10 Sequences

CASIO
Essential Materials

Introduction

These teaching materials were created with the hope of conveying to many teachers and students the appeal of scientific calculators.

(1) Change awareness (emphasizing the thinking process) and boost efficiency in learning mathematics

- By reducing the time spent on manual calculations, we can have learning with a focus on the thinking process that is more efficient.
- This reduces the aversion to mathematics caused by complicated calculations, and allows students to experience the joy of thinking, which is the essence of mathematics.

(2) Diversification of learning materials and problem-solving methods

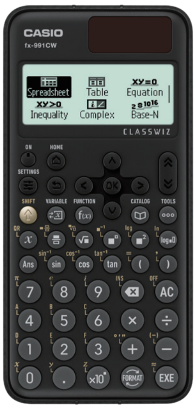
- Making it possible to do difficult calculations manually allows for diversity in learning materials and problem-solving methods.

(3) Promoting understanding of mathematical concepts

- By using the various functions of the scientific calculator in creative ways, students are able to deepen their understanding of mathematical concepts through calculations and discussions from different perspectives than before.
- This allows for exploratory learning through easy trial and error of questions.
- Listing and graphing of numerical values by means of tables allows students to discover laws and to understand visually.

Features of this book

- As well as providing first-time scientific calculator users with opportunities to learn basic scientific calculator functions from the ground up, the book also has material to show people who already use scientific calculators the appeal of scientific calculators described above.
- You can also learn about functions and techniques that are not available on conventional Casio models or other brands of scientific calculators.
- This book covers many units of high school mathematics, allowing students to learn how to use the scientific calculator as they study each topic.
- This book can be used in a variety of situations, from classroom activities to independent study and homework by students.



Better Mathematics Learning
with Scientific Calculator

Structure

Quadratic functions

TARGET To understand the concept of functions.

STUDY GUIDE

Functions

Definition of a function

For 2 variables, x and y, if the value of x is determined and if just 1 corresponding value of y is determined, then we say that y **is a function of** x.

Ex. Given $x=2$ ➡ Functions ($y=x+1$) ➡ We can determine $y=3$

How to describe functions

When y is a function of x, generally y is expressed as $f(x)$ or $g(x)$. Furthermore, in the function $y=f(x)$, then when $x=a$, the value of y is expressed as $f(a)$.

Ex. In the function $f(x)=3x+3$, when $x=1$, then when $f(1)=3\times1+3=6$ and $x=5$, then we get $f(5)=3\times5+3=18$.

Linear functions & Quadratic functions

When y is expressed as a linear equation of x, y is called the **linear function** of x, which is expressed as follows.

$$\boxed{y=ax+b \quad (a\neq0)}$$

When y is expressed as a quadratic equation of x, y is called the **quadratic function** of x, which is expressed as follows.

$$\boxed{y=ax^2+bx+c \quad (a\neq0)}$$

TARGET

Students can identify the objective to learn in each section.

STUDY GUIDE

Mathematical theorems and concepts are explained in detail. A scientific calculator is used to check and derive formulas according to the topic.

EXERCISE

1 For the quadratic function $f(x)=3x^2+2x+6$, find the various following values.

(1) $f(0)$

$f(0)=3\cdot0^2+2\cdot0+6=6$ **6**

(2) $f(1)$

$f(1)=3\cdot1^2+2\cdot1+6=11$ **11**

(3) $f(-1)$

$f(-1)=3\cdot(-1)^2+2\cdot(-1)+6=7$ **7**

(4) $f(109)$

$f(109)=3\cdot109^2+2\cdot109+6=35867$ **35867**

check

Use the FUNCTION function to calculate the value of $f(x)$.

Press ⌂, select [Calculate], press OK

Press f(x), select [Define f(x)], press OK 3 x x² + 2 x + 6 EXE

EXERCISE

Students learn basic examples based on the explanation in Study Guide.

check

Explains how to use the scientific calculator to solve problems and check answers.

PRACTICE

1 For the quadratic function $f(x)=-2x^2+4x+5$, find the various following values.

(1) $f(0)$

$f(0)=-2\cdot0^2+4\cdot0+5=5$ **5**

(2) $f(2)$

$f(2)=-2\cdot2^2+4\cdot2+5=5$ **5**

(3) $f(-3)$

$f(-3)=-2\cdot(-3)^2+4\cdot(-3)+5=-25$ **-25**

(4) $f(52)$

$f(52)=-2\cdot52^2+4\cdot52+5=-5195$ **-5195**

PRACTICE

Students can do practice problems similar to those in EXERCISE. They can also practice using the scientific calculator as they learned to in Check.

ADVANCED

3 We know that when an object is thrown vertically upward from a height of 0 m at a velocity of v m/s, the relation between the height y m of the object and the time x seconds since it was thrown is $y=-4.9x^2+vx$. So, when a ball is thrown vertically from a height of 0 m at a velocity of 9.8 m/s, find the time it takes to reach the highest point and the height of that highest point. Note that the air resistance can be ignored.

$y=-4.9x^2+9.8x=-4.9(x^2-2x)=-4.9\{(x-1)^2-1^2\}=-4.9(x-1)^2+4.9$

Therefore, the graph of this function is convex upward, the axis is $x=1$, and the vertex is $(1, 4.9)$.

Thus, the ball takes 1 second to reach its highest point, and the height of that highest point is 4.9 m.

1 second, 4.9 m

check

Press ⌂, select [Equation], press OK

Select [Polynomial], press OK, select [ax²+ bx+ c], press OK

− 4 . 9 EXE 9 . 8 EXE 0 EXE EXE

EXE EXE EXE **Press ⇧ x, scan the QR code to display a graph.**

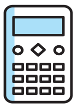
ADVANCED

Practical problems have been included in several topics. Solutions using a scientific calculator are also presented as necessary.

Other marks

Ex.	Simple examples on how to apply equations and theorems
explanation	Formulas and their supplementary explanations
proof	Proofs and checks of mathematical formulas
EXTRA Info.	Knowledge and information on formulas and other supplementary information in other units
OTHER METHODS	Alternative solutions and different verification methods for previously presented problems

Calculator mark



Where to use the scientific calculator

Colors of fonts in the teaching materials

- In STUDY GUIDE, important mathematical terms and formulas are printed in blue.
- In PRACTICE and ADVANCED the answers are printed in red.
  (Separate data is also available without the red parts, so it can be used for exercises.)

Applicable models

The applicable model is fx-991CW.
(Instructions on how to do input are for the fx-991CW, but in many cases similar calculations can be done on other models.)

Related Links

- Information and educational materials relevant to scientific calculators can be viewed on the following site.
  https://edu.casio.com
- The following video can be viewed to learn about the multiple functions of scientific calculators.
  https://www.youtube.com/playlist?list=PLRgxo9AwbIZLurUCZnrbr4cLfZdqY6aZA

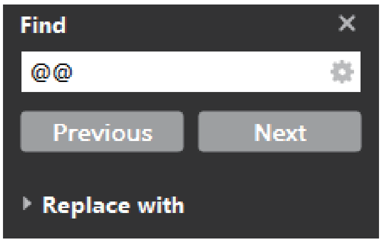
How to use PDF data

About types of data

- Data for all unit editions and data for each unit are available.
- For the above data, the PRACTICE and ADVANCED data without the answers in red is also available.

How to find where the scientific calculator is used

(1) Open a search window in the PDF Viewer.
(2) Type in "@@" as a search term.
(3) You can sequentially check where the calculator marks appear in the data.



How to search for a unit and section

(1) Search for units of data in all unit editions

- The data in all unit editions has a unit table of contents.
- Selecting a unit in the table of contents lets you jump to the first page of that unit.
- There is a bookmark on the first page of each unit, so you can jump from there also.

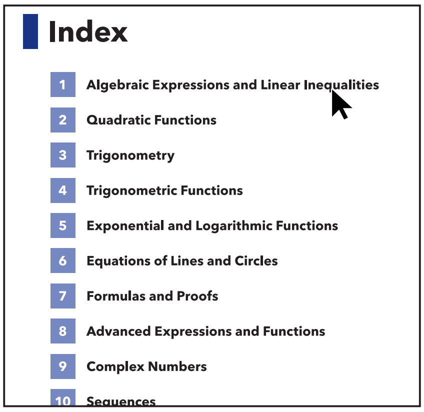
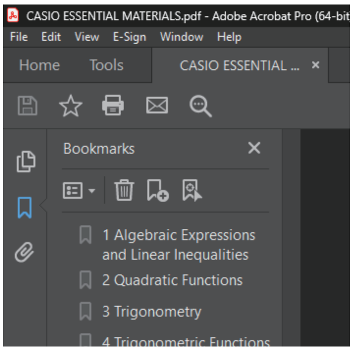


Table of contents of unit



Bookmark of unit

(2) Search for sections

- There are tables of contents for sections on the first page of units.
- Selecting a section in the table of contents takes you to the first page of that section.

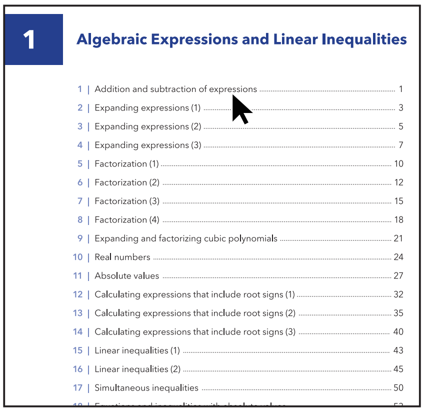


Table of contents of section

Sequences

TARGET To understand how to express a sequence and its general term.

STUDY GUIDE

General terms of sequences

Numbers arranged in a row according to a rule are called a **sequence**, and each individual number is called a **term**.

A sequence with a finite number of terms is called a **finite sequence**, and a sequence with an infinite number of terms is called an **infinite sequence**.

We say that a finite sequence has a **number of terms** and there is a **first term** and a **last term**.

Sequences are generally expressed as follows using letters with indices.

$$\boxed{a_1, a_2, a_3, \cdots\cdots, a_n, \cdots\cdots}$$

The sequence above can also be expressed as $\{a_n\}$.

When the nth term in the sequence $\{a_n\}$ is expressed in an expression as n, we say it is the **general term** of the sequence $\{a_n\}$.

Ex. Given the sequence $\{a_n\}$ of 2, 4, 6, 8, and 10 has 5 terms: $a_1=2$(first term), $a_2=4$, $a_3=6$, $a_4=8$, $a_5=10$(last term)

From $a_1=2\cdot1$, $a_2=2\cdot2$, $a_3=2\cdot3$, $a_4=2\cdot4$, and $a_5=2\cdot5$, we can estimate that the general term is $a_n=2n$.

EXERCISE

1 Find the first to 4th terms of the sequence $\{a_n\}$ whose general term is expressed by the following expression.

(1) $a_n=3n+2$

When the first term is $n=1$, we get $a_1=3\cdot1+2=5$

When the 2nd term is $n=2$, we get $a_2=3\cdot2+2=8$

When the 3rd term $n=3$, we get $a_3=3\cdot3+2=11$

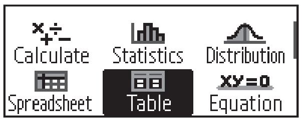
When the 4th term is $n=4$, we get $a_4=3\cdot4+2=14$

$$a_1=5, a_2=8, a_3=11, a_4=14$$

check

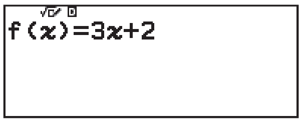
On the scientific calculator, use the Table function to confirm each term of the sequence.

Press ⌂, select [Table], press OK, then clear the previous data by pressing ⏻



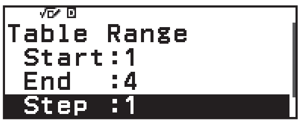
Press ⋯, select [Define f(x)/g(x)], press OK, select [Define f(x)], press OK

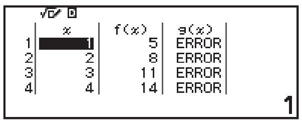
After inputting f(x)=3x+2, press EXE



Press ⋯, select [Table Range], press OK

After inputting [Start:1, End:4, Step:1], select [Execute], press EXE





(2) $a_n = n^2 - 4n$

When the first term is $n=1$, we get $a_1=1^2-4\cdot1=-3$

When the 2nd term is $n=2$, we get $a_2=2^2-4\cdot2=-4$

When the 3rd term is $n=3$, we get $a_3=3^2-4\cdot3=-3$

When the 4th term is $n=4$, we get $a_4=4^2-4\cdot4=0$

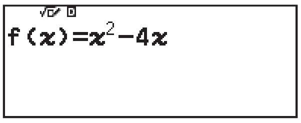
$$\boldsymbol{a_1 = -3, a_2 = -4, a_3 = -3, a_4 = 0}$$

check

Press ⌂, select [Table], press OK, then clear the previous data by pressing ⏻

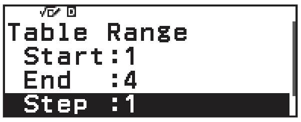
Press ⋯, select [Define $\mathrm{f}(x)/\mathrm{g}(x)$], press OK, select [Define $\mathrm{f}(x)$], press OK

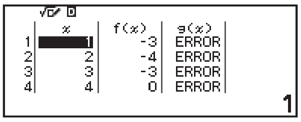
After inputting $\mathrm{f}(x)= x^2 - 4x$, press EXE



Press ⋯, select [Table Range], press OK

After inputting [Start:1, End:4, Step:1], select [Execute], press EXE





2 Estimate the general terms of the sequences $\{a_n\}$ below.

(1) $-5, -10, -15, -20, \cdots\cdots$

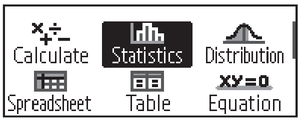
$a_1=-5\cdot1,\ a_2=-5\cdot2,\ a_3=-5\cdot3,\ a_4=-5\cdot4, \cdots\cdots$

Therefore, we can estimate the general term to be $a_n=-5n$.

$$\boldsymbol{a_n = -5n}$$

check

On the scientific calculator, use the Statistics function to confirm the general term of the sequence.

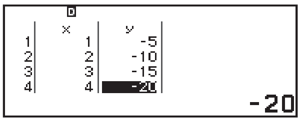


Press ⌂, select [Statistics], press OK, select [2-Variable], press OK

Press ⋯, select [Edit], press OK, select [Delete All], press OK

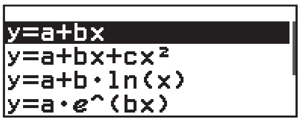
Input 1, 2, 3, and 4 in the x column, and -5, -10, -15, and -20 in the y column, respectively.

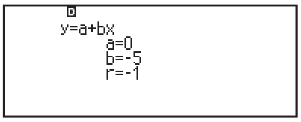
1 EXE 2 EXE 3 EXE 4 EXE ⌄ › − 5 EXE − 1 0 EXE − 1 5 EXE − 2 0 EXE EXE



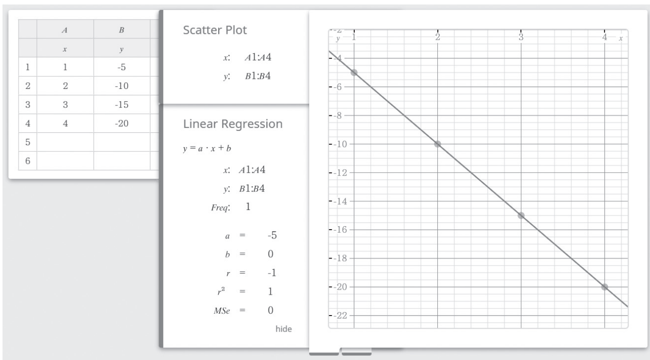
Select [Reg Results], press OK, select [y=a+bx], press OK

We can confirm that $y=-5x$.





Press ↶ ⬆ ⓧ, scan the QR code to display a graph.



(2) $1, 4, 9, 16, \cdots\cdots$

$a_1=1^2$, $a_2=2^2$, $a_3=3^2$, $a_4=4^2$, $\cdots\cdots$

Therefore, we can estimate the general term to be $a_n = n^2$.

$$a_n = n^2$$

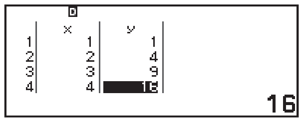
check

Press ⌂, select [Statistics], press OK, select [2-Variable], press OK

Press ⋯, select [Edit], press OK, select [Delete All], press OK

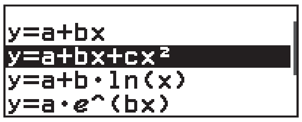
Input 1, 2, 3, and 4 in the x column, and 1, 4, 9, and 16 in the y column, respectively.

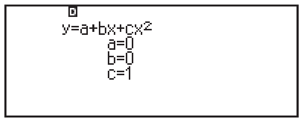
1 EXE 2 EXE 3 EXE 4 EXE ∨ > 1 EXE 4 EXE 9 EXE 1 6 EXE EXE



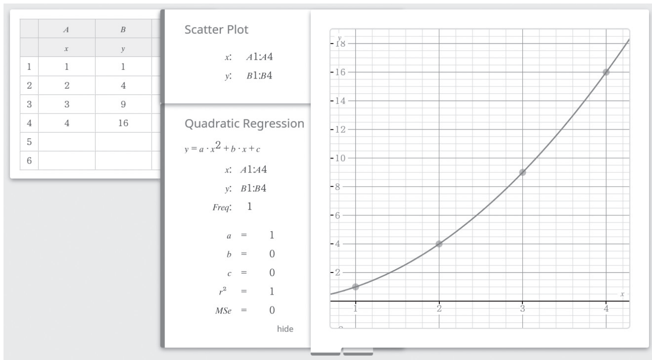
Select [Reg Results], press OK, select [y=a+bx+cx^2], press OK

We can confirm that $y=x^2$.





Press ↶ ⇧ 𝑥, scan the QR code to display a graph.



PRACTICE

1 Find the first to 4th terms of the sequence $\{a_n\}$ whose general term is expressed by the following expression.

(1) $a_n = 5n - 3$

When the first term is $n=1$**, we get** $a_1 = 5 \cdot 1 - 3 = 2$

When the 2nd term is $n=2$**, we get** $a_2 = 5 \cdot 2 - 3 = 7$

When the 3rd term is $n=3$**, we get** $a_3 = 5 \cdot 3 - 3 = 12$

When the 4th term is $n=4$**, we get** $a_4 = 5 \cdot 4 - 3 = 17$

$$a_1 = 2, a_2 = 7, a_3 = 12, a_4 = 17$$

check

Press ⌂**, select** [Table]**, press** (OK)**, then clear the previous data by pressing** ⏻

Press (⋯)**, select** [Define f(x)/g(x)]**,**

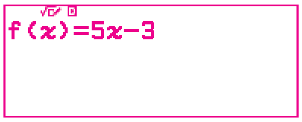
press (OK)**, select** [Define f(x)]**, press** (OK)

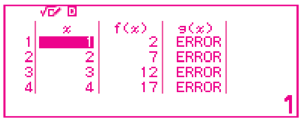
After inputting f(x)=$5x-3$**, press** (EXE)

Press (⋯)**, select** [Table Range]**, press** (OK)

After inputting [Start:1, End:4, Step:1]**,**

select [Execute]**, press** (EXE)





(2) $a_n = (-1)^n n$

When the first term is $n=1$**, we get** $a_1 = (-1)^1 \cdot 1 = -1$

When the 2nd term is $n=2$**, we get** $a_2 = (-1)^2 \cdot 2 = 2$

When the 3rd term is $n=3$**, we get** $a_3 = (-1)^3 \cdot 3 = -3$

When the 4th term is $n=4$**, we get** $a_4 = (-1)^4 \cdot 4 = 4$

$$a_1 = -1, a_2 = 2, a_3 = -3, a_4 = 4$$

check

Press ⌂**, select** [Table]**, press** (OK)**, then clear the previous data by pressing** ⏻

Press (⋯)**, select** [Define f(x)/g(x)]**,**

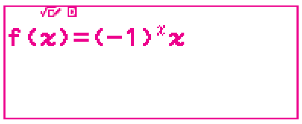
press (OK)**, select** [Define f(x)]**, press** (OK)

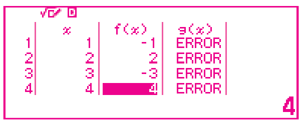
After inputting f(x)=$(-1)^x x$**, press** (EXE)

Press (⋯)**, select** [Table Range]**, press** (OK)

After inputting [Start:1, End:4, Step:1]**,**

select [Execute]**, press** (EXE)





2 Estimate the general terms of the sequences $\{a_n\}$ below.

(1) 3, 5, 7, 9, ……

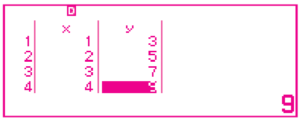
$a_1=2\cdot1+1,\ a_2=2\cdot2+1,\ a_3=2\cdot3+1,\ a_4=2\cdot4+1,\ \cdots\cdots$

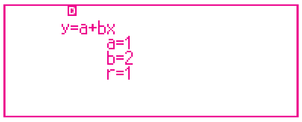
Therefore, we can estimate the general term to be $a_n=2n+1$.

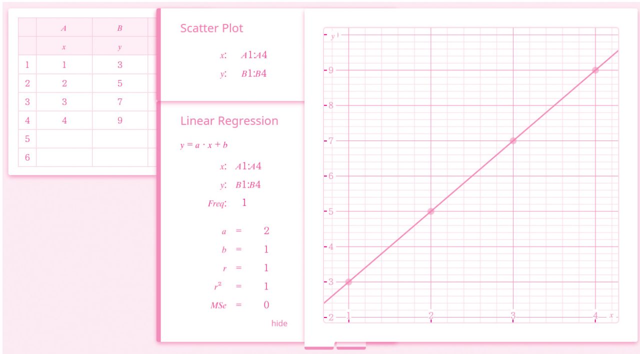
$$a_n=2n+1$$

check

Press ⌂, select [Statistics], press OK,
select [2-Variable], press OK
Press ⋯, select [Edit], press OK,
select [Delete All], press OK
After inputting 1, 2, 3, and 4 in the x column and 3, 5, 7, and 9 in the y column, respectively, press EXE
Select [Reg Results], press OK,
select [y=a+bx], press OK
We can confirm that $y=2x+1$.
Press ↩ ⇧ ⓧ, scan the QR code to display a graph.







(2) 2, 4, 8, 16, ……

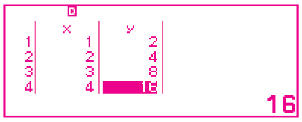
$a_1=2^1,\ a_2=2^2,\ a_3=2^3,\ a_4=2^4$

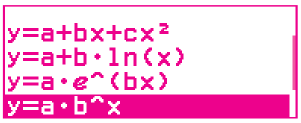
Therefore, we can estimate the general term to be $a_n=2^n$.

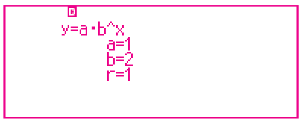
$$a_n=2^n$$

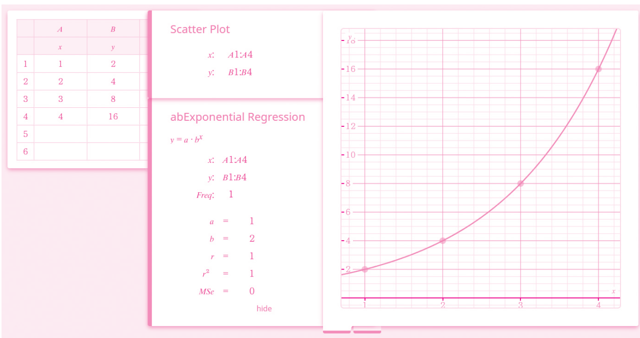
check

Press ⌂, select [Statistics], press OK,
select [2-Variable], press OK
Press ⋯, select [Edit], press OK,
select [Delete All], press OK
After inputting 1, 2, 3, and 4 in the x column, and 2, 4, 8, and 16 in the y column, respectively, press EXE
Select [Reg Results], press OK,
select [y=a·b^x], press OK
We can confirm that $y=2^x$.
Press ↩ ⇧ ⓧ, scan the QR code to display a graph.









Arithmetic progression

TARGET To understand how to express an arithmetic progression and its general term.

STUDY GUIDE

Arithmetic progression

General terms of arithmetic progressions

In a sequence in which subsequent terms are the sum of a constant number and each term, the difference between adjacent terms is constant. In this way, a sequence formed by starting at an initial term $a_1=a$ and adding a constant number d to subsequent terms is called an **arithmetic progression**, and d is called the **common difference**. We can express the general term of an arithmetic progression $\{a_n\}$ with an initial term a and a common difference d as the following **linear expression for n**.

$$\boxed{a_n = a + (n-1)d}$$

$$\underbrace{a_1 \;,\; \xrightarrow{+d} a_2 \;,\; \xrightarrow{+d} a_3 \;,\; \cdots \;,\; a_{n-1} \;,\; \xrightarrow{+d} a_n}_{(n-1)\text{ elements}} \;,\; \cdots$$

Ex. $2, 5, 8, 11, \cdots\cdots, a_n, \cdots\cdots$

$a_2-a_1=5-2=3,\ a_3-a_2=8-5=3,\ a_4-a_3=11-8=3, \cdots\cdots$

Since the difference between adjacent terms is constant, the sequence $\{a_n\}$ is an arithmetic progression with an initial term of 2 and a common difference of 3.

Arithmetic mean

When a, b, and c form an arithmetic progression in this order, the following properties hold. And, this b is called the **arithmetic mean**.

$$\boxed{2b=a+c \;\Rightarrow\; b = \frac{a+c}{2}}$$

explanation

Since the difference between a and b, and b and c is constant (common difference), then from $b-a=c-b$, we get $2b=a+c$, $b = \dfrac{a+c}{2}$.

EXERCISE

◆ Solve the following problems.

(1) Find the general term a_n of the arithmetic progression $5, 9, 13, 17, \cdots\cdots$.

Since each term is the preceding term plus 4, the common difference is 4.

The first term is 5, so the general term is $a_n = 5+(n-1)\cdot 4 = 4n+1$

$$a_n = 4n+1$$

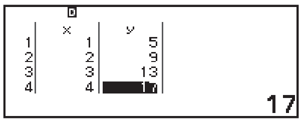
check

Press ⌂, select [Statistics], press OK, select [2-Variable], press OK

Press ⋯, select [Edit], press OK, select [Delete All], press OK

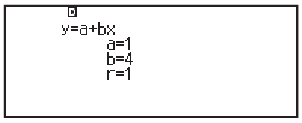
Input 1, 2, 3, and 4 in the x column, and 5, 9, 13, and 17 in the y column, respectively.

1 EXE 2 EXE 3 EXE 4 EXE ∨ > 5 EXE 9 EXE 1 3 EXE 1 7 EXE EXE

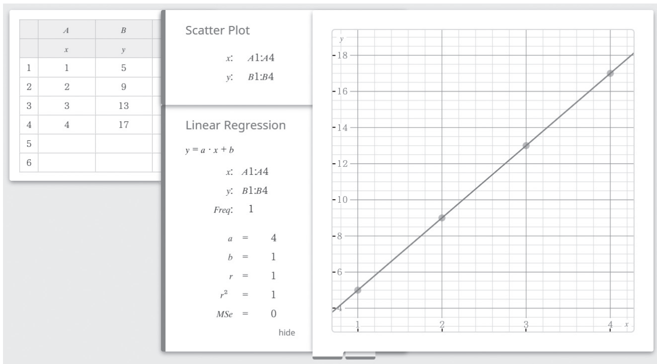


Select [Reg Results], press OK, select [y=a+bx], press OK

We can confirm that $y=4x+1$.



Press ⮌ ⇧ 𝑥, scan the QR code to display a graph.



(2) For the arithmetic progression $\{a_n\}$, find the first term and common difference when $a_3=15$ and $a_7=27$. Also, find the general term a_n.

Given the first term is a and the common difference is d and the general term is $a_n=a+(n-1)d$.

Since the 3rd term is 15, $a+(3-1)d=15$ and $a+2d=15$ ...(i)

Since the 7th term is 27, $a+(7-1)d=27$ and $a+6d=27$ ...(ii)

From (i) and (ii), we get $d=3$ and $a=9$

Therefore, $a_n=9+(n-1)\cdot 3=3n+6$

First term 9, common difference 3, and $a_n=3n+6$

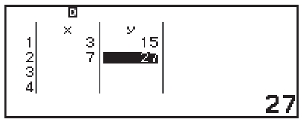
check

Press ⌂, select [Statistics], press OK, select [2-Variable], press OK

Press ⋯, select [Edit], press OK, select [Delete All], press OK

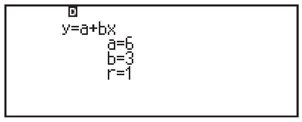
Input 3 and 7 in the x column, and 15 and 27 in the y column, respectively.

3 EXE 7 EXE ∨ > 1 5 EXE 2 7 EXE EXE

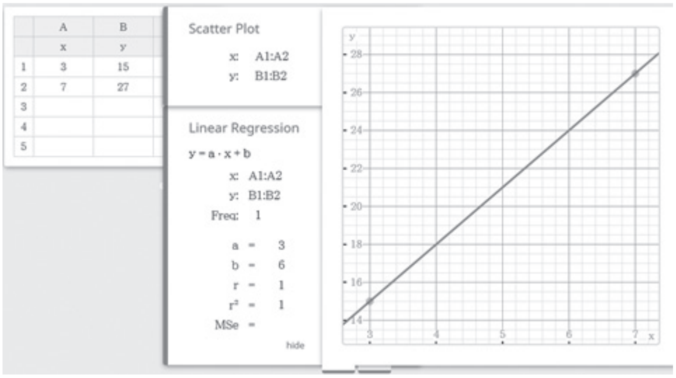


Select [Reg Results], press OK, select [y=a+bx], press OK

We can confirm that $y=3x+6$.



Press ↶ ⇧ x, scan the QR code to display a graph.



(3) When 3, x, and 8 form an arithmetic progression in this order, find the value of x.

Since 3, x, and 8 form an arithmetic progression in this order, we get $x=\dfrac{3+8}{2}=\dfrac{11}{2}$

$$\mathbf{\frac{11}{2}}$$

PRACTICE

◆ Solve the following problems.

(1) Find the general term a_n of the arithmetic progression $23, 21, 19, 17, \cdots\cdots$. Also, at what term does the first negative appear?

Since each term is the preceding term plus -2, the common difference is -2.

The first term is 23, so the general term is $a_n = 23 + (n-1)\cdot(-2) = -2n + 25$

Furthermore, the first negative value of n is the smallest natural number n that satisfies $a_n = -2n + 25 < 0$.

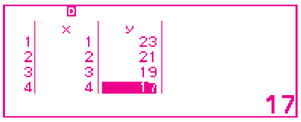
From $-2n + 25 < 0$, we get $n > \dfrac{25}{2}$, so the smallest natural number n that satisfies this is 13.

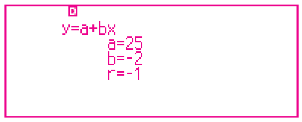
Therefore, it is the 13th term.

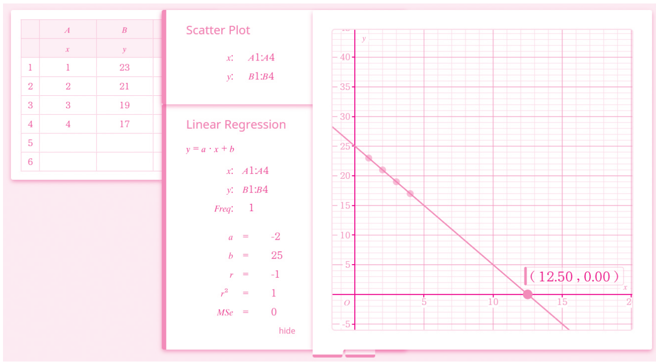
$a_n = -2n + 25$, 13th term

check

Press ⌂, select [Statistics], press (OK),
select [2-Variable], press (OK)
Press (⋯), select [Edit], press (OK),
select [Delete All], press (OK)
After inputting 1, 2, 3, and 4 in the x column, and 23, 21, 19, and 17 in the y column, respectively, press (EXE)
Select [Reg Results], press (OK),
select [y=a+bx], press (OK)
We can confirm that $y = -2x + 25$.
Press ↺ ⇧ (x), scan the QR code to display a graph.







(2) For the arithmetic progression $\{a_n\}$, find the first term and common difference when $a_4=\frac{11}{5}$ and $a_8=\frac{21}{5}$. Also, find the general term a_n.

Given the first term is a and the common difference is d and the general term is $a_n=a+(n-1)d$.

Since the 4th term is $\frac{11}{5}$, we get $a+(4-1)d=\frac{11}{5}$, $a+3d=\frac{11}{5}$ …(i)

Since the 8th term is $\frac{21}{5}$, we get $a+(8-1)d=\frac{21}{5}$, $a+7d=\frac{21}{5}$ …(ii)

From (i) and (ii), we get $d=\frac{1}{2}$, $a=\frac{7}{10}$

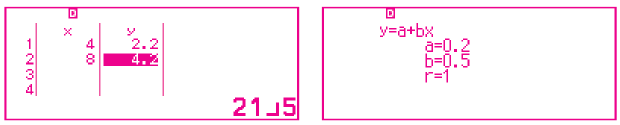
Therefore, $a_n=\frac{7}{10}+(n-1)\cdot\frac{1}{2}=\frac{1}{2}n+\frac{1}{5}$

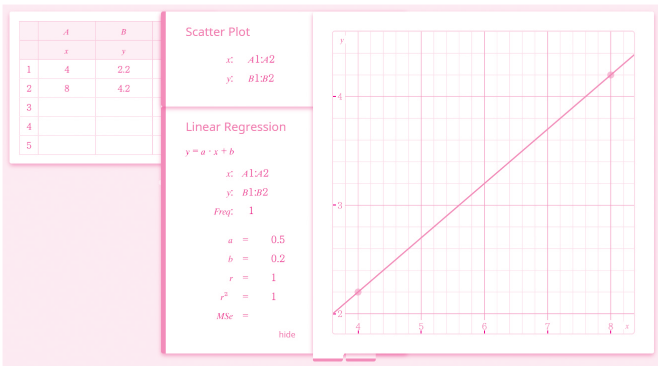
First term $\frac{7}{10}$, common difference $\frac{1}{2}$, and $a_n=\frac{1}{2}n+\frac{1}{5}$

check

Press ⌂, select [Statistics], press OK, select [2-Variable], press OK

Press ⋯, select [Edit], press OK, select [Delete All], press OK





After inputting 4 and 8 in the x column, and $\frac{11}{5}$ and $\frac{21}{5}$ in the y column, respectively, press EXE

Select [Reg Results], press OK, select [y=a+bx], press OK

We can confirm that $y=0.5x+0.2=\frac{1}{2}x+\frac{1}{5}$.

Press ↺ ⇧ ⓧ, scan the QR code to display a graph.

(3) When x, $2x+3$, and 8 form an arithmetic progression in this order, find the value of x.

Since x, $2x+3$, and 8 form an arithmetic progression in this order, we get $2x+3=\frac{x+8}{2}$, $2(2x+3)=x+8$, and $x=\frac{2}{3}$

$\frac{2}{3}$

Sums of arithmetic progressions (1)

TARGET To understand how to find the sums of arithmetic progressions.

STUDY GUIDE

Sums of arithmetic progressions

Given the sum S_n of the first term to the nth term of the arithmetic progression $\{a_n\}$, which has a first term a, a common difference d, n number of terms, and a last term l, we can express S_n by the following **quadratic equation for n.**

$$\boxed{S_n=\frac{1}{2}n(a+l)=\frac{1}{2}n\{2a+(n-1)d\}}$$

explanation

$S_n=a+(a+d)+(a+2d)+\cdots\cdots+(l-2d)+(l-d)+l$ …(i)

By arranging the terms of (i) in reverse, we get $S_n=l+(l-d)+(l-2d)+\cdots\cdots+(a+2d)+(a+d)+a$ …(ii)

From (i)+(ii), we get

$$\begin{array}{rrcccccccccccccc}
 & S_n & = & a & + & (a+d) & + & (a+2d) & + & \ldots\ldots & + & (l-2d) & + & (l-d) & + & l \\
+) & S_n & = & l & + & (l-d) & + & (l-2d) & + & \ldots\ldots & + & (a+2d) & + & (a+d) & + & a \\
\hline
 & 2S_n & = & (a+l) & + & (a+l) & + & (a+l) & + & \ldots\ldots & + & (a+l) & + & (a+l) & + & (a+l)
\end{array}$$

Therefore, from $2S_n=n(a+l)$, we can derive $S_n=\dfrac{1}{2}n(a+l)$.

By substituting $l=a+(n-1)d$ into this, we can derive $S_n=\dfrac{1}{2}n\{2a+(n-1)d\}$.

Ex. The sum of the first term to the last term of an arithmetic progression with a first term of 3, common difference of 2, and with 15 terms is $S_{15}=\dfrac{1}{2}\cdot 15\cdot\{2\cdot 3+(15-1)\cdot 2\}=255$

EXERCISE

◆ Solve the following problems.

(1) Find the sum S_n of the first term to the nth term of the arithmetic progression $7, 10, 13, 16, \cdots\cdots$

The sum S_n of the first term to the nth term of an arithmetic progression with a first term of 7 and a common difference of 3 is $S_n = \dfrac{1}{2}n\{2\cdot 7+(n-1)\cdot 3\} = \dfrac{1}{2}n(3n+11) = \dfrac{3}{2}n^2 + \dfrac{11}{2}n$

$$\boldsymbol{S_n = \frac{3}{2}n^2 + \frac{11}{2}n}$$

check

On the scientific calculator, use the Statistics function to confirm the sum of the first term to the nth term of the arithmetic progression.

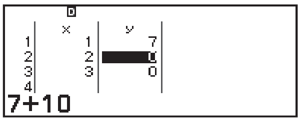
Press ⌂, select [Statistics], press OK, select [2-Variable], press OK

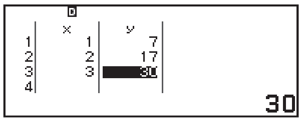
Press ⋯, select [Edit], press OK, select [Delete All], press OK

Input 1, 2, and 3 in column x, and the sum of the progression 7, 7+10, and 17+13 in column y, respectively.

1 EXE 2 EXE 3 EXE ∨ >

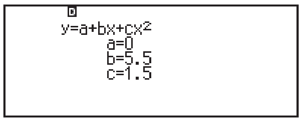
7 EXE 7 + 1 0 EXE 1 7 + 1 3 EXE EXE





Select [Reg Results], press OK, select [y=a+bx+cx^2], press OK

We can confirm that $y = 1.5x^2 + 5.5x = \dfrac{3}{2}x^2 + \dfrac{11}{2}x$.



(2) Find the sum S_{13} of an arithmetic progression with a first term of -3, a last term of 69, and with 13 terms. Also, find the common difference.

The sum S_{13} of the first term to the 13th term of an arithmetic progression with a first term of -3 and a last term of 69, and with 13 terms is $S_{13} = \dfrac{1}{2}\cdot 13(-3+69) = 429$

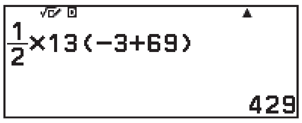
Substitute $a=-3$, $l=69$, and $n=13$ into $l = a+(n-1)d$ to get $69=-3+(13-1)d$, for $d=6$

$S_{13}=429$, common difference of 6

check

Press ⌂, select [Calculate], press OK

1 ⊟ 2 > × 1 3 (− 3 + 6 9) EXE



(3) Find the sum of the first term to the 6th term of an arithmetic progression with a first term of 7 and a common difference of -3. Also, find the sum of the 7th term to the 15th term of the arithmetic progression.

The sum S_n of the first term to the nth term of an arithmetic progression with a first term of 7 and a common difference of -3 is $S_n = \frac{1}{2}n\{2\cdot 7+(n-1)\cdot(-3)\} = \frac{1}{2}n(-3n+17)$

The sum of the first term to the 6th term is $S_6 = \frac{1}{2}\cdot 6\cdot(-3\cdot 6+17) = -3$

The sum of the 7th to the 15th terms is found by (sum of the first to 15th terms)$-$(sum of the first to 6th terms).

The sum of the first term to the 15th term is $S_{15} = \frac{1}{2}\cdot 15\cdot(-3\cdot 15+17) = -210$

Therefore, $S_{15}-S_6=-210+3=-207$

The sum of the first to the 6th term is…-3,
the sum of the 7th to the 15th term is…-207

check

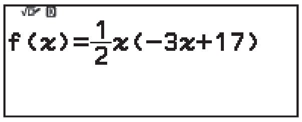
On the scientific calculator, use the FUNCTION function to register the expression to find the product of arithmetic progressions in $f(x)$.

Press ⌂, select [Calculate], press OK

Input $f(x)=\frac{1}{2}x(-3x+17)$.

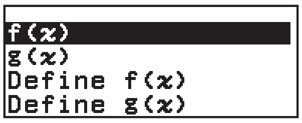
Press f(x), select [Define $f(x)$], press OK

1 ▭ 2 > x (− 3 x + 1 7) EXE



Calculate the sum of the first term to the 6th term.

Press f(x), select [$f(x)$], press OK, 6) EXE



Calculate the sum of the 7th term to the 15th term.

f(x) OK 1 5) − f(x) OK 6) EXE

PRACTICE

◆ Solve the following problems.

(1) Find the sum S_n of the first term to the nth term of the arithmetic progression $4, 11, 18, 25, \cdots\cdots$

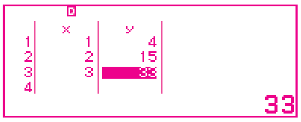
The sum S_n of the first term to the nth term of an arithmetic progression with a first term of 4 and a common difference of 7 is

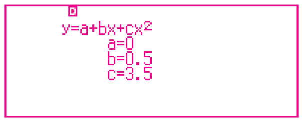
$$S_n = \frac{1}{2}n\{2\cdot 4+(n-1)\cdot 7\} = \frac{1}{2}n(7n+1) = \frac{7}{2}n^2 + \frac{1}{2}n$$

$$S_n = \frac{7}{2}n^2 + \frac{1}{2}n$$

check

Press ⌂, select [Statistics], press OK,
select [2-Variable], press OK
Press ⋯, select [Edit], press OK,
select [Delete All], press OK





After inputting 1, 2, and 3 in column x, and the sum of the progression 4, 4+11, and 15+18 in column y, respectively, press EXE

Select [Reg Results], press OK, select [y=a+bx+cx^2], press OK

We can confirm that $y = 3.5x^2 + 0.5x = \frac{7}{2}x^2 + \frac{1}{2}x$.

(2) Find the sum S_{17} of an arithmetic progression with a first term of 35, a last term of 3, and with 17 terms. Also, find the common difference.

The sum S_{17} of the first term to the 17th term of an arithmetic progression with a first term of 35 and a last term of 3, and with 17 terms is $S_{17} = \frac{1}{2}\cdot 17\cdot(35+3) = 323$

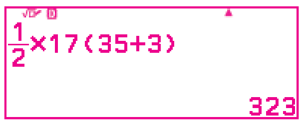
Substitute a=35, l=3, and n=17 into $l = a+(n-1)d$ to get $3=35+(17-1)d$, for $d=-2$

S_{17}=323, common difference of -2

check

Press ⌂, select [Calculate], press OK

① ⊟ ② ⊳ ⊗ ① ⑦ (③ ⑤ ⊕ ③) EXE



(3) Find the sum of the first term to the 7th term of an arithmetic progression with a first term of 1 and a common difference of 4. Also, find the sum of the 8th term to the 19th term of the arithmetic progression.

The sum S_n of the first term to the nth term of an arithmetic progression with a first term of 1 and a common difference of 4 is

$$S_n = \frac{1}{2}n\{2\cdot 1+(n-1)\cdot 4\} = n(2n-1)$$

The sum of the first term to the 7th term is $S_7=7\cdot(2\cdot 7-1)=91$
The sum of the 8th to the 19th terms is found by
(sum of the first to 19th terms)−(sum of the first to 7th terms).
The sum of the first term to the 19th term is $S_{19}=19\cdot(2\cdot 19-1)=703$
Therefore, $S_{19}-S_7=703-91=612$

The sum of the first to the 7th term is…91,
the sum of the 8th to the 19th term is…612

check

Register the expression f(x).

f(x) ∨ ∨ OK x (2 x − 1) EXE

f(x)=x(2x−1)

The sum of the first term to the 7th term

f(x) OK 7) EXE

f(7)
91

The sum of the 8th term to the 19th term

f(x) OK 1 9) − f(x) OK 7) EXE

f(19)−f(7)
612

Sums of arithmetic progressions (2)

TARGET To understand how to find the sums of sequences of various natural numbers.

STUDY GUIDE

Sums of sequences of various natural numbers

Sum of natural numbers from 1 to n

We can find the sum of natural numbers from 1 to n by using the following formula.

$$1+2+3+\cdots\cdots+n=\frac{1}{2}n(n+1)$$

explanation

Since the sequence of natural numbers, which is $1, 2, 3, \ldots\ldots, n$, is an arithmetic progression with a first term of 1, a common difference of 1, and n terms with a last term of n, we can find its sum by using the formula for the sum of arithmetic progressions as follows.

$$1+2+3+\cdots\cdots+n=\frac{1}{2}n(1+n)=\frac{1}{2}n(n+1)$$

Sum of the sequence of multiples of 2

We can find the sum of the sequence of multiples of 2 by using the following formula.

$$2+4+6+\cdots\cdots+2n=n(n+1)$$

explanation

Since the sequence of multiples of 2, which is $2, 4, 6, \ldots\ldots, 2n$, is an arithmetic progression with a first term of 2, a common difference of 2, and n terms with a last term of $2n$, we can find its sum by using the formula for the sum of arithmetic progressions as follows.

$$2+4+6+\cdots\cdots+2n=\frac{1}{2}n(2+2n)=n(n+1)$$

EXERCISE

◆ Find the following sums of the natural numbers from 1 to 100.

(1) Multiples of 3

Of the natural numbers from 1 to 100, the multiples of 3 form the sequence 3, 6, 9, 12,……, 96, 99

By rewriting this as 3·1, 3·2, 3·3, 3·4,……, 3·32, 3·33, it becomes an arithmetic progression with a first term of 3, a common difference of 3, a last term of 99, and 33 terms.

Therefore, its sum is $\frac{1}{2}\cdot 33\cdot(3+99)=1683$

1683

(2) Numbers that leave a remainder of 1 when divided by 3

Of the natural numbers from 1 to 100, the numbers that leave a remainder of 1 when divided by 3 form the sequence $3\cdot0+1, 3\cdot1+1, 3\cdot2+1, \ldots\ldots, 3\cdot33+1$

This sequence becomes an arithmetic progression with a first term of 1, a common difference of 3, and a last term of 100, and the n number of terms is $33+1=34$, to give us $n=34$.

Therefore, its sum is $\frac{1}{2}\cdot 34\cdot(1+100)=1717$

1717

(3) Numbers not divisible by 3

Let the

(Sum of numbers not divisible by 3)=(Sum of natural numbers from 1 to 100)−(Sum of multiples of 3 from 1 to 100).

(Sum of natural numbers from 1 to 100)$=1+2+3+\cdots\cdots+99+100=\frac{1}{2}\cdot100\cdot(1+100)=5050$

From (1), we know the sum of the multiples of 3 from 1 to 100 is 1683

Therefore, we find the sum is $5050-1683=3367$

3367

check

On the scientific calculator, use the VARIABLE function to, after registering the formula, input the values for A (number of terms), B (first term), and C (last term) to find the value.

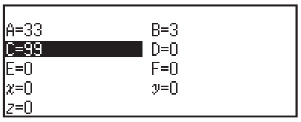
Press ⌂, select [Calculate], press OK

Input the formula $\frac{1}{2}\mathrm{A}(\mathrm{B}+\mathrm{C})$ for the sum of an arithmetic progression.

① ⊟ ② ▷ ⇧ ④ (⇧ ⑤ ⊕ ⇧ ⑥)

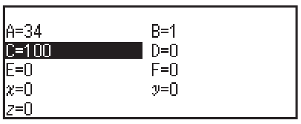
(1) In the VARIABLE screen, input [A=33, B=3, C=99], and then calculate.

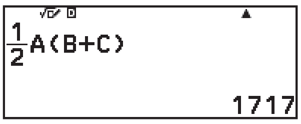
VAR ③ ③ EXE ▷ ③ EXE ▷ ⑨ ⑨ EXE ↺ EXE



(2) In the same way, input [A=34, B=1, C=100], and then calculate.

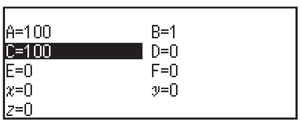
VAR ③ ④ EXE ▷ ① EXE ▷ ① ⓪ ⓪ EXE ↺ EXE

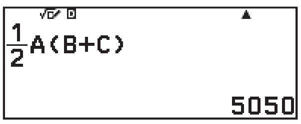




(3) In the same way, input [A=100, B=1, C=100], and then calculate.

VAR ① ⓪ ⓪ EXE ▷ ① EXE ▷ ① ⓪ ⓪ EXE ↺ EXE





PRACTICE

◆ Find the following sums of the natural numbers from 1 to 200.

(1) Multiples of 4

Of the natural numbers from 1 to 200, the multiples of 4 form the sequence 4, 8, 12, 16,……, 196, 200

By rewriting this as $4\cdot1, 4\cdot2, 4\cdot3, 4\cdot4,……, 4\cdot49, 4\cdot50$, it becomes an arithmetic progression with a first term of 4, a common difference of 4, a last term of 200, and 50 terms.

Therefore, its sum is $\frac{1}{2}\cdot50\cdot(4+200)=5100$

5100

(2) Numbers that leave a remainder of 3 when divided by 4

Of the natural numbers from 1 to 200, the numbers that leave a remainder of 3 when divided by 4 form the sequence $4\cdot0+3, 4\cdot1+3, 4\cdot2+3, ……, 4\cdot49+3$

This sequence becomes an arithmetic progression with a first term of 3, a common difference of 4, and a last term of 199, and the n number of terms is $49+1=50$, to give us $n=50$.

Therefore, its sum is $\frac{1}{2}\cdot50\cdot(3+199)=5050$

5050

(3) Numbers not divisible by 4

Let the (Sum of numbers not divisible by 4)
=(Sum of natural numbers from 1 to 200)−(Sum of multiples of 4 from 1 to 200).

(Sum of natural numbers from 1 to 200)$=1+2+3+\cdots+199+200=\frac{1}{2}\cdot200\cdot(1+200)=20100$

From (1), we know the sum of the multiples of 4 from 1 to 200 is 5100

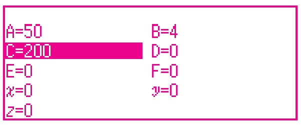
Therefore, we find the sum is $20100-5100=15000$

15000

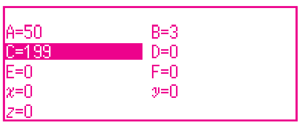
check

1 ⊟ 2 > ⇧ 4 (⇧ 5 + ⇧ 6)

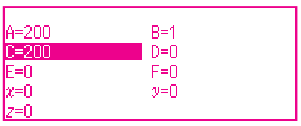
(1)
⇄ 5 0 EXE > 4 EXE > 2 0 0 EXE   ↶ EXE



(2)
⇄ 5 0 EXE > 3 EXE > 1 9 9 EXE   ↶ EXE



(3)
⇄ 2 0 0 EXE > 1 EXE > 2 0 0 EXE   ↶ EXE



Geometric progressions

TARGET To understand how to express a geometric progression and its general term.

STUDY GUIDE

Geometric progressions

General terms of geometric progressions

Consider a sequence in which subsequent terms are the product of a constant number and each term. When we start at a first term of $a_1=a$, then the sequence we get by multiplying each element by a constant number r is called a **geometric progression**. In this case, r is called the **common ratio**. We can express the general term of a geometric progression $\{a_n\}$ with an initial term a and a common ratio r as follows.

$$\boxed{a_n = ar^{n-1}}$$

$$a_1 \xrightarrow{\times r} a_2 \xrightarrow{\times r} a_3 \;,\; \cdots \;,\; a_{n-1} \xrightarrow{\times r} a_n \;,\; \cdots$$

$$\underbrace{\qquad\qquad\qquad\qquad\qquad\qquad}_{(n-1)\text{ elements}}$$

Ex. $1, 3, 9, 27, \cdots\cdots, a_n, \cdots\cdots$

$$\frac{a_2}{a_1}=\frac{3}{1}=3, \frac{a_3}{a_2}=\frac{9}{3}=3, \frac{a_4}{a_3}=\frac{27}{9}=3, \cdots\cdots$$

Since the ratio between adjacent terms is constant, the sequence $\{a_n\}$ is a geometric progression with an initial term of 1 and a common ratio of 3.

Geometric mean

When a, b, and c form a geometric progression in this order, the following properties hold. And, this b is called the **geometric mean**.

$$\boxed{b^2 = ac \quad (\text{However, } a\neq 0,\ b\neq 0, \text{ and } c\neq 0)}$$

explanation

Given a common ratio r, then from $b=ar, c=ar^2$ we get $b^2=a^2r^2, ac=a^2r^2$, so $b^2=ac$

EXERCISE

◆ Solve the following problems.

(1) Find the general term a_n of the geometric progression $3, 6, 12, 24, \cdots\cdots$.

Since each term is the preceding term multiplied by 2, the common ratio is 2.

Since the first term is 3, the general term is $a_n = 3 \cdot 2^{n-1}$

$$\boldsymbol{a_n = 3 \cdot 2^{n-1}}$$

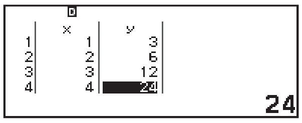
check

Press (HOME), select [Statistics], press (OK), select [2-Variable], press (OK)

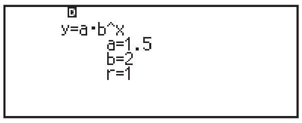
Press (⋯), select [Edit], press (OK), select [Delete All], press (OK)

Input 1, 2, 3, and 4 in the x column, and 3, 6, 12, and 24 in the y column, respectively.

1 EXE 2 EXE 3 EXE 4 EXE ∨ > 3 EXE 6 EXE 1 2 EXE 2 4 EXE EXE

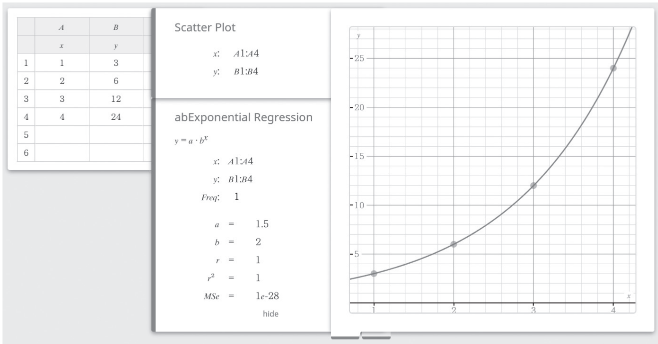


Select [Reg Results], press (OK), select [y=a·b^x], press (OK)



We can confirm that $y = 1.5 \cdot 2^x = \dfrac{3}{2} \cdot 2^x = 3 \cdot 2^{x-1}$.

Press (BACK) (SHIFT) (x), scan the QR code to display a graph.



(2) For the geometric progression $\{a_n\}$, in which the common ratio is positive, find the first term and the common ratio when $a_2=6$ and $a_4=54$. Also, find the general term a_n.

Given the first term is a and the common ratio is r and the general term is $a_n = ar^{n-1}$.

Since the 2nd term is 6, we get $ar^{2-1} = 6, ar = 6 \quad \ldots(\mathrm{i})$

Since the 4th term is 54, we get $ar^{4-1} = 54, ar^3 = 54 \quad \ldots(\mathrm{ii})$

From (i) and (ii), we get $r^2=9$ and $r>0$, so $r=3$

In this case, $a=2$

Therefore, $a_n = 2 \cdot 3^{n-1}$

First term 2, common ratio 3, and $a_n = 2 \cdot 3^{n-1}$

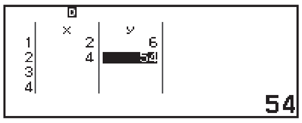
check

Press (HOME), select [Statistics], press (OK), select [2-Variable], press (OK)

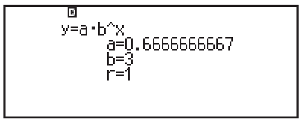
Press (⋯), select [Edit], press (OK), select [Delete All], press (OK)

Input 2 and 4 in the x column, and 6 and 54 in the y column, respectively.

(2) (EXE) (4) (EXE) (∨) (>) (6) (EXE) (5) (4) (EXE) (EXE)

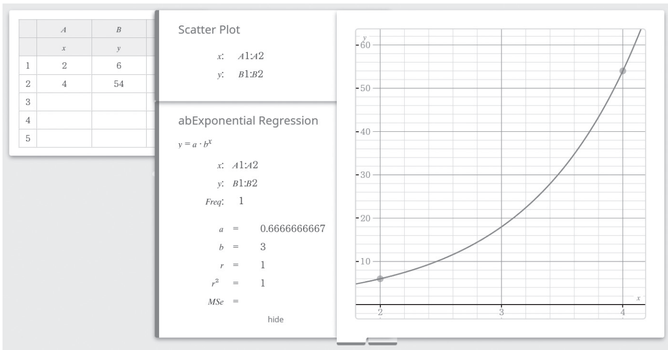


Select [Reg Results], press (OK), select [y=a·b^x], press (OK)



We can confirm that $y = 0.\dot{6} \cdot 3^x = \dfrac{2}{3} \cdot 3^x = 2 \cdot 3^{x-1}$.

Press (↩) (⇧) (x), scan the QR code to display a graph.



(3) When 3, x, and 12 form a geometric progression in this order, find the value of x.

Since 3, x, and 12 form a geometric progression in this order, we get $x^2=3\cdot12=36$

Therefore, $x=\pm6$

± 6

PRACTICE

Solve the following problems.

(1) Find the general term a_n of the geometric progression $16, 8, 4, 2, \cdots\cdots$.

Since each term is the preceding term multiplied by $\frac{1}{2}$, the common ratio is $\frac{1}{2}$.

Since the first term is 16, the general term is $a_n = 16 \cdot \left(\frac{1}{2}\right)^{n-1}$

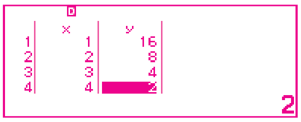
$$a_n = 16 \cdot \left(\frac{1}{2}\right)^{n-1}$$

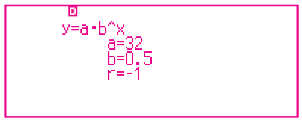
check

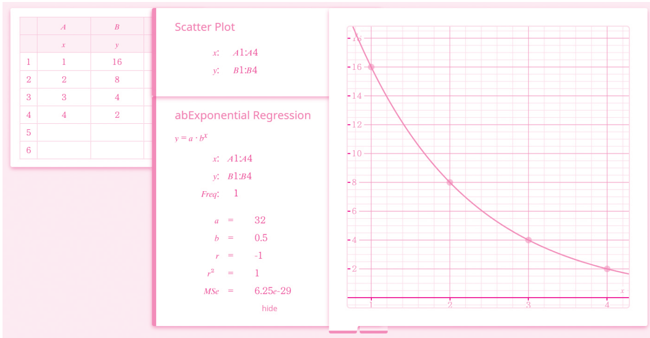
Press (home), select [Statistics], press (OK), select [2-Variable], press (OK)
Press (menu), select [Edit], press (OK), select [Delete All], press (OK)
After inputting $1, 2, 3,$ and 4 in the x column, and $16, 8, 4,$ and 2 in the y column, respectively, press (EXE)
Select [Reg Results], press (OK), select [y=a·b^x], press (OK)
We can confirm that

$$y = 32 \cdot 0.5^x = 32 \cdot \left(\frac{1}{2}\right) \cdot \left(\frac{1}{2}\right)^{x-1}$$

$$= 16 \cdot \left(\frac{1}{2}\right)^{x-1}.$$







Press (back) (shift) (x), scan the QR code to display a graph.

(2) For the geometric progression $\{a_n\}$, find the first term and common ratio when $a_3=20$ and $a_6=160$. Also, find the general term a_n.

Given the first term is a and the common ratio is r and the general term is $a_n = ar^{n-1}$.

Since the 3rd term is 20, we get $ar^{3-1} = 20, ar^2 = 20 \quad \ldots$(i)

Since the 6th term is 160, we get $ar^{6-1} = 160, ar^5 = 160 \quad \ldots$(ii)

From (i) and (ii), we get $r^3 = 8, r = 2$

In this case, $a=5$

Therefore, $a_n = 5 \cdot 2^{n-1}$

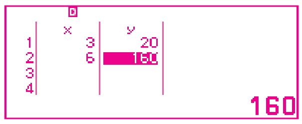
First term 5, common ratio 2, and $a_n = 5 \cdot 2^{n-1}$

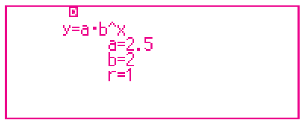
check

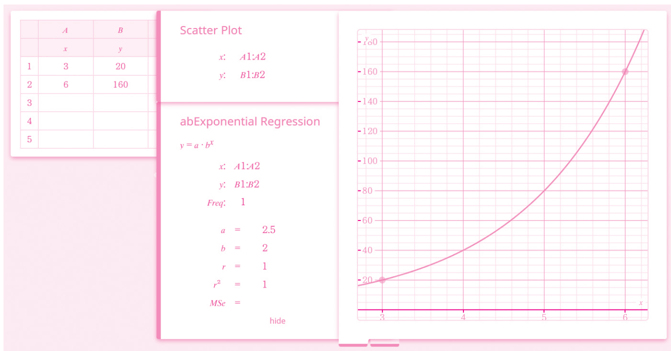
Press ⌂, select [Statistics], press OK,
select [2-Variable], press OK
Press ⋯, select [Edit], press OK,
select [Delete All], press OK
After inputting 3 and 6 in the x column, and 20 and 160 in the y column, respectively, press EXE
Select [Reg Results], press OK,
select [y=a·b^x], press OK
We can confirm that

$$y = 2.5 \cdot 2^x = \frac{5}{2} \cdot 2^x = 5 \cdot 2^{x-1}.$$

Press ⮌ ⇧ ⓧ, scan the QR code to display a graph.







(3) When 25, x, and 1 form a geometric progression in this order, find the value of x.

Since 25, x, and 1 form a geometric progression in this order, we get

$x^2 = 25 \cdot 1 = 25$

Therefore, $x=\pm 5$

± 5

Sums of geometric progressions (1)

TARGET To understand how to find the sum of geometric progressions.

STUDY GUIDE

Sums of geometric progressions

Given S_n is the sum of the first term to the nth term of a geometric progression $\{a_n\}$ with a first term of a and a common ratio of r, then we can express S_n as follows.

(i) When $r \neq 1$, $S_n = \dfrac{a(1-r^n)}{1-r} = \dfrac{a(r^n-1)}{r-1}$

(ii) When $r=1$, $S_n = na$

explanation

$S_n = a + ar + ar^2 + \cdots\cdots + ar^{n-2} + ar^{n-1}$ ...(i)

Multiply both sides of (i) by r to get $rS_n = ar + ar^2 + ar^3 + \cdots\cdots + ar^{n-1} + ar^n$ ...(ii)

From (i)−(ii), we get

$$\begin{array}{rcccccccccccccc} & S_n & = & a & + & ar & + & ar^2 & + & \cdots\cdots & + & ar^{n-2} & + & ar^{n-1} & & \\ -) & rS_n & = & & & ar & + & ar^2 & + & \cdots\cdots & + & ar^{n-2} & + & ar^{n-1} & + & ar^n \\ \hline & S_n - rS_n & = & a & & & & & & & & & & & - & ar^n \end{array}$$

Therefore, from $(1-r)S_n = a(1-r^n)$, when $r \neq 1$, we can derive $S_n = \dfrac{a(1-r^n)}{1-r}$.

By multiplying the numerator and denominator of the right-hand side of this formula by -1, we can derive $S_n = \dfrac{a(r^n-1)}{r-1}$.

We simply use them appropriately, so that when $r>1$, then $S_n = \dfrac{a(r^n-1)}{r-1}$, and when $r<1$, then $S_n = \dfrac{a(1-r^n)}{1-r}$.

Also, when $r=1$, we can derive $S_n = a+a+a+\cdots\cdots+a = na$.

Ex. The sum of the first term to the last term of a geometric progression with a first term of 3, common ratio of 2, and with 6 terms, is $S_6 = \dfrac{3(2^6-1)}{2-1} = 189$

EXERCISE

1 Find the sum of the first term to the 4th term of a geometric progression with a first term of 5 and a common difference of -4.

$S_4 = \dfrac{5\{1-(-4)^4\}}{1-(-4)} = -255$

−255

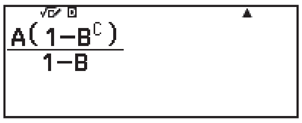
check

On the scientific calculator, use the VARIABLE function to, after registering the formula, input the values for A (first term), B (common ratio), and C (number of terms) to find the value.

Press ⌂, select [Calculate], press OK

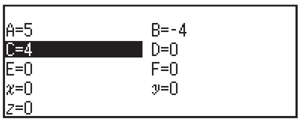
Input the formula $\dfrac{\mathrm{A}(1-\mathrm{B}^{\mathrm{C}})}{1-\mathrm{B}}$ for the sum of a geometric progression.

[SHIFT] [4] [(] [1] [−] [SHIFT] [5] [■□] [SHIFT] [6] [▷] [)] [▭] [1] [−] [SHIFT] [5]



In the VARIABLE screen, input [A=5, B=−4, C=4], and then calculate.

[VARIABLE] [5] [EXE] [▷] [−] [4] [EXE] [▷] [4] [EXE] [↶] [EXE]



2 Solve the following problems with regards to the geometric progression $4, 12, 36, 108, \ldots\ldots$.

(1) Find the sum S_n of the first term to the nth term.

The sum S_n of the first term to the nth term of a geometric progression with a first term of 4 and a common ratio of 3, is $S_n = \dfrac{4(3^n - 1)}{3-1} = 2(3^n - 1)$

$$S_n = 2(3^n - 1)$$

(2) Find the sum of the first term to the 8th term.

The sum of the first term to the 8th term is $S_8 = 2(3^8 - 1) = 13120$

13120

(3) Find the sum of the 6th term to the 10th term.

(Sum of the 6th to the 10th terms)=(Sum of the first to the 10th terms)−(Sum of the first to the 5th terms)=$S_{10} - S_5$.

Therefore, we find a sum of

$$S_{10} - S_5 = 2(3^{10} - 1) - 2(3^5 - 1) = 2\cdot 3^{10} - 2\cdot 3^5 = 2\cdot 3^5 \cdot (3^5 - 1) = 2\cdot 243 \cdot 242 = 117612$$

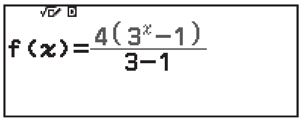
117612

check

On the scientific calculator, use the FUNCTION function to register the expression to find the product of geometric progressions in f(x).

Input f(x)=$\dfrac{4(3^x - 1)}{3-1}$.

[f(x)] [▽] [▽] [OK] [4] [(] [3] [■□] [x] [▷] [−] [1] [)] [▭] [3] [−] [1] [EXE]

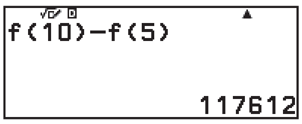


(2) Calculate the sum of the first term to the 8th term.

[f(x)] [OK] [8] [)] [EXE]

(3) Calculate the sum of the 6th term to the 10th term.

[f(x)] [OK] [1] [0] [)] [−] [f(x)] [OK] [5] [)] [EXE]



3 Find the common ratio of a geometric progression with a first term of 2 and a sum of the first term to the 3rd term of 62.

Let r be the common ratio.

When $r=1$, the sum of the first term to the 3rd term is $3\cdot 2=6$, which is inappropriate.

When $r\neq 1$, the sum of the first term to the 3rd term is 62, so $\frac{2(1-r^3)}{1-r}=62$

We can solve this for $\frac{2(1-r)(1+r+r^2)}{1-r}=62, 2(1+r+r^2)=62, r^2+r-30=0, (r-5)(r+6)=0, r=5,-6$

5, −6

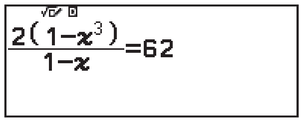
check

On the scientific calculator, use the $\mathrm{Equation}$ function to find the common ratio of the geometric progression where $a=2$ and $S_3=62$.

Press ⌂, select $[\mathrm{Equation}]$, press OK, select $[\mathrm{Solver}]$, press OK

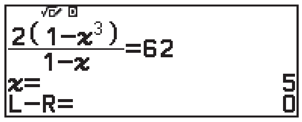
Input the equality $\frac{2(1-x^3)}{1-x}=62$.

② (① ⊖ ⓧ ■ ③ ⊳) ⊟ ① ⊖ ⓧ ⊳ ⇧ (⑥ ② EXE



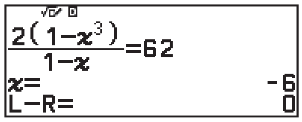
To find a positive solution, input the initial value $(*)x=2$.

② EXE EXE



To find a negative solution, input the initial value $x=-2$.

↺ ⊖ ② EXE EXE



*Initial value: The value input when you start a calculation. Input a value that seems close to the solution to the problem.

PRACTICE

1 Find the sum of the first term to the 6th term of a geometric progression with a first term of 3 and a common difference of 5.

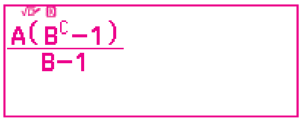
$$S_6=\frac{3(5^6-1)}{5-1}=11718$$

11718

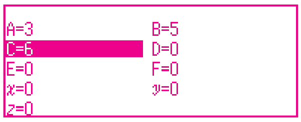
check

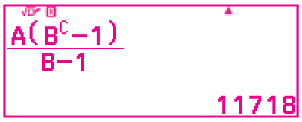
Press ⌂, select [Calculate], press OK

⇧ 4 (⇧ 5 ▪ ⇧ 6 > − 1) ▭ ⇧ 5 − 1



⇄x 3 EXE > 5 EXE > 6 EXE ↶ EXE





2 Solve the following problems with regards to the geometric progression $7, 14, 28, 56, \ldots\ldots$.

(1) Find the sum S_n of the first term to the nth term.

The sum S_n of the first term to the nth term of a geometric progression with a first term of 7 and a common ratio of 2, is $S_n=\frac{7(2^n-1)}{2-1}=7(2^n-1)$

$$S_n=7(2^n-1)$$

(2) Find the sum of the first term to the 11th term.

The sum of the first term to the 11th term is $S_{11}=7(2^{11}-1)=14329$

14329

(3) Find the sum of the 9th term to the 16th term.

(Sum of the 9th to the 16th term)

=(Sum of the first to the 16th term)−(Sum of the first to the 8th term)=$S_{16}-S_8$.

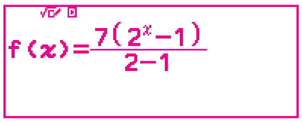
Therefore, we find a sum of

$$S_{16}-S_8=7(2^{16}-1)-7(2^8-1)=7\cdot 2^{16}-7\cdot 2^8=7\cdot 2^8\cdot(2^8-1)=7\cdot 256\cdot 255$$
$$=456960$$

456960

check

f(x) ∨ ∨ OK 7 (2 ▪ x > − 1) ▭ 2 − 1 EXE

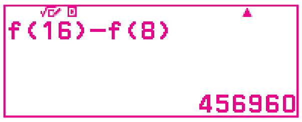


(2)

f(x) OK 1 1) EXE

(3)

f(x) OK 1 6) − f(x) OK 8) EXE



3 Find the common ratio of a geometric progression with a first term of 4 and a sum of the first term to the 3rd term of 28.

Let r be the common ratio.

When $r=1$, the sum of the first term to the 3rd term is $3 \cdot 4=12$, which is inappropriate.

When $r \neq 1$, the sum of the first term to the 3rd term is 28, so $\dfrac{4(1-r^3)}{1-r}=28$

We can solve this for

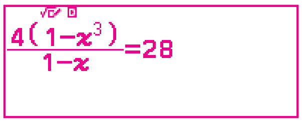
$$\frac{4(1-r)(1+r+r^2)}{1-r}=28, 4(1+r+r^2)=28, r^2+r-6=0, (r-2)(r+3)=0,$$
$$r=2, -3$$

$$2, -3$$

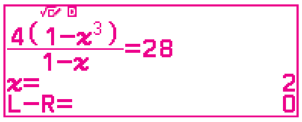
check

Press ⌂, select [Equation], press OK, select [Solver], press OK

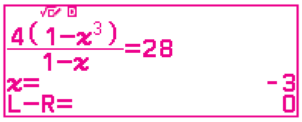
4 (1 − x ■ 3 >) ▤ 1 − x > ⇧ (2 8 EXE



2 EXE EXE



↩ − 2 EXE EXE



Sums of geometric progressions (2)

TARGET To understand how to calculate compound interest.

STUDY GUIDE

Calculating compound interest

The system of depositing money in a bank to accrue interest over fixed periods of time, and then adding that to the principal so the total can be used to calculate the interest for the next period, is called a **compound interest system**.

Ex. We deposited $\$10$ at the beginning of the year to earn compound interest every year at a 5% annual interest rate. Given we do not withdraw any interest, find the total of the principal and interest after 10 years.

After 1 year: (Principal)$\times$($1+$percentage of interest)$=10(1+0.05)(\$)$…Principal after 1 year

After 2 years: $10(1+0.05)\times(1+0.05)=10(1+0.05)^2(\$)$

After 3 years: $10(1+0.05)^2\times(1+0.05)=10(1+0.05)^3(\$)$,……

After 10 years: $10(1+0.05)^{10}(\$)$

Therefore, $10(1+0.05)^{10}\simeq16.29(\$)$

10×1.05^10
16.28894627

From this, we can express the total of the principal and interest on a principal A (currency unit) held for n years at an annual compound interest of $r(\%)$ as follows.

$$\mathrm{A}\left(1+\frac{r}{100}\right)^n$$

EXERCISE

1 Solve the following problems.

(1) If you deposit $\$100$ at an annual 10% compound interest, what will the total $ amount of the principal and interest be after 8 years? Round off your answer to the 2nd decimal place.

$\mathrm{A}=100$, $r=10$, and $n=8$, so $100\left(1+\dfrac{10}{100}\right)^8=100\cdot1.1^8\simeq214.36\,(\$)$

100×1.1^8
214.358881

$\$214.36$

(2) If you deposit $50 at an annual 8% compound interest, after how many years will the total amount of the principal and interest exceed $100?

The total amount after x years for a $50 principal at an annual interest of 8% will be $50\left(1+\dfrac{8}{100}\right)^x$ ($)

Find the lowest natural number x that satisfies $50\left(1+\dfrac{8}{100}\right)^x \geq 100$ such that it will exceed $100.

On the scientific calculator, use the Table function to calculate the total amount of the principal and interest after x years.

Press ⌂, select $[\mathrm{Table}]$, press OK, then clear the previous data by pressing ⏻

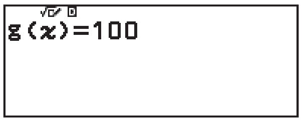
Press ⋯, select $[\mathrm{Define}\ \mathrm{f}(x)/\mathrm{g}(x)]$, press OK, select $[\mathrm{Define}\ \mathrm{f}(x)]$, press OK

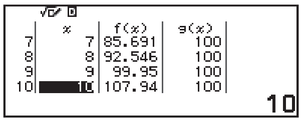
After inputting $\mathrm{f}(x) = 50 \times 1.08^x$, press EXE

In the same way, input $\mathrm{g}(x) = 100$.

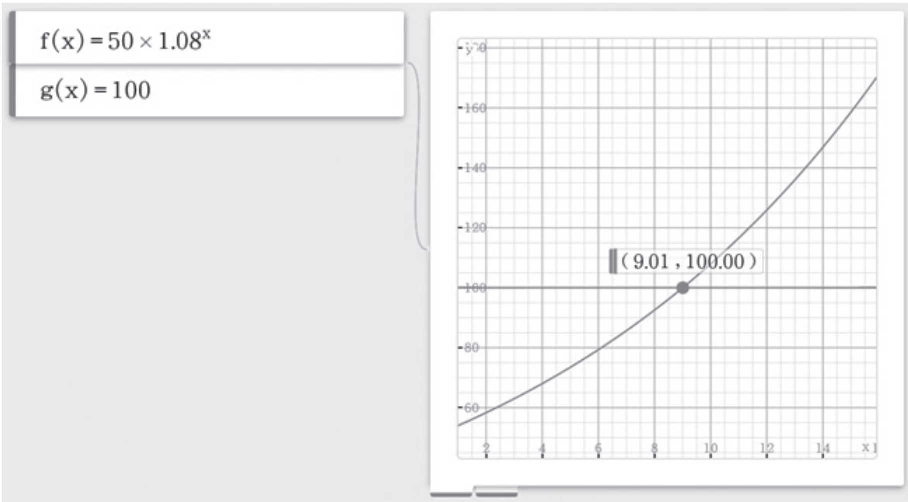
Press ⋯, select $[\mathrm{Table\ Range}]$, press OK

After inputting $[\mathrm{Start{:}1,\ End{:}16,\ Step{:}1}]$, select $[\mathrm{Execute}]$, press EXE





From the table, we can confirm that the first time it exceeds 100 is when $x=10$. Therefore, the answer is after 10 years.



After 10 years

(3) What is the minimum annual interest rate at which the sum of the principal and the interest will exceed \$150 if \$100 is deposited for 10 years at compound interest. Round off your answer to the 2nd decimal place.

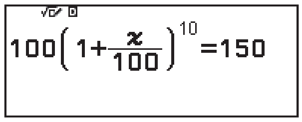
The total amount after 10 years for a \$100 principal at an annual interest of r% will be $100\left(1+\dfrac{r}{100}\right)^{10}$ (\$)

Find the r that satisfies $100\left(1+\dfrac{r}{100}\right)^{10}=150$ such that this becomes \$150.

On the scientific calculator, use the Equation function to calculate the value of x that satisfies $100\left(1+\dfrac{x}{100}\right)^{10}=150$.

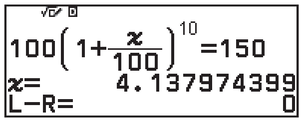
Press [HOME], select [Equation], press [OK], select [Solver], press [OK]

[1] [0] [0] [(] [1] [+] [x] [▭/▭] [1] [0] [0] [>] [)] [■□] [1] [0] [>] [⬆] [(] [1] [5] [0] [EXE]

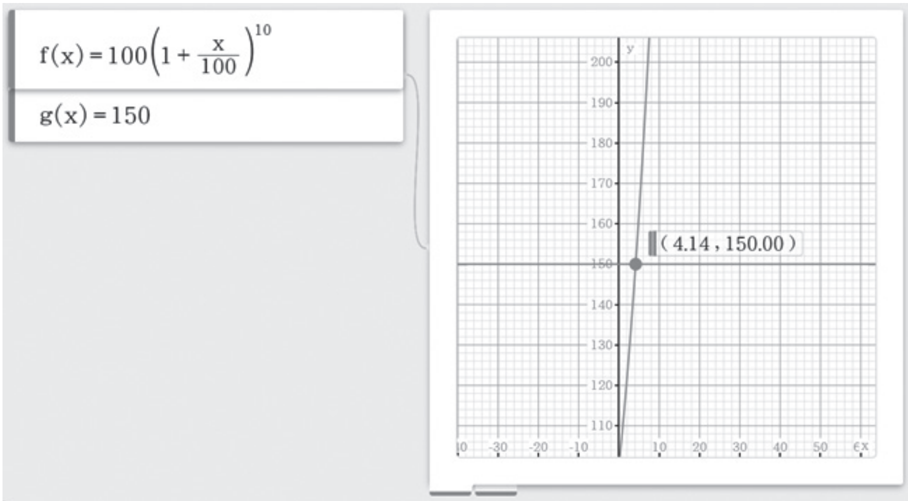


To find a positive solution, input the initial value x=1.

[1] [EXE] [EXE]



Therefore, from x=4.137…, we get 4.14%



4.14%

PRACTICE

◆ Solve the following problems.

(1) If you deposit $80 at an annual 7% compound interest, what will the total $ amount of the principal and interest be after 12 years? Round off your answer to the 2nd decimal place.

A=80, r=7, and n=12, so $80\left(1+\frac{7}{100}\right)^{12}=80\cdot 1.07^{12}\simeq 180.18$ ($)

$180.18

(2) If you deposit $100 at an annual 6% compound interest, after how many years will the total amount of the principal and interest exceed $160?

The total amount after x years for a $100 principal at an annual interest of 6% will be $100\left(1+\frac{6}{100}\right)^{x}$ ($)

Find the lowest natural number x that satisfies $100\left(1+\frac{6}{100}\right)^{x}\geq 160$ **such that it will exceed $160.**

On the scientific calculator, use the Table function to calculate the total amount of the principal and interest after x years.

Press ⌂, select [Table], press OK, then clear the previous data by pressing ⏻

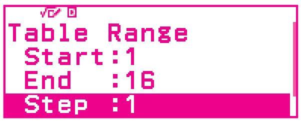
Press ⋯, select [Define f(x)/g(x)], press OK, select [Define f(x)], press OK

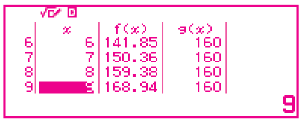
After inputting f(x)=100×1.06^x, press EXE

In the same way, input g(x)=160.

Press ⋯, select [Table Range], press OK

After inputting [Start:1, End:16, Step:1], select [Execute], press EXE





From the table, the first time it exceeds 160 is when x=9.

Therefore, the answer is after 9 years.

After 9 years

(3) What is the minimum annual interest rate at which the sum of the principal and the interest will exceed $\$160$ if $\$70$ is deposited for 9 years at compound interest. Round off your answer to the 2nd decimal place.

The total amount after 9 years for a $\$70$ principal at an annual interest of $r\%$ will be

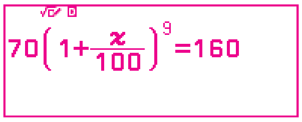
$$70\left(1+\frac{r}{100}\right)^9 \ (\$)$$

Find the r that satisfies $70\left(1+\frac{r}{100}\right)^9=160$ such that this becomes $\$160$.

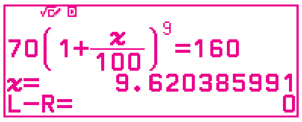
On the scientific calculator, use the Equation function to calculate the value of x that satisfies $70\left(1+\frac{x}{100}\right)^9=160$.

Press ⌂, select [Equation], press OK, select [Solver], press OK

7 0 (1 + x ▭/▭ 1 0 0 >) ▪□ 9 > ⇧ (1 6 0 EXE



1 EXE EXE



Therefore, from $x=9.620\ldots$, we get 9.62%

9.62%

ADVANCED

STUDY GUIDE

Total of principal and interest of fixed-term savings

The total amount of principal and interest accrued by the end of n years of saving $\$\mathrm{A}$ at the beginning of each year in a compound interest account earning $r\%$ annual interest can be expressed as follows.

$$\mathrm{A}\,\frac{\left(1+\frac{r}{100}\right)\left\{\left(1+\frac{r}{100}\right)^{n}-1\right\}}{\frac{r}{100}}$$

explanation

The total amount after 1 year will be $\mathrm{A}\left(1+\frac{r}{100}\right)(\$)$

The total amount after 2 years is $\mathrm{A}\left(1+\frac{r}{100}\right)^{2}+\mathrm{A}\left(1+\frac{r}{100}\right)(\$)$ because the total amount increased after 1 year by $r\%$ more savings of $\$\mathrm{A}$

The total amount after 3 years is $\mathrm{A}\left(1+\frac{r}{100}\right)^{3}+\mathrm{A}\left(1+\frac{r}{100}\right)^{2}+\mathrm{A}\left(1+\frac{r}{100}\right)(\$)$ because the total amount increased after 2 years by $r\%$ more savings of $\$\mathrm{A}$

$\vdots$

The total amount after n years is $\mathrm{A}\left(1+\frac{r}{100}\right)^{n}+\mathrm{A}\left(1+\frac{r}{100}\right)^{n-1}+\cdots\cdots+\mathrm{A}\left(1+\frac{r}{100}\right)^{2}+\mathrm{A}\left(1+\frac{r}{100}\right)(\$)$ because the total amount increased after $(n-1)$ years by $r\%$ more savings of $\$\mathrm{A}$

Therefore, the total amount at the end of the year in n years is the sum of the first term to the nth term of a geometric progression with a first term of $\mathrm{A}\left(1+\frac{r}{100}\right)$ and common ratio of $\left(1+\frac{r}{100}\right)$.

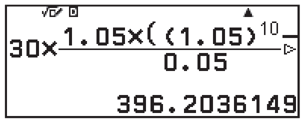
Therefore, from the formula for the sum of a geometric progression, we get

$$\frac{\mathrm{A}\left(1+\frac{r}{100}\right)\left\{\left(1+\frac{r}{100}\right)^{n}-1\right\}}{\left(1+\frac{r}{100}\right)-1}=\mathrm{A}\,\frac{\left(1+\frac{r}{100}\right)\left\{\left(1+\frac{r}{100}\right)^{n}-1\right\}}{\frac{r}{100}}.$$

EXERCISE

1 What is the total \$ amount of principal and interest accrued by the end of 10 years of saving \$$30$ at the beginning of each year in a compound interest account earning 5% annual interest? Round off your answer to the 2nd decimal place.

By using $A=30$, $r=5$, and $n=10$ in the formula, then



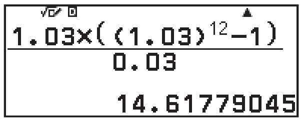
$$30\cdot\frac{\left(1+\frac{5}{100}\right)\left\{\left(1+\frac{5}{100}\right)^{10}-1\right\}}{\frac{5}{100}}\simeq 396.20(\$)$$

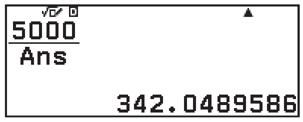
\$396.20

2 Given you want to save the same amount of money at the beginning of each year in a compound interest account earning 3% annual interest to save a total of principal and interest that exceeds \$$5000$ at the end of the year in 12 years. What is the minimum \$ amount you should save every year? Answer in whole numbers.

By saving \$$A$ every year, then by the end of the 12 years, the total principal and interest will be

$$A\frac{\left(1+\frac{3}{100}\right)\left\{\left(1+\frac{3}{100}\right)^{12}-1\right\}}{\frac{3}{100}}$$





We have enough if it exceeds \$$5000$, so the answer is $A\geq 342.04\ldots$.

Therefore, we should save more than \$$343$ each time.

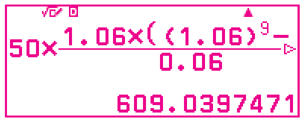
\$343 or more

PRACTICE

1 What is the total \$ amount of principal and interest accrued by the end of 9 years of saving \$$50$ at the beginning of each year in a compound interest account earning 6% annual interest? Round off your answer to the 2nd decimal place.

By using $A=50$, $r=6$, and $n=9$ in the formula, then

$$50\cdot\frac{\left(1+\frac{6}{100}\right)\left\{\left(1+\frac{6}{100}\right)^{9}-1\right\}}{\frac{6}{100}}\simeq 609.04(\$)$$

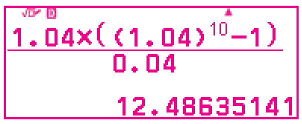


\$609.04

2 Given you want to save the same amount of money at the beginning of each year in a compound interest account earning 4% annual interest to save a total of principal and interest that exceeds \$$8000$ at the end of the year in 10 years. What is the minimum \$ amount you should save every year? Answer in whole numbers.

By saving \$$A$ every year, then by the end of the 10 years, the total principal and interest will be

$$A\frac{\left(1+\frac{4}{100}\right)\left\{\left(1+\frac{4}{100}\right)^{10}-1\right\}}{\frac{4}{100}}$$



We have enough if it exceeds \$$8000$, so the answer is $A\geq 640.69\ldots$.

Therefore, we should save \$$641$ or more each time.

\$641 or more

Summation symbol $\sum$

TARGET To understand the meaning of the $\sum$ symbol for sums of sequences.

STUDY GUIDE

$\sum$ symbol for sums of sequences

Summation symbol $\sum$

We use the symbol $\sum$ (sigma) to express the sum of a sequence. For the sequence $\{a_n\}$, we can express the sum of the first term to the nth term as shown below.

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \cdots\cdots + a_n$$

The symbol $\sum_{k=1}^{n} a_k$ expresses the sum of adding all the values of a_n, in which k is converted to $1, 2, 3, \ldots\ldots, n$.

Ex. (1) $\sum_{k=1}^{5} a_k = a_1 + a_2 + a_3 + a_4 + a_5$

(2) $\sum_{k=1}^{n} (2k-1) = (2\cdot1-1) + (2\cdot2-1) + (2\cdot3-1) + \cdots\cdots + (2n-1)$

(3) $\sum_{k=1}^{n-1} a_k = a_1 + a_2 + a_3 + \cdots\cdots + a_{n-2} + a_{n-1}$

Sums of various sequences

The sums of the various sequences are all expressed using $\sum$, as follows.

(i) $\sum_{k=1}^{n} c = nc$ (c is a constant)

(ii) $\sum_{k=1}^{n} k = \frac{1}{2}n(n+1)$

(iii) $\sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1)$

(iv) $\sum_{k=1}^{n} k^3 = \left\{\frac{1}{2}n(n+1)\right\}^2$

(v) $\sum_{k=1}^{n} ar^{k-1} = \frac{a(1-r^n)}{1-r} = \frac{a(r^n-1)}{r-1}$ $(r \neq 1)$

explanation

(i) Sum of the sequence $a_n = c$ $\quad \sum_{k=1}^{n} c = c + c + c + \cdots\cdots + c = nc$

(ii) Sum of the sequence $a_n = n$ (sum of natural numbers) $\quad \sum_{k=1}^{n} k = 1 + 2 + 3 + \cdots\cdots + n = \frac{1}{2}n(n+1)$

(iii) Sum of the sequence $a_n = n^2$ (sum of the squares of natural numbers)

$$\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^2 + \cdots\cdots + n^2 = \frac{1}{6}n(n+1)(2n+1)$$

(iv) Sum of the sequence $a_n = n^3$ (sum of the cubes of natural numbers)

$$\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + 3^3 + \cdots\cdots + n^3 = \left\{\frac{1}{2}n(n+1)\right\}^2$$

(v) Sum of the geometric progression with first term a and a common ratio of r $(r \neq 1)$

$$\sum_{k=1}^{n} ar^{k-1} = a + ar + ar^2 + \cdots\cdots + ar^{n-1} = \frac{a(1-r^n)}{1-r} = \frac{a(r^n - 1)}{r-1}$$

Ex. (1) $\sum_{k=1}^{10} k = \frac{1}{2} \cdot 10 \cdot (10+1) = 55$

(2) $\sum_{k=1}^{n-1} k = \frac{1}{2}(n-1)\{(n-1)+1\} = \frac{1}{2}n(n-1)$

proof

On the scientific calculator, use the Spreadsheet function to derive the formula (iii) $\sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1)$ from the data. (Theoretical derivation is omitted)

Press ⌂, select [Spreadsheet], press OK, then clear the previous data by pressing ⏻

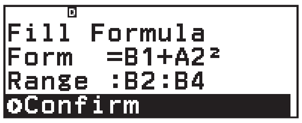
After inputting [A1:1, A2:2, A3:3, and A4:4] respectively, press EXE, move to [B1]

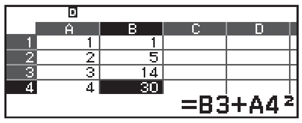
After inputting [B1:1^2], press EXE

Press ⊙⊙⊙, select [Fill Formula], press OK

After inputting [Form=B1+A2^2], press EXE

After inputting [Range:B2:B4], press EXE, select [Confirm], press OK

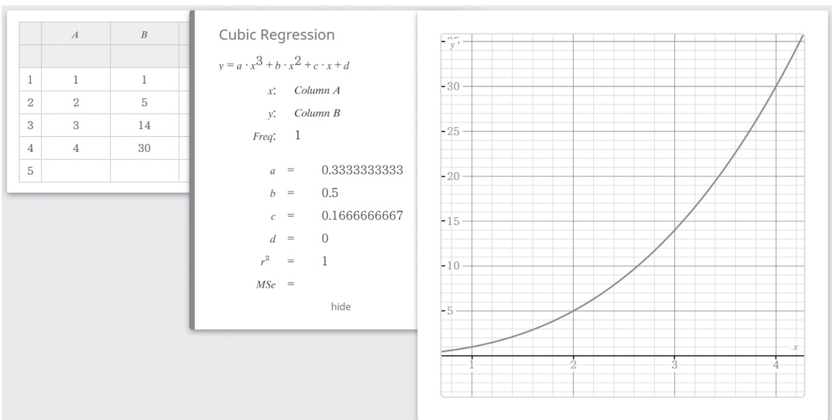




Press ⬆ (x), scan the QR code to display the data.

In addition, you can use the following procedure to change the settings at ClassPad.net.

Tap the coordinates sheet, tap column A and column B, tap [Statistics], tap [Regression], and then tap [Cubic Regression].

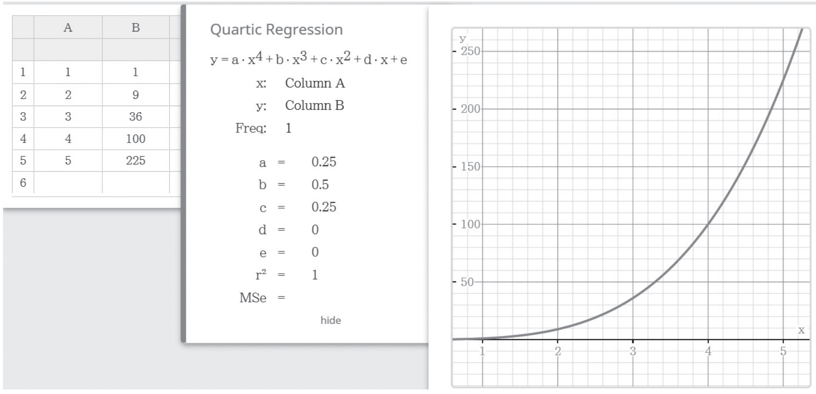


The displayed formula $y = \frac{1}{3}x^3 + \frac{1}{2}x^2 + \frac{1}{6}x = \frac{1}{6}x(x+1)(2x+1)$ is derived from the functional relation (the relation in which one amount is the function of the other amount) of column A and B from the data.

In the same way, on the scientific calculator, use the $\mathrm{Spreadsheet}$ function to derive the formula $\mathrm{(iv)}$

$$\sum_{k=1}^{n} k^3 = \left\{\frac{1}{2}n(n+1)\right\}^2.$$

(Only the results of entering data up to the 5th row of columns A and B, and then selecting $\mathrm{Quartic\ Regression}$, are described here.)



Therefore, $y = \dfrac{1}{4}x^4 + \dfrac{1}{2}x^3 + \dfrac{1}{4}x^2 = \dfrac{1}{4}x^2(x^2 + 2x + 1) = \left\{\dfrac{1}{2}x(x+1)\right\}^2$

Properties of $\sum$

The following properties hold for sequences consisting of the sum of terms in the sequences $\{a_n\}$ and $\{b_n\}$, and for sequences in which each term is multiplied by a constant.

(i) $$\sum_{k=1}^{n}(a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$$

(ii) $$\sum_{k=1}^{n}(a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$$

(iii) $$\sum_{k=1}^{n} ca_k = c\sum_{k=1}^{n} a_k$$ **(c is a constant)**

explanation

(i) $\displaystyle\sum_{k=1}^{n}(a_k + b_k) = (a_1 + b_1) + (a_2 + b_2) + \cdots\cdots + (a_n + b_n) = (a_1 + a_2 + \cdots\cdots + a_n) + (b_1 + b_2 + \cdots\cdots + b_n)$

$$= \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$$

(ii) This can be explained in the same way as (i).

(iii) $\displaystyle\sum_{k=1}^{n} ca_k = (ca_1 + ca_2 + \cdots\cdots + ca_n) = c(a_1 + a_2 + \cdots\cdots + a_n) = c\sum_{k=1}^{n} a_k$

EXERCISE

◆ Use $\sum$ to express the sum of the following sequences and calculate their sums.

(1) $3+8+13+18+\cdots\cdots+73$

This is the sum of $k=1$ to $k=15$ where $a_k = 5k-2$.

$$3+8+13+18+\cdots\cdots+73 = \sum_{k=1}^{15}(5k-2) = 5\sum_{k=1}^{15}k - \sum_{k=1}^{15}2 = 5\cdot\frac{1}{2}\cdot 15\cdot(15+1) - 15\cdot 2 = 570$$

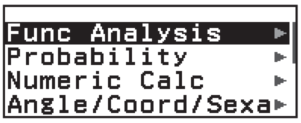
$$\boldsymbol{\sum_{k=1}^{15}(5k-2),\ 570}$$

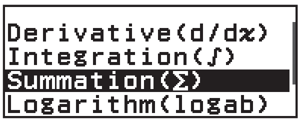
check

Use the scientific calculator to calculate the sum of the sequence expressed by the symbol $\sum$.

Press [HOME], select [Calculate], press [OK]

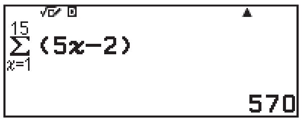
Press [CATALOG], select [Func Analysis], press [OK], select [Summation($\sum$)], press [OK]





In the displayed screen, you can find the sum of the sequences by inputting general terms $5x-2$ and $x=1$, and the number of terms 15.

⑤ ⓧ ⊖ ② ⓥ ① ⓐ ① ⑤ [EXE]



(2) $4+7+10+13+\cdots\cdots+(3n+1)$

This is the sum of $k=1$ to $k=n$ where $a_k = 3k+1$.

$$4+7+10+13+\cdots\cdots+(3n+1) = \sum_{k=1}^{n}(3k+1) = 3\sum_{k=1}^{n}k + \sum_{k=1}^{n}1 = 3\cdot\frac{1}{2}n(n+1) + n\cdot 1 = \frac{3n^2+5n}{2}$$

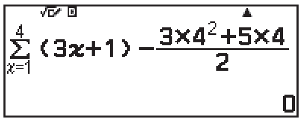
$$\boldsymbol{\sum_{k=1}^{n}(3k+1),\ \frac{3n^2+5n}{2}}$$

check

Confirm that the sums up to the 4th term are consistent.

[CATALOG] [OK] [∨] [∨] [OK] [3] [x] [+] [1] [∨] [1] [∧] [4] [<] [<] [<]

[−] [FRACTION] [3] [×] [4] [x²] [+] [5] [×] [4] [∨] [2] [EXE]

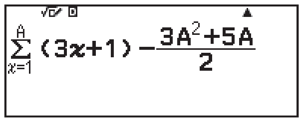


OTHER METHODS

On the scientific calculator, use the VARIABLE function to confirm the sums up to term A are consistent.

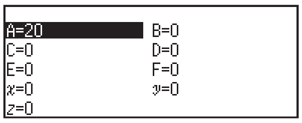
Input $\displaystyle\sum_{x=1}^{A}(3x+1)-\frac{3A^2+5A}{2}$.

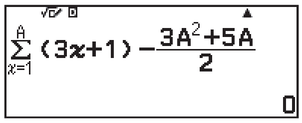
[Catalog] [OK] [∨] [∨] [OK]   [3] [x] [+] [1] [∨] [1] [∧] [Shift] [4] [<] [<] [<]
[−] [Fraction] [3] [Shift] [4] [x^2] [+] [5] [Shift] [4] [∨] [2]



In the VARIABLE screen, input [A=20], and then calculate.

[VARIABLE] [2] [0] [EXE]   [Back] [EXE]





Given A=20, you can confirm that the sums up to the 20th term are consistent.

Also, by changing the input value of A, you can confirm the value of A is derived arbitrarily.

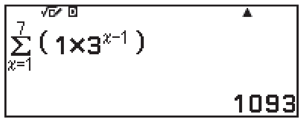
(3) $1+3+9+27+\cdots\cdots+729$

This is the sum of $k=1$ to $k=7$ where $a_k=1\cdot 3^{k-1}$.

$$1+3+9+27+\cdots\cdots+729=\sum_{k=1}^{7}1\cdot 3^{k-1}=\frac{3^7-1}{3-1}=1093$$

$$\boldsymbol{\sum_{k=1}^{7}3^{k-1},1093}$$

[Catalog] [OK] [∨] [∨] [OK]   [1] [×] [3] [$x^{\blacksquare}$] [x] [−] [1] [∨] [1] [∧] [7] [EXE]



(4) $1\cdot2+2\cdot3+3\cdot4+4\cdot5+\cdots\cdots+n(n+1)$

This is the sum of $k=1$ to $k=n$ where $a_k=k(k+1)$.

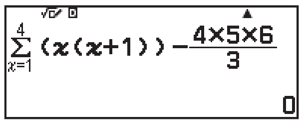
$$\begin{aligned}1\cdot2+2\cdot3+3\cdot4+4\cdot5+\cdots\cdots+n(n+1)&=\sum_{k=1}^{n}k(k+1)=\sum_{k=1}^{n}k^2+\sum_{k=1}^{n}k=\frac{1}{6}n(n+1)(2n+1)+\frac{1}{2}n(n+1)\\&=\frac{n(n+1)(n+2)}{3}\end{aligned}$$

$$\boldsymbol{\sum_{k=1}^{n}k(k+1),\frac{n(n+1)(n+2)}{3}}$$

check

Confirm that the sums up to the 4th term are consistent.

[Catalog] [OK] [∨] [∨] [OK]   [x] [(] [x] [+] [1] [)] [∨] [1] [∧] [4] [<] [<] [<]
[−] [Fraction] [4] [×] [5] [×] [6] [∨] [3] [EXE]



PRACTICE

◆ Use Σ to express the sum of the following sequences and calculate their sums.

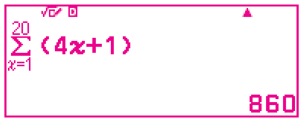
(1) $5+9+13+17+\cdots\cdots+81$

This is the sum of k=1 to k=20 where $a_k = 4k+1$.

$$5+9+13+17+\cdots\cdots+81=\sum_{k=1}^{20}(4k+1)=4\sum_{k=1}^{20}k+\sum_{k=1}^{20}1=4\cdot\frac{1}{2}\cdot 20\cdot(20+1)+20\cdot 1$$
$$=860$$

$$\sum_{k=1}^{20}(4k+1),860$$

(📖) (OK) (∨) (∨) (OK) (4) (x) (+) (1) (∨) (1) (∧) (2) (0) (EXE)



(2) $2+9+16+23+\cdots\cdots+$(nth term)

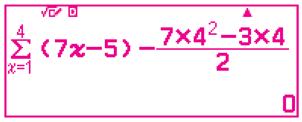
The arithmetic progression has a first term of 2 and a common difference of 7, so the general term is $a_n=2+(n-1)\cdot 7=7n-5$

This is the sum of k=1 to $k=n$ where $a_k = 7k-5$.

$$2+9+16+23+\cdots\cdots+(7n-5)=\sum_{k=1}^{n}(7k-5)=7\sum_{k=1}^{n}k-\sum_{k=1}^{n}5$$
$$=7\cdot\frac{1}{2}n(n+1)-n\cdot 5=\frac{7n^2-3n}{2}$$

$$\sum_{k=1}^{n}(7k-5),\frac{7n^2-3n}{2}$$

(📖) (OK) (∨) (∨) (OK) (7) (x) (−) (5) (∨) (1) (∧) (4) (<) (<) (<)
(−) (▤) (7) (×) (4) (■²) (−) (3) (×) (4) (∨) (2) (EXE)



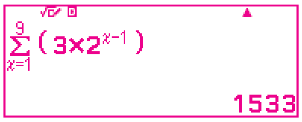
(3) 3+6+12+24+⋯⋯+768

This is the sum of k=1 **to** k=9 **where** $a_k = 3 \cdot 2^{k-1}$.

$$3+6+12+24+\cdots\cdots+768=\sum_{k=1}^{9} 3 \cdot 2^{k-1} = \frac{3(2^9-1)}{2-1} = 1533$$

$$\sum_{k=1}^{9} 3 \cdot 2^{k-1}, 1533$$

[📖] [OK] [∨] [∨] [OK]   [3] [×] [2] [$\blacksquare^{\square}$] [x] [−] [1] [∨] [1] [∧] [9] [EXE]



(4) 2·3+3·4+4·5+5·6+⋯⋯+(nth term)

General term $a_n=\{2+(n-1)\cdot 1\}\{3+(n-1)\cdot 1\}=(n+1)(n+2)$

This is the sum of k=1 **to** $k=n$ **where** $a_k = (k+1)(k+2)$.

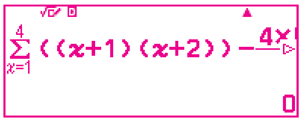
$$2 \cdot 3+3 \cdot 4+4 \cdot 5+5 \cdot 6+\cdots\cdots+(n+1)(n+2)$$
$$=\sum_{k=1}^{n}(k+1)(k+2)=\sum_{k=1}^{n}k^2+3\sum_{k=1}^{n}k+\sum_{k=1}^{n}2$$
$$=\frac{1}{6}n(n+1)(2n+1)+3\cdot\frac{1}{2}n(n+1)+2n=\frac{n(n^2+6n+11)}{3}$$

$$\sum_{k=1}^{n}(k+1)(k+2), \frac{n(n^2+6n+11)}{3}$$

[📖] [OK] [∨] [∨] [OK]

[(] [x] [+] [1] [)] [(] [x] [+] [2] [)] [∨] [1] [∧] [4] [<] [<] [<]

[−] [$\frac{\blacksquare}{\square}$] [4] [×] [(] [4] [$\blacksquare^2$] [+] [6] [×] [4] [+] [1] [1] [)] [∨] [3] [EXE]



Progression of differences

TARGET To understand the progression of differences.

STUDY GUIDE

Progression of differences

Progression of differences

We can derive the relational expression $\boldsymbol{b_n = a_{n+1} - a_n}$ from between the $\{a_n\}$ and $\{b_n\}$ sequences. In this case, we call it a **progression of differences** of the sequence $\{b_n\}$ and the sequence $\{a_n\}$.

Ex.

$\{a_n\}$: 1 , 2 , 4 , 7 , 11 , $\cdots$

$\{b_n\}$: 1 , 2 , 3 , 4 , $\cdots$

$a_5 = a_1 + (b_1 + b_2 + b_3 + b_4) = 1 + (1+2+3+4) = 11$

Sums and general terms of progression of differences

If finding the general term of a sequence $\{a_n\}$ is difficult, then we can examine the progression of differences $\{b_n\}$ of the sequence $\{a_n\}$, where $\{b_n\}$ may be a sequence whose sum can be found, such as an arithmetic progression or geometric progression. In this case, by finding the sum of the first term to the $(n-1)$ term of the progression of differences $\{b_n\}$, we can find the general term of the sequence $\{a_n\}$.

When $n \geq 2$ and the difference sequence of the sequence $\{a_n\}$ is $\{b_n\}$, then the following relation holds.

$$\boxed{a_n = a_1 + \sum_{k=1}^{n-1} b_k}$$

$\{a_n\}$: a_1 , a_2 , a_3 , a_4 , $\cdots$, a_{n-1} , a_n

$\{b_n\}$: $\underbrace{b_1 \;,\; b_2 \;,\; b_3 \;,\; \cdots \;,\; b_{n-1}}_{(n-1)\text{ elements}}$

explanation

How to find the general term a_n of a progression of differences

$$\begin{array}{rrcrcl}
 & \cancel{a_2} & - & a_1 & = & b_1 \\
 & \cancel{a_3} & - & \cancel{a_2} & = & b_2 \\
 & \cancel{a_4} & - & \cancel{a_3} & = & b_3 \\
 & & \vdots & & & \\
+) & a_n & - & \cancel{a_{n-1}} & = & b_{n-1} \\
\hline
 & a_n & - & a_1 & = & b_1 + b_2 + b_3 + \cdots + b_{n-1}
\end{array}$$

From this, $a_n = a_1 + (b_1 + b_2 + b_3 + \cdots\cdots + b_{n-1}) = a_1 + \sum_{k=1}^{n-1} b_k \quad (n \geq 2)$

EXERCISE

◆ Find the general terms a_n of the sequences $\{a_n\}$ below.

(1) $1, 3, 6, 10, 15, \cdots\cdots$

Given a progression of differences $\{b_n\}$, then $\{b_n\}$ is $2, 3, 4, 5, \cdots\cdots$, so $b_n = 2 + (n-1)\cdot 1 = n+1$

Given $n \geq 2$

$$a_n = a_1 + \sum_{k=1}^{n-1} b_k = 1 + \sum_{k=1}^{n-1}(k+1) = 1 + \sum_{k=1}^{n-1}k + \sum_{k=1}^{n-1}1 = 1 + \frac{1}{2}(n-1)n + n - 1 = \frac{n(n+1)}{2}$$

When $n=1$, this becomes $\frac{1\cdot 2}{2} = 1$, which is consistent with $a_1=1$, so it also holds when $n=1$.

Therefore, $a_n = \frac{n(n+1)}{2}$

$$\boldsymbol{a_n = \frac{n(n+1)}{2}}$$

OTHER METHODS

$$a_n = a_1 + \sum_{k=1}^{n-1} b_k = 1 + \sum_{k=1}^{n-1}(k+1) = 1 + (2+3+4+\cdots\cdots+n) = 1+2+3+4+\cdots\cdots+n = \frac{n(n+1)}{2}$$

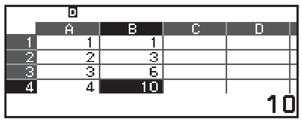
This method can also be used.

check

Press (⌂), select [Spreadsheet], press (OK), then clear the previous data by pressing (⏻)

After inputting [A1:1, A2:2, A3:3, and A4:4] respectively, press (EXE), move to [B1].

After inputting [B1:1, B2:3, B3:6, and B4:10] respectively, press (EXE)



	A	B	C	D
1	1	1		
2	2	3		
3	3	6		
4	4	10		

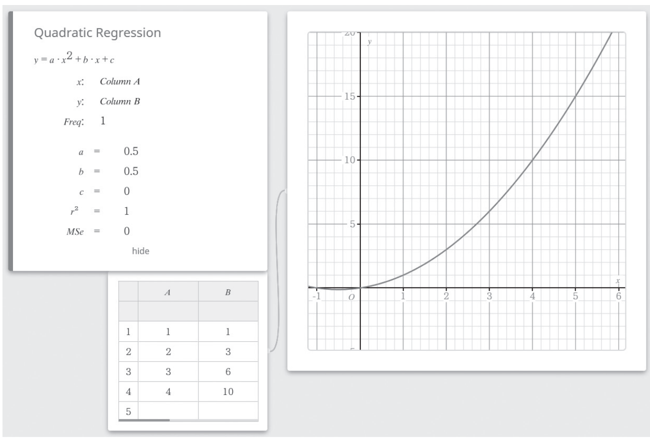
10

Press (⇧) (x), scan the QR code to display the data.

In addition, you can use the following procedure to change the settings at ClassPad.net.

Tap the coordinates sheet, tap column A and column B, tap [Statistics], tap [Regression], and then tap [Quadratic Regression].

We can confirm the functional relation $y = 0.5x^2 + 0.5x = \frac{1}{2}x(x+1)$ of columns A and B from the data.



(2) $2, 3, 5, 9, 17, \cdots\cdots$

Given a progression of differences $\{b_n\}$, then $\{b_n\}$ is $1, 2, 4, 8, \cdots\cdots$, so $b_n = 1 \cdot 2^{n-1}$

Given $n \geq 2$

$$a_n = a_1 + \sum_{k=1}^{n-1} b_k = 2 + \sum_{k=1}^{n-1} 1 \cdot 2^{k-1} = 2 + \frac{2^{n-1} - 1}{2 - 1} = 2^{n-1} + 1$$

When $n=1$, this becomes $2^0 + 1 = 2$, which is consistent with $a_1=2$, so it also holds when $n=1$.

Therefore, $a_n = 2^{n-1} + 1$

$$\boldsymbol{a_n = 2^{n-1} + 1}$$

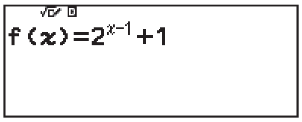
check

On the scientific calculator, use the Table function to confirm each term of the sequence.

Press ⌂, select $[\mathrm{Table}]$, press OK, then clear the previous data by pressing ⏻

Press ⋯, select $[\mathrm{Define\ f}(x)/\mathrm{g}(x)]$, press OK, select $[\mathrm{Define\ f}(x)]$, press OK

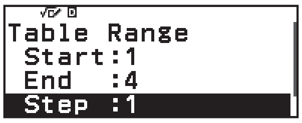
After inputting $\mathrm{f}(x) = 2^{x-1} + 1$, press EXE

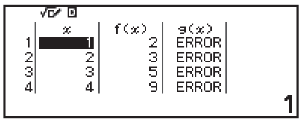


Press ⋯, select $[\mathrm{Table\ Range}]$, press OK

After inputting $[\mathrm{Start{:}1, End{:}4, Step{:}1}]$,

select $[\mathrm{Execute}]$, press EXE





PRACTICE

◆ Find the general terms a_n of the sequences $\{a_n\}$ below.

(1) $1, 5, 11, 19, 29, \cdots\cdots$

Given a progression of differences $\{b_n\}$, then $\{b_n\}$ is $4, 6, 8, 10, \cdots\cdots$, so

$b_n = 4 + (n-1)\cdot 2 = 2(n+1)$

Given $n \geqq 2$

$$a_n = a_1 + \sum_{k=1}^{n-1} b_k = 1 + \sum_{k=1}^{n-1} 2(k+1) = 1 + 2\left(\sum_{k=1}^{n-1} k + \sum_{k=1}^{n-1} 1\right)$$

$$= 1 + 2\left\{\frac{1}{2}(n-1)n + n - 1\right\} = n^2 + n - 1$$

When $n=1$, this becomes $1^2 + 1 - 1 = 1$, which is consistent with $a_1=1$, so it also holds when $n=1$.

Therefore, $a_n = n^2 + n - 1$

$$a_n = n^2 + n - 1$$

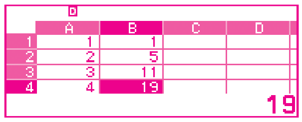
OTHER METHODS

In this case, $\sum_{k=1}^{n-1} b_k$ is the sum of the first term to the $(n-1)$ term of an arithmetic progression with a first term of 4 and a common difference of 2, so

$$a_n = a_1 + \sum_{k=1}^{n-1} b_k = 1 + \frac{1}{2}(n-1)\{2\cdot 4 + (n-1-1)\cdot 2\} = n^2 + n - 1$$

check

Press ⌂, select [Spreadsheet], press OK, then clear the previous data by pressing ⏻



After inputting [A1:1, A2:2, A3:3, and A4:4] respectively, press EXE, move to [B1].

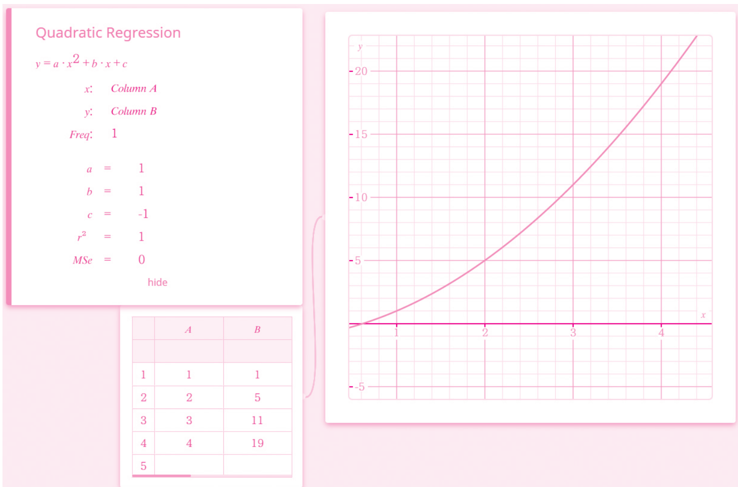
After inputting [B1:1, B2:5, B3:11, and B4:19] respectively, press EXE

Press ⇧ x, scan the QR code to display the data.

In addition, you can use the following procedure to change the settings at ClassPad.net.

Tap the coordinates sheet, tap column A and column B, tap [Statistics], tap [Regression], and then tap [Quadratic Regression].

We can confirm the functional relation $y = x^2 + x - 1$ of columns A and B from the data.



(2) 1, 2, 5, 14, 41, ……

Given a progression of differences $\{b_n\}$, then $\{b_n\}$ is 1, 3, 9, 27,……, so $b_n = 1 \cdot 3^{n-1}$

Given $n \geq 2$

$$a_n = a_1 + \sum_{k=1}^{n-1} b_k = 1 + \sum_{k=1}^{n-1} 1 \cdot 3^{k-1} = 1 + \frac{3^{n-1} - 1}{3 - 1} = \frac{1}{2} \cdot 3^{n-1} + \frac{1}{2}$$

When $n=1$, this becomes $\frac{1}{2} \cdot 3^{1-1} + \frac{1}{2} = 1$, which is consistent with $a_1 = 1$, so it also holds when $n=1$.

Therefore, $a_n = \frac{1}{2} \cdot 3^{n-1} + \frac{1}{2}$

$$a_n = \frac{1}{2} \cdot 3^{n-1} + \frac{1}{2}$$

check

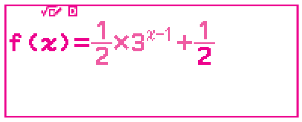
Press ⌂, select [Table], press OK, then clear the previous data by pressing ⏻

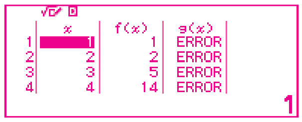
Press ⋯, select [Define f(x)/g(x)], press OK, select [Define f(x)], press OK

After inputting f(x)=$\frac{1}{2} \times 3^{x-1} + \frac{1}{2}$, press EXE

Press ⋯, select [Table Range], press OK

After inputting [Start:1, End:4, Step:1], select [Execute], press EXE





Sums and general terms of sequences

TARGET To understand the relation between general terms and the sums of sequences.

STUDY GUIDE

Sums and general terms of sequences

For some sequences, the general term is unknowable, but instead we can find the sum of the terms up to the nth term. In this case, we can find the general term from **the difference between the sum up to the nth term and the sum up to the $(n-1)$**th term.

How to find the general term of a sequence whose sum is known

Given S_n is the sum of the first term to the nth term of the sequence $\{a_n\}$, then when $n \geq 2$, we can find the difference between S_n and S_{n-1} from $S_n = a_1 + a_2 + a_3 + \cdots\cdots + a_n$, as shown below.

$$
\begin{array}{rrcl}
 & S_n & = & a_1 + a_2 + a_3 + \ldots\ldots + a_{n-1} + a_n \\
-) & S_{n-1} & = & a_1 + a_2 + a_3 + \ldots\ldots + a_{n-1} \\
\hline
 & S_n - S_{n-1} & = & a_n
\end{array}
$$

This shows that the general term a_n of the sequence $\{a_n\}$ is obtained by the difference $S_n - S_{n-1}$ between the sum S_{n-1} up to the nth term and the sum S_n up to the $(n-1)$th.

Note that in this case, when $n=1$, the left side of the expression above becomes $S_1 - S_0$, and it does not hold. That is to say, $n \geq 2$ is a condition for deriving the above expression. When $n=1$, then we derive $S_1 = a_1$.

From the above, we get the following relation between the general term and the sum of a sequence.

(i) When $n=1$, then $a_1 = S_1$

(ii) When $n \geq 2$, then $a_n = S_n - S_{n-1}$

EXERCISE

◆ Find the general term a_n of the sequence $\{a_n\}$ whose sum S_n of the first term to the nth term is given by the following expression.

(1) $S_n = 3n^2 + 4n$

When $n \geq 2$, $a_n = S_n - S_{n-1} = 3n^2 + 4n - \{3(n-1)^2 + 4(n-1)\} = 6n + 1$ …(i)

When $n=1$, then $a_1 = S_1 = 3 \cdot 1^2 + 4 \cdot 1 = 7$, and by substituting this into (i) for $n=1$, it is equivalent to $6 \cdot 1 + 1 = 7$, so (i) also holds when $n=1$.

Therefore, $a_n = 6n+1$

$$a_n = 6n + 1$$

check

Press ⌂, select [Spreadsheet], press OK, then clear the previous data by pressing ⏻

After inputting [A1:1, A2:2, A3:3, and A4:4] respectively, press EXE, move to [B1].

Press ⋯, select [Fill Formula], press OK

After inputting [Form$=3A1^2+4A1$], press EXE

After inputting [Range:B1:B4], press EXE,

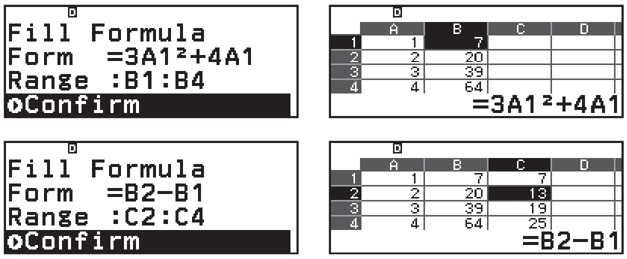
select [Confirm], press OK, move to [C1]

After inputting [C1:7], press EXE

Press ⋯, select [Fill Formula], press OK

After inputting [Form=B2−B1], press EXE

After inputting [Range:C2:C4], press EXE, select [Confirm], press OK

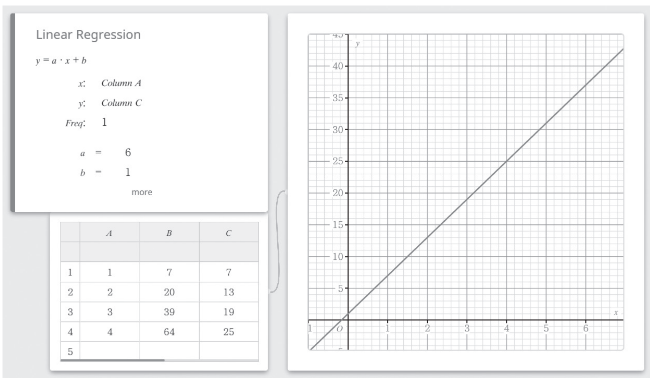


Press ⇧ x, scan the QR code to display the data.

In addition, you can use the following procedure to change the settings at ClassPad.net.

Tap the coordinates sheet, tap column A and column C, tap [Statistics], tap [Regression], and then tap [Linear Regression].

We can confirm the functional relation $y=6x+1$ of columns A and C from the data.



(2) $S_n = 3^n - 1$

When $n \geq 2$, $a_n = S_n - S_{n-1} = 3^n - 1 - (3^{n-1} - 1) = (3-1)3^{n-1} = 2 \cdot 3^{n-1}$ $\ldots(\mathrm{i})$

When $n=1$, then $a_1 = S_1 = 3^1 - 1 = 2$, and by substituting this into (i) for $n=1$, it is equivalent to $2 \cdot 3^{1-1} = 2$, so (i) also holds when $n=1$.

Therefore, $a_n = 2 \cdot 3^{n-1}$

$$\boldsymbol{a_n = 2 \cdot 3^{n-1}}$$

check

Press ⌂, select [Spreadsheet], press OK, then clear the previous data by pressing ⏻

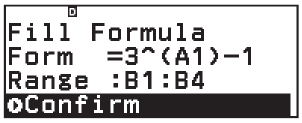
After inputting [A1:1, A2:2, A3:3, and A4:4] respectively, press EXE, move to [B1].

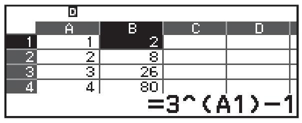
Press ⋯, select [Fill Formula], press OK

After inputting [Form$= 3^{(\mathrm{A1})} - 1$], press EXE

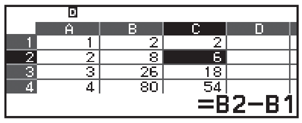
After inputting [Range:B1:B4], press EXE,

select [Confirm], press OK, move to [C1]





After inputting [C1:2], press EXE



Press ⋯, select [Fill Formula], press OK

After inputting [Form=B2−B1], press EXE

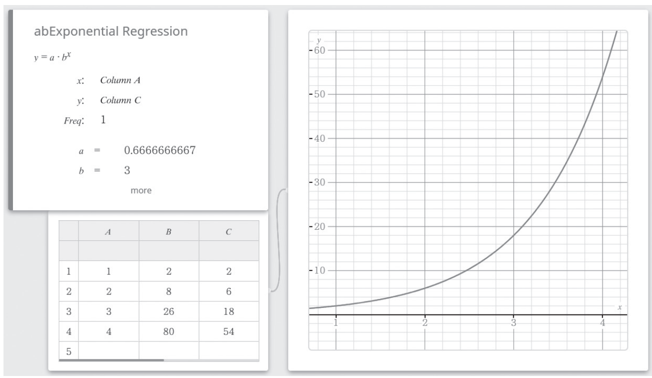
After inputting [Range:C2:C4], press EXE, select [Confirm], press OK

Press ⇧ x, scan the QR code to display the data.

In addition, you can use the following procedure to change the settings at ClassPad.net.

Tap the coordinates sheet, tap column A and column C, tap [Statistics], tap [Regression], and then tap [abExponential Regression].

We can confirm the functional relation $y = \dfrac{2}{3} \cdot 3^x = 2 \cdot 3^{x-1}$ of columns A and C from the data.



PRACTICE

◆ Find the general term a_n of the sequence $\{a_n\}$ whose sum S_n of the first term to the nth term is given by the following expression.

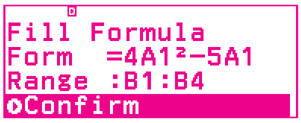
(1) $S_n = 4n^2 - 5n$

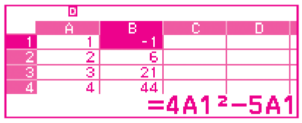
When $n \geq 2$, $a_n = S_n - S_{n-1} = 4n^2 - 5n - \{4(n-1)^2 - 5(n-1)\} = 8n - 9$ ...(i)
When $n=1$, then $a_1 = S_1 = 4 \cdot 1^2 - 5 \cdot 1 = -1$, and by substituting this into (i) for $n=1$, it is equivalent to $8 \cdot 1 - 9 = -1$, so (i) also holds when $n=1$.
Therefore, $a_n = 8n - 9$

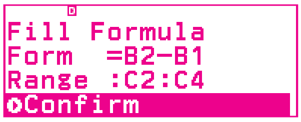
$$a_n = 8n - 9$$

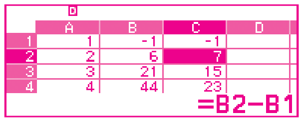
check

Press ⌂, select [Spreadsheet], press OK, then clear the previous data by pressing ⏻
After inputting [A1:1, A2:2, A3:3, and A4:4] respectively, press EXE, move to [B1].
Press ⋯, select [Fill Formula], press OK
After inputting [Form=4A1² − 5A1], press EXE
After inputting [Range:B1:B4], press EXE, select [Confirm], press OK, move to [C1]
After inputting [C1:−1], press EXE
Press ⋯, select [Fill Formula], press OK
After inputting [Form=B2−B1], press EXE
After inputting [Range:C2:C4], press EXE, select [Confirm], press OK

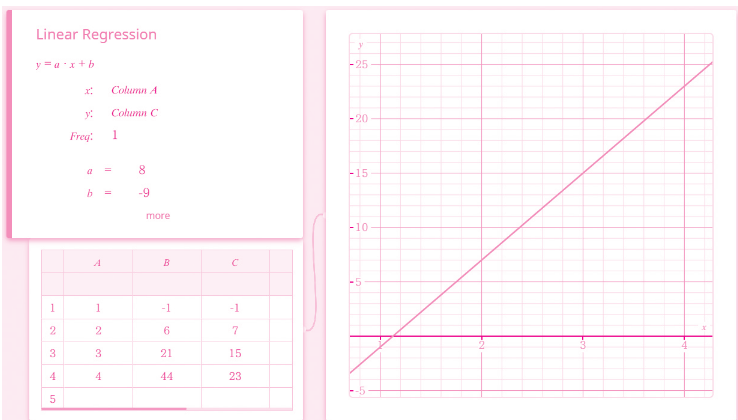








Press ⇧ x, scan the QR code to display the data.
In addition, you can use the following procedure to change the settings at ClassPad.net.
Tap the coordinates sheet, tap column A and column C, tap [Statistics], tap [Regression], and then tap [Linear Regression].
We can confirm the functional relation $y=8x-9$ of columns A and C from the data.



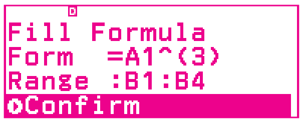
(2) $S_n = n^3$

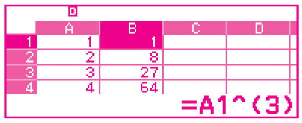
When $n \geq 2$, $a_n = S_n - S_{n-1} = n^3 - (n-1)^3 = 3n^2 - 3n + 1$ ...(i)
When $n=1$, then $a_1 = S_1 = 1^3 = 1$, and by substituting this into (i) for $n=1$, it is equivalent to $3 \cdot 1^2 - 3 \cdot 1 + 1 = 1$, so (i) also holds when $n=1$.
Therefore, $a_n = 3n^2 - 3n + 1$

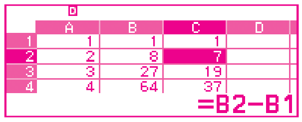
$$a_n = 3n^2 - 3n + 1$$

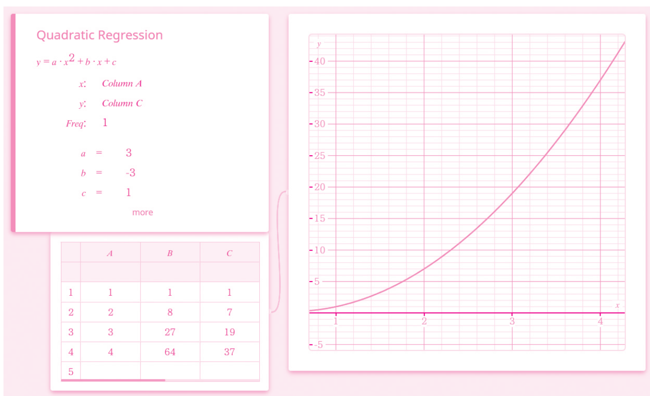
check

Press ⌂, select [Spreadsheet], press OK, then clear the previous data by pressing ⏻
After inputting [A1:1, A2:2, A3:3, and A4:4] respectively, press EXE, move to [B1].
Press ⋯, select [Fill Formula], press OK
After inputting [Form= A 1^(3)], press EXE
After inputting [Range:B1:B4], press EXE, select [Confirm], press OK, move to [C1]
After inputting [C1:1], press EXE
Press ⋯, select [Fill Formula], press OK
After inputting [Form=B2−B1], press EXE
After inputting [Range:C2:C4], press EXE, select [Confirm], press OK
Press ⇧ x, scan the QR code to display the data.
In addition, you can use the following procedure to change the settings at ClassPad.net.
Tap the coordinates sheet, tap column A and column C, tap [Statistics], tap [Regression], and then tap [Quadratic Regression].
We can confirm the functional relation $y = 3x^2 - 3x + 1$ of columns A and C from the data.









Sums of various sequences (1)

TARGET To understand how to find the sum of sequences expressed as fractions.

STUDY GUIDE

Sums of sequences expressed as fractions

Partial fraction decomposition

The sum of a sequence in which each term is expressed as a fraction may be found by transforming each term into the difference of the fraction by using the following formula.

$$\frac{1}{(k+a)(k+b)}=\frac{1}{b-a}\left(\frac{1}{k+a}-\frac{1}{k+b}\right) \quad \text{(where, } a \neq b\text{)}$$

Note that partial fraction decomposition is the inverse calculation of reduction to a common denominator.

Ex. $$\frac{1}{(k+3)(k+4)}=\frac{1}{4-3}\left(\frac{1}{k+3}-\frac{1}{k+4}\right)=\frac{1}{k+3}-\frac{1}{k+4}$$

Rationalization of denominators of irrational numbers

The sum of a sequence that contains a root sign in the denominator of each term may be found by rationalizing the denominator as shown below.

$$\frac{1}{\sqrt{k+c}+\sqrt{k+d}}=\frac{\sqrt{k+c}-\sqrt{k+d}}{c-d} \quad \text{(where, } c \neq d\text{)}$$

Ex. $$\frac{1}{\sqrt{k+2}+\sqrt{k+1}}=\frac{\sqrt{k+2}-\sqrt{k+1}}{(\sqrt{k+2}+\sqrt{k+1})(\sqrt{k+2}-\sqrt{k+1})}=\frac{\sqrt{k+2}-\sqrt{k+1}}{2-1}=\sqrt{k+2}-\sqrt{k+1}$$

EXERCISE

◆ Find the sums of the sequences below.

(1) $\dfrac{1}{1\cdot 2}+\dfrac{1}{2\cdot 3}+\dfrac{1}{3\cdot 4}+\cdots\cdots+\dfrac{1}{n(n+1)}$

The kth term in this sequence can be expressed as $\dfrac{1}{k(k+1)}=\dfrac{1}{k}-\dfrac{1}{k+1}$.

Therefore, we find a sum of

$$\sum_{k=1}^{n}\frac{1}{k(k+1)}=\sum_{k=1}^{n}\left(\frac{1}{k}-\frac{1}{k+1}\right)=\left(\frac{1}{1}-\cancel{\frac{1}{2}}\right)+\left(\cancel{\frac{1}{2}}-\cancel{\frac{1}{3}}\right)+\left(\cancel{\frac{1}{3}}-\cancel{\frac{1}{4}}\right)+\cdots\cdots+\left(\cancel{\frac{1}{n}}-\frac{1}{n+1}\right)$$
$$=1-\frac{1}{n+1}=\frac{n}{n+1}$$

$$\boldsymbol{\frac{n}{n+1}}$$

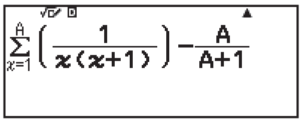
check

On the scientific calculator, use the VARIABLE function to confirm the sums up to term A are consistent.

Press [HOME], select [Calculate], press [OK]

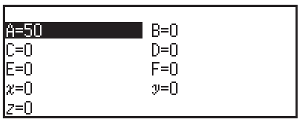
Input $\displaystyle\sum_{x=1}^{A}\frac{1}{x(x+1)}-\frac{A}{A+1}$.

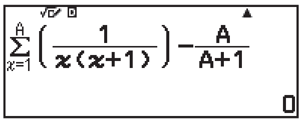
[HOME] [OK] [∨] [∨] [OK] [1] [▭/▭] [x] [(] [x] [+] [1] [)] [∨] [1] [∧] [SHIFT] [4] [<] [<] [<]
[−] [SHIFT] [4] [▭/▭] [SHIFT] [4] [+] [1]



In the VARIABLE screen, input [A=50], and then calculate.

[VARIABLE] [5] [0] [EXE] [BACK] [EXE]





We can confirm this holds when A=50. (We can confirm any value by changing the input value of A.)

(2) $\dfrac{1}{\sqrt{3}+\sqrt{2}}+\dfrac{1}{2+\sqrt{3}}+\dfrac{1}{\sqrt{5}+2}+\cdots\cdots+\dfrac{1}{\sqrt{n+2}+\sqrt{n+1}}$

The kth term in this sequence can be

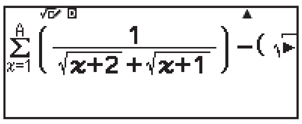
expressed as $\dfrac{1}{\sqrt{k+2}+\sqrt{k+1}}=\dfrac{\sqrt{k+2}-\sqrt{k+1}}{(\sqrt{k+2}+\sqrt{k+1})(\sqrt{k+2}-\sqrt{k+1})}=\dfrac{\sqrt{k+2}-\sqrt{k+1}}{2-1}=\sqrt{k+2}-\sqrt{k+1}$.

Therefore, we find a sum of

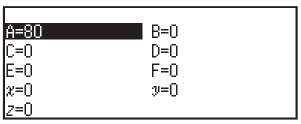
$$\sum_{k=1}^{n}\frac{1}{\sqrt{k+2}+\sqrt{k+1}}=\sum_{k=1}^{n}(\sqrt{k+2}-\sqrt{k+1})$$
$$=(\cancel{\sqrt{3}}-\sqrt{2})+(\cancel{2}-\cancel{\sqrt{3}})+(\cancel{\sqrt{5}}-\cancel{2})+\cdots\cdots+(\sqrt{n+2}-\cancel{\sqrt{n+1}})$$
$$=\sqrt{n+2}-\sqrt{2}$$

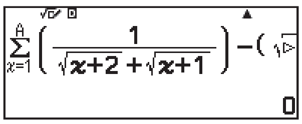
$$\boldsymbol{\sqrt{n+2}-\sqrt{2}}$$

[HOME] [OK] [∨] [∨] [OK] [1] [▭/▭] [√▭] [x] [+] [2] [>] [+] [√▭] [x] [+] [1] [∨] [1] [∧] [SHIFT] [4] [<] [<] [<]
[−] [(] [√▭] [SHIFT] [4] [+] [2] [>] [−] [√▭] [2] [>] [)]



[VARIABLE] [8] [0] [EXE] [BACK] [EXE]





We can confirm this holds when A=80. (We can confirm any value by changing the input value of A.)

◆ Find the sums of the sequences below.

(1) $\dfrac{1}{2\cdot3}+\dfrac{1}{3\cdot4}+\dfrac{1}{4\cdot5}+\cdots\cdots+\dfrac{1}{(n+1)(n+2)}$

The kth term in this sequence can be expressed as $\dfrac{1}{(k+1)(k+2)}=\dfrac{1}{k+1}-\dfrac{1}{k+2}$.

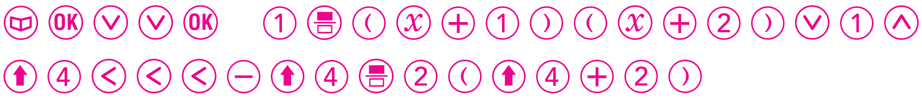
Therefore, we find a sum of

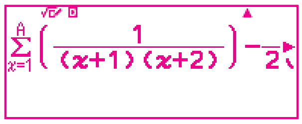
$$\sum_{k=1}^{n}\frac{1}{(k+1)(k+2)}=\sum_{k=1}^{n}\left(\frac{1}{k+1}-\frac{1}{k+2}\right)$$

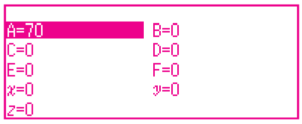
$$=\left(\frac{1}{2}-\cancel{\frac{1}{3}}\right)+\left(\cancel{\frac{1}{3}}-\cancel{\frac{1}{4}}\right)+\left(\cancel{\frac{1}{4}}-\cancel{\frac{1}{5}}\right)+\cdots\cdots+\left(\cancel{\frac{1}{n+1}}-\frac{1}{n+2}\right)$$

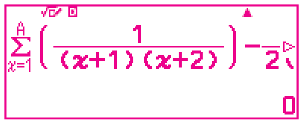
$$=\frac{1}{2}-\frac{1}{n+2}=\frac{n}{2(n+2)}$$

$$\frac{n}{2(n+2)}$$









We can confirm this holds when $A=70$. (We can confirm any value by changing the input value of A.)

(2) $\dfrac{1}{\sqrt{3}+1}+\dfrac{1}{\sqrt{5}+\sqrt{3}}+\dfrac{1}{\sqrt{7}+\sqrt{5}}+\cdots\cdots+\dfrac{1}{\sqrt{2n+1}+\sqrt{2n-1}}$

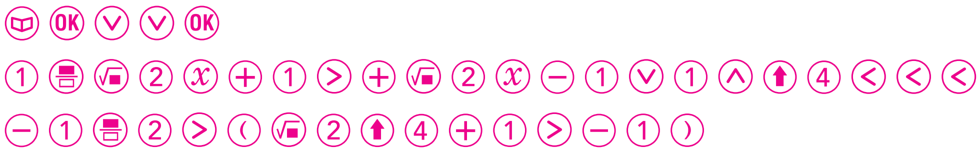
The kth term in this sequence can be expressed as

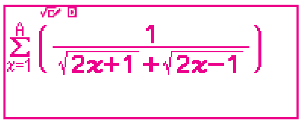
$$\frac{1}{\sqrt{2k+1}+\sqrt{2k-1}}=\frac{\sqrt{2k+1}-\sqrt{2k-1}}{(\sqrt{2k+1}+\sqrt{2k-1})(\sqrt{2k+1}-\sqrt{2k-1})}$$
$$=\frac{\sqrt{2k+1}-\sqrt{2k-1}}{1-(-1)}=\frac{1}{2}(\sqrt{2k+1}-\sqrt{2k-1}).$$

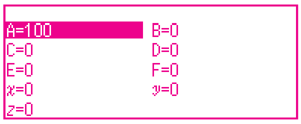
Therefore, we find a sum of

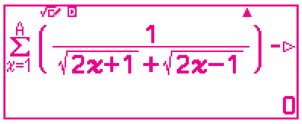
$$\sum_{k=1}^{n}\frac{1}{\sqrt{2k+1}+\sqrt{2k-1}}=\sum_{k=1}^{n}\frac{1}{2}(\sqrt{2k+1}-\sqrt{2k-1})=\frac{1}{2}\sum_{k=1}^{n}(\sqrt{2k+1}-\sqrt{2k-1})$$
$$=\frac{1}{2}\{(\cancel{\sqrt{3}}-1)+(\cancel{\sqrt{5}}-\cancel{\sqrt{3}})+(\cancel{\sqrt{7}}-\cancel{\sqrt{5}})+\cdots\cdots+(\sqrt{2n+1}-\cancel{\sqrt{2n-1}})\}$$
$$=\frac{1}{2}(\sqrt{2n+1}-1)$$

$$\frac{1}{2}(\sqrt{2n+1}-1)$$









We can confirm this holds when $A=100$. **(We can confirm any value by changing the input value of** A.**)**

Sums of various sequences (2)

TARGET To understand how to find the sum of sequence as (arithmetic progression) × (geometric progression).

STUDY GUIDE

Sum of a sequence as (arithmetic progression) × (geometric progression)

Consider how to find $S_n = 1\cdot 1 + 2\cdot 2 + 3\cdot 2^2 + 4\cdot 2^3 + \cdots\cdots + (n-1)\cdot 2^{n-2} + n\cdot 2^{n-1}$.

This sequence multiplies the following 2 sequences in the form of (arithmetic progression) × (geometric progressions).

Arithmetic progression : 1 , 2 , 3 , 4 , …… , $n-1$, n

Geometric progressions : 1 , 2 , 2^2 , 2^3 , …… , 2^{n-2} , 2^{n-1}

The sum of these sequences can be found by calculating $\boldsymbol{S_n - rS_n}$ where r is the common ratio of the geometric sequence.

$$\begin{array}{rrl} & S_n = & 1\cdot 1 + 2\cdot 2 + 3\cdot 2^2 + 4\cdot 2^3 + \cdots\cdots + (n-1)\cdot 2^{n-2} + n\cdot 2^{n-1} \\ -\,) & 2S_n = & \qquad 1\cdot 2 + 2\cdot 2^2 + 3\cdot 2^3 + \cdots\cdots + (n-2)\cdot 2^{n-2} + (n-1)\cdot 2^{n-1} + n\cdot 2^n \\ \hline & -S_n = & 1 + 2 + 2^2 + 2^3 + \cdots\cdots + 2^{n-2} + 2^{n-1} - n\cdot 2^n \end{array}$$

By using the formula of the sum of geometric sequences on this, we get

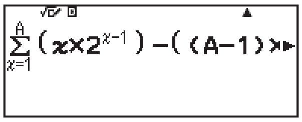
$$-S_n = (1 + 2 + 2^2 + 2^3 + \cdots\cdots + 2^{n-2} + 2^{n-1}) - n\cdot 2^n = \frac{2^n - 1}{2-1} - n\cdot 2^n = (1-n)\cdot 2^n - 1$$

Therefore, we can find $S_n = (n-1)\cdot 2^n + 1$.

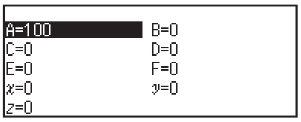
proof

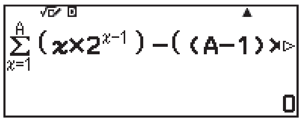
On the scientific calculator, use the VARIABLE function to confirm the sums up to term A are consistent.

[CATALOG] [OK] [▽] [▽] [OK] [x] [×] [2] [■□] [x] [−] [1] [▽] [1] [△] [⇧] [4] [◁] [◁] [◁]
[−] [(] [(] [⇧] [4] [−] [1] [)] [×] [2] [■□] [⇧] [4] [▷] [+] [1] [)]



[VARIABLE] [1] [0] [0] [EXE] [↶] [EXE]





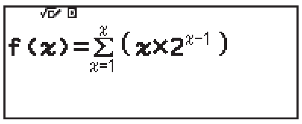
We can confirm this holds when A=100. (We can confirm any value by changing the input value of A.)

OTHER METHODS

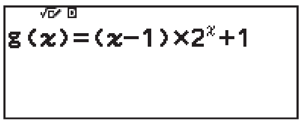
Press [HOME], select [Table], press [OK], then clear the previous data by pressing [ON]

Press [TOOLS], select [Define f(x)/g(x)], press [OK], select [Define f(x)], press [OK]

After inputting $f(x) = \sum_{x=1}^{x}(x \times 2^{x-1})$, press [EXE]

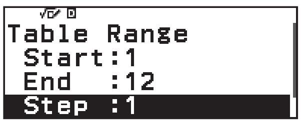


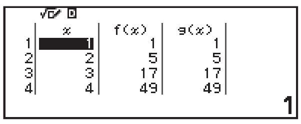
In the same way, input $g(x) = (x-1)\times 2^x + 1$.

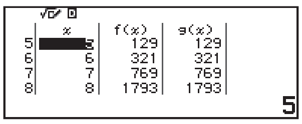


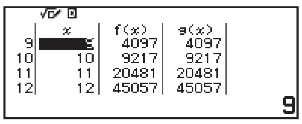
Press [TOOLS], select [Table Range], press [OK]

After inputting [Start:1, End:12, Step:1], select [Execute], press [EXE]









EXERCISE

◆ Find $S_n=1\cdot1+3\cdot3+5\cdot3^2+7\cdot3^3+\cdots\cdots+(2n-3)\cdot3^{n-2}+(2n-1)\cdot3^{n-1}$.

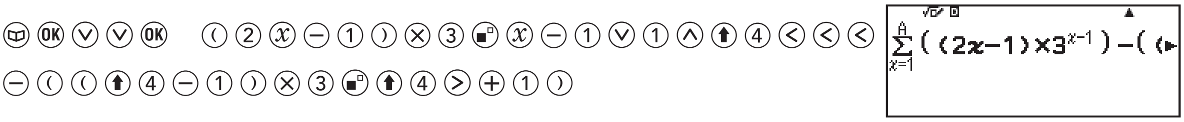
$$S_n=1\cdot1+3\cdot3+5\cdot3^2+7\cdot3^3+\cdots\cdots+(2n-3)\cdot3^{n-2}+(2n-1)\cdot3^{n-1}\quad\ldots(\mathrm{i})$$
$$3S_n=\qquad 1\cdot3+3\cdot3^2+5\cdot3^3+\cdots\cdots+(2n-5)\cdot3^{n-2}+(2n-3)\cdot3^{n-1}+(2n-1)\cdot3^n\quad\ldots(\mathrm{ii})$$

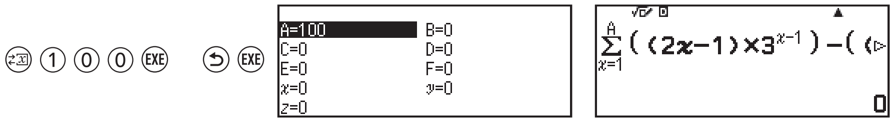
From $(\mathrm{i})-(\mathrm{ii})$, we get $-2S_n=1\cdot1+2\cdot3+2\cdot3^2+2\cdot3^3+\cdots\cdots+2\cdot3^{n-2}+2\cdot3^{n-1}-(2n-1)\cdot3^n$

$$=1+2(3+3^2+3^3+\cdots\cdots+3^{n-2}+3^{n-1})-(2n-1)\cdot3^n$$
$$=1+2\cdot\frac{3(3^{n-1}-1)}{3-1}-(2n-1)\cdot3^n=-2\{(n-1)\cdot3^n+1\}$$

Therefore, $S_n=(n-1)\cdot3^n+1$

$$\mathbf{S_n=(n-1)\cdot3^n+1}$$





We can confirm this holds when $\mathrm{A}=100$. (We can confirm any value by changing the input value of A.)

PRACTICE

◆ Find $S_n=1\cdot5+2\cdot5^2+3\cdot5^3+4\cdot5^4+\cdots\cdots+(n-1)\cdot5^{n-1}+n\cdot5^n$.

$$S_n=1\cdot5+2\cdot5^2+3\cdot5^3+4\cdot5^4+\cdots\cdots+(n-1)\cdot5^{n-1}+\qquad n\cdot5^n\quad\ldots(\mathrm{i})$$
$$5S_n=\qquad 1\cdot5^2+2\cdot5^3+3\cdot5^4+\cdots\cdots+(n-2)\cdot5^{n-1}+(n-1)\cdot5^n+n\cdot5^{n+1}\quad\ldots(\mathrm{ii})$$

From $(\mathrm{i})-(\mathrm{ii})$, we get

$$-4S_n=1\cdot5+1\cdot5^2+1\cdot5^3+1\cdot5^4+\cdots\cdots+1\cdot5^{n-1}+1\cdot5^n-n\cdot5^{n+1}$$
$$=\frac{5(5^n-1)}{5-1}-n\cdot5^{n+1}=-\frac{(4n-1)\cdot5^{n+1}+5}{4}$$

Therefore, $S_n=\dfrac{(4n-1)\cdot5^{n+1}+5}{16}$

$$S_n=\frac{(4n-1)\cdot5^{n+1}+5}{16}$$

check

Press ⌂**, select** [Table]**, press** OK**, then clear the previous data by pressing** ⏻

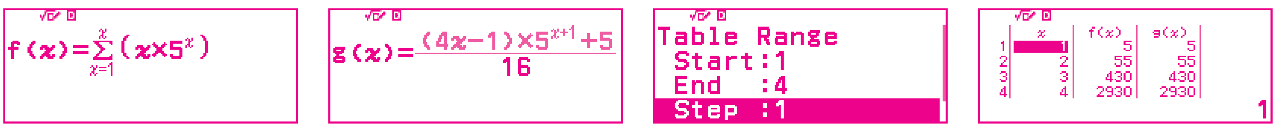
Press ⋯**, select** [Define $\mathrm{f}(x)/\mathrm{g}(x)$]**, press** OK**, select** [Define $\mathrm{f}(x)$]**, press** OK

After inputting $\mathrm{f}(x)=\sum_{x=1}^{x}(x\times5^x)$**, press** EXE

In the same way, input $\mathrm{g}(x)=\dfrac{(4x-1)\times5^{x+1}+5}{16}$.

Press ⋯**, select** [Table Range]**, press** OK

After inputting [Start:1, End:4, Step:1]**, select** [Execute]**, press** EXE



Grouping sequences

TARGET To understand grouped sequences.

STUDY GUIDE

Grouping sequences

A sequence that is split into groups according to some rule is called a **grouped sequence**.

Ex. $\underbrace{1}_{\text{1st group}} \mid \underbrace{2, 3}_{\text{2nd group}} \mid \underbrace{4, 5, 6}_{\text{3rd group}} \mid \cdots\cdots$

The grouped sequence above can be summarized as follows.

Group	1st group	2nd group	3rd group	$\cdots\cdots$	$(n-1)$th group	nth group
Sequences	1	$2, 3$	$4, 5, 6$	$\cdots\cdots$		(ii)
First term	1	2	4	$\cdots\cdots$		(i)
Number of elements	1 element	2 elements	3 elements	$\cdots\cdots$	$(n-1)$ elements	n elements

(i) Until the last number in the $(n-1)$th group, there are $1+2+3+\cdots\cdots+(n-1)=\dfrac{1}{2}n(n-1)$ (elements) terms.

The first number in the nth group is the $\left\{\dfrac{1}{2}n(n-1)+1\right\}$th term.

The kth term in this sequence is $a_k=k$, so $a_{\frac{1}{2}n(n-1)+1}=\dfrac{1}{2}n(n-1)+1$

(ii) The nth group is an arithmetic progression with a first term of $\left\{\dfrac{1}{2}n(n-1)+1\right\}$, a common difference of 1, and there are n terms, so the numbers arranged in the nth group are

$$\frac{1}{2}n(n-1)+1,\ \frac{1}{2}n(n-1)+2,\ \frac{1}{2}n(n-1)+3,\ \cdots\cdots,\ \frac{1}{2}n(n-1)+n$$

EXERCISE

◆ Given natural numbers are arranged in ascending order and divided so that the nth group contains n elements. In this case, solve the following problems.

$1 \mid 2, 3 \mid 4, 5, 6 \mid 7, 8, 9, 10 \mid \ldots\ldots$

(1) Find the first number and the last number in the 12th group.

Until the last number in the 11th group, there are $1+2+3+\cdots\cdots+11=\frac{1}{2}\cdot 11\cdot(11+1)=66$ (elements) terms.

The first number in the 12th group is 67 terms from the beginning. Since the general term of this sequence is a_k, such that $a_k=k$, we get $a_{67}=67$

Since the last number of the 12th group is $1+2+3+\cdots\cdots+12=\frac{1}{2}\cdot 12\cdot(12+1)=78$ (terms) from the beginning, we get $a_{78}=78$

First number…67, last number…78

(2) Find the first number in the nth group and the sum of the numbers in the nth group.

The last number in the $(n-1)$th group is $1+2+3+\cdots\cdots+(n-1)=\frac{1}{2}n(n-1)$.

Therefore, the first number in the nth group is $\left\{\frac{1}{2}n(n-1)+1\right\}$ terms from the beginning.

Since the general term of this sequence is a_k, such that $a_k=k$, we get $a_{\frac{1}{2}n(n-1)+1}=\frac{1}{2}n(n-1)+1$

The numbers arranged in the nth group are an arithmetic progression with a first term of $\left\{\frac{1}{2}n(n-1)+1\right\}$, a common difference of 1, and there are n terms, so the sum of those numbers is

$$\frac{1}{2}\times(\text{number of terms})\times\{2\times(\text{first term})+(n-1)\times(\text{common difference})\}=\frac{1}{2}n\left[2\left\{\frac{1}{2}n(n-1)+1\right\}+(n-1)\cdot 1\right]=\frac{1}{2}n(n^2+1)$$

First number… $\frac{1}{2}n(n-1)+1$, sum… $\frac{1}{2}n(n^2+1)$

PRACTICE

◆ Given odd numbers are arranged in ascending order and divided so that the nth group contains $(2n-1)$ numbers. In this case, solve the following problems.

$1 \mid 3, 5, 7 \mid 9, 11, 13, 15, 17 \mid 19, 21, 23, 25, 27, 29, 31 \mid \ldots\ldots$

(1) Find the first number and the last number in the 15th group.

Until the last number in the 14th group, there are

$1+3+5+\cdots\cdots+(2\cdot 14-1)=\dfrac{1}{2}\cdot 14\cdot\{1+(2\cdot 14-1)\}=196$ **(elements) terms.**

The first number in the 15th group is 197 terms from the beginning.

Since the general term of this sequence is a_k, such that $a_k=2k-1$, we get

$a_{197}=2\cdot 197-1=393$

The last number in the 15th group is

$1+3+5+\cdots\cdots+(2\cdot 15-1)=\dfrac{1}{2}\cdot 15\cdot\{1+(2\cdot 15-1)\}=225$ **(terms) from the beginning**

Therefore, $a_{225}=2\cdot 225-1=449$

First number…393, last number…449

(2) Find the first number in the nth group and the sum of the numbers in the nth group.

The last number in the $(n-1)$th group is

$$1+3+5+\cdots\cdots+\{2\cdot(n-1)-1\}=\sum_{k=1}^{n-1}(2k-1)=2\cdot\frac{1}{2}(n-1)n-(n-1)$$
$$=(n-1)^2$$

Therefore, the first number in the nth group is $\{(n-1)^2+1\}$ terms from the beginning.

Since the general term of this sequence is a_k, such that $a_k=2k-1$, we get

$a_{(n-1)^2+1}=2\{(n-1)^2+1\}-1=2n^2-4n+3$

The numbers arranged in the nth group are an arithmetic progression with a first term of $(2n^2-4n+3)$, a common difference of 2, and there are $(2n-1)$ terms, so the sum of those numbers is

$\dfrac{1}{2}\times$**(number of terms)**$\times[2\times$**(first term)**$+\{$**(number of terms)**$-1\}\times$**(common difference)**$]$

$$=\frac{1}{2}\cdot(2n-1)[2(2n^2-4n+3)+\{(2n-1)-1\}\cdot 2]$$
$$=(2n-1)(2n^2-2n+1)$$

First number…$2n^2-4n+3$, sum…$(2n-1)(2n^2-2n+1)$

Recurrence formula (1)

TARGET To understand recurrence formulas for arithmetic progressions, geometric progressions, and progressions of differences.

STUDY GUIDE

Recurrence formula

Recurrence formulas for arithmetic progressions and geometric progressions

Given we can derive that $a_{n+1}=2a_n+1$ is between the 2 terms a_n and a_{n+1} that are adjacent to $\{a_n\}$, then by stating that $a_1=1$, we can determine that $a_2=2a_1+1=2\cdot1+1=3$, $a_3=2a_2+1=2\cdot3+1=7$, $a_4=2a_3+1=2\cdot7+1=15$,$\cdots\cdots$for each element in the sequence.

Equations that show us a rule that determines the 1 way to get the next term from the previous term in this way are called **recurrence formulas**.

Given a first term of a_1, a common difference of d, or a common ratio of r, we can express the general term and recurrence formula of arithmetic progressions and geometric progressions as follows.

	Recurrence formula	General term
Arithmetic progression	$a_{n+1}=a_n+d$	$a_n=a_1+(n-1)d$
Geometric progression	$a_{n+1}=ra_n$	$a_n=a_1r^{n-1}$

How to find the general term of a progression of differences

When a recurrence formula is stated as $a_{n+1}=a_n+f(n)$ (formula expressing n), then we can use a progression of differences in the following procedure to find the general term.

(1) Transform it to $a_{n+1}-a_n=f(n)$, and consider $f(n)$ to be the progression of differences $\{b_n\}$ of the sequence $\{a_n\}$.

(2) Find a_n from the equation $a_n=a_1+\sum_{k=1}^{n-1}b_k$ $(n\geq2)$ for finding the general term of the sequence $\{a_n\}$ from the progression of differences $\{b_n\}$.

(3) Confirm whether the formula found in (2) also holds when $n=1$, and then find the general term.

EXERCISE

1 Find the 2nd to 5th terms given the first term and recurrence formula are as follows.

(1) $a_1=2,\ a_{n+1}=4a_n+5$

$a_2=4a_1+5=4\cdot2+5=13,\ a_3=4a_2+5=4\cdot13+5=57,\ a_4=4a_3+5=4\cdot57+5=233,\ a_5=4a_4+5=4\cdot233+5=937$

$$\boldsymbol{a_2=13,\ a_3=57,\ a_4=233,\ a_5=937}$$

(2) $a_1=5,\ a_{n+1}=2a_n-3$

$a_2=2a_1-3=2\cdot5-3=7,\ a_3=2a_2-3=2\cdot7-3=11,\ a_4=2a_3-3=2\cdot11-3=19,\ a_5=2a_4-3=2\cdot19-3=35$

$$\boldsymbol{a_2=7,\ a_3=11,\ a_4=19,\ a_5=35}$$

(3) $a_1=1,\ a_{n+1}=a_n{}^2+1$

$a_2=a_1{}^2+1=1^2+1=2,\ a_3=a_2{}^2+1=2^2+1=5,\ a_4=a_3{}^2+1=5^2+1=26,\ a_5=a_4{}^2+1=26^2+1=677$

$$\boldsymbol{a_2=2,\ a_3=5,\ a_4=26,\ a_5=677}$$

check

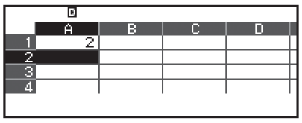
Press ⌂, select [Spreadsheet], press OK, then clear the previous data by pressing ⏻

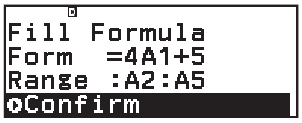
(1) After inputting [A1:2], press EXE

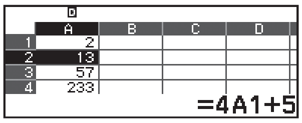
Press ⋯, select [Fill Formula], press OK

After inputting [Form=4A1+5], press EXE

After inputting [Range:A2:A5], press EXE, select [Confirm], press OK







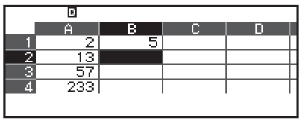
(2) When the sheet is displayed, move to [B1].

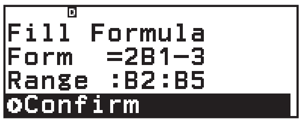
After inputting [B1:5], press EXE

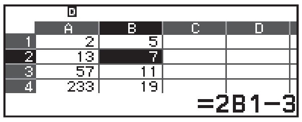
Press ⋯, select [Fill Formula], press OK

After inputting [Form=2B1−3], press EXE

After inputting [Range:B2:B5], press EXE, select [Confirm], press OK







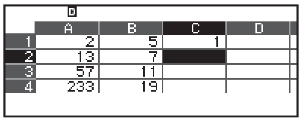
(3) When the sheet is displayed, move to [C1].

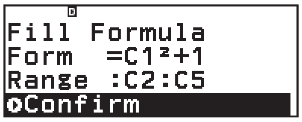
After inputting [C1:1], press EXE

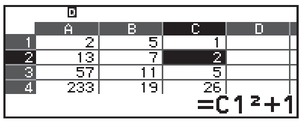
Press ⋯, select [Fill Formula], press OK

After inputting [Form=C1²+1], press EXE

After inputting [Range:C2:C5], press EXE, select [Confirm], press OK







2 Find the general term the sequence $\{a_n\}$ given the first term and recurrence formula are as follows.

(1) $a_1=7,\ a_{n+1}=a_n-3$

The sequence $\{a_n\}$ is an arithmetic progression with a first term of 7 and a common difference of -3,

so $a_n=7+(n-1)\cdot(-3)=-3n+10$

$$\boldsymbol{a_n=-3n+10}$$

(2) $a_1=5,\ a_{n+1}=2a_n$

The sequence $\{a_n\}$ is a geometric progression with a first term of 5 and a common ratio of 2, so $a_n=5\cdot 2^{n-1}$

$$\boldsymbol{a_n=5\cdot 2^{n-1}}$$

(3) $a_1=3,\ a_{n+1}=a_n+2n$

From $a_{n+1}-a_n=2n$, then for the progression of differences $\{b_n\}$ of the sequence $\{a_n\}$, we get $b_n=2n$.

Given $n\geq 2$, then $a_n=a_1+\sum_{k=1}^{n-1}b_k=3+\sum_{k=1}^{n-1}2k=3+2\cdot\frac{1}{2}(n-1)n=n^2-n+3$

This also holds when $n=1$ and $1^2-1+3=3$.

Therefore, the general term is $a_n=n^2-n+3$

$$\boldsymbol{a_n=n^2-n+3}$$

check

Press ⌂, select [Spreadsheet], press OK, then clear the previous data by pressing ⏻

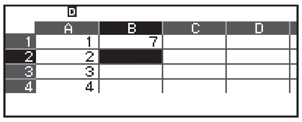
After inputting [A1:1, A2:2, A3:3, and A4:4] respectively, press EXE, move to [B1].

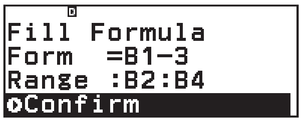
(1) After inputting [B1:7], press EXE

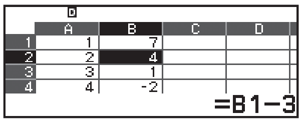
Press ⋯, select [Fill Formula], press OK

After inputting [Form=B1−3], press EXE

After inputting [Range:B2:B4], press EXE, select [Confirm], press OK







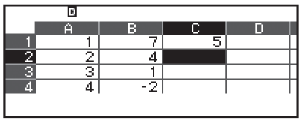
(2) When the sheet is displayed, move to [C1].

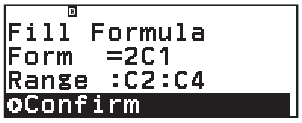
After inputting [C1:5], press EXE

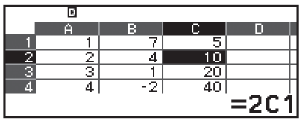
Press ⋯, select [Fill Formula], press OK

After inputting [Form=2C1], press EXE

After inputting [Range:C2:C4], press EXE, select [Confirm], press OK







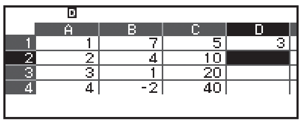
(3) When the sheet is displayed, move to [D1].

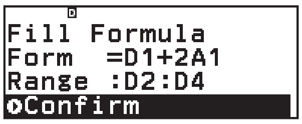
After inputting [D1:3], press EXE

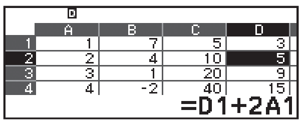
Press ⋯, select [Fill Formula], press OK

After inputting [Form=D1+2A1], press EXE

After inputting [Range:D2:D4], press EXE, select [Confirm], press OK



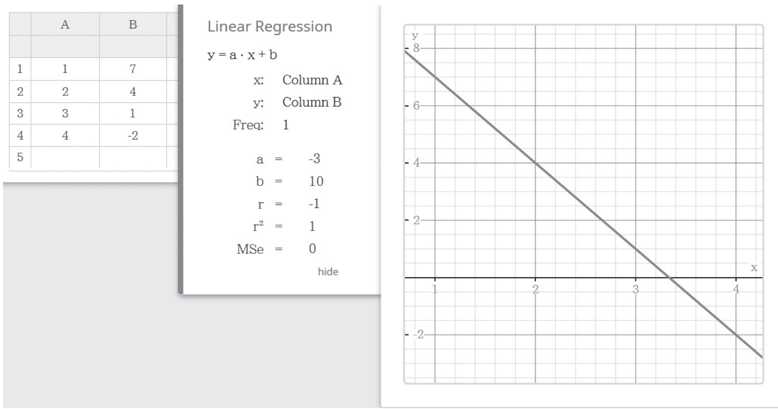




Press ⇧ 𝑥, scan the QR code to display the data. (Continued on the next page.)

(1) Tap column A and column B, tap [$\mathrm{Statistics}$], [$\mathrm{Regression}$], and [$\mathrm{Linear\ Regression}$] in this order.

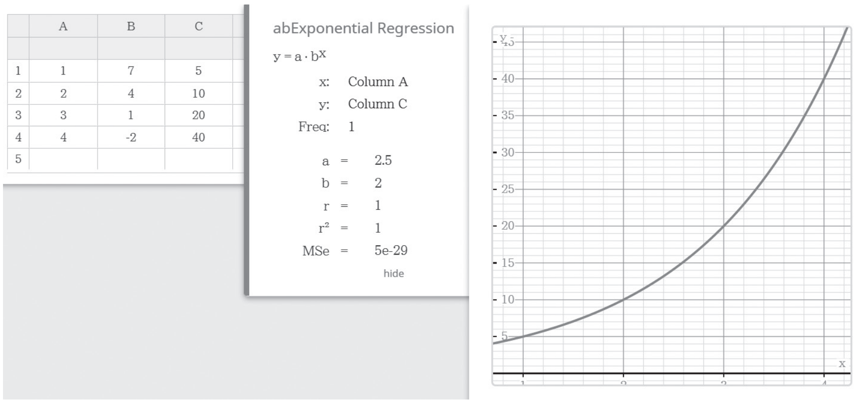
We can confirm the functional relation $y=-3x+10$ of columns A and B.



	A	B
1	1	7
2	2	4
3	3	1
4	4	-2
5		

(2) Tap column A and column C, tap [$\mathrm{Statistics}$], [$\mathrm{Regression}$], and [$\mathrm{abExponential\ Regression}$] in this order.

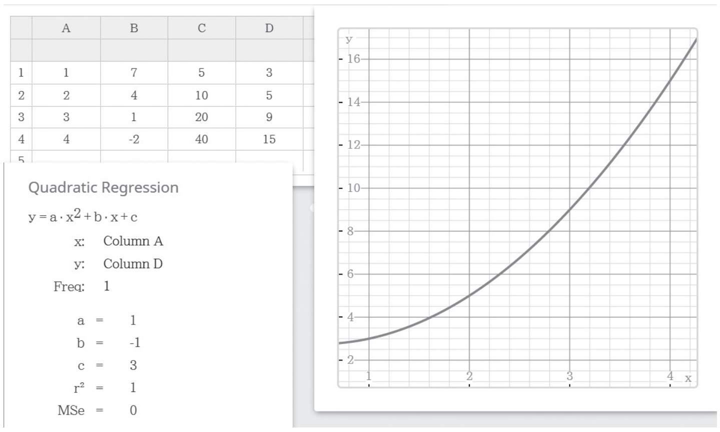
We can confirm the functional relation $y=\frac{5}{2}\cdot 2^x=5\cdot 2^{x-1}$ of columns A and C.



	A	B	C
1	1	7	5
2	2	4	10
3	3	1	20
4	4	-2	40
5			

(3) Tap column A and column D, tap [$\mathrm{Statistics}$], [$\mathrm{Regression}$], and [$\mathrm{Quadratic\ Regression}$] in this order.

We can confirm the functional relation $y=x^2-x+3$ of columns A and D.



	A	B	C	D
1	1	7	5	3
2	2	4	10	5
3	3	1	20	9
4	4	-2	40	15

PRACTICE

1 Find the 2nd to 5th terms given the first term and recurrence formula are as follows.

(1) $a_1=-4$, $a_{n+1}=-2a_n+1$

$a_2=-2a_1+1=-2\cdot(-4)+1=9$, $a_3=-2a_2+1=-2\cdot9+1=-17$,

$a_4=-2a_3+1=-2\cdot(-17)+1=35$, $a_5=-2a_4+1=-2\cdot35+1=-69$

$a_2=9$, $a_3=-17$, $a_4=35$, $a_5=-69$

(2) $a_1=3$, $a_{n+1}=2a_n-1$

$a_2=2a_1-1=2\cdot3-1=5$, $a_3=2a_2-1=2\cdot5-1=9$,

$a_4=2a_3-1=2\cdot9-1=17$, $a_5=2a_4-1=2\cdot17-1=33$

$a_2=5$, $a_3=9$, $a_4=17$, $a_5=33$

check

Press ⌂**, select** [Spreadsheet]**, press** OK**, then clear the previous data by pressing** ⏻

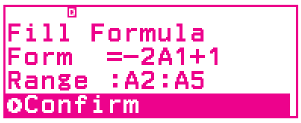
(1) **After inputting** [A1:−4]**, press** EXE

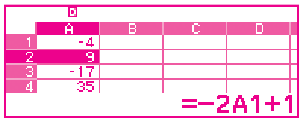
Press ⋯**, select** [Fill Formula]**, press** OK

After inputting [Form=−2A1+1]**, press** EXE

After inputting [Range:A2:A5]**, press** EXE**,**

select [Confirm]**, press** OK





(2) **When the sheet is displayed,**

move to [B1]**.**

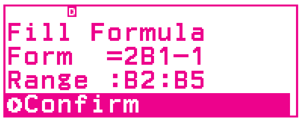
After inputting [B1:3]**, press** EXE

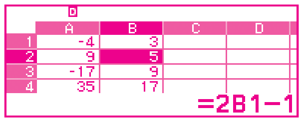
Press ⋯**, select** [Fill Formula]**, press** OK

After inputting [Form=2B1−1]**, press** EXE

After inputting [Range:B2:B5]**, press** EXE**,**

select [Confirm]**, press** OK





2 Find the general term the sequence $\{a_n\}$ given the first term and recurrence formula are as follows.

(1) $a_1=4,\ a_{n+1}=a_n+7$

The sequence $\{a_n\}$ is an arithmetic progression with a first term of 4 and a common difference of 7, so $a_n=4+(n-1)\cdot 7=7n-3$

$$a_n=7n-3$$

(2) $a_1=2,\ a_{n+1}=\frac{1}{2}a_n$

The sequence $\{a_n\}$ is a geometric progression with a first term of 2 and a common ratio of $\frac{1}{2}$, so $a_n=2\cdot\left(\frac{1}{2}\right)^{n-1}$

$$a_n=2\cdot\left(\frac{1}{2}\right)^{n-1}$$

(3) $a_1=-5,\ a_{n+1}=a_n+4n$

From $a_{n+1}-a_n=4n$, then for the progression of differences $\{b_n\}$ of the sequence $\{a_n\}$, we get $b_n=4n$.

Given $n\geq 2$, then

$$a_n=a_1+\sum_{k=1}^{n-1}b_k=-5+\sum_{k=1}^{n-1}4k=-5+4\cdot\frac{1}{2}(n-1)n=2n^2-2n-5$$

This also holds when $n=1$ and $2\cdot 1^2-2\cdot 1-5=-5$.

Therefore, the general term is $a_n=2n^2-2n-5$

$$a_n=2n^2-2n-5$$

check

Press ⌂, select [Spreadsheet], press OK, then clear the previous data by pressing ⏻

After inputting [A1:1, A2:2, A3:3, and A4:4] respectively, press EXE, move to [B1].

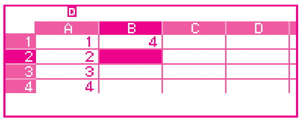
(1) **After inputting [B1:4], press EXE**

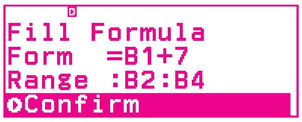
Press ⋯, select [Fill Formula], press OK

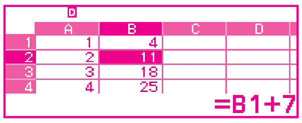
After inputting [Form=B1+7], press EXE

After inputting [Range:B2:B4], press EXE,

select [Confirm], press OK







(2) **When the sheet is displayed,**

move to [C1].

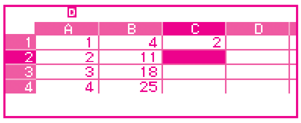
After inputting [C1:2], press EXE

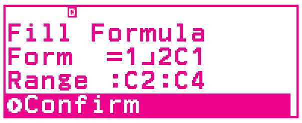
Press ⋯, select [Fill Formula], press OK

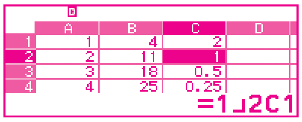
After inputting [Form=$\frac{1}{2}$C1], press EXE

After inputting [Range:C2:C4], press EXE,

select [Confirm], press OK







(3) **When the sheet is displayed,**

move to [D1].

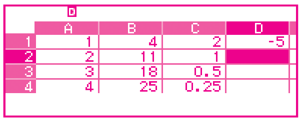
After inputting [D1:−5], press EXE

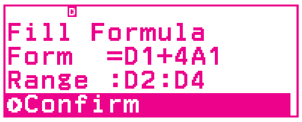
Press ⋯, select [Fill Formula], press OK

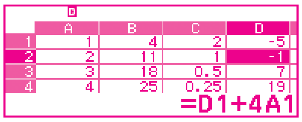
After inputting [Form=D1+4A1], press EXE

After inputting [Range:D2:D4], press EXE,

select [Confirm], press OK



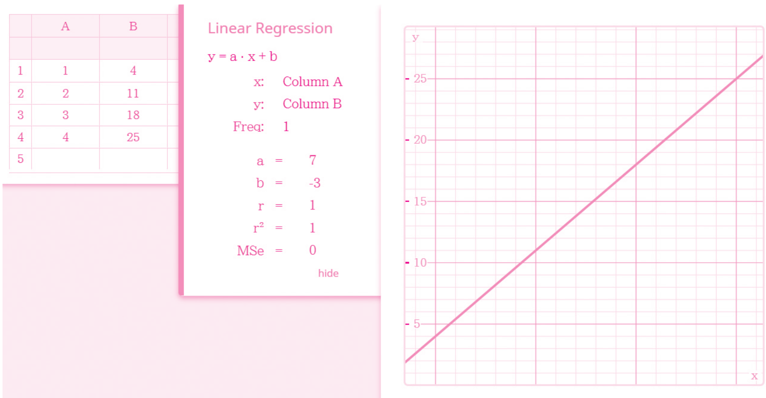




Press ⇧ x, scan the QR code to display the data. **(Continued on the next page.)**

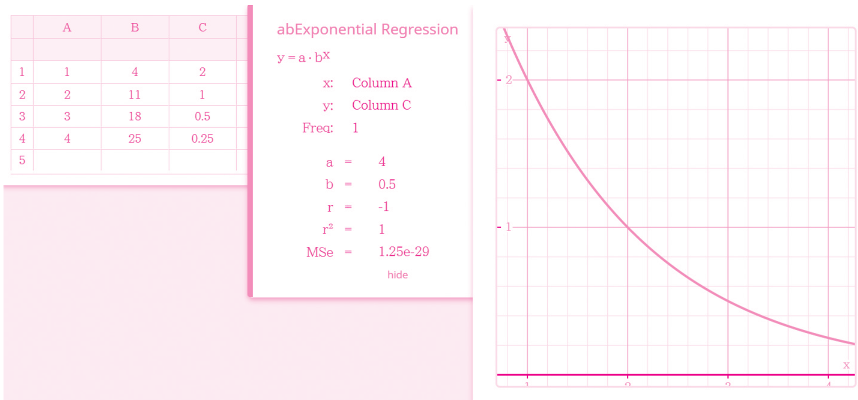
(1) **Tap column** A **and column** B**, tap** [Statistics], [Regression], **and** [Linear Regression] **in this order.**

We can confirm the functional relation $y = 7x - 3$ **of columns** A **and** B**.**



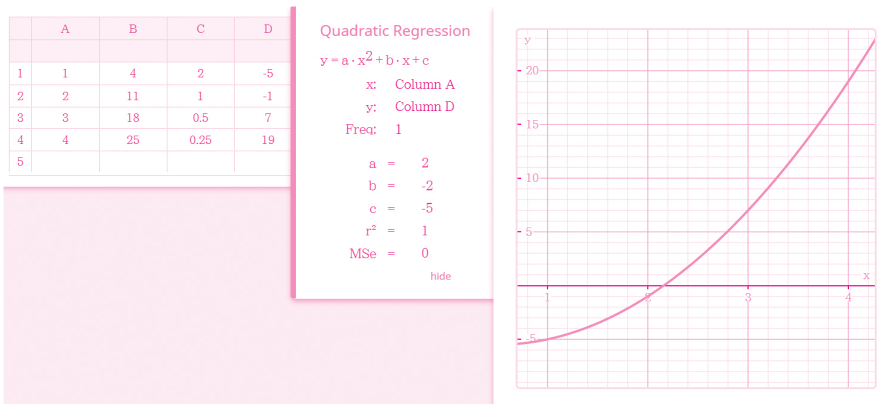
(2) **Tap column** A **and column** C**, tap** [Statistics], [Regression], **and** [abExponential Regression] **in this order.**

We can confirm the functional relation $y = 4 \cdot \left(\dfrac{1}{2}\right)^{x} = 2 \cdot \left(\dfrac{1}{2}\right)^{x-1}$ **of columns** A **and** C**.**



(3) **Tap column** A **and column** D**, tap** [Statistics], [Regression], **and** [Quadratic Regression] **in this order.**

We can confirm the functional relation $y = 2x^2 - 2x - 5$ **of columns** A **and** D**.**



Recurrence formula (2)

TARGET To understand how to find the general term of a recurrence formula in the $a_{n+1}=pa_n+q$ format.

STUDY GUIDE

Recurrence formula in $a_{n+1}=pa_n+q$ format

Consider how to find the general term given that the recurrence formula of the sequence $\{a_n\}$ is expressed as

$a_{n+1}=pa_n+q$ (p and q are constants, and $p\neq 1$ and $q\neq 0$) …(i).

If (i) can be transformed to $a_{n+1}-c=p(a_n-c)$ …(ii), then we can consider the sequence $\{a_n-c\}$ to be a geometric progression with a first term of a_1-c and a common ratio of p, from which we can find the general term a_n.

Assume we can convert the recurrence formula $a_{n+1}=pa_n+q$ …(i) to $a_{n+1}-c=p(a_n-c)$ …(ii).

By transforming to (ii), we get $a_{n+1}=pa_n-pc+c$ …(iii)

By comparing (iii) and (i), we get $q=-pc+c$. That is to say, we get $c=pc+q$ …(iv).

This formula (iv) is the same as formula (i) but with the a_{n+1} and a_n replaced by c, and we call this formula (iv) a **characteristic equation**.

We can use this formula to solve recurrence equations in the format of $a_{n+1}=pa_n+q$, as shown below.

> **(1) Find the solution c of $c=pc+q$ for the characteristic equation of $a_{n+1}=pa_n+q$.**
> **(2) Find the general term of $\{a_n-c\}$, and find the general term of $\{a_n\}$.**

EXERCISE

◆ Find the general terms a_n of the sequences $\{a_n\}$ below.

(1) $a_1=1$, $a_{n+1}=3a_n+2$

Solve $c=3c+2$ to get $c=-1$

Therefore, we can transform $a_{n+1}=3a_n+2$ to $a_{n+1}+1=3(a_n+1)$.

This shows that the sequence $\{a_n+1\}$ is a geometric progression with a first term of $a_1+1=2$ and a common ratio of 3.

Therefore, from $a_n+1=2\cdot 3^{n-1}$, we get $a_n=2\cdot 3^{n-1}-1$

$$a_n=2\cdot 3^{n-1}-1$$

(2) $a_1=3$, $a_{n+1}=-4a_n+1$

Solve $c=-4c+1$ to get $c=\frac{1}{5}$

Therefore, we can transform $a_{n+1}=-4a_n+1$ to $a_{n+1}-\frac{1}{5}=-4\left(a_n-\frac{1}{5}\right)$.

This shows that the sequence $\left\{a_n-\frac{1}{5}\right\}$ is a geometric progression with a first term of $a_1-\frac{1}{5}=\frac{14}{5}$ and a common ratio of -4.

Therefore, from $a_n-\frac{1}{5}=\frac{14}{5}\cdot(-4)^{n-1}$, we get $a_n=\frac{14}{5}\cdot(-4)^{n-1}+\frac{1}{5}$

$$a_n=\frac{14}{5}\cdot(-4)^{n-1}+\frac{1}{5}$$

check

Press ⌂, select [Spreadsheet], press OK, then clear the previous data by pressing ⏻

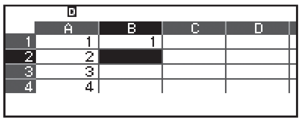
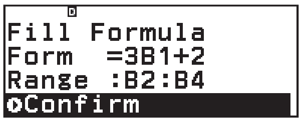
After inputting [A1:1, A2:2, A3:3, and A4:4] respectively, press EXE, move to [B1].

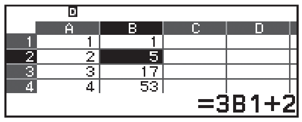
(1) After inputting [B1:1], press EXE

Press ⋯, select [Fill Formula], press OK

After inputting [Form=3B1+2], press EXE

After inputting [Range:B2:B4], press EXE, select [Confirm], press OK, move to [C1].

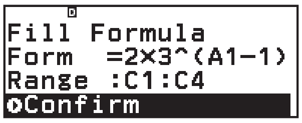



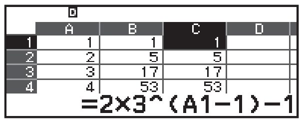


Press ⋯, select [Fill Formula], press OK

After inputting [Form= $2\times3^{(A1-1)}-1$], press EXE

After inputting [Range:C1:C4], press EXE, select [Confirm], press OK, move to [D1].





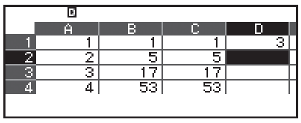
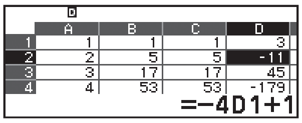
We can confirm that the first term to the 4th term of the recurrence formulas in B1 to B4 and the first term to the 4th term of the general terms in C1 to C4 are the same.

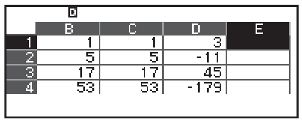
(2) After inputting [D1:3], press EXE

Press ⋯, select [Fill Formula], press OK

After inputting [Form=−4D1+1], press EXE

After inputting [Range:D2:D4], press EXE, select [Confirm], press OK, move to [E1].

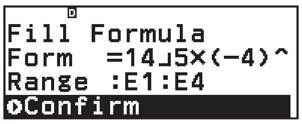



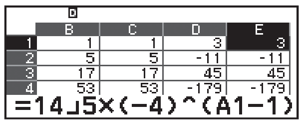


Press ⋯, select [Fill Formula], press OK

After inputting [Form= $\frac{14}{5}\times(-4)^{(A1-1)}+\frac{1}{5}$], press EXE

After inputting [Range:E1:E4], press EXE, select [Confirm], press OK





We can confirm that the first term to the 4th term of the recurrence formulas in D1 to D4 and the first term to the 4th term of the general terms in E1 to E4 are the same.

PRACTICE

◆ Find the general terms a_n of the sequences $\{a_n\}$ below.

(1) $a_1=1,\ a_{n+1}=2a_n+1$

Solve $c=2c+1$ to get $c=-1$

Therefore, we can transform $a_{n+1}=2a_n+1$ to $a_{n+1}+1=2(a_n+1)$.

This shows that the sequence $\{a_n+1\}$ is a geometric progression with a first term of $a_1+1=2$ and a common ratio of 2.

Therefore, from $a_n+1=2\cdot 2^{n-1}$, we get $a_n=2^n-1$

$$a_n=2^n-1$$

(2) $a_1=\frac{3}{2},\ a_{n+1}=\frac{1}{2}a_n+\frac{1}{2}$

Solve $c=\frac{1}{2}c+\frac{1}{2}$ to get $c=1$

Therefore, we can transform $a_{n+1}=\frac{1}{2}a_n+\frac{1}{2}$ to $a_{n+1}-1=\frac{1}{2}(a_n-1)$.

This shows that the sequence $\{a_n-1\}$ is a geometric progression with a first term of $a_1-1=\frac{1}{2}$ and a common ratio of $\frac{1}{2}$.

Therefore, from $a_n-1=\frac{1}{2}\cdot\left(\frac{1}{2}\right)^{n-1}$, we get $a_n=\left(\frac{1}{2}\right)^n+1$

$$a_n=\left(\frac{1}{2}\right)^n+1$$

check

Press ⌂, select [Spreadsheet], press OK, then clear the previous data by pressing ⏻.

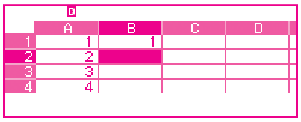
After inputting [A1:1, A2:2, A3:3, and A4:4] respectively, press EXE, move to [B1].

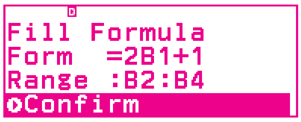
(1) After inputting [B1:1], press EXE.

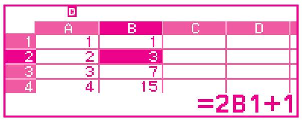
Press ⋯, select [Fill Formula], press OK.

After inputting [Form=2B1+1], press EXE.

After inputting [Range:B2:B4], press EXE, select [Confirm], press OK, move to [C1].



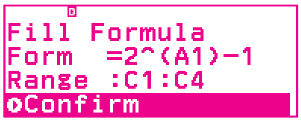


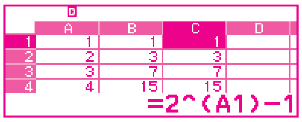


Press ⋯, select [Fill Formula], press OK.

After inputting [Form$=2^{(A1)}-1$], press EXE.

After inputting [Range:C1:C4], press EXE, select [Confirm], press OK, move to [D1].





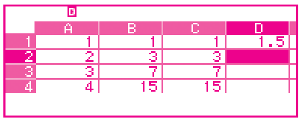
We can confirm that the first term to the 4th term of the recurrence formulas in B1 to B4 and the first term to the 4th term of the general terms in C1 to C4 are the same.

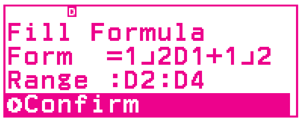
(2) After inputting [D1:$\frac{3}{2}$], press EXE.

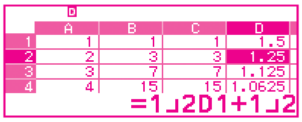
Press ⋯, select [Fill Formula], press OK.

After inputting [Form $=\frac{1}{2}$D1$+\frac{1}{2}$], press EXE.

After inputting [Range:D2:D4], press EXE, select [Confirm], press OK, move to [E1].



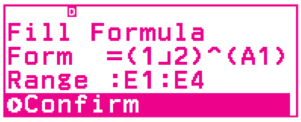


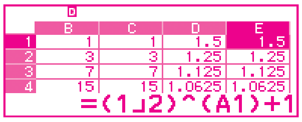


Press ⋯, select [Fill Formula], press OK.

After inputting [Form$=\left(\frac{1}{2}\right)^{(A1)}+1$], press EXE.

After inputting [Range:E1:E4], press EXE, select [Confirm], press OK.





We can confirm that the first term to the 4th term of the recurrence formulas in D1 to D4 and the first term to the 4th term of the general terms in E1 to E4 are the same.

Various recurrence formulas (1)

TARGET To understand about shapes and recurrence formulas.

STUDY GUIDE

Shapes and recurrence formulas

We can use recurrence formulas to solve problems related to shapes, such as finding the number of **domains** created by drawing n number of straight lines on a plane, or finding the length of the sides of a square made by connecting the midpoints of the sides of a square.

Ex. Assume there are n number of straight lines drawn on a plane, no 2 of which are parallel, and no 3 of which cross at 1 point.

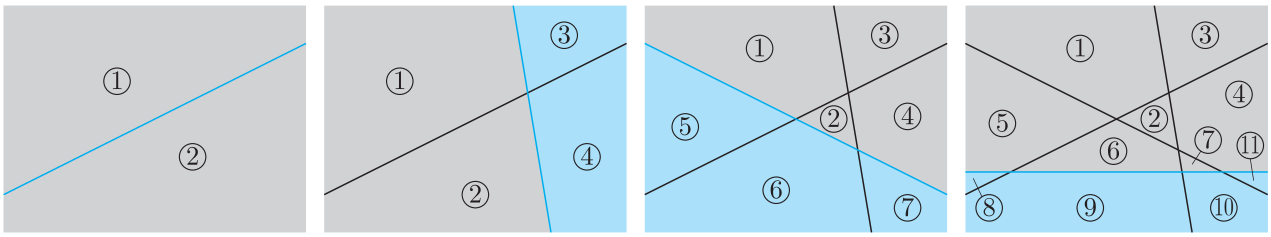
Consider how these n straight lines divide the plane into a_n domains.

(i) Drawing 1 line on a plane divides it into 2 domains.

(ii) Adding 1 line to (i) adds 2 new domains.

(iii) Adding 1 line to (ii) adds 3 new domains.

(iv) Adding 1 line to (iii) adds 4 new domains.



$a_1=2$ $a_2=a_1+2=2+2=4$ $a_3=a_2+3=4+3=7$ $a_4=a_3+4=7+4=11$

When the $(n+1)$th line l is added, the line l crosses each of the existing n lines at 1 point, forming an n number of new intersections. Therefore, by drawing a line l, the domains increase by $(n+1)$. That is to say, we can see that the number of domains a_{n+1} partitioned by $(n+1)$ lines is $(n+1)$ more than the number of domains a_n partitioned by n lines.

Therefore, we get $\boldsymbol{a_{n+1}=a_n+n+1}$.

EXERCISE

◆ Assume there are n number of straight lines drawn on a plane, no 2 of which are parallel, and no 3 of which cross at 1 point. Answer the following problems given how these n straight lines divide the plane into a_n domains.

(1) Find a_1, a_2, a_3, and a_4.

Drawing 1 line on a plane divides it into 2 domains, $a_1=2$

Adding 1 more line produces an increase of 2 new domains, so $a_2=a_1+2=2+2=4$

Adding 1 line to this produces an increase of 3 new domains, so $a_3=a_2+3=4+3=7$

Adding 1 more line produces an increase of 4 new domains, so $a_4=a_3+4=7+4=11$

$$\boldsymbol{a_1=2, a_2=4, a_3=7, a_4=11}$$

(2) Find the relation derived from a_{n+1} and a_n.

From the results of $a_2=a_1+2$, $a_3=a_2+3$ and $a_4=a_3+4$, we get $a_{n+1}=a_n+n+1$

$$\boldsymbol{a_{n+1}=a_n+n+1}$$

(3) Find a_n.

From (2) the progression of differences $\{b_n\}$ of the sequence $\{a_n\}$, we get $b_n=n+1$.

Given $n\geq 2$

$$a_n=a_1+\sum_{k=1}^{n-1} b_k=2+\sum_{k=1}^{n-1}(k+1)=2+\frac{1}{2}(n-1)n+(n-1)=\frac{1}{2}n^2+\frac{1}{2}n+1$$

When $n=1$, this becomes $\frac{1}{2}\cdot 1^2+\frac{1}{2}\cdot 1+1=2$, which is consistent with $a_1=2$, so it also holds when $n=1$.

Therefore, $a_n=\frac{1}{2}n^2+\frac{1}{2}n+1$

$$\boldsymbol{a_n=\frac{1}{2}n^2+\frac{1}{2}n+1}$$

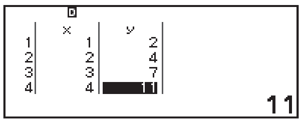
check

Press ⌂, select [Statistics], press OK, select [2-Variable], press OK

Press ⋯, select [Edit], press OK, select [Delete All], press OK

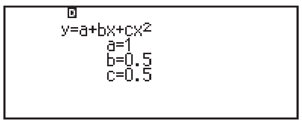
Input 1, 2, 3, and 4 in the x column, and 2, 4, 7, and 11 in the y column, respectively.

① EXE ② EXE ③ EXE ④ EXE ∨ ＞ ② EXE ④ EXE ⑦ EXE ① ① EXE EXE



Select [Reg Results], press OK, select [$y=a+bx+cx^2$], press OK

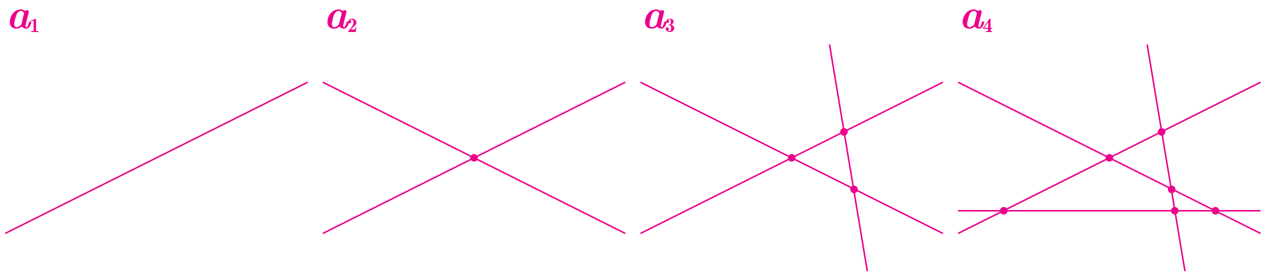
We can confirm that $y=0.5x^2+0.5x+1=\frac{1}{2}x^2+\frac{1}{2}x+1$.



PRACTICE

◆ Assume there are n number of straight lines drawn on a plane, no 2 of which are parallel, and no 3 of which cross at 1 point. Answer the following problems given a_n is the number of intersections possible from these n straight lines.

(1) Find a_1, a_2, a_3, and a_4.



If we draw 1 straight line on a plane, there can be no intersections, so $a_1=0$

Adding 1 line produces 1 intersection, so $a_2=a_1+1=0+1=1$

Adding 1 more line to this produces an increase of 2 intersections, so $a_3=a_2+2=1+2=3$

Adding yet 1 more line produces an increase of 3 intersections, so $a_4=a_3+3=3+3=6$

$$a_1=0, a_2=1, a_3=3, a_4=6$$

(2) Find a_n.

From the results of $a_2=a_1+1$, $a_3=a_2+2$, and $a_4=a_3+3$, we get $a_{n+1}=a_n+n$.

Let the progression of differences be $\{b_n\}$, such that $b_n=n$.

Given $n\geq 2$

$$a_n=a_1+\sum_{k=1}^{n-1}b_k=0+\sum_{k=1}^{n-1}k=\frac{1}{2}(n-1)n=\frac{1}{2}n^2-\frac{1}{2}n$$

When $n=1$, this becomes $\frac{1}{2}\cdot 1^2-\frac{1}{2}\cdot 1=0$, which is consistent with $a_1=0$, so it also holds when $n=1$.

Therefore, $a_n=\frac{1}{2}n^2-\frac{1}{2}n$

$$a_n=\frac{1}{2}n^2-\frac{1}{2}n$$

check

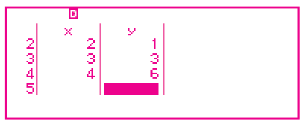
Press ⌂, select [Statistics], press OK, select [2-Variable], press OK

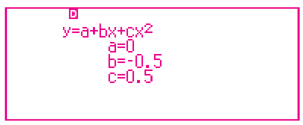
Press ⋯, select [Edit], press OK, select [Delete All], press OK

After inputting 1, 2, 3, and 4 in the x column, and 0, 1, 3, and 6 in the y column, respectively, press EXE

Select [Reg Results], press OK, select [y=a+bx+cx²], press OK

We can confirm that $y=0.5x^2-0.5x=\frac{1}{2}x^2-\frac{1}{2}x$.





Various recurrence formulas (2)

TARGET To understand about probability and recurrence formulas.

STUDY GUIDE

Probability and recurrence formulas

In some cases, we can use recurrence formulas to consider problems related to **probability**.

EXERCISE

◆ Given 1 die is rolled n times, then the probability that a 5 appears an odd number of times is p_n. In this case, solve the following problems.

(1) Find p_1.

Since p_1 is the probability that a 5 appears 1 time after 1 roll, we get $p_1 = \frac{1}{6}$

$$\boldsymbol{p_1 = \frac{1}{6}}$$

(2) Use p_n to express p_{n+1}.

When a die is rolled $(n+1)$ times, a 5 appears an odd number of times in the following 2 cases.

(i) After n rolls, a 5 appearing an odd number of times is (probability p_n), and on the $(n+1)$th time something other than a 5 appears.

(ii) After n rolls, a 5 appearing an even number of times is (probability $(1-p_n)$), and on the $(n+1)$th time a 5 appears.

Therefore, $p_{n+1} = p_n \cdot \frac{5}{6} + (1-p_n) \cdot \frac{1}{6} = \frac{2}{3} p_n + \frac{1}{6}$

$$\boldsymbol{p_{n+1} = \frac{2}{3} p_n + \frac{1}{6}}$$

(3) Find p_n.

Solve the characteristic equation of $c = \frac{2}{3}c + \frac{1}{6}$ for $p_{n+1} = \frac{2}{3}p_n + \frac{1}{6}$, to get $c = \frac{1}{2}$

Therefore, $p_{n+1} = \frac{2}{3}p_n + \frac{1}{6}$ can be transformed to $p_{n+1} - \frac{1}{2} = \frac{2}{3}\left(p_n - \frac{1}{2}\right)$.

This shows that the sequence $\left\{p_n - \frac{1}{2}\right\}$ is a geometric progression with a first term of $p_1 - \frac{1}{2} = -\frac{1}{3}$ and a common ratio of $\frac{2}{3}$.

Therefore, $p_n - \frac{1}{2} = -\frac{1}{3} \cdot \left(\frac{2}{3}\right)^{n-1}$ and $p_n = -\frac{1}{3} \cdot \left(\frac{2}{3}\right)^{n-1} + \frac{1}{2}$

$$\boldsymbol{p_n = -\frac{1}{3} \cdot \left(\frac{2}{3}\right)^{n-1} + \frac{1}{2}}$$

check

Press ⌂, select [Spreadsheet], press OK, then clear the previous data by pressing ⏻

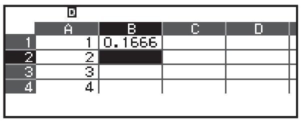
After inputting [A1:1, A2:2, A3:3, and A4:4] respectively, press EXE, move to [B1].

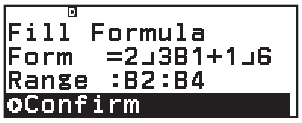
After inputting [B1: $\frac{1}{6}$], press EXE

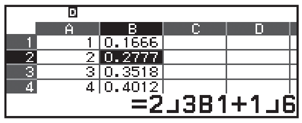
Press ⋯, select [Fill Formula], press OK

After inputting [Form= $\frac{2}{3}$ B1+ $\frac{1}{6}$], press EXE

After inputting [Range:B2:B4], press EXE, select [Confirm], press OK, move to [C1].



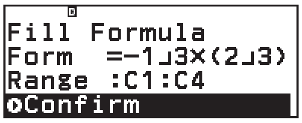


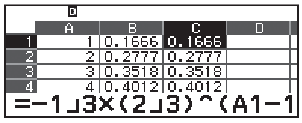


Press ⋯, select [Fill Formula], press OK

After inputting [Form= $-\frac{1}{3}\times\left(\frac{2}{3}\right)^{(A1-1)}+\frac{1}{2}$], press EXE

After inputting [Range:C1:C4], press EXE, select [Confirm], press OK





We can confirm that the first term to the 4th term of the recurrence formulas in B1 to B4 and the first term to the 4th term of the general terms in C1 to C4 are the same.

PRACTICE

◆ Given 1 die is rolled n times, then the probability that a multiple of 3 appears an odd number of times is p_n . In this case, solve the following problems.

(1) Find p_1.

Since p_1 is the probability that a multiple of 3 appears 1 time after 1 roll, we get

$$p_1=\frac{2}{6}=\frac{1}{3}$$

$$p_1=\frac{1}{3}$$

(2) Use p_n to express p_{n+1} .

When a die is rolled $(n+1)$ times, a multiple of 3 appears an odd number of times in the following 2 cases.

(i) After n rolls, a multiple of 3 appearing an odd number of times is (probability p_n), and on the $(n+1)$th time something other than a multiple of 3 appears.

(ii) After n rolls, a multiple of 3 appearing an even number of times is (probability $(1-p_n)$), and on the $(n+1)$th time a multiple of 3 appears.

Therefore, $p_{n+1}=p_n\cdot\frac{4}{6}+(1-p_n)\cdot\frac{2}{6}=\frac{1}{3}p_n+\frac{1}{3}$

$$p_{n+1}=\frac{1}{3}p_n+\frac{1}{3}$$

(3) Find p_n.

Solve the characteristic equation of $c=\frac{1}{3}c+\frac{1}{3}$ for $p_{n+1}=\frac{1}{3}p_n+\frac{1}{3}$, to get $c=\frac{1}{2}$

Therefore, $p_{n+1}=\frac{1}{3}p_n+\frac{1}{3}$ can be transformed to $p_{n+1}-\frac{1}{2}=\frac{1}{3}\left(p_n-\frac{1}{2}\right)$.

This shows that the sequence $\left\{p_n-\frac{1}{2}\right\}$ is a geometric progression with a first term of $p_1-\frac{1}{2}=-\frac{1}{6}$ and a common ratio of $\frac{1}{3}$.

Therefore, $p_n-\frac{1}{2}=-\frac{1}{6}\cdot\left(\frac{1}{3}\right)^{n-1}$, $p_n=-\frac{1}{6}\cdot\left(\frac{1}{3}\right)^{n-1}+\frac{1}{2}$

$$p_n=-\frac{1}{6}\cdot\left(\frac{1}{3}\right)^{n-1}+\frac{1}{2}$$

check

Press ⌂, select [Spreadsheet], press OK, then clear the previous data by pressing ⏻

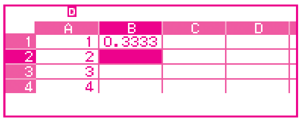
After inputting [A1:1, A2:2, A3:3, and A4:4] respectively, press EXE, move to [B1].

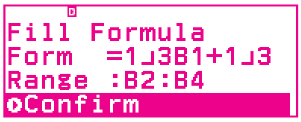
After inputting [B1: $\frac{1}{3}$], press EXE

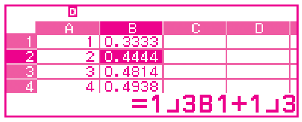
Press ⋯, select [Fill Formula], press OK

After inputting [Form= $\frac{1}{3}$B1+ $\frac{1}{3}$], press EXE

After inputting [Range:B2:B4], press EXE, select [Confirm], press OK, move to [C1].



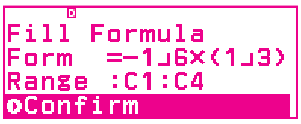


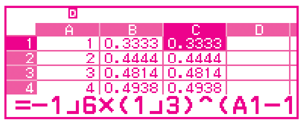


Press ⋯, select [Fill Formula], press OK

After inputting [Form= $-\frac{1}{6}\times\left(\frac{1}{3}\right)^{(A1-1)}+\frac{1}{2}$], press EXE

After inputting [Range:C1:C4], press EXE, select [Confirm], press OK





We can confirm that the first term to the 4th term of the recurrence formulas in B1 to B4 and the first term to the 4th term of the general terms in C1 to C4 are the same.

Various recurrence formulas (3)

TARGET To understand about recurrence formulas between 3 adjacent terms.

STUDY GUIDE

Recurrence formula between 3 adjacent terms

Recurrence formula between 3 adjacent terms

The recurrence formula determined by the relation of 3 adjacent terms a_n , a_{n+1} , and a_{n+2} , is called a **recurrence formula between 3 adjacent terms**.

Ex. We can find the 3rd and subsequent terms of a sequence $\{a_n\}$, which is determined by $a_1=1$, $a_2=1$, and $a_{n+2}=a_{n+1}+a_n$, as follows.

By using a recurrence formula, when $n=1$, then $a_3=a_2+a_1=1+1=2$, when $n=2$, then $a_4=a_3+a_2=2+1=3$, when $n=3$, then $a_5=a_4+a_3=3+2=5$, when $n=4$, then $a_6=a_5+a_4=5+3=8$, when $n=5$, then $a_7=a_6+a_5=8+5=13,\cdots\cdots$

check

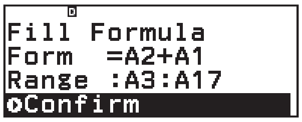
Press (HOME), select [Spreadsheet], press (OK), then clear the previous data by pressing (ON)

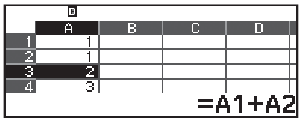
After inputting [A1:1 and A2:1] respectively, press (EXE)

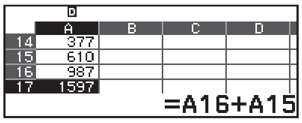
Press (OPTN), select [Fill Formula], press (OK)

After inputting [Form$=$A2$+$A1], press (EXE)

After inputting [Range:A3:A17], press (EXE), select [Confirm], press (OK)







The sequence shown in column A is $1, 1, 2, 3, 5, 8, 13, 21, 34, 55,\ldots\ldots$

The rule is that the preceding 2 terms are added to get the next term.

This sequence is called the Fibonacci sequence, and it often occurs in the natural world and the STEAM fields of science, engineering, and art.

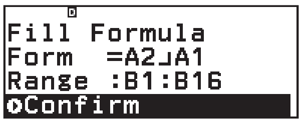
Furthermore, a characteristic that appears in this sequence is that as n increases, $\dfrac{a_{n+1}}{a_n}\left(=\dfrac{\text{(subsequent term)}}{\text{(preceding term)}}\right)$ approaches the fixed value $(1.618\cdots\cdots)$, which is called the golden ratio.

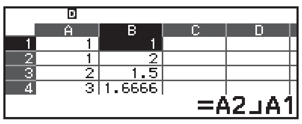
Move to [B1], press (OPTN), select [Fill Formula], press (OK)

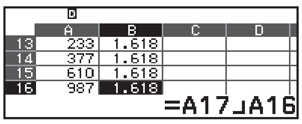
After inputting [Form$=\dfrac{\text{A2}}{\text{A1}}$], press (EXE)

After inputting [Range:B1:B16], press (EXE),

select [Confirm], press (OK)







How to find the general term of recurrence formula between 3 adjacent terms

The general term of a sequence determined by a_1, a_2, $a_{n+2}+pa_{n+1}+qa_n=0$ can be found as follows.

Consider the characteristic equation $x^2+px+q=0$ with a_{n+2}, a_{n+1}, and a_n of $a_{n+2}+pa_{n+1}+qa_n=0$ for x^2, x, and 1 respectively.

Given the 2 solutions of this quadratic equation are α and $\beta(\alpha\neq\beta)$, and since $\alpha+\beta=-p$ and $\alpha\beta=q$, then

$a_{n+2}+pa_{n+1}+qa_n=0$ gives us $a_{n+2}-(\alpha+\beta)a_{n+1}+\alpha\beta a_n=0$, which can be transformed in the following 2 ways.

$a_{n+2}-\alpha a_{n+1}-\beta a_{n+1}+\beta\cdot\alpha a_n=0 \rightarrow a_{n+2}-\alpha a_{n+1}=\beta(a_{n+1}-\alpha a_n)$ …(i)

$a_{n+2}-\beta a_{n+1}-\alpha a_{n+1}+\alpha\cdot\beta a_n=0 \rightarrow a_{n+2}-\beta a_{n+1}=\alpha(a_{n+1}-\beta a_n)$ …(ii)

From (i), the sequence $\{a_{n+1}-\alpha a_n\}$ is a geometric progression with a first term of $a_2-\alpha a_1$ and a common ratio of β,

so $a_{n+1}-\alpha a_n=(a_2-\alpha a_1)\beta^{n-1}$ $\cdots$(iii)

From (ii), the sequence $\{a_{n+1}-\beta a_n\}$ is a geometric progression with a first term of $a_2-\beta a_1$ and a common ratio of α,

so $a_{n+1}-\beta a_n=(a_2-\beta a_1)\alpha^{n-1}$ $\cdots$(iv)

From (iii)−(iv), we get $(\beta-\alpha)a_n=(a_2-\alpha a_1)\beta^{n-1}-(a_2-\beta a_1)\alpha^{n-1}$.

From $\alpha\neq\beta$, we can find $a_n=\dfrac{1}{\beta-\alpha}\{(a_2-\alpha a_1)\beta^{n-1}-(a_2-\beta a_1)\alpha^{n-1}\}$.

EXERCISE

1 Find a_3, a_4, a_5, and a_6 for the sequence $\{a_n\}$ determined by the following recurrence formulas.

(1) $a_1=1$, $a_2=2$, $a_{n+2}-4a_{n+1}+3a_n=0$

The given recurrence formula is transformed to $a_{n+2}=4a_{n+1}-3a_n$.

$a_3=4a_2-3a_1=4\cdot2-3\cdot1=5$, $a_4=4a_3-3a_2=4\cdot5-3\cdot2=14$, $a_5=4a_4-3a_3=4\cdot14-3\cdot5=41$, $a_6=4a_5-3a_4=4\cdot41-3\cdot14=122$

$\boldsymbol{a_3=5,\ a_4=14,\ a_5=41,\ a_6=122}$

(2) $a_1=1$, $a_2=4$, $a_{n+2}+3a_{n+1}+2a_n=0$

The given recurrence formula is transformed to $a_{n+2}=-3a_{n+1}-2a_n$.

$a_3=-3a_2-2a_1=-3\cdot4-2\cdot1=-14$, $a_4=-3a_3-2a_2=-3\cdot(-14)-2\cdot4=34$, $a_5=-3a_4-2a_3=-3\cdot34-2\cdot(-14)=-74$,

$a_6=-3a_5-2a_4=-3\cdot(-74)-2\cdot34=154$

$\boldsymbol{a_3=-14,\ a_4=34,\ a_5=-74,\ a_6=154}$

check

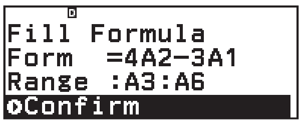
Press Ⓗ, select [Spreadsheet], press OK, then clear the previous data by pressing ⏻

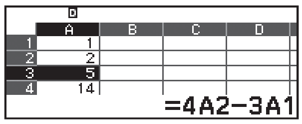
(1) After inputting [A1:1 and A2:2] respectively, press EXE

Press ⋯, select [Fill Formula], press OK

After inputting [Form=4A2−3A1], press EXE

After inputting [Range:A3:A6], press EXE, select [Confirm], press OK, move to [B1].



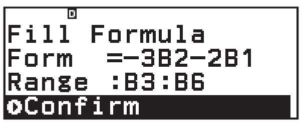


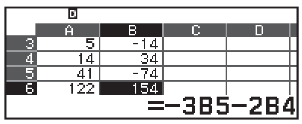
(2) After inputting [B1:1 and B2:4] respectively, press EXE

Press ⋯, select [Fill Formula], press OK

After inputting [Form=−3B2−2B1], press EXE

After inputting [Range:B3:B6], press EXE, select [Confirm], press OK





We can confirm the 3rd term to the 6th term of (1) in A3 to A6 and the 3rd term to the 6th term of (2) in B3 to B6.

2 Find the general term a_n of the sequence determined by $a_1=2$, $a_2=3$ and $a_{n+2}-5a_{n+1}+6a_n=0$.

By solving the characteristic equation of $x^2-5x+6=0$, we get $(x-2)(x-3)=0$, $x=2, 3$

Therefore, $a_{n+2}-5a_{n+1}+6a_n=0$ can be transformed in the following 2 ways.

$a_{n+2}-2a_{n+1}=3(a_{n+1}-2a_n)$ …(i)

$a_{n+2}-3a_{n+1}=2(a_{n+1}-3a_n)$ …(ii)

From (i), the sequence $\{a_{n+1}-2a_n\}$ is a geometric progression with a first term of $a_2-2a_1=3-2\cdot2=-1$ and a common ratio of 3, so $a_{n+1}-2a_n=-1\cdot3^{n-1}$ ⋯(iii)

From (ii), the sequence $\{a_{n+1}-3a_n\}$ is a geometric progression with a first term of $a_2-3a_1=3-3\cdot2=-3$ and a common ratio of 2, so $a_{n+1}-3a_n=-3\cdot2^{n-1}$ ⋯(iv)

From (iii)$-$(iv), we get $a_n=-1\cdot3^{n-1}-(-3\cdot2^{n-1})$.

Therefore, $a_n=-3^{n-1}+3\cdot2^{n-1}$

$$\boldsymbol{a_n=-3^{n-1}+3\cdot2^{n-1}}$$

check

Press HOME, select [Spreadsheet], press OK, then clear the previous data by pressing ⏻

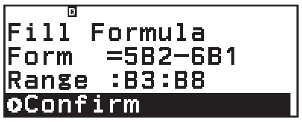
After inputting [A1:1, A2:2, A3:3, A4:4, A5:5, A6:6, A7:7, and A8:8] respectively, press EXE, move to [B1].

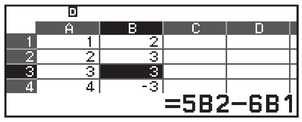
After inputting [B1:2 and B2:3] respectively, press EXE

Press ⋯, select [Fill Formula], press OK

After inputting [Form=5B2−6B1], press EXE

After inputting [Range:B3:B8], press EXE, select [Confirm], press OK, move to [C1].

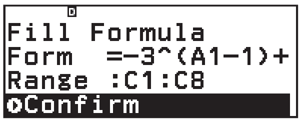


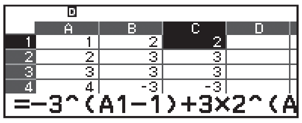


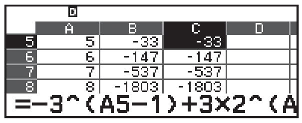
Press ⋯, select [Fill Formula], press OK

After inputting [Form$=-3^{(\mathrm{A1}-1)}+3\times2^{(\mathrm{A1}-1)}$], press EXE

After inputting [Range:C1:C8], press EXE, select [Confirm], press OK







We can confirm the first term to the 8th term of the recurrence formulas in B1 to B8 and the first term to the 8th term of the general terms in C1 to C8 are the same.

PRACTICE

1 Find a_3, a_4, a_5, and a_6 for the sequence $\{a_n\}$ determined by the following recurrence formulas.

(1) $a_1=2,\ a_2=3,\ a_{n+2}+a_{n+1}-6a_n=0$

The given recurrence formula is transformed to $a_{n+2}=-a_{n+1}+6a_n$.

$a_3=-a_2+6a_1=-3+6\cdot2=9,\ a_4=-a_3+6a_2=-9+6\cdot3=9,\ a_5=-a_4+6a_3=-9+6\cdot9=45,$

$a_6=-a_5+6a_4=-45+6\cdot9=9$

$$a_3=9,\ a_4=9,\ a_5=45,\ a_6=9$$

(2) $a_1=1,\ a_2=2,\ a_{n+2}-2a_{n+1}-8a_n=0$

The given recurrence formula is transformed to $a_{n+2}=2a_{n+1}+8a_n$.

$a_3=2a_2+8a_1=2\cdot2+8\cdot1=12,\ a_4=2a_3+8a_2=2\cdot12+8\cdot2=40,$

$a_5=2a_4+8a_3=2\cdot40+8\cdot12=176,\ a_6=2a_5+8a_4=2\cdot176+8\cdot40=672$

$$a_3=12,\ a_4=40,\ a_5=176,\ a_6=672$$

check

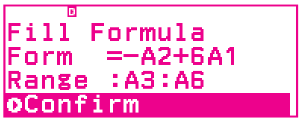
Press ⌂, select [Spreadsheet], press OK, then clear the previous data by pressing ⏻

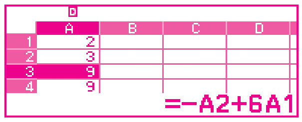
(1) After inputting [A1:2 and A2:3] respectively, press EXE

Press ⋯, select [Fill Formula], press OK

After inputting [Form=−A2+6A1] press EXE

After inputting [Range:A3:A6], press EXE, select [Confirm], press OK, move to [B1].



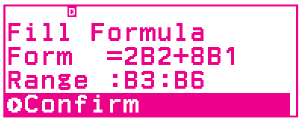


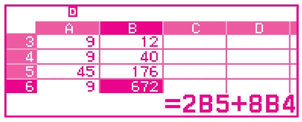
(2) After inputting [B1:1 and B2:2] respectively, press EXE

Press ⋯, select [Fill Formula], press OK

After inputting [Form=2B2+8B1], press EXE

After inputting [Range:B3:B6], press EXE, select [Confirm], press OK





We can confirm the 3rd term to the 6th term of (1) in A3 to A6 and the 3rd term to the 6th term of (2) in B3 to B6.

2 Find the general term a_n of the sequence determined by $a_1=1$, $a_2=6$, $a_{n+2}-7a_{n+1}+12a_n=0$.

By solving the characteristic equation of $x^2-7x+12=0$, we get $(x-3)(x-4)=0$ and $x=3, 4$

Therefore, $a_{n+2}-7a_{n+1}+12a_n=0$ can be transformed in the following 2 ways.

$a_{n+2}-3a_{n+1}=4(a_{n+1}-3a_n)$ ...(i)

$a_{n+2}-4a_{n+1}=3(a_{n+1}-4a_n)$ ...(ii)

From (i), the sequence $\{a_{n+1}-3a_n\}$ is a geometric progression with a first term of $a_2-3a_1=6-3\cdot1=3$ and a common ratio of 4, so $a_{n+1}-3a_n=3\cdot4^{n-1}$ ···(iii)

From (ii), the sequence $\{a_{n+1}-4a_n\}$ is a geometric progression with a first term of $a_2-4a_1=6-4\cdot1=2$ and a common ratio of 3, so $a_{n+1}-4a_n=2\cdot3^{n-1}$ ···(iv)

From (iii)−(iv), we get $a_n=3\cdot4^{n-1}-2\cdot3^{n-1}$

$$a_n=3\cdot4^{n-1}-2\cdot3^{n-1}$$

check

Press ⌂, select [Spreadsheet], press OK, then clear the previous data by pressing ⏻

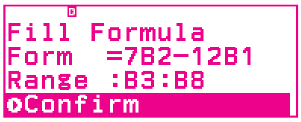
After inputting [A1:1, A2:2, A3:3, A4:4, A5:5, A6:6, A7:7, and A8:8] respectively, press EXE, move to [B1].

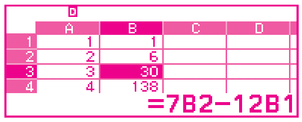
After inputting [B1:1 and B2:6] respectively, press EXE

Press ⋯, select [Fill Formula], press OK

After inputting [Form=7B2−12B1], press EXE

After inputting [Range:B3:B8], press EXE, select [Confirm], press OK, move to [C1].

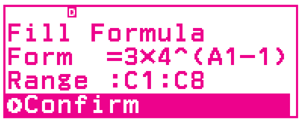


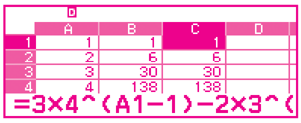


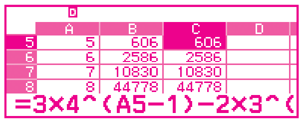
Press ⋯, select [Fill Formula], press OK

After inputting [Form=$3\times4^{(A1-1)}-2\times3^{(A1-1)}$], press EXE

After inputting [Range:C1:C8], press EXE, select [Confirm], press OK







We can confirm that the first term to the 8th term of the recurrence formulas in B1 to B8 and the first term to the 8th term of the general terms in C1 to C8 are the same.

Mathematical induction (1)

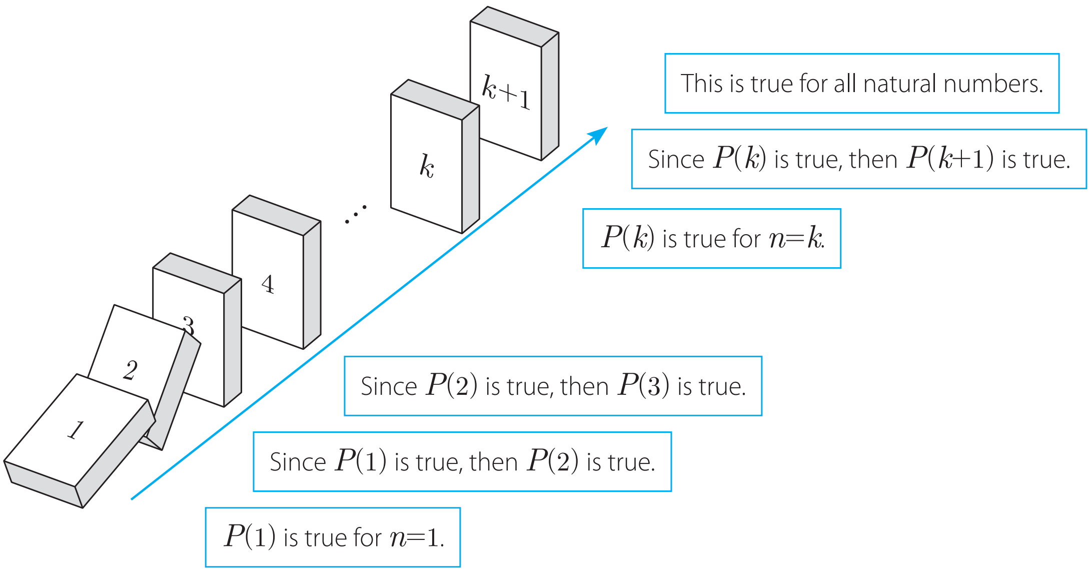
TARGET To understand proofs using mathematical induction.

STUDY GUIDE

Mathematical induction

Mathematical induction is a method to prove that a proposition $P(n)$ that includes a natural number n holds for all natural numbers n. With mathematical induction, we can do proofs in 2 steps.

(i) Show that $P(1)$ is true for $n=1$.
(ii) Assume that $P(k)$ is true for $n=k$
$\Rightarrow$ Show that $P(k+1)$ is true for $n=k+1$.
From (i) and (ii), it is true for all natural numbers.



The "falling dominoes" shown above are a good way to visualize mathematical induction.

EXERCISE

1 Let n be a natural number. Use mathematical induction to prove $1+3+5+\cdots\cdots+(2n-1)=n^2$.

[Proof]

Given $1+3+5+\cdots\cdots+(2n-1)=n^2 \quad \cdots(\mathrm{A})$.

(i) When $n=1$

The (left side) of $(\mathrm{A})=1$ and the (right side) of $(\mathrm{A})=1^2=1$, so (A) is true.

(ii) When $n=k$

Assume that (A) is true.

Specifically, $1+3+5+\cdots\cdots+(2k-1)=k^2$ is true.

When $n=k+1$

(Left side) $=1+3+5+\cdots\cdots+(2k-1)+\{2(k+1)-1\}=k^2+\{2(k+1)-1\}=k^2+2k+1=(k+1)^2$

Therefore, (A) is also true when $n=k+1$.

From (i) and (ii) above, (A) is true for all natural numbers.

2 Let n be a natural number. Use mathematical induction to prove the inequality $2^n>n$ is true.

[Proof]

Given $2^n>n \quad \cdots(\mathrm{A})$.

(i) When $n=1$

The (left side) of $(\mathrm{A})=2^1=2$, and the (right side) of $(\mathrm{A})=1$, so (A) is true.

(ii) When $n=k$

Assume that (A) is true.

Specifically, $2^k>k$ is true.

When $n=k+1$, then we simply show that $2^{k+1}>k+1$ is true.

(left side)−(right side)$=2^{k+1}-(k+1)=2\cdot 2^k-(k+1)>2\cdot k-(k+1)=k-1\geq 0$

Therefore, the (left side)−(right side)>0, so (A) is also true when $n=k+1$.

From (i) and (ii) above, (A) is true for all natural numbers.

PRACTICE

1 Let n be a natural number. Use mathematical induction to prove $2+4+6+\cdots\cdots+2n=n(n+1)$.

[Proof]

Given $2+4+6+\cdots\cdots+2n=n(n+1)\quad\cdots(\mathrm{A})$.

(i) **When** $n=1$

The (left side) of $(\mathrm{A})=2$ **and the (right side) of** $(\mathrm{A})=1\cdot(1+1)=2$, **so** (A) **is true.**

(ii) **When** $n=k$

Assume that (A) **is true.**

Specifically, $2+4+6+\cdots\cdots+2k=k(k+1)$ **is true.**

When $n=k+1$

$$(\text{Left side})=2+4+6+\cdots\cdots+2k+2(k+1)=k(k+1)+2(k+1)$$
$$=(k+1)(k+2)=(k+1)\{(k+1)+1\}$$

Therefore, (A) **is also true when** $n=k+1$.

From (i) and (ii) above, (A) **is true for all natural numbers.**

2 Let n be a natural number. Use mathematical induction to prove the inequality $3^n>n+1$ is true.

[Proof]

Given $3^n>n+1\quad\cdots(\mathrm{A})$.

(i) **When** $n=1$

The (left side) of $(\mathrm{A})=3^1=3$ **and the (right side) of** $(\mathrm{A})=1+1=2$, **so** (A) **is true.**

(ii) **When** $n=k$

Assume that (A) **is true.**

Specifically, $3^k>k+1$ **is true.**

When $n=k+1$, **then we simply show that** $3^{k+1}>(k+1)+1$ **is true.**

$$(\text{left side})-(\text{right side})$$
$$=3^{k+1}-\{(k+1)+1\}$$
$$=3\cdot3^k-(k+2)>3\cdot(k+1)-(k+2)=2k+1>0$$

Therefore, (A) **is also true when** $n=k+1$.

From (i) and (ii) above, (A) **is true for all natural numbers.**

Mathematical induction (2)

TARGET To understand how to find general terms from recurrence formulas and to do proofs of properties of whole numbers by using mathematical induction.

STUDY GUIDE

Proof of properties of whole numbers

By using mathematical induction, we can prove the properties of whole numbers.

The proof that a natural number N is a multiple of a can be shown in the following format.

$N=a\times$(whole numbers)

How to find general terms from recurrence formulas

By using mathematical induction, we can prove that general terms deduced from a recursion formula are correct.

Substitute $n=1, 2, 3, \ldots$ into the recursion formula to deduce the general term, then prove it correct by mathematical induction.

EXERCISE

1 Given n is a natural number, solve the following problems with regards to the sequence $a_n=2^n+5^{n-1}$.

(1) Find a_1, a_2, a_3, and a_4.

$a_1=2^1+5^{1-1}=2+1=3, a_2=2^2+5^{2-1}=4+5=9, a_3=2^3+5^{3-1}=8+25=33,$

$a_4=2^4+5^{4-1}=16+125=141$

$a_1=3$, $a_2=9$, $a_3=33$, $a_4=141$

check

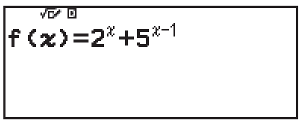
Press (HOME), select [Table], press (OK), then clear the previous data by pressing (AC)

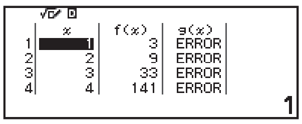
Press (OPTN), select [Define f(x)/g(x)], press (OK), select [Define f(x)], press (OK)

After inputting f(x)$=2^x+5^{x-1}$, press (EXE)

Press (OPTN), select [Table Range], press (OK)

After inputting [Start:1, End:4, Step:1], select [Execute], press (EXE)





(2) From the result of (1), deduce that a_n is a multiple of some number, and prove it using mathematical induction.

[Proof]

From $a_1=3=3\cdot1$, $a_2=9=3\cdot3$, $a_3=33=3\cdot11$, and $a_4=141=3\cdot47$, we can deduce that a_n is a multiple of 3.

(i) When $n=1$

From $a_1=2^1+5^{1-1}=2+1=3$, we know that a_1 is a multiple of 3.

(ii) When $n=k$

Assuming that a_k is a multiple of 3, show that $a_k=2^k+5^{k-1}=3m$ (m is a whole number).

When $n=k+1$

$a_{k+1}=2^{k+1}+5^{(k+1)-1}=2\cdot2^k+5\cdot5^{k-1}=2(3m-5^{k-1})+5\cdot5^{k-1}=2\cdot3m+3\cdot5^{k-1}=3(2m+5^{k-1})$

Since $2m+5^{k-1}$ is a whole number, a_{k+1} is a multiple of 3.

Therefore, when $n=k+1$, it is also a multiple of 3.

From (i) and (ii) above, a_n is a multiple of 3.

2 Solve the following problems with regards to a sequence $\{a_n\}$ that satisfies $a_1=2$ and $a_{n+1}=na_n-n^2+2$.

(1) Deduce the general term of $\{a_n\}$.

Substitute $n=1$, 2, and 3 into the recurrence formula.

$a_2=1\cdot a_1-1^2+2=1\cdot2-1^2+2=3, a_3=2\cdot a_2-2^2+2=2\cdot3-2^2+2=4,$

$a_4=3\cdot a_3-3^2+2=3\cdot4-3^2+2=5$

Therefore, we can deduce the general term to be $a_n=n+1$.

$$\boldsymbol{a_n=n+1}$$

(2) Use mathematical induction to prove that the general term in (1) is correct.

[Proof]

Prove that the general term is $a_n=n+1$.

(i) When $n=1$

From $a_1=1+1=2$, this is true when $n=1$.

(ii) When $n=k$

Assume that $a_k=k+1$.

When $n=k+1$, the recurrence formula gives

$a_{k+1}=ka_k-k^2+2=k(k+1)-k^2+2=k+2=(k+1)+1$

Therefore, it is also true when $n=k+1$.

From (i) and (ii) above, the general term is $a_n=n+1$ with regard to all whole numbers n.

check

Press ⌂, select [Spreadsheet], press OK, then clear the previous data by pressing ⏻

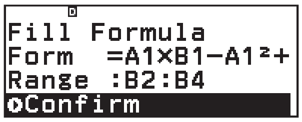
After inputting [A1:1, A2:2, A3:3, and A4:4] respectively, press EXE, move to [B1]

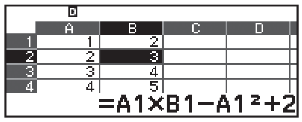
After inputting [B1:2], press EXE

Press ⋯, select [Fill Formula], press OK

After inputting [Form=A1×B1−A1^2+2], press EXE

After inputting [Range:B2:B4], press EXE, select [Confirm], press OK

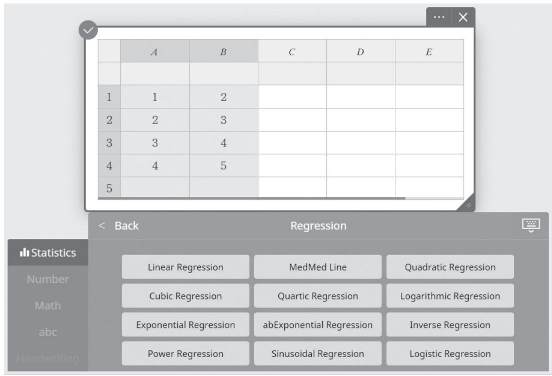


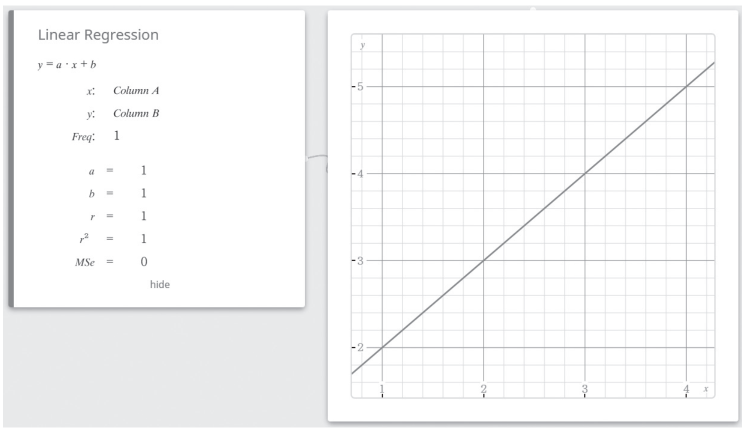


Press ⇧ 𝑥, scan the QR code to display the data.

Tap column A and column B, tap [Statistics], tap [Regression], and then tap [Linear Regression].

The deduction formulas are displayed.





PRACTICE

1 Let n be a natural number. Use mathematical induction to prove $2n^3+n$ is a multiple of 3.

[Proof]

(i) When $n=1$

From $2\cdot1^3+1=3$, it is true when $n=1$.

(ii) When $n=k$

Assuming it is true, show $2k^3+k=3m$ (m is a whole number).

When $n=k+1$, $$2(k+1)^3+(k+1)=(2k^3+k)+6k^2+6k+3$$
$$=3m+6k^2+6k+3$$
$$=3(m+2k^2+2k+1)$$

Since $m+2k^2+2k+1$ is a whole number, this is a multiple of 3.

Therefore, when $n=k+1$, it is also a multiple of 3.

From (i) and (ii) above, $2n^3+n$ is a multiple of 3 with regard to all whole numbers n.

2 Deduce the general term of a sequence $\{a_n\}$ that satisfies $a_1=1$ and $(a_{n+1}-1)a_n=2n(2n-1)$, then use mathematical induction to prove it is correct.

[Proof]

Substitute $n=1$, 2, and 3 into the recurrence formula.

$$(a_2-1)a_1=2\cdot1(2\cdot1-1),(a_2-1)\cdot1=2,a_2=3$$
$$(a_3-1)a_2=2\cdot2(2\cdot2-1),(a_3-1)\cdot3=12,a_3=5$$
$$(a_4-1)a_3=2\cdot3(2\cdot3-1),(a_4-1)\cdot5=30,a_4=7$$

Therefore, we can deduce the general term to be $a_n=2n-1$.

(i) When $n=1$

From $a_1=2\cdot1-1=1$, this is true when $n=1$.

(ii) When $n=k$

Assume that $a_k=2k-1$.

When $n=k+1$, the recurrence formula gives

$$(a_{k+1}-1)a_k=2k(2k-1),(a_{k+1}-1)(2k-1)=2k(2k-1)$$

From $2k-1\neq0$, we get $a_{k+1}-1=2k, a_{k+1}=2k+1=2(k+1)-1$

Therefore, it is also true when $n=k+1$.

From (i) and (ii) above, the general term is $a_n=2n-1$ with regard to all whole numbers n.

CASIO Essential Materials

Publisher: CASIO Institute for Educational Development

Date of Publication: 2023/12/22 (1st edition)

https://edu.casio.com