3}$$

$$\cos\theta=\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}=\frac{5\sqrt{3}}{5\cdot2\sqrt{3}}=\frac{1}{2}$$

Since $0°\leq\theta\leq180°$, we get $\theta=60°$

$60°$

check

Register $\vec{a}=(0,-5)$.

[ooo] [OK] [OK] [OK] [OK] [0] [EXE] [(−)] [5] [EXE] [EXE]

Register $\vec{b}=(-3,-\sqrt{3})$.

[ooo] [∨] [OK] [OK] [OK] [OK] [(−)] [3] [EXE] [(−)] [√■] [3] [)] [EXE] [EXE]

(1) Calculate and verify the value of $\vec{a}\cdot\vec{b}$.

[📖] [OK] [∨] [OK] [📖] [OK] [OK] [OK] [📖] [OK] [∨] [∨] [OK] [EXE]
[Ans] [(−)] [5] [√■] [3] [)] [EXE]

(2) Calculate the angle θ formed by $\vec{a}$ and $\vec{b}$.

[📖] [OK] [OK] [∨] [∨] [OK] [📖] [OK] [∨] [OK] [⬆] [)] [📖] [OK] [∨] [∨] [OK] [)] [EXE]

ADVANCED

2 Find a vector $\vec{b}$ with a magnitude of $2\sqrt{6}$ such that $\vec{a}=(1,1)$ forms an angle of $30°$.

Given $\vec{b}=(x,y)$**, then from** $|\vec{b}|=2\sqrt{6}$ **, we get**

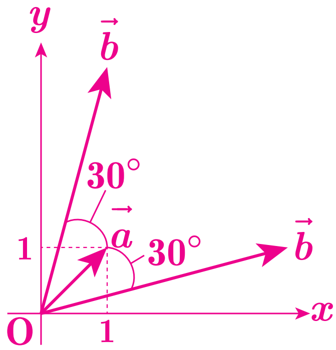
$|\vec{b}|^2=24 \Rightarrow x^2+y^2=24 \quad \ldots ①$

Also, the angle formed by $\vec{a}=(1,1)$ **and** $\vec{b}=(x,y)$ **is 30°,**

so that

$\vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|\cos 30° \Rightarrow 1\cdot x+1\cdot y=\sqrt{2}\cdot 2\sqrt{6}\cdot\dfrac{\sqrt{3}}{2}$

$\Rightarrow y=6-x \quad \ldots ②$



By substituting ② into ①, we get

$x^2+(6-x)^2=24 \Rightarrow x^2-6x+6=0$**, therefore** $x=3\pm\sqrt{3}$

From ②, we get $y=6-(3\pm\sqrt{3})=3\mp\sqrt{3}$

Therefore $(x,y)=(3\pm\sqrt{3},3\mp\sqrt{3})$

$\vec{b}=(3\pm\sqrt{3},3\mp\sqrt{3})$ **(double sign same order)**

check

Register $\vec{a}=(1,1)$**.**

○○○ OK OK OK OK 1 EXE 1 EXE EXE

Register $\vec{b}=(3+\sqrt{3},3-\sqrt{3})$**.**

○○○ ∨ OK OK OK OK 3 + √■ 3) EXE 3 − √■ 3) EXE EXE

Register $\vec{c}=(3-\sqrt{3},3+\sqrt{3})$**.**

○○○ ∨ ∨ OK OK OK OK 3 − √■ 3) EXE 3 + √■ 3) EXE EXE

Calculate the angle θ **formed by** $\vec{a}$ **and** $\vec{b}$**.**

📖 OK OK ∨ ∨ OK 📖 OK ∨ OK ⬆) 📖 OK ∨ ∨ OK) EXE

Calculate the angle θ **formed by** $\vec{a}$ **and** $\vec{c}$**.**

📖 OK OK ∨ ∨ OK 📖 OK ∨ OK ⬆) 📖 OK ∨ ∨ ∨ OK) EXE

Perpendicular vector conditions

TARGET To understand perpendicular vector conditions.

STUDY GUIDE

Perpendicular vector conditions

Regarding 2 vectors $\vec{a}=(a_1,a_2),\vec{b}=(b_1,b_2)$ that are not $\vec{0}$, when the angle formed by $\vec{a}$ and $\vec{b}$ is 90°, then $\vec{a}$ and $\vec{b}$ are said to be **perpendicular**, and are expressed as $\vec{a}\perp\vec{b}$. When $\vec{a}$ and $\vec{b}$ are perpendicular, the following holds.

$$\boxed{\vec{a}\perp\vec{b} \iff \vec{a}\cdot\vec{b}=0 \iff a_1b_1+a_2b_2=0}$$

explanation

If $\vec{a}$ and $\vec{b}$ are perpendicular, then the angle formed by $\vec{a}$ and $\vec{b}$ is 90°.

Therefore, because $\cos 90°=0$, we get $\vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|\cos 90^\circ=0$.

So, the inner product from the components is expressed as $\vec{a}\cdot\vec{b}=a_1b_1+a_2b_2$, giving us $a_1b_1+a_2b_2=0$.

EXERCISE

◆ Of the 3 vectors $\vec{a}=(4,-3),\vec{b}=(2,1),\vec{c}=(3,4)$, find the 2 vectors that are perpendicular to each other, and show it by using the "⊥" symbol.

$\vec{a}\cdot\vec{b}=4\cdot2-3\cdot1=5\neq0$

$\vec{b}\cdot\vec{c}=2\cdot3+1\cdot4=10\neq0$

$\vec{c}\cdot\vec{a}=3\cdot4+4\cdot(-3)=0$

Therefore, we get $\vec{c}\perp\vec{a}$.

$\vec{c}\perp\vec{a}$

PRACTICE

◆ Of the 3 vectors $\vec{a}=(1,4),\vec{b}=(6,-2),\vec{c}=(2,6)$, find the 2 vectors that are perpendicular to each other, and show it by using the "⊥" symbol.

$\vec{a}\cdot\vec{b}=1\cdot6+4\cdot(-2)=-2\neq0$

$\vec{b}\cdot\vec{c}=6\cdot2-2\cdot6=0$

$\vec{c}\cdot\vec{a}=2\cdot1+6\cdot4=26\neq0$

Therefore, we get $\vec{b}\perp\vec{c}$.

$\vec{b}\perp\vec{c}$

How to find perpendicular vectors

TARGET To understand how to find perpendicular vectors by using perpendicular vector conditions.

STUDY GUIDE

How to find perpendicular vectors

From the following conditions, we can find perpendicular vectors for 2 vectors $\vec{a} = (a_1, a_2), \vec{b} = (b_1, b_2)$ that are not $\vec{0}$.

$$\boldsymbol{\vec{a} \perp \vec{b} \iff \vec{a} \cdot \vec{b} = 0 \iff a_1 b_1 + a_2 b_2 = 0}$$

EXERCISE

1 Find a value for x such that the 2 vectors $\vec{a} = (x+1, -3), \vec{b} = (2, x-1)$ are perpendicular.

From $\vec{a} \perp \vec{b}$, we get $\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 = 0$.

Therefore, we get $(x+1) \cdot 2 - 3 \cdot (x-1) = 0$.

Solving this gives us x=5.

x=5

2 Find a vector $\vec{b}$ that is perpendicular to $\vec{a} = (3, -4)$ with a magnitude of 5.

Given $\vec{b} = (x, y)$.

From $\vec{a} \perp \vec{b}$, we get $\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 = 0$, and from $3 \cdot x - 4 \cdot y = 0$, we get $y = \frac{3}{4}x \quad \ldots(\text{i})$

And, from $|\vec{b}| = 5$, we get $|\vec{b}|^2 = 25$, such that $x^2 + y^2 = 25 \quad \ldots(\text{ii})$

By assigning (i) to (ii), we get $x^2 + \left(\frac{3}{4}x\right)^2 = 25, \frac{25}{16}x^2 = 25, x^2 = 16, x = \pm 4$.

Assign this value to (i). When x=4 then y=3, and when x=-4 then y=-3.

Therefore, $\vec{b} = (4, 3), (-4, -3)$

$$\boldsymbol{\vec{b} = (4, 3), (-4, -3)}$$

PRACTICE

1 Find a value for x such that the 2 vectors $\vec{a}=(-3,2x+3),\vec{b}=(x,1)$ are perpendicular.

From $\vec{a}\perp\vec{b}$, we get $\vec{a}\cdot\vec{b}=a_1b_1+a_2b_2=0$.
Therefore, we get $-3\cdot x+(2x+3)\cdot 1=0$.
Solving this gives us $x=3$.

$x=3$

2 Find a vector $\vec{b}$ that is perpendicular to $\vec{a}=(3,1)$ with a magnitude of $\sqrt{10}$.

Given $\vec{b}=(x,y)$.
From $\vec{a}\perp\vec{b}$, we get $\vec{a}\cdot\vec{b}=a_1b_1+a_2b_2=0$, and from $3\cdot x+1\cdot y=0$, we get
$y=-3x\quad \ldots$(i)
And, from $|\vec{b}|=\sqrt{10}$, we get $|\vec{b}|^2=10$, such that $x^2+y^2=10\quad \ldots$(ii)
By assigning (i) to (ii), we get $x^2+(-3x)^2=10, 10x^2=10, x^2=1, x=\pm 1$.
Assign this value to (i). When $x=1$ then $y=-3$, and when $x=-1$ then $y=3$.
Therefore, $\vec{b}=(1,-3),(-1,3)$

$\vec{b}=(1,-3),(-1,3)$

3 Find a unit vector $\vec{b}$ that is perpendicular to $\vec{a}=(1,1)$. Note that, a unit vector is a vector with a magnitude of 1.

Given $\vec{b}=(x,y)$.
From $\vec{a}\perp\vec{b}$, we get $\vec{a}\cdot\vec{b}=a_1b_1+a_2b_2=0$, and from $1\cdot x+1\cdot y=0$, we get
$y=-x\quad \ldots$(i)
And, from $|\vec{b}|=1$, we get $|\vec{b}|^2=1$, such that $x^2+y^2=1\quad \ldots$(ii)
By assigning (i) to (ii), we get $x^2+(-x)^2=1, 2x^2=1, x^2=\frac{1}{2}, x=\pm\frac{\sqrt{2}}{2}$.
Assign this value to (i). When $x=\frac{\sqrt{2}}{2}$ then $y=-\frac{\sqrt{2}}{2}$, and when $x=-\frac{\sqrt{2}}{2}$ then $y=\frac{\sqrt{2}}{2}$.
Therefore, $\vec{b}=\left(\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}\right),\left(-\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)$

$\vec{b}=\left(\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}\right),\left(-\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)$

Properties of inner products

TARGET To understand various calculation methods that use the properties of inner products.

STUDY GUIDE

Properties of inner products

The following properties hold for the inner product of 2 vectors $\vec{a}$ and $\vec{b}$. This can be considered in the same way as calculating the product of a literal expression.

(1) $\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{a}$ **Commutative law**

(2) $(k\vec{a})\cdot\vec{b}=\vec{a}\cdot(k\vec{b})=k(\vec{a}\cdot\vec{b})$ **Associative law**

(3) $\vec{a}\cdot(\vec{b}+\vec{c})=\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}$ **Distributive law**

$(\vec{a}+\vec{b})\cdot\vec{c}=\vec{a}\cdot\vec{c}+\vec{b}\cdot\vec{c}$

(4) $\vec{a}\cdot\vec{a}=|\vec{a}|^2$ **Magnitude of vectors**

(5) $|\vec{a}+\vec{b}|^2=|\vec{a}|^2+2\vec{a}\cdot\vec{b}+|\vec{b}|^2$ **Magnitude of the sum of vectors**

EXERCISE

◆ When $|\vec{a}|=2, |\vec{b}|=3, \vec{a}\cdot\vec{b}=1$, find the following values.

(1) $(\vec{a}+\vec{b})\cdot(\vec{a}+2\vec{b})$

$=|\vec{a}|^2+3\vec{a}\cdot\vec{b}+2|\vec{b}|^2$

$=2^2+3\cdot1+2\cdot3^2$

$=25$

$\underline{25}$

(2) $|\vec{a}-\vec{b}|$

$|\vec{a}-\vec{b}|^2=|\vec{a}|^2-2\vec{a}\cdot\vec{b}+|\vec{b}|^2$

$=2^2-2\cdot1+3^2$

$=11$

From $|\vec{a}-\vec{b}|\geq0$, we get $|\vec{a}-\vec{b}|=\sqrt{11}$.

$\underline{\sqrt{11}}$

PRACTICE

◆ When $|\vec{a}|=1, |\vec{b}|=2, |2\vec{a}+\vec{b}|=2\sqrt{3}$, find the following values.

(1) $\vec{a}\cdot\vec{b}$

$|2\vec{a}+\vec{b}|=2\sqrt{3}$

$|2\vec{a}+\vec{b}|^2=12$

$4|\vec{a}|^2+4\vec{a}\cdot\vec{b}+|\vec{b}|^2=12$

$4\cdot1+4\vec{a}\cdot\vec{b}+4=12$

$\vec{a}\cdot\vec{b}=1$

1

(2) $(2\vec{a}+\vec{b})\cdot(\vec{a}+2\vec{b})$

$(2\vec{a}+\vec{b})\cdot(\vec{a}+2\vec{b})$

$=2\vec{a}\cdot\vec{a}+2\vec{a}\cdot2\vec{b}+\vec{b}\cdot\vec{a}+\vec{b}\cdot2\vec{b}$

$=2|\vec{a}|^2+5\vec{a}\cdot\vec{b}+2|\vec{b}|^2$

$=2\cdot1+5\cdot1+2\cdot4$

$=15$

15

Position vectors, and internal dividing points and external dividing points

TARGET To understand position vectors of internal dividing points and external dividing points.

STUDY GUIDE

Position vectors

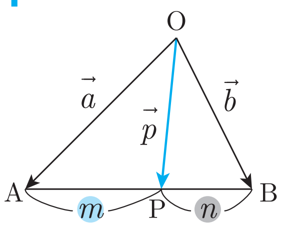
When a point O is fixed on a plane, the position of a point P on the same plane can be expressed as a vector $\overrightarrow{\mathrm{OP}}=\vec{p}$.
When doing this, we say $\vec{p}$ is the **position vector** of point P, which has the point O as a reference point. The position vector of a point P is expressed as $\mathrm{P}(\vec{p})$.

Position vectors of internal dividing points and external dividing points

For 2 points $\mathrm{A}(\vec{a})$ and $\mathrm{B}(\vec{b})$, where P is a point internally dividing the line segment AB into $m:n$, and Q is the point externally dividing the line segment AB into $m:n$, the respective position vectors $\vec{p}$ and $\vec{q}$ are expressed as follows.

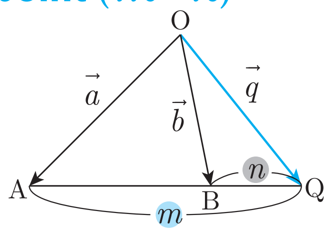
Internal dividing point

$$\vec{p}=\frac{n\vec{a}+m\vec{b}}{m+n}$$



External dividing point $(m \neq n)$

$$\vec{q}=\frac{-n\vec{a}+m\vec{b}}{m-n}$$



explanation

Position vectors of internal dividing points

$\mathrm{AP}:\mathrm{AB}=m:(m+n)$, such that $\overrightarrow{\mathrm{AP}}=\dfrac{m}{m+n}\overrightarrow{\mathrm{AB}}$

Therefore, $\vec{p}-\vec{a}=\dfrac{m}{m+n}(\vec{b}-\vec{a})$

This gives us $\vec{p}=\left(1-\dfrac{m}{m+n}\right)\vec{a}+\dfrac{m}{m+n}\vec{b}$.

$$=\frac{n\vec{a}+m\vec{b}}{m+n}$$

Position vectors of external dividing points

When $m>n$

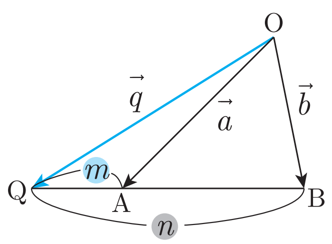
$\mathrm{AQ}:\mathrm{AB}=m:(m-n)$, such that $\overrightarrow{\mathrm{AQ}}=\dfrac{m}{m-n}\overrightarrow{\mathrm{AB}}$

Therefore, $\vec{q}-\vec{a}=\dfrac{m}{m-n}(\vec{b}-\vec{a})$

This gives us $\vec{q}=\left(1-\dfrac{m}{m-n}\right)\vec{a}+\dfrac{m}{m-n}\vec{b}$.

$$=\frac{-n\vec{a}+m\vec{b}}{m-n}$$

When $m<n$



EXERCISE

◆ For 2 points $\mathrm{A}(\vec{a})$ and $\mathrm{B}(\vec{b})$, express the respective position vectors $\vec{a}$ and $\vec{b}$ of the following points.

(1) Point $\mathrm{P}(\vec{p})$ internally dividing line segment AB by 3:2

Assign the values m=3 and n=2 to $\vec{p}=\dfrac{n\vec{a}+m\vec{b}}{m+n}$.

$$\vec{p}=\frac{2\vec{a}+3\vec{b}}{3+2}=\frac{2}{5}\vec{a}+\frac{3}{5}\vec{b}$$

$$\frac{2}{5}\vec{a}+\frac{3}{5}\vec{b}$$

(2) Point $\mathrm{Q}(\vec{q})$ externally dividing line segment AB by 3:2

Assign the values m=3 and n=2 to $\vec{q}=\dfrac{-n\vec{a}+m\vec{b}}{m-n}$.

$$\vec{q}=\frac{-2\vec{a}+3\vec{b}}{3-2}=-2\vec{a}+3\vec{b}$$

$$-2\vec{a}+3\vec{b}$$

PRACTICE

1 For 2 points $\mathrm{A}(\vec{a})$ and $\mathrm{B}(\vec{b})$, express the respective position vectors $\vec{a}$ and $\vec{b}$ of the following points.

(1) Point $\mathrm{P}(\vec{p})$ internally dividing line segment AB by 5:2

$$\vec{p}=\frac{2\vec{a}+5\vec{b}}{5+2}=\frac{2}{7}\vec{a}+\frac{5}{7}\vec{b}$$

$$\frac{2}{7}\vec{a}+\frac{5}{7}\vec{b}$$

(2) Point $\mathrm{Q}(\vec{q})$ externally dividing line segment AB by 1:4

$$\vec{q}=\frac{-4\vec{a}+\vec{b}}{1-4}=\frac{4}{3}\vec{a}-\frac{1}{3}\vec{b}$$

$$\frac{4}{3}\vec{a}-\frac{1}{3}\vec{b}$$

2 For 3 points $\mathrm{A}(\vec{a})$, $\mathrm{B}(\vec{b})$, and $\mathrm{C}(\vec{c})$, where a point $\mathrm{P}(\vec{p})$ internally divides line segment AB by 1:2, and a point $\mathrm{Q}(\vec{q})$ externally divides a line segment AC by 2:1, solve the following problems.

(1) Use $\vec{a}$, $\vec{b}$, and $\vec{c}$ to express the position vectors $\vec{p}$ and $\vec{q}$ of points P and Q respectively.

$$\vec{p}=\frac{2\vec{a}+\vec{b}}{1+2}=\frac{2}{3}\vec{a}+\frac{1}{3}\vec{b}$$

$$\vec{q}=\frac{-\vec{a}+2\vec{c}}{2-1}=-\vec{a}+2\vec{c}$$

$$\vec{p}=\frac{2}{3}\vec{a}+\frac{1}{3}\vec{b},\vec{q}=-\vec{a}+2\vec{c}$$

(2) Use $\vec{a}$, $\vec{b}$, and $\vec{c}$ to express the position vector $\vec{r}$ of the point R, which is externally dividing the line segment PQ by 1:2.

$$\vec{r}=\frac{-2\vec{p}+\vec{q}}{1-2}=2\vec{p}-\vec{q}=2\left(\frac{2}{3}\vec{a}+\frac{1}{3}\vec{b}\right)-(-\vec{a}+2\vec{c})=\frac{7}{3}\vec{a}+\frac{2}{3}\vec{b}-2\vec{c}$$

$$\vec{r}=\frac{7}{3}\vec{a}+\frac{2}{3}\vec{b}-2\vec{c}$$

EXTRA Info.

On the scientific calculator, use the Vector function to calculate the x and y components of the internal dividing point and external dividing point of the position vectors.

In this section, use the scientific calculator to calculate the x and y components of the internal dividing point and external dividing point to confirm that the formula holds from the positional relation on the coordinate plane.

EXERCISE

◆ Given $\vec{a}=(-6,8), \vec{b}=(2,4)$, show the components of the following position vectors for the 2 points $\mathrm{A}(\vec{a})$ and $\mathrm{B}(\vec{b})$.

(1) Point $\mathrm{P}(\vec{p})$ internally dividing line segment AB by 1:3

Assign the values m=1 and n=3 to $\vec{p}=\dfrac{n\vec{a}+m\vec{b}}{m+n}$.

$$\vec{p}=\frac{3\vec{a}+\vec{b}}{1+3}=\frac{3}{4}\vec{a}+\frac{\vec{b}}{4}=(-4,7)$$

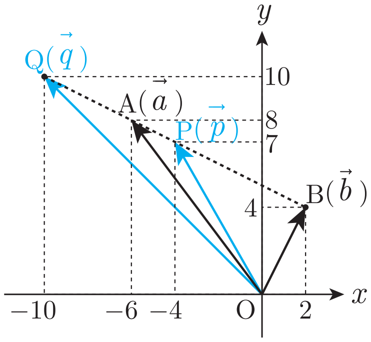
$(-4,7)$

(2) Point $\mathrm{Q}(\vec{q})$ externally dividing line segment AB by 1:3

Assign the values m=1 and n=−3 to $\vec{q}=\dfrac{n\vec{a}+m\vec{b}}{m+n}$.

$$\vec{q}=\frac{-3\vec{a}+\vec{b}}{1-3}=\frac{3}{2}\vec{a}-\frac{\vec{b}}{2}=(-10,10)$$

$(-10,10)$

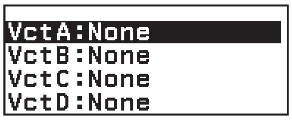


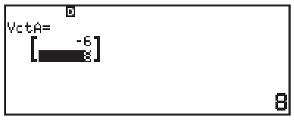
check

Press ⌂, select [Vector], press OK

Register $\vec{a}=(-6,8)$.

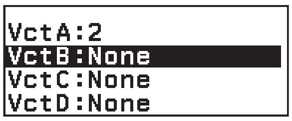
⋯ OK OK OK OK (−) 6 EXE 8 EXE EXE

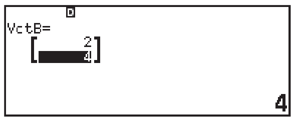




Register $\vec{b}=(2,4)$.

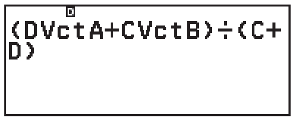
⋯ ∨ OK OK OK OK 2 EXE 4 EXE EXE





Input the formula for the internal dividing point and external dividing point.

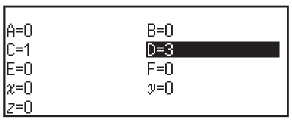
(⇧ 1 ▭ OK ∨ OK + ⇧ 6 ▭ OK ∨ ∨ OK) ÷ (⇧ 6 + ⇧ 1)



(1) Calculate the components of the point $\mathrm{P}(\vec{p})$.

Input the ratio values C=1 and D=3

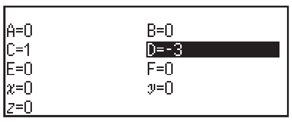
x ∨ 1 EXE > 3 EXE ↺ EXE



(2) Calculate the components of the point $\mathrm{Q}(\vec{q})$.

Input the ratio values C=1 and D=−3

↺ x ∨ 1 EXE > (−) 3 EXE ↺ EXE



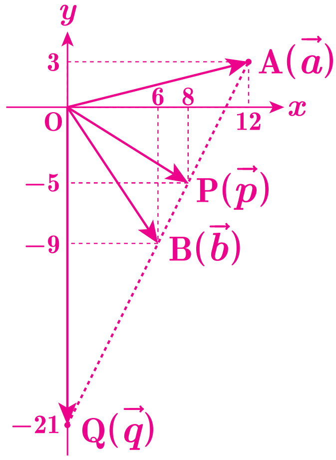
PRACTICE

◆ Given $\vec{a}=(12,3), \vec{b}=(6,-9)$, show the components of the following position vectors for the 2 points $\mathrm{A}(\vec{a})$ and $\mathrm{B}(\vec{b})$.

(1) Point $\mathrm{P}(\vec{p})$ internally dividing line segment AB by 2:1

Assign the values m=2 and n=1 to $\vec{p}=\dfrac{n\vec{a}+m\vec{b}}{m+n}$.

$$\vec{p}=\frac{\vec{a}+2\vec{b}}{2+1}=\frac{1}{3}\vec{a}+\frac{2}{3}\vec{b}=(8,-5)$$



(8,−5)

(2) Point $\mathrm{Q}(\vec{q})$ externally dividing line segment AB by 2:1

Assign the values m=2 and n=−1 to $\vec{q}=\dfrac{n\vec{a}+m\vec{b}}{m+n}$.

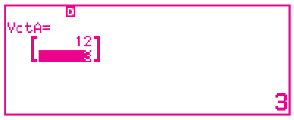
$$\vec{q}=\frac{-\vec{a}+2\vec{b}}{2-1}=-\vec{a}+2\vec{b}=(0,-21)$$

(0,−21)

check

Register $\vec{a}=(12,3)$.

(○○○) (OK) (OK) (OK) (OK)   (1) (2) (EXE) (3) (EXE) (EXE)

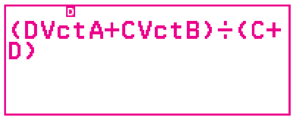


Register $\vec{b}=(6,-9)$.

(○○○) (∨) (OK) (OK) (OK) (OK)   (6) (EXE) (−) (9) (EXE) (EXE)

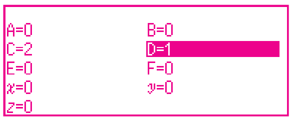
Input the formula for the internal dividing point and external dividing point.

(( ) (⇧) (1) (📖) (OK) (∨) (OK) (+) (⇧) (6) (📖) (OK) (∨) (∨) (OK) ( )) (÷) (( ) (⇧) (6) (+) (⇧) (1) ( ))



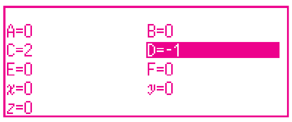
(1) Calculate the components of the point $\mathrm{P}(\vec{p})$.

(⇄x̄) (∨) (2) (EXE) (>) (1) (EXE)   (↺) (EXE)



(2) Calculate the components of the point $\mathrm{Q}(\vec{q})$.

(↺)   (⇄x̄) (∨) (2) (EXE) (>) (−) (1) (EXE)   (↺) (EXE)



Position vectors of the midpoint and center of gravity

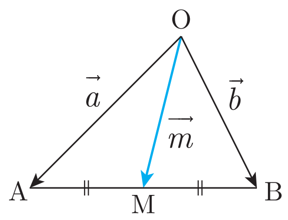
TARGET To understand position vectors of the midpoint and center of gravity.

STUDY GUIDE

Position vectors of the midpoint

For 2 points $\mathrm{A}(\vec{a})$ and $\mathrm{B}(\vec{b})$, given the midpoint M of the line segment AB, we can express its position vector $\vec{m}$ as shown on the right.

$$\vec{m}=\frac{\vec{a}+\vec{b}}{2}$$



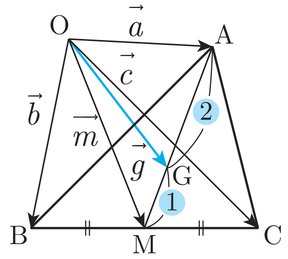
explanation

This can be derived by assigning $m=1$ and $n=1$ to the formula $\vec{p}=\dfrac{n\vec{a}+m\vec{b}}{m+n}$ for finding an internal dividing point.

Position vectors of the center of gravity

For 3 points $\mathrm{A}(\vec{a})$, $\mathrm{B}(\vec{b})$, and $\mathrm{C}(\vec{c})$ at the vertices of the triangle $\triangle\mathrm{ABC}$, we can express the position vector $\vec{g}$ of the center of gravity G as shown on the right.

$$\vec{g}=\frac{\vec{a}+\vec{b}+\vec{c}}{3}$$



explanation

In the diagram on the right, from AG:GM=2:1, we get

$$\vec{g}=\frac{\vec{a}+2\vec{m}}{2+1}=\frac{1}{3}\vec{a}+\frac{2}{3}\vec{m}=\frac{1}{3}\vec{a}+\frac{2}{3}\left(\frac{1}{2}\vec{b}+\frac{1}{2}\vec{c}\right)=\frac{1}{3}\vec{a}+\frac{1}{3}\vec{b}+\frac{1}{3}\vec{c}$$

EXERCISE

1 For 3 points $\mathrm{A}(\vec{a})$, $\mathrm{B}(\vec{b})$, and $\mathrm{C}(\vec{c})$, where a point $\mathrm{P}(\vec{p})$ internally divides line segment AB by 1:2, and a point $\mathrm{Q}(\vec{q})$ externally divides a line segment AC by 2:1, use $\vec{a}$, $\vec{b}$, and $\vec{c}$ to express the position vector $\vec{m}$ of midpoint M of line segment PQ.

$$\vec{p}=\frac{2\vec{a}+\vec{b}}{1+2}=\frac{2}{3}\vec{a}+\frac{1}{3}\vec{b},\ \vec{q}=\frac{-\vec{a}+2\vec{c}}{2-1}=-\vec{a}+2\vec{c}$$

$$\vec{m}=\frac{\vec{p}+\vec{q}}{2}=\frac{1}{2}\left(\frac{2}{3}\vec{a}+\frac{1}{3}\vec{b}-\vec{a}+2\vec{c}\right)=-\frac{1}{6}\vec{a}+\frac{1}{6}\vec{b}+\vec{c}$$

$$\vec{m}=-\frac{1}{6}\vec{a}+\frac{1}{6}\vec{b}+\vec{c}$$

2 Given $\triangle ABC$ with sides AB, BC, and CA with respective midpoints of P, Q, and R, the center of gravity G_1 of $\triangle ABC$ and the center of gravity G_2 of $\triangle PQR$ are the same. This is proved in the following way. Fill in the blanks [a] to [e] with the appropriate expression.

[Proof]

Given $A(\vec{a}), B(\vec{b}), C(\vec{c}), P(\vec{p}), Q(\vec{q}), R(\vec{r})$.

By expressing $\vec{p}$, $\vec{q}$, and $\vec{r}$ respectively as $\vec{a}$, $\vec{b}$, and $\vec{c}$, we get $\vec{p}$=[a], $\vec{q}$=[b], and $\vec{r}$=[c].

Here, the position vectors of the centers of gravity of $\triangle ABC$ and $\triangle PQR$ are $\vec{g_1}$ and $\vec{g_2}$ respectively.

By expressing $\vec{g_1}$ as $\vec{a}$, $\vec{b}$, and $\vec{c}$, we get $\vec{g_1}$=[d].

By expressing $\vec{g_2}$ as $\vec{a}$, $\vec{b}$, and $\vec{c}$, we get $\vec{g_2}$=[e]=$\frac{1}{3}$([a]+[b]+[c])=[d].

Therefore, because $\vec{g_1}=\vec{g_2}$, the center of gravity G_1 of $\triangle ABC$ and the center of gravity G_2 of $\triangle PQR$ are the same.

[a]… $\frac{1}{2}\vec{a}+\frac{1}{2}\vec{b}$, [b]… $\frac{1}{2}\vec{b}+\frac{1}{2}\vec{c}$, [c]… $\frac{1}{2}\vec{c}+\frac{1}{2}\vec{a}$, [d]… $\frac{\vec{a}+\vec{b}+\vec{c}}{3}$, [e]… $\frac{\vec{p}+\vec{q}+\vec{r}}{3}$

PRACTICE

1 For 3 points $A(\vec{a})$, $B(\vec{b})$, and $C(\vec{c})$, where the midpoint of line segment AB is $P(\vec{p})$ and $Q(\vec{q})$ externally divides line segment BC by 3:2, use $\vec{a}$, $\vec{b}$, and $\vec{c}$ to express the position vector $\vec{m}$ of midpoint M of line segment PQ.

$$\vec{p}=\frac{\vec{a}+\vec{b}}{2}=\frac{1}{2}\vec{a}+\frac{1}{2}\vec{b}, \vec{q}=\frac{-2\vec{b}+3\vec{c}}{3-2}=-2\vec{b}+3\vec{c}$$

$$\vec{m}=\frac{\vec{p}+\vec{q}}{2}=\frac{1}{2}\left(\frac{1}{2}\vec{a}+\frac{1}{2}\vec{b}-2\vec{b}+3\vec{c}\right)=\frac{1}{4}\vec{a}-\frac{3}{4}\vec{b}+\frac{3}{2}\vec{c}$$

$$\vec{m}=\frac{1}{4}\vec{a}-\frac{3}{4}\vec{b}+\frac{3}{2}\vec{c}$$

2 Given $\triangle ABC$ with sides AB, BC, and CA, which are divided internally by 2:1 at P, Q, and R respectively, prove that the center of gravity G_1 of $\triangle ABC$ and the center of gravity G_2 of $\triangle PQR$ are the same.

[Proof]

Given $A(\vec{a}), B(\vec{b}), C(\vec{c}), P(\vec{p}), Q(\vec{q}), R(\vec{r})$.

$$\vec{p}=\frac{1}{3}\vec{a}+\frac{2}{3}\vec{b}, \vec{q}=\frac{1}{3}\vec{b}+\frac{2}{3}\vec{c}, \vec{r}=\frac{1}{3}\vec{c}+\frac{2}{3}\vec{a}$$

Here, the position vectors of the centers of gravity of $\triangle ABC$ and $\triangle PQR$ are $\vec{g_1}, \vec{g_2}$ respectively.

$$\vec{g_1}=\frac{\vec{a}+\vec{b}+\vec{c}}{3}$$

$$\vec{g_2}=\frac{\vec{p}+\vec{q}+\vec{r}}{3}=\frac{1}{3}\left(\frac{1}{3}\vec{a}+\frac{2}{3}\vec{b}+\frac{1}{3}\vec{b}+\frac{2}{3}\vec{c}+\frac{1}{3}\vec{c}+\frac{2}{3}\vec{a}\right)=\frac{\vec{a}+\vec{b}+\vec{c}}{3}$$

Therefore, because $\vec{g_1}=\vec{g_2}$, the center of gravity G_1 of $\triangle ABC$ and the center of gravity G_2 of $\triangle PQR$ are the same.

Conditions of figures

TARGET To understand the conditions for 3 points to be on a straight line.

STUDY GUIDE

Parallel vector conditions

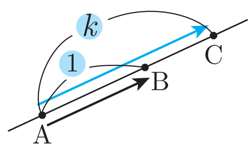
For 2 vectors, $\vec{a}$ and $\vec{b}$ that are not $\vec{0}$, when $\vec{a}$ and $\vec{b}$ are parallel, then the following holds.

When $\vec{a} \neq \vec{0}, \vec{b} \neq \vec{0}$,
we get $\vec{a} \parallel \vec{b} \iff \vec{a} = k\vec{b}$, in which there exists the real number k.

Conditions for 3 points to be on a straight line

For 3 different points A, B, and C, when the 3 points A, B, and C are on a straight line, $\overrightarrow{\mathrm{AC}}$ can be expressed as a real multiple of $\overrightarrow{\mathrm{AB}}$. This condition is called a **collinear condition**.

Given 3 points, A, B, and C, are on a straight line.
We get $\iff$ $\overrightarrow{\mathrm{AC}} = k\overrightarrow{\mathrm{AB}}$, in which there exists the real number k.



EXERCISE

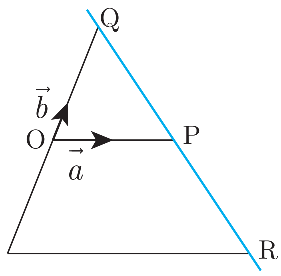
1 When $\overrightarrow{\mathrm{OP}} = 2\vec{a}, \overrightarrow{\mathrm{OQ}} = 3\vec{b}, \overrightarrow{\mathrm{OR}} = 4\vec{a} - 3\vec{b}$, then the 3 points P, Q, and R are on a straight line. This is proved in the following way. Fill in the blanks [a] to [e] with the appropriate term or expression. Provided that $\vec{a} \neq \vec{0}, \vec{b} \neq \vec{0}, \vec{a} \not\parallel \vec{b}$.

[Proof]

$\overrightarrow{\mathrm{PQ}} = \overrightarrow{\mathrm{OQ}} -$[a]$=$[b]

$\overrightarrow{\mathrm{PR}} =$[c]$-$[d]$=$[e]

Therefore, because $\overrightarrow{\mathrm{PR}} = -\overrightarrow{\mathrm{PQ}}$, the 3 points P, Q, and R are on a straight line.



[a]…$\overrightarrow{\mathrm{OP}}$, [b]…$3\vec{b} - 2\vec{a}$, [c]…$\overrightarrow{\mathrm{OR}}$, [d]…$\overrightarrow{\mathrm{OP}}$, [e]…$2\vec{a} - 3\vec{b}$

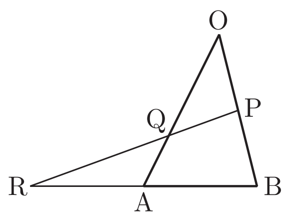
2 For $\triangle\mathrm{OAB}$, when side OB has a midpoint P, and side OA is divided internally by $2{:}1$ at point Q, and side AB is divided externally by $1{:}2$ at point R, then, that the 3 points P, Q, and R are on a straight line is proved in the following way. Fill in the blanks [a] to [e] with the appropriate term or expression.

[Proof]

$\overrightarrow{\mathrm{OP}} = \frac{1}{2}\overrightarrow{\mathrm{OB}}$, $\overrightarrow{\mathrm{OQ}} =$[a], $\overrightarrow{\mathrm{OR}} =$[b]

Therefore, we get $\overrightarrow{\mathrm{PQ}} = \overrightarrow{\mathrm{OQ}} - \overrightarrow{\mathrm{OP}} =$[c], $\overrightarrow{\mathrm{PR}} = \overrightarrow{\mathrm{OR}} - \overrightarrow{\mathrm{OP}} =$[d].

This gives us, $\overrightarrow{\mathrm{PR}} =$[e], so the 3 points P, Q, and R are on a straight line.

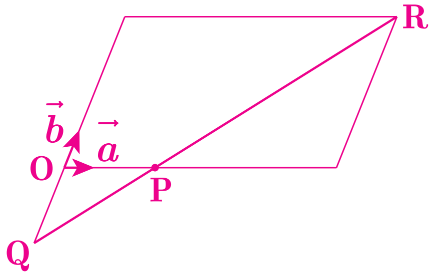


[a]…$\frac{2}{3}\overrightarrow{\mathrm{OA}}$, [b]…$2\overrightarrow{\mathrm{OA}} - \overrightarrow{\mathrm{OB}}$, [c]…$\frac{2}{3}\overrightarrow{\mathrm{OA}} - \frac{1}{2}\overrightarrow{\mathrm{OB}}$, [d]…$2\overrightarrow{\mathrm{OA}} - \frac{3}{2}\overrightarrow{\mathrm{OB}}$, [e]…$3\overrightarrow{\mathrm{PQ}}$

PRACTICE

1 Prove that when $\overrightarrow{OP} = 3\vec{a}, \overrightarrow{OQ} = -2\vec{b}, \overrightarrow{OR} = 9\vec{a} + 4\vec{b}$, then the 3 points P, Q, and R are on a straight line. Provided that $\vec{a} \neq \vec{0}, \vec{b} \neq \vec{0}, \vec{a} \not\parallel \vec{b}$.

[Proof]



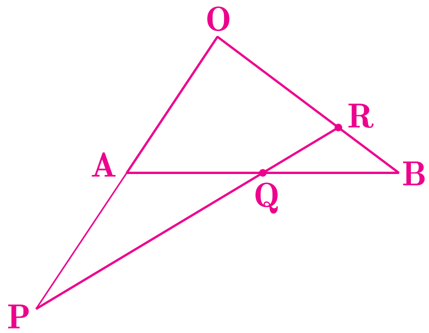
$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = -2\vec{b} - 3\vec{a}$$

$$\overrightarrow{PR} = \overrightarrow{OR} - \overrightarrow{OP} = (9\vec{a} + 4\vec{b}) - 3\vec{a} = 6\vec{a} + 4\vec{b}$$

Therefore, because $\overrightarrow{PR} = -2\overrightarrow{PQ}$, the 3 points P, Q, and R are on a straight line.

2 For $\triangle OAB$, when side OA is divided externally by 2:1 at point P, and side AB has a midpoint Q, and side OB is divided internally by 2:1 at point R, prove that the 3 points P, Q, and R are on a straight line.

[Proof]



$$\overrightarrow{OP} = 2\overrightarrow{OA}, \overrightarrow{OQ} = \frac{1}{2}\overrightarrow{OA} + \frac{1}{2}\overrightarrow{OB}, \overrightarrow{OR} = \frac{2}{3}\overrightarrow{OB}$$

Therefore,

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \frac{1}{2}\overrightarrow{OA} + \frac{1}{2}\overrightarrow{OB} - 2\overrightarrow{OA}$$
$$= -\frac{3}{2}\overrightarrow{OA} + \frac{1}{2}\overrightarrow{OB}$$

$$\overrightarrow{PR} = \overrightarrow{OR} - \overrightarrow{OP} = \frac{2}{3}\overrightarrow{OB} - 2\overrightarrow{OA} = -2\overrightarrow{OA} + \frac{2}{3}\overrightarrow{OB}$$

This gives us, $\overrightarrow{PR} = \frac{4}{3}\overrightarrow{PQ}$, so the 3 points P, Q, and R are on a straight line.

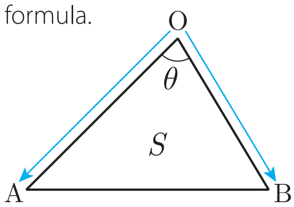
Formula for the area of a triangle

TARGET To understand how to find the area of a triangle by using vectors.

STUDY GUIDE

Formula for vectors and the area of a triangle

Given the area of $\triangle OAB$ is S, and $\overrightarrow{OA}=(a_1,a_2), \overrightarrow{OB}=(b_1,b_2)$, we can derive the following formula.



$$S=\frac{1}{2}\sqrt{|\overrightarrow{OA}|^2|\overrightarrow{OB}|^2-(\overrightarrow{OA}\cdot\overrightarrow{OB})^2}$$
$$=\frac{1}{2}|a_1b_2-a_2b_1|$$

explanation

Given $\angle AOB=\theta$.

$$S=\frac{1}{2}OA\cdot OB\cdot\sin\theta=\frac{1}{2}|\overrightarrow{OA}||\overrightarrow{OB}|\sin\theta \quad \triangleleft \quad \sin^2\theta+\cos^2\theta=1, \sin\theta>0, \sin\theta=\sqrt{1-\cos^2\theta}$$

$$=\frac{1}{2}|\overrightarrow{OA}||\overrightarrow{OB}|\sqrt{1-\cos^2\theta}=\frac{1}{2}\sqrt{|\overrightarrow{OA}|^2|\overrightarrow{OB}|^2-|\overrightarrow{OA}|^2|\overrightarrow{OB}|^2\cos^2\theta}$$

$$=\frac{1}{2}\sqrt{|\overrightarrow{OA}|^2|\overrightarrow{OB}|^2-(|\overrightarrow{OA}||\overrightarrow{OB}|\cos\theta)^2}=\frac{1}{2}\sqrt{|\overrightarrow{OA}|^2|\overrightarrow{OB}|^2-(\overrightarrow{OA}\cdot\overrightarrow{OB})^2} \quad \triangleleft \quad |\overrightarrow{OA}||\overrightarrow{OB}|\cos\theta=\overrightarrow{OA}\cdot\overrightarrow{OB}$$

$$S=\frac{1}{2}\sqrt{|\overrightarrow{OA}|^2|\overrightarrow{OB}|^2-(\overrightarrow{OA}\cdot\overrightarrow{OB})^2}=\frac{1}{2}\sqrt{(a_1^2+a_2^2)(b_1^2+b_2^2)-(a_1b_1+a_2b_2)^2}$$

$$=\frac{1}{2}\sqrt{a_1^2b_2^2-2a_1b_1a_2b_2+a_2^2b_1^2}=\frac{1}{2}\sqrt{(a_1b_2-a_2b_1)^2}=\frac{1}{2}|a_1b_2-a_2b_1|$$

EXERCISE

1 For $\triangle OAB$, when $|\overrightarrow{OA}|=5, |\overrightarrow{OB}|=2, \overrightarrow{OA}\cdot\overrightarrow{OB}=6$, find the area S of $\triangle OAB$.

$$S=\frac{1}{2}\sqrt{|\overrightarrow{OA}|^2|\overrightarrow{OB}|^2-(\overrightarrow{OA}\cdot\overrightarrow{OB})^2}=\frac{1}{2}\sqrt{5^2\cdot2^2-6^2}=4$$

$S=4$

2 Given 3 points $A(-1,1)$, $B(1,-2)$, and $C(3,2)$ are vertices, find the area S of $\triangle ABC$.

$$\overrightarrow{AB}=(1+1,-2-1)=(2,-3), \overrightarrow{AC}=(3+1,2-1)=(4,1)$$

$$S=\frac{1}{2}|a_1b_2-a_2b_1|=\frac{1}{2}|2\cdot1-(-3)\cdot4|=7$$

$S=7$

PRACTICE

1 For $\triangle OAB$, when $|\overrightarrow{OA}|=3, |\overrightarrow{OB}|=2, \overrightarrow{OA}\cdot\overrightarrow{OB}=2$, find the area S of $\triangle OAB$.

$$S=\frac{1}{2}\sqrt{|\overrightarrow{OA}|^2|\overrightarrow{OB}|^2-(\overrightarrow{OA}\cdot\overrightarrow{OB})^2}=\frac{1}{2}\sqrt{3^2\cdot 2^2-2^2}=2\sqrt{2}$$

$$S=2\sqrt{2}$$

2 Given 3 points $A(5,3)$, $B(2,4)$, and $C(7,-1)$ are vertices, find the area S of $\triangle ABC$.

$$\overrightarrow{AB}=(2-5,4-3)=(-3,1), \overrightarrow{AC}=(7-5,-1-3)=(2,-4)$$

$$S=\frac{1}{2}|a_1b_2-a_2b_1|=\frac{1}{2}|(-3)\cdot(-4)-1\cdot 2|=5$$

$$S=5$$

3 For $\triangle OAB$, when $|\overrightarrow{OA}|=\sqrt{2}, |\overrightarrow{OB}|=\sqrt{6}, |\overrightarrow{OA}-\overrightarrow{OB}|=2$, solve the following problems.

(1) Find the value of $\overrightarrow{OA}\cdot\overrightarrow{OB}$.

$$|\overrightarrow{OA}-\overrightarrow{OB}|=2, |\overrightarrow{OA}-\overrightarrow{OB}|^2=4, |\overrightarrow{OA}|^2-2\overrightarrow{OA}\cdot\overrightarrow{OB}+|\overrightarrow{OB}|^2=4,$$

$$2-2\overrightarrow{OA}\cdot\overrightarrow{OB}+6=4, 2\overrightarrow{OA}\cdot\overrightarrow{OB}=4, \overrightarrow{OA}\cdot\overrightarrow{OB}=2$$

$$\overrightarrow{OA}\cdot\overrightarrow{OB}=2$$

(2) Find the area S of $\triangle OAB$.

$$S=\frac{1}{2}\sqrt{|\overrightarrow{OA}|^2|\overrightarrow{OB}|^2-(\overrightarrow{OA}\cdot\overrightarrow{OB})^2}=\frac{1}{2}\sqrt{2\cdot 6-2^2}=\sqrt{2}$$

$$S=\sqrt{2}$$

Spatial vectors

TARGET To understand how to express spatial vectors.

STUDY GUIDE

Spatial vector operations

Spatial vector operations are defined in the same was as planar vectors.

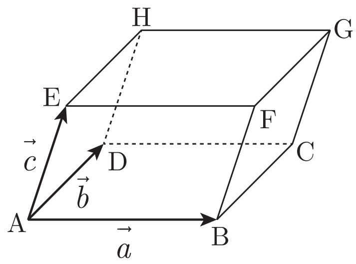
Solving spatial vectors

Given 4 points, O, A, B, and C that are not on the same plane, when $\overrightarrow{\mathrm{OA}}=\vec{a}, \overrightarrow{\mathrm{OB}}=\vec{b}, \overrightarrow{\mathrm{OC}}=\vec{c}$, then all vectors $\vec{p}$ can be expressed in 1 line using real numbers s, t, and u.

$$\boxed{\vec{p}=s\vec{a}+t\vec{b}+u\vec{c}}$$

EXERCISE

◆ For a parallelepiped ABCD−EFGH, given $\overrightarrow{\mathrm{AB}}=\vec{a}, \overrightarrow{\mathrm{AD}}=\vec{b}, \overrightarrow{\mathrm{AE}}=\vec{c}$, express the following vectors by using $\vec{a}$, $\vec{b}$, and $\vec{c}$.



(1) $\overrightarrow{\mathrm{BG}}$

$\overrightarrow{\mathrm{BG}}=\overrightarrow{\mathrm{AH}}=\overrightarrow{\mathrm{AD}}+\overrightarrow{\mathrm{DH}}=\overrightarrow{\mathrm{AD}}+\overrightarrow{\mathrm{AE}}=\vec{b}+\vec{c}$

$\vec{b}+\vec{c}$

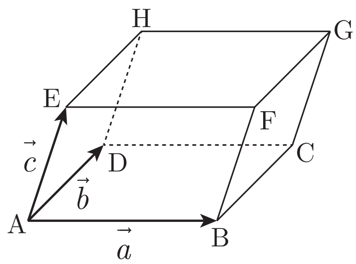
(2) $\overrightarrow{\mathrm{EC}}$

$\overrightarrow{\mathrm{EC}}=\overrightarrow{\mathrm{EA}}+\overrightarrow{\mathrm{AB}}+\overrightarrow{\mathrm{BC}}=-\vec{c}+\vec{a}+\vec{b}=\vec{a}+\vec{b}-\vec{c}$

$\vec{a}+\vec{b}-\vec{c}$

PRACTICE

◆ For a parallelepiped ABCD−EFGH, given $\overrightarrow{\mathrm{AB}}=\vec{a}, \overrightarrow{\mathrm{AD}}=\vec{b}, \overrightarrow{\mathrm{AE}}=\vec{c}$, express the following vectors by using $\vec{a}$, $\vec{b}$, and $\vec{c}$.



(1) $\overrightarrow{\mathrm{HC}}$

$\overrightarrow{\mathrm{HC}}=\overrightarrow{\mathrm{EB}}=\overrightarrow{\mathrm{EA}}+\overrightarrow{\mathrm{AB}}=-\vec{c}+\vec{a}=\vec{a}-\vec{c}$

$\vec{a}-\vec{c}$

(2) $\overrightarrow{\mathrm{HB}}$

$\overrightarrow{\mathrm{HB}}=\overrightarrow{\mathrm{HE}}+\overrightarrow{\mathrm{EA}}+\overrightarrow{\mathrm{AB}}=-\vec{b}-\vec{c}+\vec{a}=\vec{a}-\vec{b}-\vec{c}$

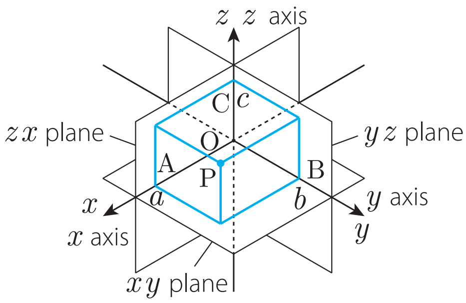
$\vec{a}-\vec{b}-\vec{c}$

Spatial coordinates

TARGET To understand how to express spatial coordinates.

STUDY GUIDE

Spatial coordinates



As shown in the diagram on the right, 3 planes (xy plane, yz plane, and zx plane) defined using 3 coordinate axes (x axis, y axis, and z axis) that are orthogonal to each other and share a fixed point O as an origin are called **coordinate planes**. When coordinates a, b, and c are on respective axes at points A, B, and C, such that A, B, and C are the points where the 3 coordinate axes cross and are perpendicular planes to each coordinate axis passing through a point P, then we say that a set of 3 real numbers (a, b, and c) are the **coordinates** of the point P, and express it as $\mathbf{P(a,b,c)}$. Furthermore, we say the real numbers a, b, and c are respectively the **x coordinate**, **y coordinate**, and **z coordinate**, and that the space these coordinates define is called a **coordinate space**.

Distance between 2 points in space

By using the following formula, we can find the distance between 2 points in relation to the coordinate origin O $(0, 0, 0)$ and 2 points $A(x_1, y_1, z_1), B(x_2, y_2, z_2)$.

$$\text{(1)}\quad AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

$$\text{(2)}\quad OA = \sqrt{x_1{}^2 + y_1{}^2 + z_1{}^2}$$

EXERCISE

◆ Take 2 points $A(1,3,2), B(2,1,-1)$. Now, find the distance AB between the 2 points A and B.

$AB = \sqrt{(2-1)^2 + (1-3)^2 + (-1-2)^2} = \sqrt{14}$

$AB = \sqrt{14}$

PRACTICE

◆ Take 3 points $A(-1,-1,1), B(2,-1,4), C(5,1,-2)$. Now, solve the following problems.

(1) Find the distance AB between the 2 points A and B.

$AB = \sqrt{(2+1)^2 + (-1+1)^2 + (4-1)^2} = 3\sqrt{2}$

$AB = 3\sqrt{2}$

(2) Given vertices at 3 points A, B, and C, answer what kind of triangle $\triangle ABC$ is.

$BC = \sqrt{(5-2)^2 + (1+1)^2 + (-2-4)^2} = 7$

$CA = \sqrt{(-1-5)^2 + (-1-1)^2 + (1+2)^2} = 7$

Therefore, it is an isosceles triangle for which BC=CA.

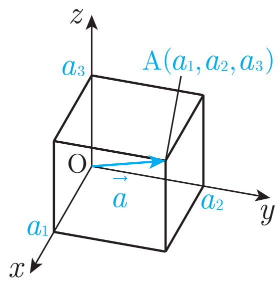
Isosceles triangle for which BC=CA

Components of spatial vectors

TARGET To understand the components of vectors in three-dimensional space.

STUDY GUIDE

Components of spatial vectors



As shown in the diagram on the right, for the spatial vector $\vec{a}$, when (a_1, a_2, a_3) is the coordinate of point A such that $\vec{a} = \overrightarrow{\mathrm{OA}}$, we say that the 3 real numbers a_1, a_2, a_3 are respectively the **x component**, **y component**, and **z component** of $\vec{a}$, and the **component expression** is $\vec{a} = (a_1, a_2, a_3)$.

For 2 vectors $\vec{a} = (a_1, a_2, a_3), \vec{b} = (b_1, b_2, b_3)$, we can say the following, which is the same as in a plane.

Equality $\vec{a} = \vec{b} \iff a_1 = b_1, a_2 = b_2, a_3 = b_3$

Magnitude $|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

Adding $\vec{a} + \vec{b} = (a_1, a_2, a_3) + (b_1, b_2, b_3) = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$

Subtracting

$\vec{a} - \vec{b} = (a_1, a_2, a_3) - (b_1, b_2, b_3) = (a_1 - b_1, a_2 - b_2, a_3 - b_3)$

Real multiples $k(a_1, a_2, a_3) = (ka_1, ka_2, ka_3)$ (k **is a real number**)

Components of spatial points and vectors

We can find the component expression of $\overrightarrow{\mathrm{AB}}$ for the origin O and 2 points $\mathrm{A}(a_1, a_2, a_3), \mathrm{B}(b_1, b_2, b_3)$. Because $\overrightarrow{\mathrm{OA}} = (a_1, a_2, a_3), \overrightarrow{\mathrm{OB}} = (b_1, b_2, b_3)$, we get $\overrightarrow{\mathrm{AB}} = \overrightarrow{\mathrm{OB}} - \overrightarrow{\mathrm{OA}} = (b_1, b_2, b_3) - (a_1, a_2, a_3) = (b_1 - a_1, b_2 - a_2, b_3 - a_3)$, from which we derive the following formulas.

(1) $\overrightarrow{\mathrm{AB}} = (b_1 - a_1, b_2 - a_2, b_3 - a_3)$

(2) $|\overrightarrow{\mathrm{AB}}| = \sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2 + (b_3 - a_3)^2}$

EXERCISE

1 When $\vec{a} = (2, 1, -3), \vec{b} = (4, -2, 1)$, solve the following problems.

(1) Express the components of $2\vec{a} + \vec{b}$.

$$\begin{aligned} 2\vec{a} + \vec{b} &= 2(2, 1, -3) + (4, -2, 1) \\ &= (4, 2, -6) + (4, -2, 1) \\ &= (4 + 4, 2 - 2, -6 + 1) \\ &= (8, 0, -5) \end{aligned}$$

$2\vec{a} + \vec{b} = (8, 0, -5)$

(2) Find $|\vec{a} - \vec{b}|$.

$$\begin{aligned} \vec{a} - \vec{b} &= (2, 1, -3) - (4, -2, 1) \\ &= (2 - 4, 1 + 2, -3 - 1) \\ &= (-2, 3, -4) \end{aligned}$$

$$\begin{aligned} |\vec{a} - \vec{b}| &= \sqrt{(-2)^2 + 3^2 + (-4)^2} \\ &= \sqrt{29} \end{aligned}$$

$|\vec{a} - \vec{b}| = \sqrt{29}$

2 Given the 2 points $A(5,-3,-4), B(2,-7,1)$, express the components of $\overrightarrow{AB}$ and find the magnitudes.

$\overrightarrow{AB} = (2-5, -7+3, 1+4) = (-3,-4,5)$

$|\overrightarrow{AB}| = \sqrt{(-3)^2+(-4)^2+5^2} = 5\sqrt{2}$

$$\overrightarrow{AB} = (-3,-4,5), |\overrightarrow{AB}| = 5\sqrt{2}$$

PRACTICE

1 When $\vec{a} = (-3,-5,2), \vec{b} = (-1,4,3)$, solve the following problems.

(1) Express the components of $3\vec{a}-2\vec{b}$.

$$\begin{aligned} 3\vec{a}-2\vec{b} &= 3(-3,-5,2)-2(-1,4,3) \\ &= (-9,-15,6)+(2,-8,-6) \\ &= (-9+2, -15-8, 6-6) \\ &= (-7,-23,0) \end{aligned}$$

$$3\vec{a}-2\vec{b} = (-7,-23,0)$$

(2) Find $|2\vec{a}+\vec{b}|$.

$$\begin{aligned} 2\vec{a}+\vec{b} &= 2(-3,-5,2)+(-1,4,3) \\ &= (-6,-10,4)+(-1,4,3) \\ &= (-6-1, -10+4, 4+3) \\ &= (-7,-6,7) \end{aligned}$$

$$\begin{aligned} |2\vec{a}+\vec{b}| &= \sqrt{(-7)^2+(-6)^2+7^2} \\ &= \sqrt{134} \end{aligned}$$

$$|2\vec{a}+\vec{b}| = \sqrt{134}$$

2 Given the 2 points $A(1,6,-2), B(-3,5,2)$, express the components of $\overrightarrow{AB}$ and find the magnitudes.

$\overrightarrow{AB} = (-3-1, 5-6, 2+2) = (-4,-1,4)$

$|\overrightarrow{AB}| = \sqrt{(-4)^2+(-1)^2+4^2} = \sqrt{33}$

$$\overrightarrow{AB} = (-4,-1,4), |\overrightarrow{AB}| = \sqrt{33}$$

EXTRA Info.

On the scientific calculator, use the Vector function to calculate the magnitude and the components of spatial vectors.

EXERCISE

◆ Given $\vec{a}=(-5, 2, 1)$ and $\vec{b}=(-3, 4, -2)$ solve the following questions.

(1) Calculate $\vec{a}+2\vec{b}$.

$$\begin{aligned}\vec{a}+2\vec{b} &= (-5,2,1)+2(-3,4,-2)\\ &= (-5,2,1)+(-6,8,-4)\\ &= (-5-6,2+8,1-4)\\ &= (-11,10,-3)\end{aligned}$$

$(-11, 10, -3)$

(2) Find $|2\vec{a}-\vec{b}|$.

$$\begin{aligned}2\vec{a}-\vec{b} &= 2(-5,2,1)-(-3,4,-2)\\ &= (-10,4,2)+(3,-4,2)\\ &= (-10+3,4-4,2+2)\\ &= (-7,0,4)\end{aligned}$$

$$\begin{aligned}|2\vec{a}-\vec{b}| &= \sqrt{(-7)^2+0^2+4^2}\\ &= \sqrt{65}\end{aligned}$$

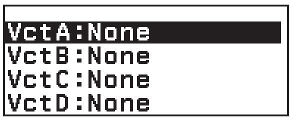
$\sqrt{65}$

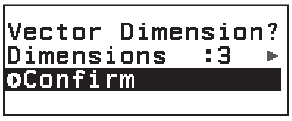
check

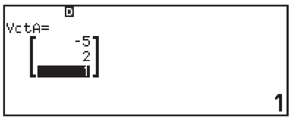
Press [HOME], select [Vector], press [OK]

Register $\vec{a}=(-5, 2, 1)$.

Press [MENU], select [VctA], press [OK], select [Dimensions], press [OK], select [3 Dimensions], press [OK], select [Confirm], press [OK], [−] [5] [EXE] [2] [EXE] [1] [EXE] [EXE]

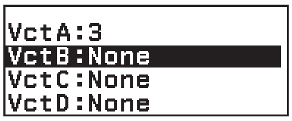


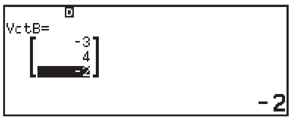




Register $\vec{b}=(-3, 4, -2)$.

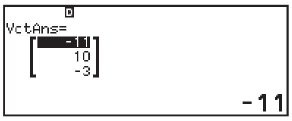
[MENU] [⌄] [OK] [OK] [⌄] [OK] [OK]   [−] [3] [EXE] [4] [EXE] [−] [2] [EXE] [EXE]





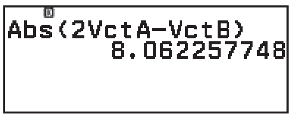
(1) Calculate the components of $\vec{a}+2\vec{b}$.

[CATALOG] [OK] [⌄] [OK] [+] [2] [CATALOG] [OK] [⌄] [⌄] [OK] [EXE]

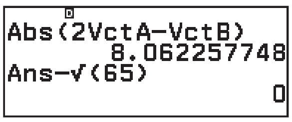


(2) Calculate and verify the value of $|2\vec{a}-\vec{b}|$.

[AC]   [CATALOG] [⌄] [⌄] [⌄] [OK] [OK] [2] [CATALOG] [OK] [⌄] [OK] [−] [CATALOG] [OK] [⌄] [⌄] [OK] [)] [EXE]



[Ans] [−] [√] [6] [5] [)] [EXE]



PRACTICE

1 Given $\vec{a}=(2,-4,7)$ and $\vec{b}=(1,5,-8)$, calculate $3\vec{a}+2\vec{b}$.

$$\begin{aligned}3\vec{a}+2\vec{b}&=3(2,-4,7)+2(1,5,-8)\\&=(6,-12,21)+(2,10,-16)\\&=(6+2,-12+10,21-16)\\&=(8,-2,5)\end{aligned}$$

$(8,-2,5)$

check

Register $\vec{a}=(2,-4,7)$.

⋯ OK OK ∨ OK OK 2 EXE − 4 EXE 7 EXE EXE

```
VctA=
[ 2
 -4
  7 ]
          7
```

Register $\vec{b}=(1,5,-8)$.

⋯ ∨ OK OK ∨ OK OK 1 EXE 5 EXE − 8 EXE EXE

```
VctB=
[ 1
  5
 -8 ]
         -8
```

Calculate the components of $3\vec{a}+2\vec{b}$.

3 📖 OK ∨ OK + 2 📖 OK ∨ ∨ OK EXE

```
3VctA+2VctB
```

```
VctAns=
[ 8
 -2
  5 ]
          8
```

2 Given 2 points $A(-4,3,-5)$ and $B(-9,-2,1)$, express the components of $\overrightarrow{AB}$ and find its magnitude.

$$\overrightarrow{AB}=(-9+4,-2-3,1+5)=(-5,-5,6)$$

$$|\overrightarrow{AB}|=\sqrt{(-5)^2+(-5)^2+6^2}=\sqrt{86}$$

$\overrightarrow{AB}=(-5,-5,6), |\overrightarrow{AB}|=\sqrt{86}$

check

Register $\vec{a}=(-4,3,-5)$.

⋯ OK OK ∨ OK OK − 4 EXE 3 EXE − 5 EXE EXE

```
VctA=
[-4
  3
 -5 ]
         -5
```

Register $\vec{b}=(-9,-2,1)$.

⋯ ∨ OK OK ∨ OK OK − 9 EXE − 2 EXE 1 EXE EXE

```
VctB=
[-9
 -2
  1 ]
          1
```

Calculate and verify the value of $|\overrightarrow{AB}|$.

📖 ∨ ∨ ∨ OK OK 📖 OK ∨ ∨ OK − 📖 OK ∨ OK) EXE

```
Abs(VctB-VctA)
        9.273618495
```

Ans − √■ 8 6) EXE

```
Abs(VctB-VctA)
        9.273618495
Ans-√(86)
                  0
```

Internal dividing points and external dividing points

TARGET To understand position vectors of internal dividing points and external dividing points in three-dimensional space.

STUDY GUIDE

Position vectors of internal dividing points and external dividing points

To define an origin point O in space for the position of an arbitrary point P is the same as in a plane, take a point O as the reference for $\overrightarrow{\mathrm{OP}} = \vec{p}$, and define the **position vector** $\vec{p}$ for point P, and express it as $\mathrm{P}\,(\vec{p}\,)$.

The following holds both when in space and in a plane. For 2 points $\mathrm{A}(\vec{a}\,)$ and $\mathrm{B}(\vec{b}\,)$, let $\mathrm{P}(\vec{p}\,)$ be the point at which the line segment AB is internally divided by $m{:}n$, and let $\mathrm{Q}(\vec{q}\,)$ be the point at which the line segment AB is externally divided by $m{:}n$, such that the midpoint of line segment AB is $\mathrm{M}(\vec{m}\,)$. Furthermore, for 3 points $\mathrm{A}(\vec{a}\,)$, $\mathrm{B}(\vec{b}\,)$, and $\mathrm{C}(\vec{c}\,)$ at the vertices of the triangle $\triangle\mathrm{ABC}$, the center of gravity is $\mathrm{G}(\vec{g}\,)$.

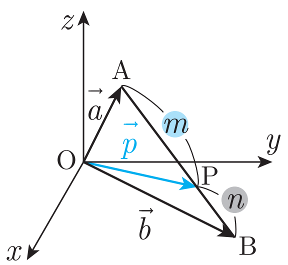
(1) $\overrightarrow{\mathrm{AB}} = \vec{b} - \vec{a}$

(2) **Internal dividing point** $\vec{p} = \dfrac{n\vec{a} + m\vec{b}}{m+n}$

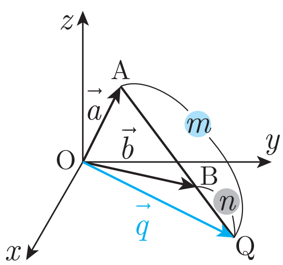
(3) **External dividing point** $\vec{q} = \dfrac{-n\vec{a} + m\vec{b}}{m-n}$

(4) **Midpoint** $\vec{m} = \dfrac{\vec{a} + \vec{b}}{2}$ (5) **Center of gravity** $\vec{g} = \dfrac{\vec{a} + \vec{b} + \vec{c}}{3}$

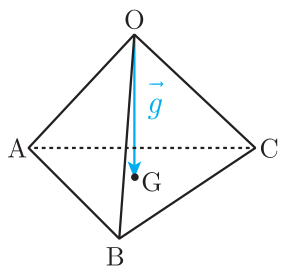
Internal dividing point



External dividing point

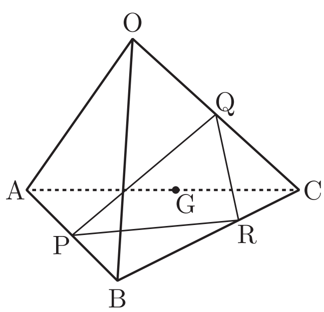


Center of gravity



EXERCISE

◆ For the tetrahedron OABC on the right, the midpoints on sides AB and OC are P and Q respectively, side BC is divided internally by $2{:}1$ at point R, and $\triangle\mathrm{PQR}$ has a center of gravity G. Given $\overrightarrow{\mathrm{OA}} = \vec{a}, \overrightarrow{\mathrm{OB}} = \vec{b}, \overrightarrow{\mathrm{OC}} = \vec{c}$, express $\overrightarrow{\mathrm{OG}}$ by using $\vec{a}$, $\vec{b}$, and $\vec{c}$.



$$\overrightarrow{\mathrm{OP}} = \frac{1}{2}\vec{a} + \frac{1}{2}\vec{b}$$

$$\overrightarrow{\mathrm{OQ}} = \frac{1}{2}\vec{c}$$

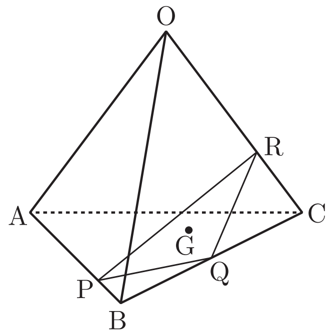
$$\overrightarrow{\mathrm{OR}} = \frac{\vec{b} + 2\vec{c}}{2+1} = \frac{1}{3}\vec{b} + \frac{2}{3}\vec{c}$$

$$\overrightarrow{\mathrm{OG}} = \frac{1}{3}\overrightarrow{\mathrm{OP}} + \frac{1}{3}\overrightarrow{\mathrm{OQ}} + \frac{1}{3}\overrightarrow{\mathrm{OR}} = \frac{1}{3}\left(\frac{1}{2}\vec{a} + \frac{1}{2}\vec{b}\right) + \frac{1}{3}\left(\frac{1}{2}\vec{c}\right) + \frac{1}{3}\left(\frac{1}{3}\vec{b} + \frac{2}{3}\vec{c}\right) = \frac{1}{6}\vec{a} + \frac{5}{18}\vec{b} + \frac{7}{18}\vec{c}$$

$$\overrightarrow{\mathrm{OG}} = \frac{1}{6}\vec{a} + \frac{5}{18}\vec{b} + \frac{7}{18}\vec{c}$$

PRACTICE

1 For the tetrahedron OABC on the right, side AB is divided internally by 3:1 at point P, the midpoint on side BC is Q, side OC is divided internally by 2:1 at point R, and $\triangle$PQR has a center of gravity G. Given $\overrightarrow{OA}=\vec{a}, \overrightarrow{OB}=\vec{b}, \overrightarrow{OC}=\vec{c}$, express the following vectors by using $\vec{a}$, $\vec{b}$, and $\vec{c}$.



(1) $\overrightarrow{OP}$

$$\overrightarrow{OP}=\frac{\vec{a}+3\vec{b}}{3+1}=\frac{1}{4}\vec{a}+\frac{3}{4}\vec{b}$$

$$\overrightarrow{OP}=\frac{1}{4}\vec{a}+\frac{3}{4}\vec{b}$$

(2) $\overrightarrow{OG}$

$$\overrightarrow{OQ}=\frac{\vec{b}+\vec{c}}{2}=\frac{1}{2}\vec{b}+\frac{1}{2}\vec{c}$$

$$\overrightarrow{OR}=\frac{2\vec{c}}{2+1}=\frac{2}{3}\vec{c}$$

$$\overrightarrow{OG}=\frac{1}{3}\overrightarrow{OP}+\frac{1}{3}\overrightarrow{OQ}+\frac{1}{3}\overrightarrow{OR}=\frac{1}{3}\left(\frac{1}{4}\vec{a}+\frac{3}{4}\vec{b}\right)+\frac{1}{3}\left(\frac{1}{2}\vec{b}+\frac{1}{2}\vec{c}\right)+\frac{1}{3}\left(\frac{2}{3}\vec{c}\right)$$

$$=\frac{1}{12}\vec{a}+\frac{5}{12}\vec{b}+\frac{7}{18}\vec{c}$$

$$\overrightarrow{OG}=\frac{1}{12}\vec{a}+\frac{5}{12}\vec{b}+\frac{7}{18}\vec{c}$$

2 Given the tetrahedron OABC with sides OA, AB, BC, and OC with respective midpoints of P, Q, R, and S, prove that the center of gravity G_1 of $\triangle$ABS and the center of gravity G_2 of $\triangle$PQR are the same.

[Proof]

Given $\overrightarrow{OA}=\vec{a}, \overrightarrow{OB}=\vec{b}, \overrightarrow{OC}=\vec{c}$.

$$\overrightarrow{OG_1}=\frac{1}{3}\overrightarrow{OA}+\frac{1}{3}\overrightarrow{OB}+\frac{1}{3}\overrightarrow{OS}$$

$$=\frac{1}{3}\vec{a}+\frac{1}{3}\vec{b}+\frac{1}{3}\left(\frac{1}{2}\vec{c}\right)=\frac{1}{3}\vec{a}+\frac{1}{3}\vec{b}+\frac{1}{6}\vec{c}$$

$$\overrightarrow{OG_2}=\frac{1}{3}\overrightarrow{OP}+\frac{1}{3}\overrightarrow{OQ}+\frac{1}{3}\overrightarrow{OR}$$

$$=\frac{1}{3}\left(\frac{1}{2}\vec{a}\right)+\frac{1}{3}\left(\frac{1}{2}\vec{a}+\frac{1}{2}\vec{b}\right)+\frac{1}{3}\left(\frac{1}{2}\vec{b}+\frac{1}{2}\vec{c}\right)=\frac{1}{3}\vec{a}+\frac{1}{3}\vec{b}+\frac{1}{6}\vec{c}$$

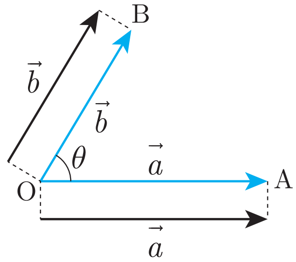
Therefore, because $\overrightarrow{OG_1}=\overrightarrow{OG_2}$, the center of gravity G_1 of $\triangle$ABS and the center of gravity G_2 of $\triangle$PQR are the same.

Inner products of spatial vectors

TARGET To understand the inner products of vectors in three-dimensional space.

STUDY GUIDE

Inner products of spatial vectors



The same as in a plane, when the angle θ formed by 2 spatial vectors $\vec{a}$ and $\vec{b}$ that are not $\vec{0}$ is $(0° \leq \theta \leq 180°)$, we can define the inner product of $\vec{a}$ and $\vec{b}$ as $|\vec{a}||\vec{b}|\cos\theta$.
When $\vec{a}=\vec{0}$ or $\vec{b}=\vec{0}$, the inner product of $\vec{a}$ and $\vec{b}$ is stipulated as $\vec{a}\cdot\vec{b}=0$.
For 2 vectors $\vec{a}=(a_1,a_2,a_3), \vec{b}=(b_1,b_2,b_3)$ that are not $\vec{0}$ and given that $\vec{a}$ and $\vec{b}$ form an angle $\theta(0° \leq \theta \leq 180°)$, then, the same as when in a plane, we can say the following.

(1) Inner product $\vec{a}\cdot\vec{b}=a_1b_1+a_2b_2+a_3b_3$

(2) Cosine of angle

$$\cos\theta=\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}=\frac{a_1b_1+a_2b_2+a_3b_3}{\sqrt{a_1^2+a_2^2+a_3^2}\cdot\sqrt{b_1^2+b_2^2+b_3^2}}$$

(3) Perpendicular conditions

$$\vec{a}\perp\vec{b} \iff \vec{a}\cdot\vec{b}=0 \iff a_1b_1+a_2b_2+a_3b_3=0$$

EXERCISE

1 Find the angle $\theta(0° \leq \theta \leq 180°)$ formed by $\vec{a}=(0,-1,1), \vec{b}=(1,1,-2)$.

$|\vec{a}|=\sqrt{0^2+(-1)^2+1^2}=\sqrt{2}, |\vec{b}|=\sqrt{1^2+1^2+(-2)^2}=\sqrt{6}$

$\vec{a}\cdot\vec{b}=0\cdot1-1\cdot1+1\cdot(-2)=-3$

$$\cos\theta=\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}=\frac{-3}{\sqrt{2}\cdot\sqrt{6}}=-\frac{\sqrt{3}}{2}$$

Since $0° \leq \theta \leq 180°$, we get $\theta=150°$.

150°

2 Given $\vec{a}=(x,y,-3)$ is perpendicular to both $\vec{b}=(1,1,1)$ and $\vec{c}=(2,-1,1)$, find the value of x and y.

From $\vec{a}\perp\vec{b}$ and $\vec{a}\perp\vec{c}$, we get $\vec{a}\cdot\vec{b}=0, \vec{a}\cdot\vec{c}=0$.

$\vec{a}\cdot\vec{b}=x\cdot1+y\cdot1-3\cdot1=x+y-3=0 \quad \ldots$(i)

$\vec{a}\cdot\vec{c}=x\cdot2+y\cdot(-1)-3\cdot1=2x-y-3=0 \quad \ldots$(ii)

Solving for (i) and (ii) gives us $x=2$ and $y=1$.

$x=2, y=1$

PRACTICE

1 Find the angle $\theta(0° \le \theta \le 180°)$ formed by $\vec{a}=(-1,2,-1), \vec{b}=(-1,-1,2)$.

$$|\vec{a}|=\sqrt{(-1)^2+2^2+(-1)^2}=\sqrt{6},\ |\vec{b}|=\sqrt{(-1)^2+(-1)^2+2^2}=\sqrt{6}$$

$$\vec{a}\cdot\vec{b}=-1\cdot(-1)+2\cdot(-1)-1\cdot 2=-3$$

$$\cos\theta=\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}=\frac{-3}{\sqrt{6}\cdot\sqrt{6}}=-\frac{1}{2}$$

Since $0° \le \theta \le 180°$, we get $\theta=120°$.

$120°$

2 Given $\vec{a}=(x,y,2)$ is perpendicular to both $\vec{b}=(7,-1,6)$ and $\vec{c}=(-1,1,-3)$, find the value of x and y.

From $\vec{a}\perp\vec{b}, \vec{a}\perp\vec{c}$, we get $\vec{a}\cdot\vec{b}=0, \vec{a}\cdot\vec{c}=0$.

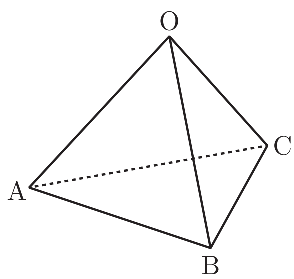
$$\vec{a}\cdot\vec{b}=x\cdot 7+y\cdot(-1)+2\cdot 6=7x-y+12=0 \quad \text{...(i)}$$

$$\vec{a}\cdot\vec{c}=x\cdot(-1)+y\cdot 1+2\cdot(-3)=-x+y-6=0 \quad \text{...(ii)}$$

Solving for (i) and (ii) gives us $x=-1$ and $y=5$.

$x=-1,\ y=5$

3 Find the inner product of the regular tetrahedron OABC with each side of length 3, as shown in the diagram on the right.



(1) $\overrightarrow{OA}\cdot\overrightarrow{OB}$

The angle formed by $\overrightarrow{OA}, \overrightarrow{OB}$ is $60°$.

$$\overrightarrow{OA}\cdot\overrightarrow{OB}=|\overrightarrow{OA}||\overrightarrow{OB}|\cos 60°=3\cdot 3\cdot\frac{1}{2}=\frac{9}{2}$$

$$\overrightarrow{OA}\cdot\overrightarrow{OB}=\frac{9}{2}$$

(2) $\overrightarrow{OB}\cdot\overrightarrow{BC}$

The angle formed by $\overrightarrow{OB}, \overrightarrow{BC}$ is $120°$.

$$\overrightarrow{OB}\cdot\overrightarrow{BC}=|\overrightarrow{OB}||\overrightarrow{BC}|\cos 120°=3\cdot 3\cdot\left(-\frac{1}{2}\right)=-\frac{9}{2}$$

$$\overrightarrow{OB}\cdot\overrightarrow{BC}=-\frac{9}{2}$$

(3) $\overrightarrow{OB}\cdot\overrightarrow{AB}$

The angle formed by $\overrightarrow{OB}, \overrightarrow{AB}$ is $60°$.

$$\overrightarrow{OB}\cdot\overrightarrow{AB}=|\overrightarrow{OB}||\overrightarrow{AB}|\cos 60°=3\cdot 3\cdot\frac{1}{2}=\frac{9}{2}$$

$$\overrightarrow{OB}\cdot\overrightarrow{AB}=\frac{9}{2}$$

EXTRA Info.

On the scientific calculator, use the Vector function to do calculations related to the inner products and angles formed by spatial vectors.

EXERCISE

1 Find the angle θ $(0^\circ \leq \theta \leq 180^\circ)$ formed by $\vec{a}=(1,-\sqrt{2},-1), \vec{b}=(-1,0,1)$.

$$|\vec{a}|=\sqrt{1^2+(-\sqrt{2})^2+(-1)^2}=2, |\vec{b}|=\sqrt{(-1)^2+0^2+1^2}=\sqrt{2}$$

$$\vec{a}\cdot\vec{b}=1\cdot(-1)-\sqrt{2}\cdot 0-1\cdot 1=-2$$

$$\cos\theta=\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}=\frac{-2}{2\cdot\sqrt{2}}=-\frac{1}{\sqrt{2}}$$

Since $0^\circ \leq \theta \leq 180^\circ$, we get $\theta=135^\circ$.

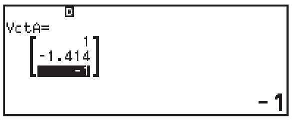
135°

check

Press [⌂], select [Vector], press [OK]

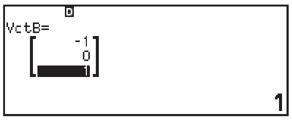
Register $\vec{a}=(1,-\sqrt{2},-1)$.

[○○○] [OK] [OK] [∨] [OK] [OK] [1] [EXE] [(−)] [√■] [2] [)] [EXE] [(−)] [1] [EXE] [EXE]



Register $\vec{b}=(-1,0,1)$.

[○○○] [∨] [OK] [OK] [∨] [OK] [OK] [(−)] [1] [EXE] [0] [EXE] [1] [EXE] [EXE]



Set the angle display to Degree.

[☰] [OK] [∨] [OK] [OK] [AC]

Calculate the angle θ formed by $\vec{a}$ and $\vec{b}$.

[⊡] [OK] [OK] [∨] [∨] [OK] [⊡] [OK] [∨] [OK] [⇧] [)] [⊡] [OK] [∨] [∨] [OK] [)] [EXE]

PRACTICE

◆ Given points $A(2\sqrt{3},0,2)$ and $B(\sqrt{3},2\sqrt{3},1)$ are in a coordinate space with an origin of O. Now, prove that triangle OAB is an equilateral triangle.

(Proof)

From $\overrightarrow{OA}=(2\sqrt{3},0,2),\overrightarrow{OB}=(\sqrt{3},2\sqrt{3},1)$,

such that $|\overrightarrow{OA}|=|\overrightarrow{OB}|=4, \overrightarrow{OA}\cdot\overrightarrow{OB}=8$,

we get $\cos\angle AOB=\dfrac{\overrightarrow{OA}\cdot\overrightarrow{OB}}{|\overrightarrow{OA}||\overrightarrow{OB}|}=\dfrac{8}{4\cdot 4}=\dfrac{1}{2}$,

since $0^\circ<\angle AOB<180^\circ$**, we get** $\angle AOB=60^\circ$**.**

Therefore, $OA=OB$**, and since the angle between them is** 60°**,**

we find that the triangle OAB **is an equilateral triangle.**

check

Register $\overrightarrow{OA}=(2\sqrt{3},0,2)$**.**

(ooo) (OK) (OK) (∨) (OK) (OK) (2) (√■) (3) ()) (EXE) (0) (EXE) (2) (EXE) (EXE)

Register $\overrightarrow{OB}=(\sqrt{3},2\sqrt{3},1)$**.**

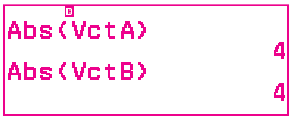
(ooo) (∨) (OK) (OK) (∨) (OK) (OK) (√■) (3) ()) (EXE) (2) (√■) (3) ()) (EXE) (1) (EXE) (EXE)

Calculate the value of $|\overrightarrow{OA}|$**.**

(📖) (∨) (∨) (∨) (OK) (OK) (📖) (OK) (∨) (OK) ()) (EXE)

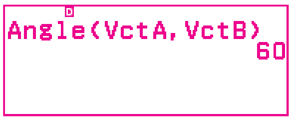
Calculate the value of $|\overrightarrow{OB}|$**.**

(📖) (∨) (∨) (∨) (OK) (OK) (📖) (OK) (∨) (∨) (OK) ()) (EXE)



Calculate the angle θ **formed by** $\overrightarrow{OA}$ **and** $\overrightarrow{OB}$**.**

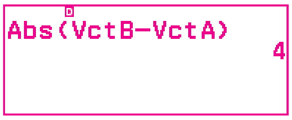
(📖) (OK) (OK) (∨) (∨) (OK) (📖) (OK) (∨) (OK) (⇧) ()) (📖) (OK) (∨) (∨) (OK) ()) (EXE)



OTHER METHODS

We can also calculate the value of $|\overrightarrow{AB}|$ **to show that** $|\overrightarrow{OA}|=|\overrightarrow{OB}|=|\overrightarrow{AB}|=4$**.**

(📖) (∨) (∨) (∨) (OK) (OK) (📖) (OK) (∨) (∨) (OK) (−) (📖) (OK) (∨) (OK) ()) (EXE)



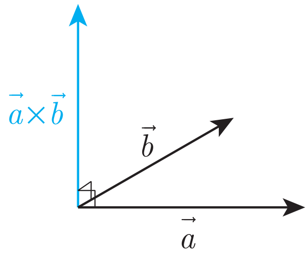
Outer products of vectors

TARGET To understand the outer products of vectors.

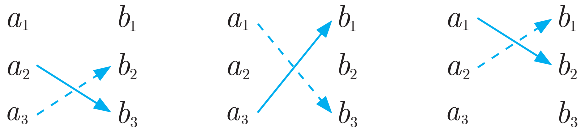
STUDY GUIDE

Components of outer products of vectors

A vector that is perpendicular to 2 vectors $\vec{a}=(a_1,a_2,a_3), \vec{b}=(b_1,b_2,b_3)$, which are not $\vec{0}$, in three-dimensional space and whose components can be found by the following formula is called the **outer product** of $\vec{a}$ and $\vec{b}$ and is expressed as $\vec{a}\times\vec{b}$.



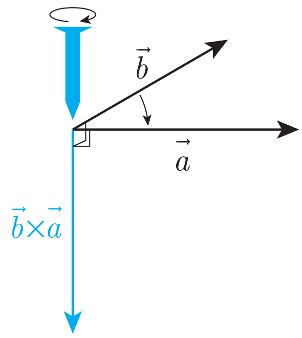
$$\boxed{\vec{a}\times\vec{b}=(a_2b_3-a_3b_2,\, a_3b_1-a_1b_3,\, a_1b_2-a_2b_1)}$$



Properties of outer products of vectors

Direction of outer products

For $\vec{a}$ and $\vec{b}$, there are 2 kinds of cross products $\vec{a}\times\vec{b}$ and $\vec{b}\times\vec{a}$, which have an inverse vector relation, from which we can derive the following expression.



$$\boxed{\vec{a}\times\vec{b}=-(\vec{b}\times\vec{a})}$$

We can find the direction of each outer product by using the right-hand screw rule (*).

* For $\vec{a}\times\vec{b}$, when facing from $\vec{a}$ to $\vec{b}$, the direction in which the screw advances when the screw is turned is the direction of the outer product.
For $\vec{b}\times\vec{a}$, when facing from $\vec{b}$ to $\vec{a}$, the direction in which the screw advances when the screw is turned is the direction of the outer product.

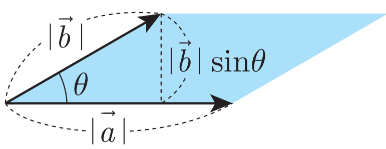
explanation

Regarding $\vec{a}=(a_1,a_2,a_3), \vec{b}=(b_1,b_2,b_3)$,

$$\vec{a}\times\vec{b}=(a_2b_3-a_3b_2, a_3b_1-a_1b_3, a_1b_2-a_2b_1)=\{-(b_2a_3-b_3a_2),-(b_3a_1-b_1a_3),-(b_1a_2-b_2a_1)\}$$
$$=-(b_2a_3-b_3a_2, b_3a_1-b_1a_3, b_1a_2-b_2a_1)=-(\vec{b}\times\vec{a})$$

Magnitude of outer products

Given θ is the angle formed by $\vec{a}=(a_1,a_2,a_3), \vec{b}=(b_1,b_2,b_3)$, then the magnitude $|\vec{a}\times\vec{b}|$ of the outer product of $\vec{a}$ and $\vec{b}$ is equal to the area of the parallelogram formed by $\vec{a}$ and $\vec{b}$, which we can find as follows.



$$\boxed{|\vec{a}\times\vec{b}|=|\vec{a}||\vec{b}|\sin\theta}$$

explanation

$$|\vec{a}\times\vec{b}|^2=(a_2b_3-a_3b_2)^2+(a_3b_1-a_1b_3)^2+(a_1b_2-a_2b_1)^2=(a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2)-(a_1b_1+a_2b_2+a_3b_3)^2$$
$$=|\vec{a}|^2|\vec{b}|^2-(\vec{a}\cdot\vec{b})^2=|\vec{a}|^2|\vec{b}|^2(1-\cos^2\theta)=|\vec{a}|^2|\vec{b}|^2\sin^2\theta \Rightarrow |\vec{a}\times\vec{b}|=|\vec{a}||\vec{b}|\sin\theta$$

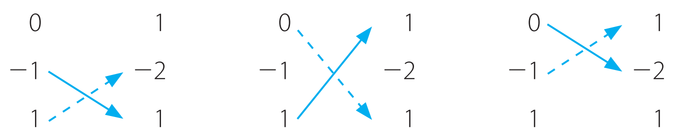
EXTRA Info.

On the scientific calculator, use the Vector function to do calculations related to the outer products of vectors.

EXERCISE

◆ When $\vec{a}=(0,-1,1)$ and $\vec{b}=(1,-2,1)$, solve the following problems.

(1) Find the value of $\vec{a}\times\vec{b}$.



$$\vec{a}\times\vec{b}=(-1\cdot1+1\cdot2,1\cdot1-0\cdot1,-0\cdot2+1\cdot1)=(1,1,1)$$

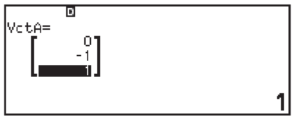
(1,1,1)

check

Press ⌂, select [Vector], press OK

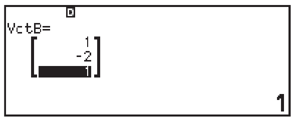
Register $\vec{a}=(0,-1,1)$.

⋯ OK OK ∨ OK OK 0 EXE (−) 1 EXE 1 EXE EXE



Register $\vec{b}=(1,-2,1)$.

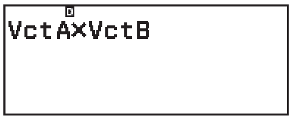
⋯ ∨ OK OK ∨ OK OK 1 EXE (−) 2 EXE 1 EXE EXE

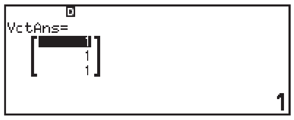


Calculate the value of $\vec{a}\times\vec{b}$.

Press Catalog OK ∨ OK, Catalog, select [Vector], press OK, select [Vector Calc], press OK, select [Cross Product], press OK,

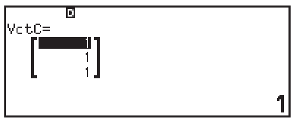
Catalog OK ∨ ∨ OK EXE





Register $\vec{c}=\vec{a}\times\vec{b}$.

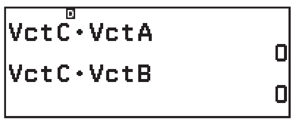
Press ⋯, select [Store], press OK, select [VctC], press OK, EXE



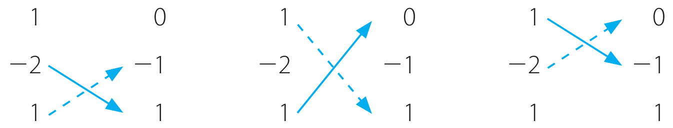
Confirm that the values of $\vec{c}\cdot\vec{a},\vec{c}\cdot\vec{b}$ are 0 (vertical).

Catalog OK ∨ ∨ ∨ OK Catalog OK OK OK Catalog OK ∨ OK EXE

Catalog OK ∨ ∨ ∨ OK Catalog OK OK OK Catalog OK ∨ ∨ OK EXE



(2) Find the value of $\vec{b}\times\vec{a}$.



$$\vec{b}\times\vec{a}=(-2\cdot1+1\cdot1,1\cdot0-1\cdot1,-1\cdot1+2\cdot0)=(-1,-1,-1)$$

$(-1,-1,-1)$

check

Calculate the value of $\vec{b}\times\vec{a}$.

AC   [CATALOG] OK ∨ ∨ OK [CATALOG] OK OK ∨ OK [CATALOG] OK ∨ OK EXE

```
VctB×VctA
```

```
VctAns=
[-1
 -1
 -1]
                -1
```

Register $\vec{d}=\vec{b}\times\vec{a}$.

Press [⋯] OK, select [VctD], press OK, EXE

```
VctA
VctB
VctC
VctD
```

```
VctD=
[-1
 -1
 -1]
                -1
```

Confirm that the values of $\vec{d}\cdot\vec{a},\vec{d}\cdot\vec{b}$ are 0 (vertical).

[CATALOG] OK ∨ ∨ ∨ ∨ OK [CATALOG] OK OK OK [CATALOG] OK ∨ OK EXE

[CATALOG] OK ∨ ∨ ∨ ∨ OK [CATALOG] OK OK OK [CATALOG] OK ∨ ∨ OK EXE

```
VctD·VctA
                 0
VctD·VctB
                 0
```

Confirm that $\vec{c}+\vec{d}$ is a null vector (magnitudes are equal).

AC   [CATALOG] OK ∨ ∨ ∨ OK + [CATALOG] OK ∨ ∨ ∨ ∨ OK EXE

```
VctC+VctD
```

```
VctAns=
[0
 0
 0]
                 0
```

(3) Find the area of the parallelogram formed by $\vec{a}$ and $\vec{b}$.

$|\vec{a}\times\vec{b}|=\sqrt{1^2+1^2+1^2}=\sqrt{3}$

(As an alternative solution, $|\vec{b}\times\vec{a}|=\sqrt{(-1)^2+(-1)^2+(-1)^2}=\sqrt{3}$ is also possible.)

$\sqrt{3}$

check

Set the angle display to Degree.

☰ OK ∨ OK OK   AC

```
◉Degree
○Radian
○Gradian
```

Calculate the angle θ formed by $\vec{a}$ and $\vec{b}$.

[CATALOG] OK OK ∨ ∨ OK [CATALOG] OK ∨ OK ⇧) [CATALOG] OK ∨ ∨ OK) EXE

```
Angle(VctA,VctB)
                30
```

Calculate and verify the value of $|\vec{a}||\vec{b}|\sin30°$.

[CATALOG] ∨ ∨ ∨ OK OK [CATALOG] OK ∨ OK) [CATALOG] ∨ ∨ ∨ OK OK [CATALOG] OK ∨ ∨ OK) sin 3 0) EXE   Ans − √ 3) EXE

```
                30
Abs(VctA)Abs(VctB
)sin(30)
       1.732050808
```

```
)sin(30)
       1.732050808
Ans−√(3)
                 0
```

PRACTICE

◆ When $\vec{a}=(-2,-2,1)$ and $\vec{b}=(3,1,-3)$, solve the following problems.

(1) Find the value of $\vec{a}\times\vec{b}$.

$$\vec{a}\times\vec{b}=(2\cdot3-1\cdot1,1\cdot3-2\cdot3,-2\cdot1+2\cdot3)$$
$$=(5,-3,4)$$

$(5,-3,4)$

(2) Find the value of $\vec{b}\times\vec{a}$.

$$\vec{b}\times\vec{a}=(1\cdot1-3\cdot2,3\cdot2-3\cdot1,-3\cdot2+1\cdot2)$$
$$=(-5,3,-4)$$

$(-5,3,-4)$

check

Register $\vec{a}=(-2,-2,1)$.

[ooo] [OK] [OK] [∨] [OK] [OK] [−] [2] [EXE] [−] [2] [EXE] [1] [EXE] [EXE]

VctA= [-2 -2 1]
1

Register $\vec{b}=(3,1,-3)$.

[ooo] [∨] [OK] [OK] [∨] [OK] [OK] [3] [EXE] [1] [EXE] [−] [3] [EXE] [EXE]

VctB= [3 1 -3]
-3

Calculate the value of $\vec{a}\times\vec{b}$.

[📖] [OK] [∨] [OK] [📖] [OK] [OK] [∨] [OK] [📖] [OK] [∨] [∨] [OK] [EXE]

VctA×VctB

VctAns= [5 -3 4]
5

Register $\vec{c}=\vec{a}\times\vec{b}$.

[ooo] [OK] [∨] [∨] [OK] [EXE]

VctA
VctB
VctC
VctD

VctC= [5 -3 4]
5

Confirm that the values of $\vec{c}\cdot\vec{a},\vec{c}\cdot\vec{b}$ are 0 (vertical).

[📖] [OK] [∨] [∨] [∨] [OK] [📖] [OK] [OK] [OK] [📖] [OK] [∨] [OK] [EXE]

[📖] [OK] [∨] [∨] [∨] [OK] [📖] [OK] [OK] [OK] [📖] [OK] [∨] [∨] [OK] [EXE]

VctC·VctA
0
VctC·VctB
0

Calculate the value of $\vec{b}\times\vec{a}$.

[AC] [📖] [OK] [∨] [∨] [OK] [📖] [OK] [OK] [∨] [OK] [📖] [OK] [∨] [OK] [EXE]

VctB×VctA

VctAns= [-5 3 -4]
-5

Register $\vec{d}=\vec{b}\times\vec{a}$.

[ooo] [OK] [∨] [∨] [∨] [OK] [EXE]

VctA
VctB
VctC
VctD

VctD= [-5 3 -4]
-5

Confirm that the values of $\vec{d}\cdot\vec{a},\vec{d}\cdot\vec{b}$ are 0 (vertical).

[📖] [OK] [∨] [∨] [∨] [∨] [OK] [📖] [OK] [OK] [OK] [📖] [OK] [∨] [OK] [EXE]

[📖] [OK] [∨] [∨] [∨] [∨] [OK] [📖] [OK] [OK] [OK] [📖] [OK] [∨] [∨] [OK] [EXE]

VctD·VctA
0
VctD·VctB
0

Confirm that $\vec{c}+\vec{d}$ is a null vector (magnitudes are equal).

[AC] [📖] [OK] [∨] [∨] [∨] [OK] [+] [📖] [OK] [∨] [∨] [∨] [∨] [OK] [EXE]

VctC+VctD

VctAns= [0 0 0]
0

(3) Find the area of the parallelogram formed by $\vec{a}$ and $\vec{b}$.

$|\vec{a}\times\vec{b}| = \sqrt{5^2+(-3)^2+4^2} = \sqrt{50} = 5\sqrt{2}$

(As an alternative solution, $|\vec{b}\times\vec{a}| = \sqrt{(-5)^2+3^2+(-4)^2} = \sqrt{50} = 5\sqrt{2}$ is also possible.)

$5\sqrt{2}$

check

Calculate the angle θ formed by $\vec{a}$ and $\vec{b}$.

[📖] [OK] [OK] [⌄] [⌄] [OK] [📖] [OK] [⌄] [OK] [⬆] [)] [📖] [OK] [⌄] [⌄] [OK] [)] [EXE]

```
Angle(VctA,VctB)
        147.2661034
```

Calculate and verify the value of $|\vec{a}||\vec{b}|\sin\mathrm{Ans}$.

[📖] [⌄] [⌄] [⌄] [OK] [OK] [📖] [OK] [⌄] [OK] [)] [📖] [⌄] [⌄] [⌄] [OK] [OK] [📖] [OK] [⌄] [⌄] [OK] [)]

[sin] [Ans] [)] [EXE]　[Ans] [−] [5] [√■] [2] [)] [EXE]

```
        147.2661034
Abs(VctA)Abs(VctB
)sin(Ans)
        7.071067812
```

```
)sin(Ans)
        7.071067812
Ans-5√(2)
                  0
```

(4) Find a vector $\vec{p}$ that is perpendicular to both $\vec{a}$ and $\vec{b}$ with a magnitude of 30.

From $\vec{a}\times\vec{b} = (5,-3,4)$, $\vec{b}\times\vec{a} = (-5,3,-4)$, $|\vec{a}\times\vec{b}| = 5\sqrt{2}$, we get

$$\vec{p} = \pm\frac{|\vec{p}|}{|\vec{a}\times\vec{b}|}(\vec{a}\times\vec{b}) = \pm\frac{30}{5\sqrt{2}}(5,-3,4) = \pm 3\sqrt{2}(5,-3,4)$$

$\pm 3\sqrt{2}(5,-3,4)$

check

Calculate the value of $|3\sqrt{2}\vec{c}|$.

[AC]　[3] [√■] [2] [)] [📖] [OK] [⌄] [⌄] [⌄] [OK] [EXE]

Press [📖] [⌄] [⌄] [⌄] [OK] [OK], [📖], **select** [Vector], **press** [OK], **select** [VctAns], **press** [OK], [)] [EXE]

```
3√(2)VctC
```

```
VctAns=
[21.213]
[-12.72]
[ 16.97]
        21.21320344
```

```
Abs(VctAns)
                 30
```

Confirm that the values of $3\sqrt{2}\vec{c}\cdot\vec{a}, 3\sqrt{2}\vec{c}\cdot\vec{b}$ are 0 (vertical).

[📖] [OK] [⌄] [⌄] [⌄] [⌄] [⌄] [OK] [📖] [OK] [OK] [OK] [📖] [OK] [⌄] [OK] [EXE]

[📖] [OK] [⌄] [⌄] [⌄] [⌄] [⌄] [OK] [📖] [OK] [OK] [OK] [📖] [OK] [⌄] [⌄] [OK] [EXE]

```
VctAns・VctA
                  0
VctAns・VctB
                  0
```

Calculate the value of $|3\sqrt{2}\vec{d}|$.

[AC]　[3] [√■] [2] [)] [📖] [OK] [⌄] [⌄] [⌄] [⌄] [OK] [EXE]

[📖] [⌄] [⌄] [⌄] [OK] [OK] [📖] [OK] [⌄] [⌄] [⌄] [⌄] [⌄] [OK] [)] [EXE]

```
3√(2)VctD
```

```
VctAns=
[-21.21]
[12.727]
[-16.97]
       -21.21320344
```

```
Abs(VctAns)
                 30
```

Confirm that the values of $3\sqrt{2}\vec{d}\cdot\vec{a}, 3\sqrt{2}\vec{d}\cdot\vec{b}$ are 0 (vertical).

[📖] [OK] [⌄] [⌄] [⌄] [⌄] [⌄] [OK] [📖] [OK] [OK] [OK] [📖] [OK] [⌄] [OK] [EXE]

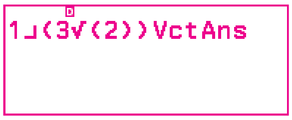
[📖] [OK] [⌄] [⌄] [⌄] [⌄] [⌄] [OK] [📖] [OK] [OK] [OK] [📖] [OK] [⌄] [⌄] [OK] [EXE]

```
VctAns・VctA
                  0
VctAns・VctB
                  0
```

OTHER METHODS

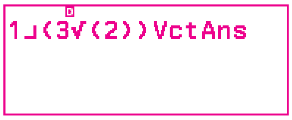
Calculate a vector with the same direction as $\vec{c}$ and a magnitude of 30.

Verify the components.



Calculate a vector with the same direction as $\vec{d}$ and a magnitude of 30.

Verify the components.



CASIO Essential Materials

Publisher: CASIO Institute for Educational Development

Date of Publication: 2023/12/22 (1st edition)

https://edu.casio.com