9 Complex Numbers

CASIO
Essential Materials

Introduction

These teaching materials were created with the hope of conveying to many teachers and students the appeal of scientific calculators.

(1) Change awareness (emphasizing the thinking process) and boost efficiency in learning mathematics

- By reducing the time spent on manual calculations, we can have learning with a focus on the thinking process that is more efficient.
- This reduces the aversion to mathematics caused by complicated calculations, and allows students to experience the joy of thinking, which is the essence of mathematics.

(2) Diversification of learning materials and problem-solving methods

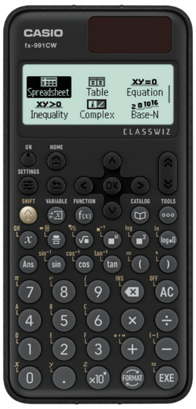
- Making it possible to do difficult calculations manually allows for diversity in learning materials and problem-solving methods.

(3) Promoting understanding of mathematical concepts

- By using the various functions of the scientific calculator in creative ways, students are able to deepen their understanding of mathematical concepts through calculations and discussions from different perspectives than before.
- This allows for exploratory learning through easy trial and error of questions.
- Listing and graphing of numerical values by means of tables allows students to discover laws and to understand visually.

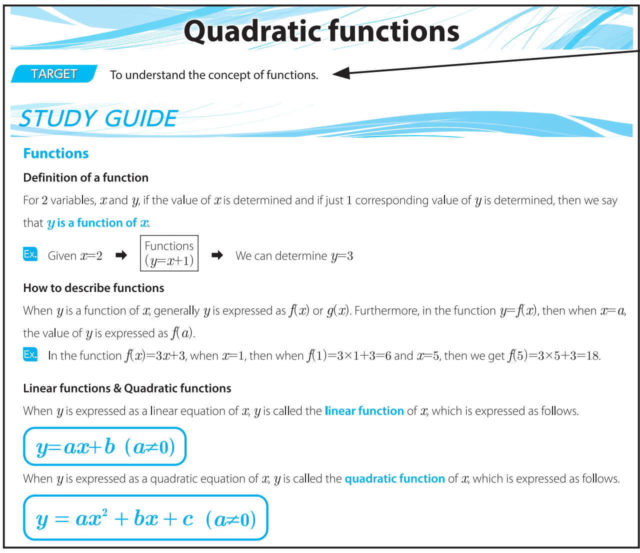
Features of this book

- As well as providing first-time scientific calculator users with opportunities to learn basic scientific calculator functions from the ground up, the book also has material to show people who already use scientific calculators the appeal of scientific calculators described above.
- You can also learn about functions and techniques that are not available on conventional Casio models or other brands of scientific calculators.
- This book covers many units of high school mathematics, allowing students to learn how to use the scientific calculator as they study each topic.
- This book can be used in a variety of situations, from classroom activities to independent study and homework by students.



Better Mathematics Learning
with Scientific Calculator

Structure



Quadratic functions

TARGET To understand the concept of functions.

STUDY GUIDE

Functions

Definition of a function

For 2 variables, x and y, if the value of x is determined and if just 1 corresponding value of y is determined, then we say that **y is a function of x.**

Ex. Given $x=2$ ➡ Functions ($y=x+1$) ➡ We can determine $y=3$

How to describe functions

When y is a function of x, generally y is expressed as $f(x)$ or $g(x)$. Furthermore, in the function $y=f(x)$, then when $x=a$, the value of y is expressed as $f(a)$.

Ex. In the function $f(x)=3x+3$, when $x=1$, then when $f(1)=3\times1+3=6$ and $x=5$, then we get $f(5)=3\times5+3=18$.

Linear functions & Quadratic functions

When y is expressed as a linear equation of x, y is called the **linear function** of x, which is expressed as follows.

$$\boxed{y=ax+b \quad (a\neq0)}$$

When y is expressed as a quadratic equation of x, y is called the **quadratic function** of x, which is expressed as follows.

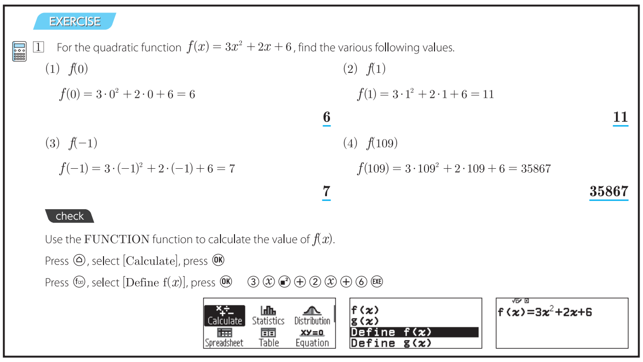
$$\boxed{y=ax^2+bx+c \quad (a\neq0)}$$

TARGET

Students can identify the objective to learn in each section.

STUDY GUIDE

Mathematical theorems and concepts are explained in detail. A scientific calculator is used to check and derive formulas according to the topic.



EXERCISE

1 For the quadratic function $f(x)=3x^2+2x+6$, find the various following values.

(1) $f(0)$

$f(0)=3\cdot0^2+2\cdot0+6=6$ **6**

(2) $f(1)$

$f(1)=3\cdot1^2+2\cdot1+6=11$ **11**

(3) $f(-1)$

$f(-1)=3\cdot(-1)^2+2\cdot(-1)+6=7$ **7**

(4) $f(109)$

$f(109)=3\cdot109^2+2\cdot109+6=35867$ **35867**

check

Use the FUNCTION function to calculate the value of $f(x)$.

Press ⌂, select [Calculate], press OK

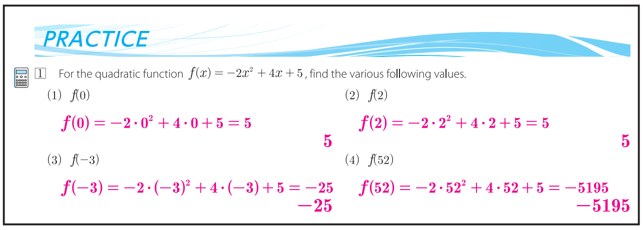
Press f(x), select [Define f(x)], press OK 3 x x² + 2 x + 6 EXE

EXERCISE

Students learn basic examples based on the explanation in Study Guide.

check

Explains how to use the scientific calculator to solve problems and check answers.



PRACTICE

1 For the quadratic function $f(x)=-2x^2+4x+5$, find the various following values.

(1) $f(0)$

$f(0)=-2\cdot0^2+4\cdot0+5=5$ **5**

(2) $f(2)$

$f(2)=-2\cdot2^2+4\cdot2+5=5$ **5**

(3) $f(-3)$

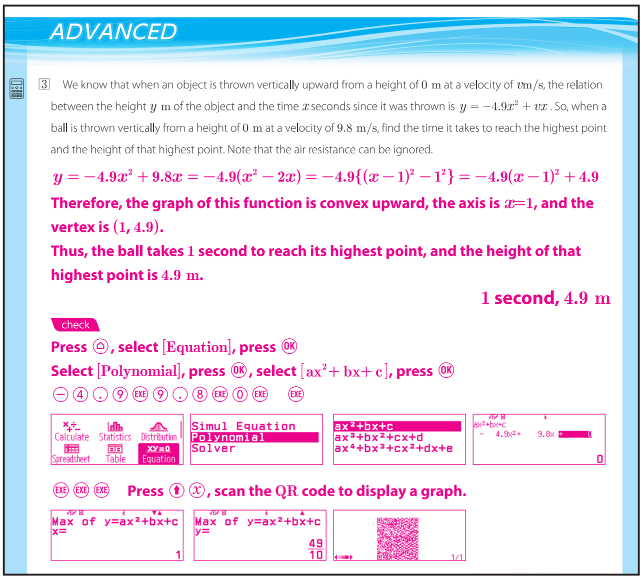
$f(-3)=-2\cdot(-3)^2+4\cdot(-3)+5=-25$ **-25**

(4) $f(52)$

$f(52)=-2\cdot52^2+4\cdot52+5=-5195$ **-5195**

PRACTICE

Students can do practice problems similar to those in EXERCISE. They can also practice using the scientific calculator as they learned to in Check.



ADVANCED

3 We know that when an object is thrown vertically upward from a height of 0 m at a velocity of v m/s, the relation between the height y m of the object and the time x seconds since it was thrown is $y=-4.9x^2+vx$. So, when a ball is thrown vertically from a height of 0 m at a velocity of 9.8 m/s, find the time it takes to reach the highest point and the height of that highest point. Note that the air resistance can be ignored.

$y=-4.9x^2+9.8x=-4.9(x^2-2x)=-4.9\{(x-1)^2-1^2\}=-4.9(x-1)^2+4.9$

Therefore, the graph of this function is convex upward, the axis is $x=1$, and the vertex is (1, 4.9).

Thus, the ball takes 1 second to reach its highest point, and the height of that highest point is 4.9 m.

1 second, 4.9 m

check

Press ⌂, select [Equation], press OK

Select [Polynomial], press OK, select [ax²+bx+c], press OK

(−) 4 . 9 EXE 9 . 8 EXE 0 EXE EXE

EXE EXE EXE **Press ⇧ x, scan the QR code to display a graph.**

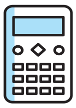
ADVANCED

Practical problems have been included in several topics. Solutions using a scientific calculator are also presented as necessary.

Other marks

Ex.	Simple examples on how to apply equations and theorems
explanation	Formulas and their supplementary explanations
proof	Proofs and checks of mathematical formulas
EXTRA Info.	Knowledge and information on formulas and other supplementary information in other units
OTHER METHODS	Alternative solutions and different verification methods for previously presented problems

Calculator mark



Where to use the scientific calculator

Colors of fonts in the teaching materials

- In STUDY GUIDE, important mathematical terms and formulas are printed in blue.
- In PRACTICE and ADVANCED the answers are printed in red.
  (Separate data is also available without the red parts, so it can be used for exercises.)

Applicable models

The applicable model is fx-991CW.
(Instructions on how to do input are for the fx-991CW, but in many cases similar calculations can be done on other models.)

Related Links

- Information and educational materials relevant to scientific calculators can be viewed on the following site.
  https://edu.casio.com
- The following video can be viewed to learn about the multiple functions of scientific calculators.
  https://www.youtube.com/playlist?list=PLRgxo9AwbIZLurUCZnrbr4cLfZdqY6aZA

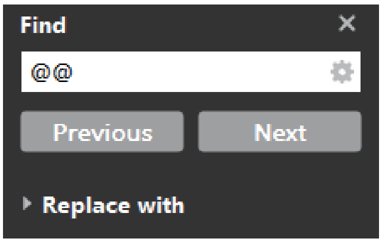
How to use PDF data

About types of data

- Data for all unit editions and data for each unit are available.
- For the above data, the PRACTICE and ADVANCED data without the answers in red is also available.

How to find where the scientific calculator is used

(1) Open a search window in the PDF Viewer.
(2) Type in "@@" as a search term.
(3) You can sequentially check where the calculator marks appear in the data.



How to search for a unit and section

(1) Search for units of data in all unit editions
- The data in all unit editions has a unit table of contents.
- Selecting a unit in the table of contents lets you jump to the first page of that unit.
- There is a bookmark on the first page of each unit, so you can jump from there also.

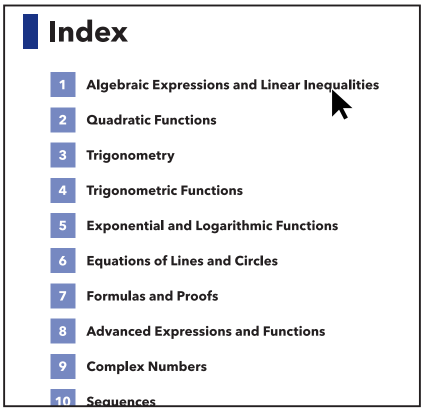


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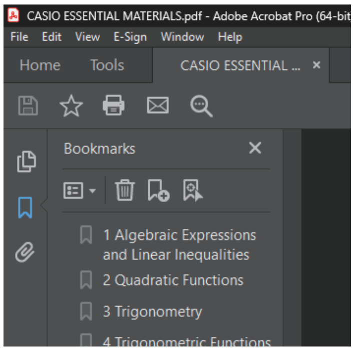
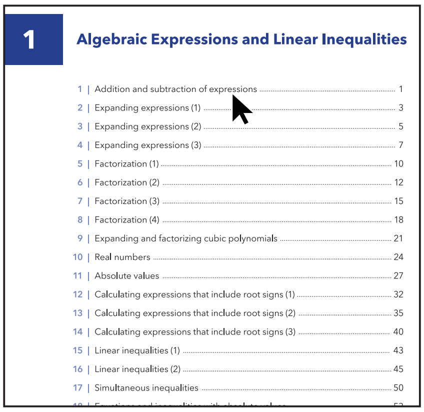
Table of contents of unit



Bookmark of unit

(2) Search for sections
- There are tables of contents for sections on the first page of units.
- Selecting a section in the table of contents takes you to the first page of that section.



1 Algebraic Expressions and Linear Inequalities

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Equality of irrational numbers

TARGET To understand the equality relation of rational numbers and irrational numbers by using proof by contradiction.

STUDY GUIDE

Equality of irrational numbers

Let a, b, c, and d be rational numbers, and let $\sqrt{l}$ be an irrational number. The relation between rational numbers and irrational numbers has the following properties.

(1) When $a+b\sqrt{l}=0$, then $a=b=0$
(2) When $a+b\sqrt{l}=c+d\sqrt{l}$, $a=c$ and $b=d$

EXERCISE

◆ Solve the following problems.

(1) Let a and b be rational numbers. When $a+b\sqrt{2}=0$, use the fact that $\sqrt{2}$ is an irrational number to prove that $a=b=0$.

[Proof] Assume $b\neq 0$.

From $a+b\sqrt{2}=0$, then $\sqrt{2}=-\dfrac{a}{b}$ $\quad\ldots(\mathrm{i})$

Since a and b are rational numbers, the right-hand side of (i) is a rational number.

However, since the left-hand side of (i) is an irrational number, this is contradictory.

Therefore, we get $b=0$. Substitute $b=0$ in $a+b\sqrt{2}=0$, such that $a=0$

This gives us $a=b=0$.

(2) Find the values of the rational numbers x and y that satisfy $(3+2\sqrt{2})x+(1-\sqrt{2})y=10$.

Transforming this to an equality gives $(3x+y-10)+(2x-y)\sqrt{2}=0$.

When x and y are rational numbers, $3x+y-10, 2x-y$ is also a rational number, but since $\sqrt{2}$ is an irrational number, from (1) we can derive $3x+y-10=0, 2x-y=0$.

Solve these as simultaneous equations to get $x=2$ and $y=4$.

$x=2,\ y=4$

PRACTICE

◆ Find the values of the rational numbers x and y that satisfy $(2+5\sqrt{3})x+(3-2\sqrt{3})y=7-11\sqrt{3}$.

Transforming this to an equality gives $(2x+3y)+(5x-2y)\sqrt{3}=7-11\sqrt{3}$.

When x and y are rational numbers, $2x+3y, 5x-2y$ is also a rational number, but since $\sqrt{3}$ is an irrational number, we can derive $2x+3y=7, 5x-2y=-11$.

Solve these as simultaneous equations to get $x=-1$ and $y=3$.

$x=-1,\ y=3$

Imaginary units i and complex numbers

TARGET To expand the range of numbers to complex numbers and to understand how to express and calculate complex numbers.

STUDY GUIDE

Imaginary units i and complex numbers

This is a new way to consider numbers that equal -1 when squared, which are expressed by the letter i. This i is called an **imaginary unit**. This gives us $i^2=-1$, where $a>0$, and is expressed as $\sqrt{-a}=\sqrt{a}i$.

Ex. The square root of -2 is $\sqrt{2}i, -\sqrt{2}i$.

Using 2 real numbers a and b and an imaginary unit i, can be considered a new number $a+bi$. These are called **complex numbers**, such that a and b are respectively called the **real part** and the **imaginary part** of the complex number $a+bi$. When $b=0$, it is a real number.

Complex numbers $a+bi$	
Real numbers $b=0$	Imaginary numbers $b\neq 0$

Addition and subtraction of complex numbers

Addition $(a+bi)+(c+di)=(a+c)+(b+d)i$
Subtraction $(a+bi)-(c+di)=(a-c)+(b-d)i$

EXERCISE

◆ Calculate the following.

(1) $\sqrt{3}\sqrt{-12}=\sqrt{3}\sqrt{12}i$
$=\sqrt{36}i$
$=6i$

$6i$

(2) $\sqrt{-2}\sqrt{-8}=\sqrt{2}i\sqrt{8}i$
$=\sqrt{16}i^2$
$=-4$

-4

(3) $(2+3i)+(3-i)=(2+3)+(3-1)i$
$=5+2i$

$5+2i$

(4) $\sqrt{-25}-\sqrt{-9}-\sqrt{-36}=\sqrt{25}i-\sqrt{9}i-\sqrt{36}i$
$=5i-3i-6i$
$=-4i$

$-4i$

PRACTICE

◆ Calculate the following.

(1) $(-7+8i)-(3-2i)$

$=(-7-3)+(8+2)i$
$=-10+10i$

$-10+10i$

(2) $\sqrt{-4}-\sqrt{-16}+\sqrt{-49}$

$=\sqrt{4}i-\sqrt{16}i+\sqrt{49}i$
$=2i-4i+7i$
$=5i$

$5i$

(3) $\sqrt{3}\sqrt{-6}+\sqrt{-2}\sqrt{-8}$

$=\sqrt{3}\sqrt{6}i+\sqrt{2}i\sqrt{8}i$
$=\sqrt{18}i+\sqrt{16}i^2$
$=3\sqrt{2}i-4$

$-4+3\sqrt{2}i$

(4) $\sqrt{-75}+2\sqrt{-2}\sqrt{6}$

$=\sqrt{75}i+2\sqrt{2}i\sqrt{6}$
$=5\sqrt{3}i+4\sqrt{3}i$
$=9\sqrt{3}i$

$9\sqrt{3}i$

Determining solutions to quadratic equations

TARGET To understand how to determine solutions from the formula for solving quadratic equations.

STUDY GUIDE

Finding the discriminant of quadratic equations

The solution of the quadratic equation $ax^2+bx+c=0$ is given by the formula on the right. When the range of numbers is expanded to complex numbers, the square root of negative numbers exists, such that the solution of a quadratic equation $ax^2+bx+c=0$ can have a real solution or an imaginary solution. To determine this, we can look at the sign of the expression b^2-4ac in the radical of the formula of the solution. This expression is called the **discriminant** and is expressed by D.

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

(a, b, and c are real numbers, and $a\neq 0$)

(1) $D>0 \iff$ **Has 2 different real roots.**
(2) $D=0 \iff$ **Has multiple roots.**
(3) $D<0 \iff$ **Has 2 different imaginary roots.**

EXERCISE

1 Solve the following quadratic equations.

(1) $x^2=-9$

$x=\pm\sqrt{-9}=\pm\sqrt{9}i=\pm 3i$

$x=\pm 3i$

(2) $x^2+5x+8=0$

$x=\dfrac{-5\pm\sqrt{5^2-4\cdot 1\cdot 8}}{2\cdot 1}=\dfrac{-5\pm\sqrt{-7}}{2}=\dfrac{-5\pm\sqrt{7}i}{2}$

$x=\dfrac{-5\pm\sqrt{7}i}{2}$

2 Determine the type of solution for the following quadratic equations.

(1) $x^2+3x+3=0$

Let the discriminant be D, such that $D=3^2-4\cdot 1\cdot 3=-3<0$

Has 2 different imaginary roots.

(2) $x^2-4x-7=0$

Let the discriminant be D, such that $D=(-4)^2-4\cdot 1\cdot(-7)=44>0$

Has 2 different real roots.

PRACTICE

1 Solve the following quadratic equations.

(1) $x^2+12=0$ $x=\pm\sqrt{-12}=\pm\sqrt{12}i=\pm 2\sqrt{3}i$

$x=\pm 2\sqrt{3}i$

(2) $5x^2+3x+1=0$ $x=\dfrac{-3\pm\sqrt{3^2-4\cdot 5\cdot 1}}{2\cdot 5}=\dfrac{-3\pm\sqrt{-11}}{10}=\dfrac{-3\pm\sqrt{11}i}{10}$

$x=\dfrac{-3\pm\sqrt{11}i}{10}$

2 Determine the type of solution for the following quadratic equations.

(1) $2x^2-5x+7=0$ **Let the discriminant be D, such that $D=(-5)^2-4\cdot 2\cdot 7=-31<0$**

Has 2 different imaginary roots.

(2) $9x^2+12x+4=0$ **Let the discriminant be D, such that $D=12^2-4\cdot 9\cdot 4=0$**

Has multiple roots.

Relation between solutions and coefficients

TARGET To understand the relation between solutions and coefficients in quadratic equations.

STUDY GUIDE

Relation between solutions and coefficients in quadratic equations

Given that 2 solutions of the quadratic equation $ax^2+bx+c=0$ are α and β, the following relation holds.

$$\boldsymbol{\alpha+\beta=-\frac{b}{a}, \alpha\beta=\frac{c}{a}}$$

explanation

Let α and β be the 2 solutions of the quadratic equation $ax^2+bx+c=0$.

$$\alpha+\beta=\frac{-b+\sqrt{b^2-4ac}}{2a}+\frac{-b-\sqrt{b^2-4ac}}{2a}=\frac{-2b}{2a}=-\frac{b}{a}$$

$$\alpha\beta=\frac{-b+\sqrt{b^2-4ac}}{2a}\cdot\frac{-b-\sqrt{b^2-4ac}}{2a}=\frac{b^2-(b^2-4ac)}{4a^2}=\frac{c}{a}$$

EXERCISE

1 Add and multiply the 2 solutions for the following quadratic equations.

(1) $x^2+4x-2=0$

Given the 2 solutions of α and β, from the relation between solutions and coefficients, we get

$$\alpha+\beta=-\frac{4}{1}=-4, \alpha\beta=\frac{-2}{1}=-2$$

Sum…-4, Product …-2

(2) $3x^2-2x=0$

Given the 2 solutions of α and β, from the relation between solutions and coefficients, we get

$$\alpha+\beta=-\frac{-2}{3}=\frac{2}{3}, \alpha\beta=\frac{0}{1}=0$$

Sum … $\frac{2}{3}$, Product …0

2 Find the value of the constant k and the solution of the equation when 1 solution of the quadratic equation $x^2-6x+k=0$ is twice as large as the other solution.

Let 1 solution be α, and the other solution be expressed as 2α. From the relation between solutions and coefficients,

$$\alpha+2\alpha=-\frac{-6}{1}=6, 3\alpha=6 \quad \ldots(\text{i})$$

$$\alpha\cdot 2\alpha=\frac{k}{1}=k, 2\alpha^2=k \quad \ldots(\text{ii})$$

From (i), we get α=2.

Substituting this into (ii) gives us k=8 or 2α=4.

k=8, the 2 solutions are …2 and 4

PRACTICE

1 Add and multiply the 2 solutions for the following quadratic equations.

(1) $x^2-9x+12=0$

Given the 2 solutions of α and β, from the relation between solutions and coefficients, we get $\alpha+\beta=-\frac{-9}{1}=9, \alpha\beta=\frac{12}{1}=12$

Sum …9, Product …12

(2) $2x^2+7x+3=0$

Given the 2 solutions of α and β, from the relation between solutions and coefficients, we get $\alpha+\beta=-\frac{7}{2}, \alpha\beta=\frac{3}{2}$

Sum …$-\frac{7}{2}$, Product …$\frac{3}{2}$

(3) $x^2+5=0$

Given the 2 solutions of α and β, from the relation between solutions and coefficients, we get $\alpha+\beta=-\frac{0}{1}=0, \alpha\beta=\frac{5}{1}=5$

Sum …0, Product …5

2 Find the value of the constant k and the solutions to the equation, respectively, which satisfy the following conditions.

(1) 1 solution of the quadratic equation $x^2-8x+k=0$ is 3 times the other solution.

Let 1 solution be α, and the other solution be expressed as 3α. From the relation between solutions and coefficients,

$$\alpha+3\alpha=-\frac{-8}{1}=8, 4\alpha=8 \quad \dots(\mathrm{i}) \qquad \alpha\cdot 3\alpha=\frac{k}{1}=k, 3\alpha^2=k \quad \dots(\mathrm{ii})$$

From (i), we get α=2. Substituting this into (ii) gives us k=12 or 3α=6.

k=12, the 2 solutions are …2 and 6

(2) The difference between the 2 solutions of the quadratic equation $x^2+kx+32=0$ is 4.

Let 1 solution be α, and the other solution be expressed as α+4. From the relation between solutions and coefficients,

$$\alpha+(\alpha+4)=-\frac{k}{1}=-k, 2\alpha+4=-k \quad \dots(\mathrm{i})$$

$$\alpha(\alpha+4)=\frac{32}{1}=32, \alpha^2+4\alpha-32=0 \quad \dots(\mathrm{ii})$$

From (ii), we get α=4 or -8.

When α=4, we assign it to (i), so that k=-12 and the other solution is 4+4=8.

Also, when α=-8, we assign it to (i), so that k=12 and the other solution is -8+4=-4.

k=-12, the 2 solutions are…4 and 8

k=12, the 2 solutions are …-4 and -8

Problems to find the values by the relation between solutions and coefficients

TARGET To understand how to find the value of expressions by using the relation between solutions and coefficients in quadratic equations.

STUDY GUIDE

Symmetry and its variants

The formula of $\alpha+\beta$ and $\alpha\beta$ is called the basic symmetry formula, and we can use it to find the value of expressions.

- $\alpha^2\beta+\alpha\beta^2=\alpha\beta(\alpha+\beta)$
- $\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta$
- $\alpha^3+\beta^3=(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)$
- $\dfrac{1}{\alpha}+\dfrac{1}{\beta}=\dfrac{\beta}{\alpha\beta}+\dfrac{\alpha}{\alpha\beta}=\dfrac{\alpha+\beta}{\alpha\beta}$

EXERCISE

◆ When we let α and β be the 2 solutions of the quadratic equation $x^2-2x+5=0$, find the value of the expressions.

(1) $\dfrac{1}{\alpha}+\dfrac{1}{\beta}$

From the relation between solutions and coefficients, we get $\alpha+\beta=-\dfrac{-2}{1}=2, \alpha\beta=\dfrac{5}{1}=5$.

Therefore, we get $\dfrac{1}{\alpha}+\dfrac{1}{\beta}=\dfrac{\alpha+\beta}{\alpha\beta}=\dfrac{2}{5}$.

$\dfrac{2}{5}$

(2) $\alpha^2+\beta^2$

Use the values of $\alpha+\beta$ and $\alpha\beta$ found in (1).

We get $\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta=2^2-2\cdot5=-6$.

-6

PRACTICE

1 When we let α and β be the 2 solutions of the quadratic equation $3x^2+4x+1=0$, find the value of the expressions.

(1) $\dfrac{1}{\alpha}+\dfrac{1}{\beta}$

From the relation between solutions and coefficients, we get

$\alpha+\beta=-\dfrac{4}{3}, \alpha\beta=\dfrac{1}{3}$.

Therefore, we get $\dfrac{1}{\alpha}+\dfrac{1}{\beta}=\dfrac{\alpha+\beta}{\alpha\beta}=\dfrac{-\frac{4}{3}}{\frac{1}{3}}=-4$.

-4

(2) $\alpha^2+\beta^2$

Use the values of $\alpha+\beta$ and $\alpha\beta$ found in (1).

$\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta=\left(-\dfrac{4}{3}\right)^2-2\cdot\dfrac{1}{3}=\dfrac{10}{9}$

$\dfrac{10}{9}$

2 When we let α and β be the 2 solutions of the quadratic equation $x^2-5x+3=0$, find the value of the expressions.

(1) $\alpha^2+\beta^2$

From the relation between solutions and coefficients, we get

$\alpha+\beta=-\dfrac{-5}{1}=5, \alpha\beta=\dfrac{3}{1}=3$.

Therefore, we get $\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta=5^2-2\cdot3=19$.

19

(2) $\alpha^3+\beta^3$

Use the values of $\alpha+\beta$ and $\alpha\beta$ found in (1).

$\alpha^3+\beta^3=(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)=5^3-3\cdot3\cdot5=80$

80

(3) $(\alpha-1)(\beta-1)$

Use the values of $\alpha+\beta$ and $\alpha\beta$ found in (1).

$(\alpha-1)(\beta-1)=\alpha\beta-(\alpha+\beta)+1=3-5+1=-1$

-1

EXTRA Info.

Use the VARIABLE function in the scientific calculator to find the value of symmetric expressions.

You can use the VARIABLE function of the scientific calculator to confirm the value of symmetric expressions.

EXERCISE

◆ When we let α and β be the 2 solutions of the quadratic equation $3x^2-6x+5=0$, find the value of the expressions.

(1) $\alpha^2+\beta^2$ (2) $\alpha^3+\beta^3$ (3) $\alpha^5+\beta^5$ (4) $\dfrac{\beta}{\alpha}+\dfrac{\alpha}{\beta}$

From the relation between solutions and coefficients, we get $\alpha+\beta=-\dfrac{-6}{3}=2, \alpha\beta=\dfrac{5}{3}$

(1) $\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta=2^2-2\cdot\dfrac{5}{3}=\dfrac{2}{3}$ **$\dfrac{2}{3}$**

(2) $\alpha^3+\beta^3=(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)=2^3-3\cdot\dfrac{5}{3}\cdot 2=-2$ **-2**

(3) $\alpha^5+\beta^5=(\alpha^3+\beta^3)(\alpha^2+\beta^2)-(\alpha\beta)^2(\alpha+\beta)=(-2)\cdot\left(\dfrac{2}{3}\right)-\left(\dfrac{5}{3}\right)^2\cdot 2=-\dfrac{62}{9}$ **$-\dfrac{62}{9}$**

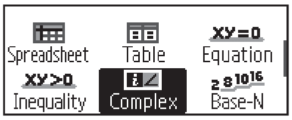
(4) $\dfrac{\beta}{\alpha}+\dfrac{\alpha}{\beta}=\dfrac{\beta^2+\alpha^2}{\alpha\beta}=\dfrac{\frac{2}{3}}{\frac{5}{3}}=\dfrac{2}{5}$ **$\dfrac{2}{5}$**

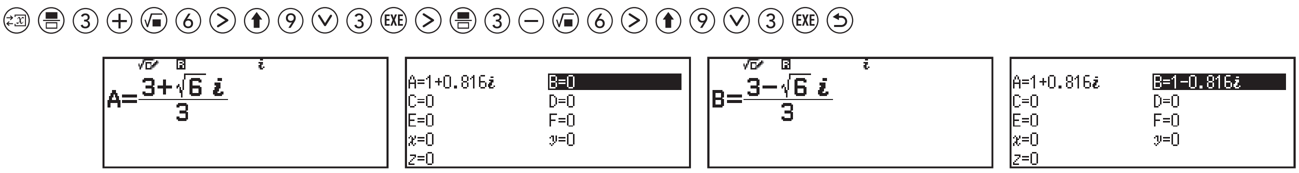
check

Use the formula of the solution in $3x^2-6x+5=0$ to get $x=\dfrac{3\pm\sqrt{6}i}{3}$

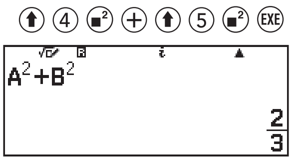
Press ⓐ, select [Complex], press (OK)

In the VARIABLE screen, input [A$=\dfrac{3+\sqrt{6}i}{3}$ and B$=\dfrac{3-\sqrt{6}i}{3}$] (values of solutions obtained above).

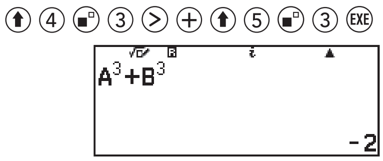




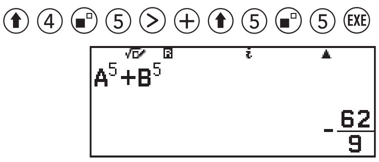
(1) Calculate the value of $\mathrm{A}^2+\mathrm{B}^2$.



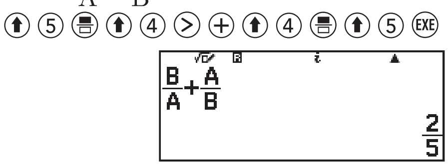
(2) Calculate the value of $\mathrm{A}^3+\mathrm{B}^3$.



(3) Calculate the value of $\mathrm{A}^5+\mathrm{B}^5$.



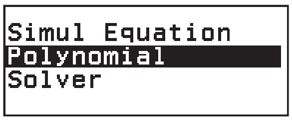
(4) Calculate the value of $\dfrac{\mathrm{B}}{\mathrm{A}}+\dfrac{\mathrm{A}}{\mathrm{B}}$.

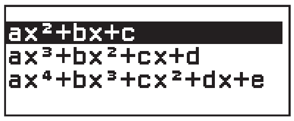


OTHER METHODS

You can use the following procedure to register the value of the solution to $3x^2-6x+5=0$.

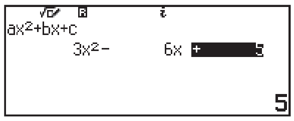
Press ⌂, select [Equation], press OK, select [Polynomial], press OK, select [ax²+bx+c], press OK





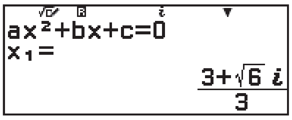
After calculating the solution to $3x^2-6x+5$, do VARIABLE registration.

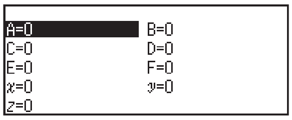
3 EXE (−) 6 EXE 5 EXE



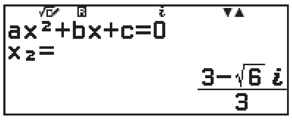
Press EXE ⇄x OK, select [Store],

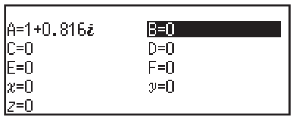
press OK (x_1 registration complete)





Press EXE ⇄x > OK OK (x_2 registration complete)





PRACTICE

◆ When we let α and β be the 2 solutions of the quadratic equation $x^2+x+2=0$, find the value of the expressions.

(1) $\alpha^2+\beta^2$ (2) $\alpha^3+\beta^3$ (3) $(\alpha-\beta)^2$ (4) $\dfrac{\beta}{\alpha^2}+\dfrac{\alpha}{\beta^2}$

From the relation between solutions and coefficients, we get $\alpha+\beta=-1$, $\alpha\beta=2$

(1) $\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta=(-1)^2-2\cdot2=-3$ $\quad -3$

(2) $\alpha^3+\beta^3=(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)=(-1)^3-3\cdot2\cdot(-1)=5$ $\quad 5$

(3) $(\alpha-\beta)^2=(\alpha+\beta)^2-4\alpha\beta=(-1)^2-4\cdot2=-7$ $\quad -7$

(4) $\dfrac{\beta}{\alpha^2}+\dfrac{\alpha}{\beta^2}=\dfrac{\beta^3+\alpha^3}{(\alpha\beta)^2}=\dfrac{5}{2^2}=\dfrac{5}{4}$ $\quad \dfrac{5}{4}$

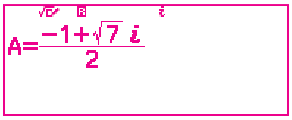
check

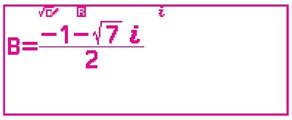
Use the formula of the solution in $x^2+x+2=0$ to get $x=\dfrac{-1\pm\sqrt{7}i}{2}$

In the VARIABLE screen, input $\left[\mathrm{A}=\dfrac{-1+\sqrt{7}i}{2}\right.$ and $\left.\mathrm{B}=\dfrac{-1-\sqrt{7}i}{2}\right]$ (values of solutions obtained above).

⇄x ⊟ (−) 1 + √ 7 ▷ ⇧ 9 ∨ 2 EXE

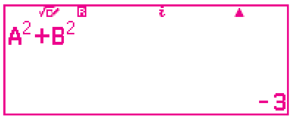
▷ ⊟ (−) 1 − √ 7 ▷ ⇧ 9 ∨ 2 EXE ↺





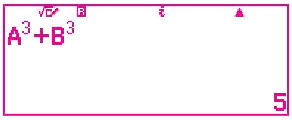
(1) Calculate the value of $\mathrm{A}^2+\mathrm{B}^2$.

⇧ 4 ■² + ⇧ 5 ■² EXE



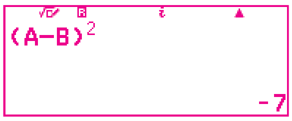
(2) Calculate the value of $\mathrm{A}^3+\mathrm{B}^3$.

⇧ 4 ■□ 3 ▷ + ⇧ 5 ■□ 3 EXE



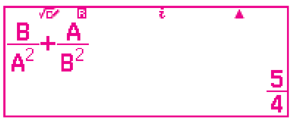
(3) Calculate the value of $(\mathrm{A}-\mathrm{B})^2$.

(⇧ 4 − ⇧ 5) ■² EXE



(4) Calculate the value of $\dfrac{\mathrm{B}}{\mathrm{A}^2}+\dfrac{\mathrm{A}}{\mathrm{B}^2}$.

⇧ 5 ⊟ ⇧ 4 ■² ▷ + ⇧ 4 ⊟ ⇧ 5 ■² EXE



Methods to solve higher-order equations

TARGET To understand how to solve higher-order equations through the factor theorem.

STUDY GUIDE

Solution by the factor theorem

Let $P(x)$ be the nth order for x. At this time, equation $P(x)=0$ is called an nth order **equation**, specifically, equations that are higher than the 3rd order are called **higher-order equations**. To solve higher-order equations, we factorize the left side, and we use the **factor theorem** shown below to find the factors of the left side.

(1) Find k where $P(k)=0$ and $P(x)=(x-k)\,Q(x)$.
(2) $Q(x)$ is found by using division.
(3) Then, factorize until $Q(x)$ is a 1st order or 2nd order expression.

2-fold roots and 3-fold roots

In the solution to the equation $(x-a)^2(x-b)=0$, we can say $x=a$ is a **2-fold root** (where $a \neq b$).
Also, as the solution to the equation $(x-a)^3=0$, we can say $x=a$ is a **3-fold root**.

EXERCISE

◆ Solve the following equations.

(1) $x^3+x^2-7x+5=0$

Let $P(x)=x^3+x^2-7x+5$.

From $P(1)=1^3+1^2-7\cdot1+5=0$, $P(x)$ has $x-1$ as a factor.

$(x-1)(x^2+2x-5)=0$

$x-1=0$ or $x^2+2x-5=0$

$x=1,-1\pm\sqrt{6}$

$$x=1,-1\pm\sqrt{6}$$

(2) $x^4-4x^2-x+2=0$

Let $P(x)=x^4-4x^2-x+2$.

From $P(-1)=(-1)^4-4(-1)^2-(-1)+2=0$, $P(x)$ has $x+1$ as a factor.

$(x+1)(x^3-x^2-3x+2)=0$

Let $Q(x)=x^3-x^2-3x+2$.

From $Q(2)=2^3-2^2-3\cdot2+2=0$, $Q(x)$ has $x-2$ as a factor.

$(x+1)(x-2)(x^2+x-1)=0$

$x=-1,2,\dfrac{-1\pm\sqrt{5}}{2}$

$$x=-1,2,\frac{-1\pm\sqrt{5}}{2}$$

PRACTICE

◆ Solve the following equations.

(1) $x^3-6x^2+9x-2=0$

Let $P(x)=x^3-6x^2+9x-2$.

From $P(2)=2^3-6\cdot 2^2+9\cdot 2-2=0$, $P(x)$ has $x-2$ as a factor.

$(x-2)(x^2-4x+1)=0$

$x=2, 2\pm\sqrt{3}$

$$x=2, 2\pm\sqrt{3}$$

(2) $x^3-x^2-2x+8=0$

Let $P(x)=x^3-x^2-2x+8$.

From $P(-2)=(-2)^3-(-2)^2-2(-2)+8=0$, $P(x)$ has $x+2$ as a factor.

$(x+2)(x^2-3x+4)=0$

$x=-2, \dfrac{3\pm\sqrt{7}i}{2}$

$$x=-2, \frac{3\pm\sqrt{7}i}{2}$$

(3) $x^4+4x^3-8x^2-35x-12=0$

Let $P(x)=x^4+4x^3-8x^2-35x-12$.

From $P(3)=3^4+4\cdot 3^3-8\cdot 3^2-35\cdot 3-12=0$, $P(x)$ has $x-3$ as a factor.

$(x-3)(x^3+7x^2+13x+4)=0$

Let $Q(x)=x^3+7x^2+13x+4$.

From $Q(-4)=(-4)^3+7(-4)^2+13(-4)+4=0$, $Q(x)$ has $x+4$ as a factor.

$(x-3)(x+4)(x^2+3x+1)=0$

$x=3, -4, \dfrac{-3\pm\sqrt{5}}{2}$

$$x=3, -4, \frac{-3\pm\sqrt{5}}{2}$$

(4) $x^4-4x^3+11x^2-14x+6=0$

Let $P(x)=x^4-4x^3+11x^2-14x+6$.

From $P(1)=1^4-4\cdot 1^3+11\cdot 1^2-14\cdot 1+6=0$, $P(x)$ has $x-1$ as a factor.

$(x-1)(x^3-3x^2+8x-6)=0$

Let $Q(x)=x^3-3x^2+8x-6$.

From $Q(1)=1^3-3\cdot 1^2+8\cdot 1-6=0$, $Q(x)$ has $x-1$ as a factor.

$(x-1)^2(x^2-2x+6)=0$

$x=1, 1\pm\sqrt{5}i$

$$x=1, 1\pm\sqrt{5}i$$

Relation between solutions and coefficients in cubic equations

TARGET To understand the relation between solutions and coefficients in cubic equations.

STUDY GUIDE

Relation between solutions and coefficients in cubic equations

Given that 3 solutions of the cubic equation $ax^3+bx^2+cx+d=0$ are α, β, and γ, the following relation holds.

$$\begin{aligned} ax^3+bx^2+cx+d &= a(x-\alpha)(x-\beta)(x-\gamma) \\ &= ax^3-a(\alpha+\beta+\gamma)x^2+a(\alpha\beta+\beta\gamma+\gamma\alpha)x-a\alpha\beta\gamma \end{aligned}$$

Compare the coefficients on both sides $b=-a(\alpha+\beta+\gamma), c=a(\alpha\beta+\beta\gamma+\gamma\alpha), d=-a\alpha\beta\gamma$.

From the above, we can derive the following.

$$\boldsymbol{\alpha+\beta+\gamma=-\frac{b}{a}, \alpha\beta+\beta\gamma+\gamma\alpha=\frac{c}{a}, \alpha\beta\gamma=-\frac{d}{a}}$$

Basically, $\alpha+\beta+\gamma$ and $\alpha\beta+\beta\gamma+\gamma\alpha$ and $\alpha\beta\gamma$ are symmetric expressions, so we can use the following formula deformation.

· $\boldsymbol{\alpha^2+\beta^2+\gamma^2=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\beta\gamma+\gamma\alpha)}$

· $\boldsymbol{\alpha^3+\beta^3+\gamma^3=(\alpha+\beta+\gamma)\{\alpha^2+\beta^2+\gamma^2-(\alpha\beta+\beta\gamma+\gamma\alpha)\}+3\alpha\beta\gamma}$

EXERCISE

◆ Solve the following problems.

(1) Let α, β, and γ be the 3 solutions of the cubic equation $x^3-2x^2+3x+1=0$. Find the value of $(\alpha+1)(\beta+1)(\gamma+1)$.

From the relation between solutions and coefficients, we get

$$\alpha+\beta+\gamma=-\frac{-2}{1}=2, \alpha\beta+\beta\gamma+\gamma\alpha=\frac{3}{1}=3, \alpha\beta\gamma=-\frac{1}{1}=-1.$$

$$\begin{aligned} (\alpha+1)(\beta+1)(\gamma+1) &= (\alpha\beta+\alpha+\beta+1)(\gamma+1) \\ &= \alpha\beta\gamma+\alpha\beta+\beta\gamma+\gamma\alpha+\alpha+\beta+\gamma+1 \\ &= -1+3+2+1 \\ &= 5 \end{aligned}$$

5

(2) When the 3 solutions to the cubic equation $x^3+ax^2+bx+2=0$ are -1, 2, and c, find the values of the constants a, b, and c.

From the relation between solutions and coefficients, we get

$-1+2+c=-a, (-1)\cdot 2+2\cdot c+c\cdot(-1)=b, (-1)\cdot 2\cdot c=-2$.

These are arranged into $a=-c-1, b=c-2, c=1$.

Therefore, we get $a=-2, b=-1, c=1$.

$a=-2$, $b=-1$, $c=1$

PRACTICE

1 Let α, β, and γ be the 3 solutions of the cubic equation $x^3+3x^2-2x-5=0$. Find the value of the following equations.

(1) $\alpha^2+\beta^2+\gamma^2$

From the relation between solutions and coefficients, we get

$$\alpha+\beta+\gamma=-3, \alpha\beta+\beta\gamma+\gamma\alpha=-2, \alpha\beta\gamma=5.$$

$$\begin{aligned}\alpha^2+\beta^2+\gamma^2&=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\beta\gamma+\gamma\alpha)\\&=(-3)^2-2(-2)\\&=13\end{aligned}$$

13

(2) $\alpha^3+\beta^3+\gamma^3$

Use the values found in (1).

$$\begin{aligned}\alpha^3+\beta^3+\gamma^3&=(\alpha+\beta+\gamma)\{\alpha^2+\beta^2+\gamma^2-(\alpha\beta+\beta\gamma+\gamma\alpha)\}+3\alpha\beta\gamma\\&=(-3)\{13-(-2)\}+3\cdot5\\&=-30\end{aligned}$$

-30

2 Let α, β, and γ be the 3 solutions of the cubic equation $x^3-4x^2-x-3=0$. Find the value of the following equations.

(1) $\dfrac{1}{\alpha}+\dfrac{1}{\beta}+\dfrac{1}{\gamma}$

From the relation between solutions and coefficients, we get

$$\alpha+\beta+\gamma=4, \alpha\beta+\beta\gamma+\gamma\alpha=-1, \alpha\beta\gamma=3.$$

$$\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}=\frac{\alpha\beta+\beta\gamma+\gamma\alpha}{\alpha\beta\gamma}=-\frac{1}{3}$$

$-\dfrac{1}{3}$

(2) $(\alpha-2)(\beta-2)(\gamma-2)$

Use the values found in (1).

$$\begin{aligned}(\alpha-2)(\beta-2)(\gamma-2)&=\alpha\beta\gamma-2(\alpha\beta+\beta\gamma+\gamma\alpha)+4(\alpha+\beta+\gamma)-8\\&=3-2(-1)+4\cdot4-8\\&=13\end{aligned}$$

13

3 When the 3 solutions to the cubic equation $x^3+ax^2+2x+b=0$ are -2, 5, and c, find the values of the constants a, b, and c.

From the relation between solutions and coefficients, we get

$$-2+5+c=-a, (-2)\cdot5+5\cdot c+c\cdot(-2)=2, (-2)\cdot5\cdot c=-b.$$

These are arranged into $a=-c-3, 3c=12, b=10c$.

Therefore, we get $a=-7, b=40, c=4$.

$a=-7, b=40, c=4$

Complex numbers

TARGET To understand the various properties of logarithms and how to add, subtract, multiply, and divide them.

STUDY GUIDE

Equality of complex numbers

When a, b, c, and d are real numbers, $a+bi=c+di \Leftrightarrow a=c$, and $b=d$, specifically, $a+bi=0 \Leftrightarrow a=0$, and $b=0$

Ex. $a+bi=4-3i \Leftrightarrow a=4, b=-3$

Conjugate complex numbers

For a complex number $z=a+bi$, we say that $a-bi$ is the **conjugate complex number** for z, which is expressed as $\overline{z}=a-bi$.

z and $\overline{z}$ are mutually conjugate complex numbers. $\overline{z}$ is also called the **conjugate complex number** of z.

Ex. (1) $z=2+5i \Rightarrow \overline{z}=2-5i$ (2) $z=-4-6i \Rightarrow \overline{z}=-4+6i$

Absolute value of complex numbers

For the complex number $z=a+bi$, we say that $\sqrt{a^2+b^2}$ is the **absolute value of z**, and express it as $|z|$.

When $z=a+bi$, then $|z|=|a+bi|=\sqrt{a^2+b^2}$

Ex. When $z=4-7i$, then $|z|=|4-7i|=\sqrt{4^2+(-7)^2}=\sqrt{65}$

4 arithmetic operations on complex numbers

The 4 arithmetic operations are defined for complex numbers the same as for real numbers, so the commutative law, associative law, and distributive law hold for addition and multiplication.

Adding $(a+bi)+(c+di)=(a+c)+(b+d)i$

Subtracting $(a+bi)-(c+di)=(a-c)+(b-d)i$

Multiplying $(a+bi)(c+di)=(ac-bd)+(bc+ad)i$

Specifically, when $(a+bi)(a-bi)=a^2+b^2$**, and**
k **are real numbers, then** $k(a+bi)=ka+kbi$

Dividing $\dfrac{a+bi}{c+di}=\dfrac{ac+bd}{c^2+d^2}+\dfrac{(bc-ad)i}{c^2+d^2}$

explanation

1 The four arithmetic operations for complex numbers are calculated by regarding i as a letter, and replacing i^2 by -1.

2 Division is calculated by multiplying the denominator and the numerator by the conjugate complex number $c-di$ of the denominator to make the denominator a real number.

3 For complex numbers α and β, $\alpha\beta=0 \Leftrightarrow \alpha=0$ or $\beta=0$

4 Imaginary numbers ($a+bi$ of $b \neq 0$) have no magnitude relationship.

EXTRA Info.

Use the scientific calculator for operations on complex numbers.

EXERCISE

1 Find values for the real numbers a and b that satisfy the following equations.

(1) $(a-3)+4i=2+(b+6)i$

$(a-3)+4i=2+(b+6)i \Leftrightarrow a-3=2, 4=b+6$

Therefore, $a=5$ and $b=-2$

$a=5, b=-2$

(2) $(2a+3b)+(5a-2b)i=4-9i$

$(2a+3b)+(5a-2b)i=4-9i \Leftrightarrow 2a+3b=4, 5a-2b=-9$

Solving this gives us $a=-1$ and $b=2$

$a=-1, b=2$

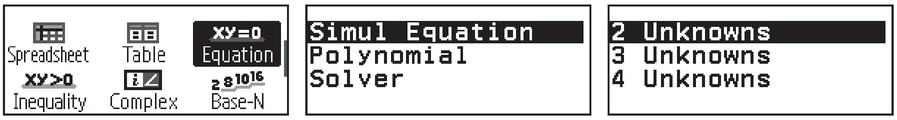
check

In this unit, Radian is used for angular units unless otherwise specified.

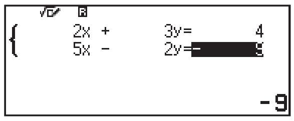
Press ⊜, select [Calc Settings], press OK, select [Angle Unit], press OK, select [Radian], press OK, AC

Use the Equation function in the scientific calculator to calculate a solution for the simultaneous equations.

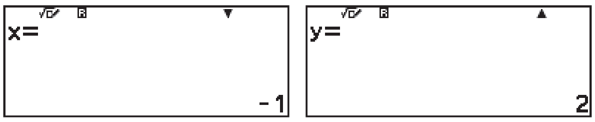
Press ⌂, select [Equation], press OK, select [Simul Equation], press OK, select [2 Unknowns], press OK



② EXE ③ EXE ④ EXE ⑤ EXE ⊖ ② EXE ⊖ ⑨ EXE



EXE EXE



2 Find the conjugate complex numbers and the absolute values of the following complex numbers.

(1) $z=2+3i$

$\bar{z}=\overline{2+3i}=2-3i$

$|z|=|2+3i|=\sqrt{2^2+3^2}=\sqrt{13}$

$$\boldsymbol{\bar{z}=2-3i,\ |z|=\sqrt{13}}$$

(2) $z=\sqrt{7}-5i$

$\bar{z}=\overline{\sqrt{7}-5i}=\sqrt{7}+5i$

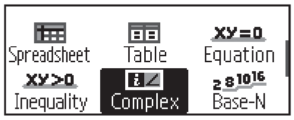
$|z|=|\sqrt{7}-5i|=\sqrt{(\sqrt{7})^2+(-5)^2}=\sqrt{32}=4\sqrt{2}$

$$\boldsymbol{\bar{z}=\sqrt{7}+5i,\ |z|=4\sqrt{2}}$$

check

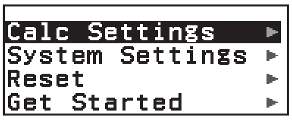
Use the Complex function in the scientific calculator to calculate the conjugate complex numbers and absolute values of the complex numbers.

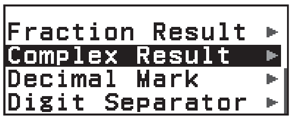
Press (HOME), select [Complex], press (OK)



Set the results of complex number calculations to be in the a+bi format.

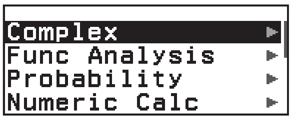
Press (SETTINGS), select [Calc Settings], press (OK), select [Complex Result], press (OK), select [a+bi], press (OK), (AC)

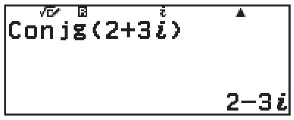




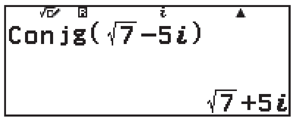
Find the conjugate complex number.

Press (CATALOG), select [Complex], press (OK), select [Conjugate], press (OK), (2) (+) (3) (SHIFT) (9) ()) (EXE)



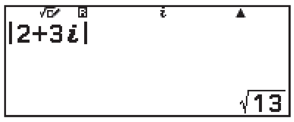


(CATALOG) (OK) (⌄) (⌄) (⌄) (OK) (√■) (7) (>) (−) (5) (SHIFT) (9) ()) (EXE)

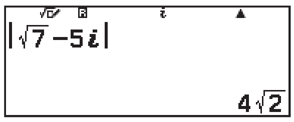


Find the absolute value.

Press (AC), (CATALOG), select [Numeric Calc], press (OK), select [Absolute Value], press (OK), (2) (+) (3) (SHIFT) (9) (EXE)



(CATALOG) (⌄) (⌄) (⌄) (OK) (OK) (√■) (7) (>) (−) (5) (SHIFT) (9) (EXE)



3 Calculate the following.

(1) $(2+3i)+(5-4i)$

$= (2+5)+(3-4)i = 7-i$

$7-i$

(2) $(2+3i)-(5-4i)$

$= (2-5)+(3+4)i = -3+7i$

$-3+7i$

(3) $(2+3i)(5-4i)$

$= \{2\cdot 5+3\cdot(-4)\cdot i^2\}+\{3\cdot 5+2\cdot(-4)\}i = 22+7i$

$22+7i$

(4) $\dfrac{2+3i}{5-4i}$

$= \dfrac{2+3i}{5-4i}\cdot\dfrac{5+4i}{5+4i} = \dfrac{(10-12)+(15+8)i}{5^2+4^2} = -\dfrac{2}{41}+\dfrac{23}{41}i$

$-\dfrac{2}{41}+\dfrac{23}{41}i$

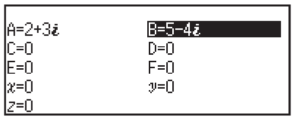
check

Use the VARIABLE function in the scientific calculator to confirm the results of the four arithmetic operations on complex numbers.

Press ⌂, select [Complex], press OK.

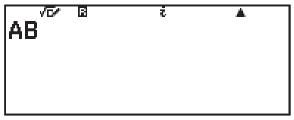
In the VARIABLE screen, input [A=2+3i and B=5−4i].

VARIABLE 2 + 3 ⇧ 9 EXE > 5 − 4 ⇧ 9 EXE ↶



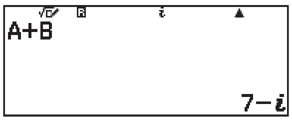
Make AB the basic form of the calculation.

⇧ 4 ⇧ 5



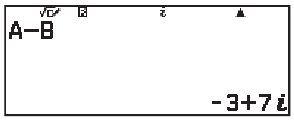
(1) Add.

< + EXE



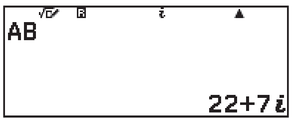
(2) Subtract.

↶ < ⌫ − EXE



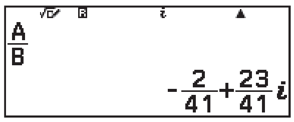
(3) Multiply.

↶ < ⌫ EXE



(4) Divide.

↶ < ÷ EXE



PRACTICE

◆ Calculate the following.

(1) $(2-7i)+(-6+5i)$

$= (2-6)+(-7+5)i = -4-2i$

$-4-2i$

(2) $(2-7i)-(-6+5i)$

$= (2+6)+(-7-5)i = 8-12i$

$8-12i$

(3) $(2-7i)(-6+5i)$

$= \{2\cdot(-6)-7\cdot5\cdot i^2\}+\{(-7)\cdot(-6)+2\cdot5\}i = 23+52i$

$23+52i$

(4) $\dfrac{2-7i}{-6+5i}$

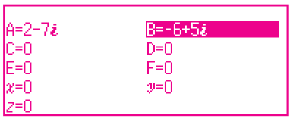
$$= \frac{2-7i}{-6+5i}\cdot\frac{-6-5i}{-6-5i} = \frac{(-12-35)+(42-10)i}{6^2+5^2} = -\frac{47}{61}+\frac{32}{61}i$$

$-\dfrac{47}{61}+\dfrac{32}{61}i$

check

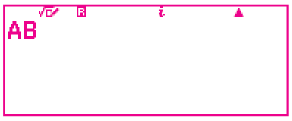
In the VARIABLE screen, input [A=2−7i and B=−6+5i].

[⇄x] [2] [⊖] [7] [⬆] [9] [EXE] [>] [⊖] [6] [+] [5] [⬆] [9] [EXE] [↩]



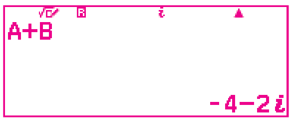
Make AB the basic form of the calculation.

[⬆] [4] [⬆] [5]



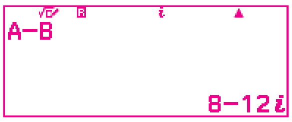
(1) Add.

[<] [+] [EXE]



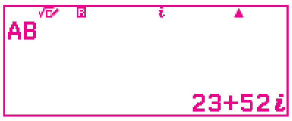
(2) Subtract.

[↩] [<] [⌫] [⊖] [EXE]



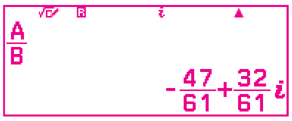
(3) Multiply.

[↩] [<] [⌫] [EXE]



(4) Divide.

[↩] [<] [÷] [EXE]



The complex number plane

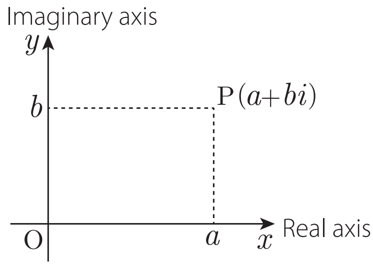
TARGET To understand the concept of expressing complex numbers on a plane.

STUDY GUIDE

The complex number plane

When a point (a, b) on the coordinate plane is made to correspond to a complex number $z=a+bi$ (a,b are real numbers), the complex number and the point on the coordinate plane correspond 1-to-1.

The plane on which complex numbers are visualized is called the **complex number plane**, and we say the x axis is the **real axis** and the y axis is the **imaginary axis**.



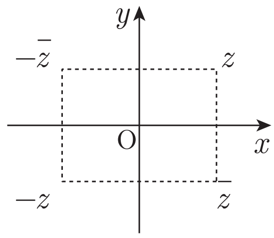
The point P that represents the complex number z is written as the point P(z) or the point z. The point that represents 0 is the origin O.

The complex numbers z, $-z$, $\bar{z}$, and $-\bar{z}$ have the following relationship on the complex number plane.

(a) Point z and point $-z$ are symmetric about the origin

(b) Point z and point $\bar{z}$ are symmetric about the real axis

(c) Point z and point $-\bar{z}$ are symmetric about the imaginary axis



Addition, subtraction, and real multiples of complex numbers

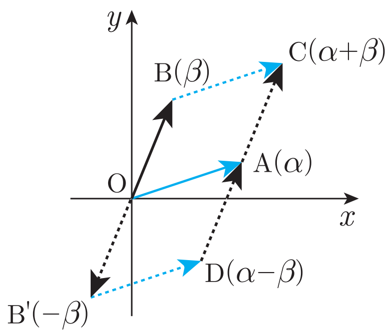
Let the points that represents the 2 complex numbers $\alpha=a+bi$ and $\beta=c+di$ be $\mathrm{A}(\alpha)$, and $\mathrm{B}(\beta)$.

The point $\mathrm{C}(\alpha+\beta)$ represented by the sum $\alpha+\beta$ is the 4th vertex of the parallelogram with OA and OB as 2 sides.

The point $\mathrm{D}(\alpha-\beta)$ represented by the difference $\alpha-\beta$ is the 4th vertex of the parallelogram with OA and OB' as 2 sides.

(However, given $\mathrm{B}'(-\beta)$.)

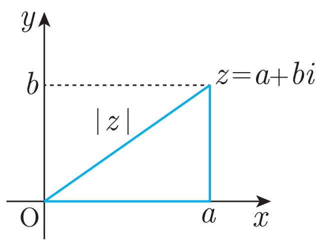
Furthermore, the point represented by the real multiples $k\alpha$ is a point on the line OA.



Absolute value of complex numbers

The absolute value $|z|$ of complex numbers is the distance between the point z and the origin O.

The absolute value of a complex number has the following properties.



(1) When $z=a+bi$, then $|z|=|a+bi|=\sqrt{a^2+b^2}$

(2) $|z|=|-z|=|\bar{z}|=|-\bar{z}|$

(3) $z\bar{z}=|z|^2=a^2+b^2$

(4) $|\alpha\beta|=|\alpha||\beta|$, $\left|\dfrac{\alpha}{\beta}\right|=\dfrac{|\alpha|}{|\beta|}$

EXTRA Info.

Use the scientific calculator to verify complex numbers on the complex number plane.

EXERCISE

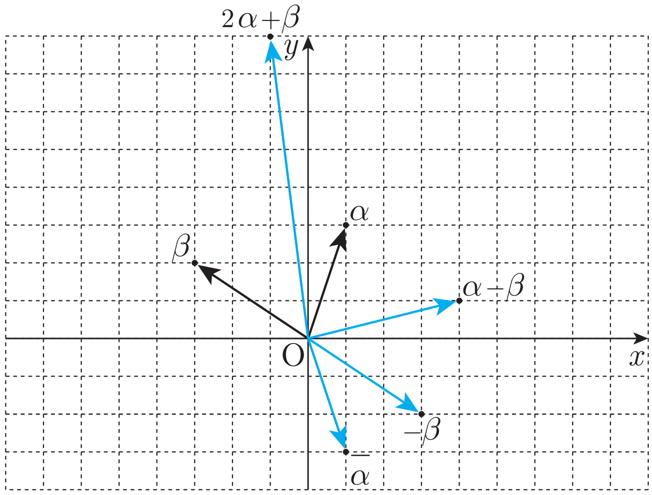
◆ When $\alpha = 1+3i, \beta = -3+2i$, solve the following problems.

(1) Draw the points that represent the complex numbers $\overline{\alpha}$, $-\beta$, $2\alpha+\beta$, and $\alpha-\beta$.

$\overline{\alpha} = 1-3i, -\beta = 3-2i$

$2\alpha+\beta = 2(1+3i)+(-3+2i) = -1+8i$

$\alpha-\beta = 1+3i-(-3+2i) = 4+i$



(2) Find the values of $|\alpha|$, $|\beta|$, $|\alpha\beta|$, and $\left|\dfrac{\alpha}{\beta}\right|$. Also, from those results, confirm that $|\alpha\beta| = |\alpha||\beta|$, $\left|\dfrac{\alpha}{\beta}\right| = \dfrac{|\alpha|}{|\beta|}$ hold.

$|\alpha| = |1+3i| = \sqrt{1^2+3^2} = \sqrt{10}$

$|\beta| = |-3+2i| = \sqrt{(-3)^2+2^2} = \sqrt{13}$

$|\alpha\beta| = |(1+3i)(-3+2i)| = |-9-7i| = \sqrt{(-9)^2+(-7)^2} = \sqrt{130}$

$(= \sqrt{10}\sqrt{13} = |\alpha||\beta|)$

$$\left|\frac{\alpha}{\beta}\right| = \left|\frac{1+3i}{-3+2i}\right| = \left|\frac{1+3i}{-3+2i}\cdot\frac{-3-2i}{-3-2i}\right| = \left|\frac{(1+3i)(-3-2i)}{(-3)^2+2^2}\right| = \left|\frac{3-11i}{13}\right| = \frac{\sqrt{3^2+(-11)^2}}{13} = \frac{\sqrt{130}}{13}$$

$$\left(= \frac{\sqrt{10}}{\sqrt{13}} = \frac{|\alpha|}{|\beta|}\right)$$

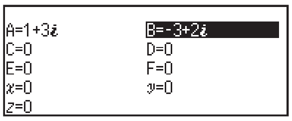
Therefore, we can confirm that $|\alpha\beta| = |\alpha||\beta|$, $\left|\dfrac{\alpha}{\beta}\right| = \dfrac{|\alpha|}{|\beta|}$.

$$\boldsymbol{|\alpha| = \sqrt{10}, |\beta| = \sqrt{13}, |\alpha\beta| = \sqrt{130}, \left|\frac{\alpha}{\beta}\right| = \frac{\sqrt{130}}{13}}$$

check

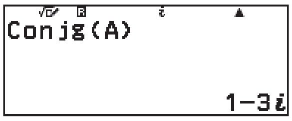
In the $\mathrm{VARIABLE}$ screen, input [$\mathrm{A}=1+3i$ and $\mathrm{B}=-3+2i$].

[VAR] 1 + 3 [SHIFT] 9 [EXE] > − 3 + 2 [SHIFT] 9 [EXE] [BACK]



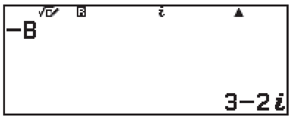
(1) Calculate the value of $\overline{\alpha}$.

[CATALOG] [OK] ∨ ∨ ∨ [OK] [SHIFT] 4) [EXE]



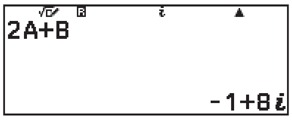
Calculate the value of $-\beta$.

− [SHIFT] 5 [EXE]



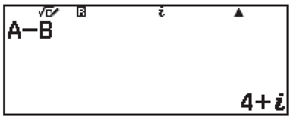
Calculate the value of $2\alpha+\beta$.

2 [SHIFT] 4 + [SHIFT] 5 [EXE]



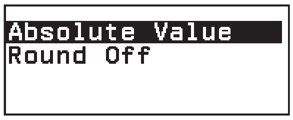
Calculate the value of $\alpha-\beta$.

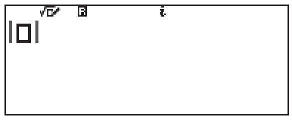
[SHIFT] 4 − [SHIFT] 5 [EXE]



(2) Input the symbol for absolute values.

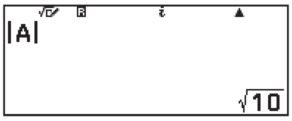
Press [CATALOG], select [$\mathrm{Numeric\ Calc}$], press [OK], select [$\mathrm{Absolute\ Value}$], press [OK]





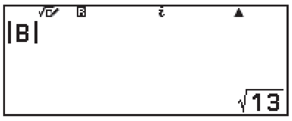
Calculate the value of $|\alpha|$.

[SHIFT] 4 [EXE]



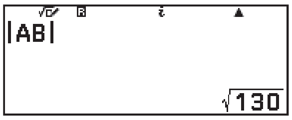
Calculate the value of $|\beta|$.

[BACK] < [DEL] [SHIFT] 5 [EXE]



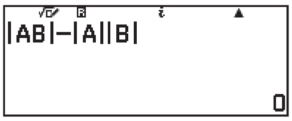
Calculate the value of $|\alpha\beta|$.

[BACK] < < [SHIFT] 4 [EXE]



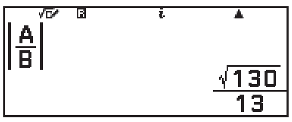
Confirm that $|\alpha\beta|=|\alpha||\beta|$.

[BACK] − [CATALOG] ∨ ∨ ∨ [OK] [OK] [SHIFT] 4 > [CATALOG] ∨ ∨ ∨ [OK] [OK] [SHIFT] 5 [EXE]



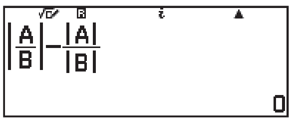
Calculate the value of $\left|\frac{\alpha}{\beta}\right|$.

[BACK] [DEL] [DEL] [DEL] [DEL] [DEL] < < [FRAC] [EXE]



Confirm that $\left|\frac{\alpha}{\beta}\right|=\frac{|\alpha|}{|\beta|}$.

[BACK] − [FRAC] [CATALOG] ∨ ∨ ∨ [OK] [OK] [SHIFT] 4 ∨ [CATALOG] ∨ ∨ ∨ [OK] [OK] [SHIFT] 5 [EXE]

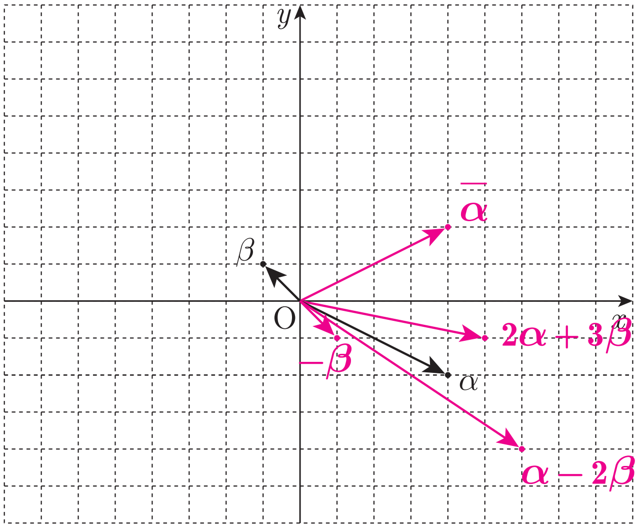


PRACTICE

◆ When $\alpha=4-2i$ and $\beta=-1+i$, solve the following problems.

(1) Draw the points that represent the complex numbers $\overline{\alpha}$, $-\beta$, $2\alpha+3\beta$, and $\alpha-2\beta$.

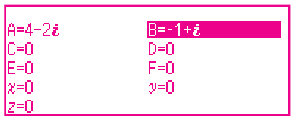
$$\overline{\alpha}=4+2i, -\beta=1-i$$
$$2\alpha+3\beta=2(4-2i)+3(-1+i)=5-i$$
$$\alpha-2\beta=(4-2i)-2(-1+i)=6-4i$$



check

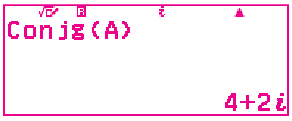
In the VARIABLE screen, input [$A=4-2i$ and $B=-1+i$].

⇄ 4 − 2 ⇧ 9 EXE > − 1 + ⇧ 9 EXE ↺



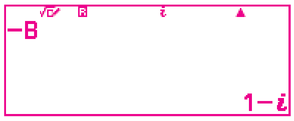
Calculate the value of $\overline{\alpha}$.

📖 OK ∨ ∨ ∨ OK ⇧ 4) EXE



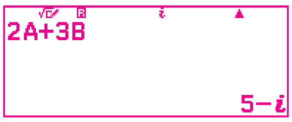
Calculate the value of $-\beta$.

− ⇧ 5 EXE



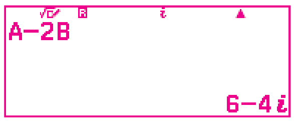
Calculate the value of $2\alpha+3\beta$.

2 ⇧ 4 + 3 ⇧ 5 EXE



Calculate the value of $\alpha-2\beta$.

⇧ 4 − 2 ⇧ 5 EXE



(2) Find the values of $|\alpha|$, $|\beta|$, $|\alpha\beta|$, and $\left|\dfrac{\alpha}{\beta}\right|$. Also, from those results, confirm that $|\alpha\beta|=|\alpha||\beta|$, $\left|\dfrac{\alpha}{\beta}\right|=\dfrac{|\alpha|}{|\beta|}$ hold.

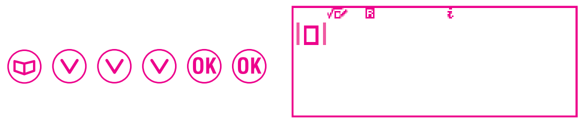
$$|\alpha|=|4-2i|=\sqrt{4^2+(-2)^2}=\sqrt{20}=2\sqrt{5}$$
$$|\beta|=|-1+i|=\sqrt{(-1)^2+1^2}=\sqrt{2}$$
$$|\alpha\beta|=|(4-2i)(-1+i)|=|-2+6i|=\sqrt{(-2)^2+6^2}=\sqrt{40}=2\sqrt{10}$$
$$(=\sqrt{20}\sqrt{2}=|\alpha||\beta|)$$
$$\left|\frac{\alpha}{\beta}\right|=\left|\frac{4-2i}{-1+i}\right|=\left|\frac{4-2i}{-1+i}\cdot\frac{-1-i}{-1-i}\right|=\left|\frac{(4-2i)(-1-i)}{(-1)^2+1^2}\right|=\left|\frac{-6-2i}{2}\right|=|-3-i|=\sqrt{10}$$
$$\left(=\frac{\sqrt{20}}{\sqrt{2}}=\frac{|\alpha|}{|\beta|}\right)$$

We can confirm that $|\alpha\beta|=|\alpha||\beta|$, $\left|\dfrac{\alpha}{\beta}\right|=\dfrac{|\alpha|}{|\beta|}$.

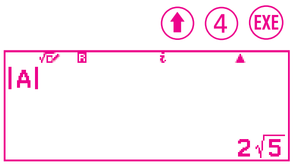
$$|\alpha|=2\sqrt{5},\ |\beta|=\sqrt{2},\ |\alpha\beta|=2\sqrt{10},\ \left|\frac{\alpha}{\beta}\right|=\sqrt{10}$$

check

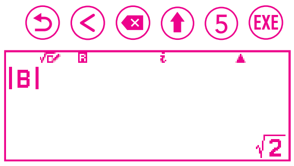
Input the symbol for absolute values.



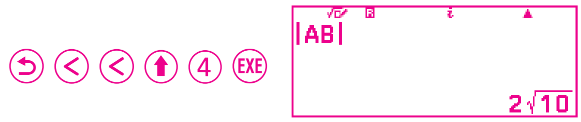
Calculate the value of $|\alpha|$.



Calculate the value of $|\beta|$.



Calculate the value of $|\alpha\beta|$.



Confirm that $|\alpha\beta|=|\alpha||\beta|$.

Calculate the value of $\left|\dfrac{\alpha}{\beta}\right|$.

Confirm that $\left|\dfrac{\alpha}{\beta}\right|=\dfrac{|\alpha|}{|\beta|}$.

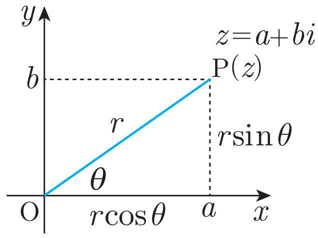
Polar form of complex numbers

TARGET To understand the polar forms of complex numbers.

STUDY GUIDE

Polar forms

For the complex number $z=a+bi$, we can see from the figure on the right that from $a=r\cos\theta, b=r\sin\theta$ (a and b are real numbers), we get the following expressions, which are called the **polar forms** of the complex number z.



$$a+bi=r(\cos\theta+i\sin\theta)$$

However, $r=|z|=\sqrt{a^2+b^2}$ $(r>0)$

θ **is the angle that satisfies** $\cos\theta=\dfrac{a}{r}, \sin\theta=\dfrac{b}{r}$.

The angle θ formed by the half-line OP and the positive part of the real axis is called the **deflection angle** of z, and is expressed as $\arg z$.

The deflection angle θ is determined only 1 time in the range of $0\leq\theta<2\pi$.

Generally, it is $\arg z=\theta+2\pi\times n(n=0,\pm1,\pm2,\cdots)$.

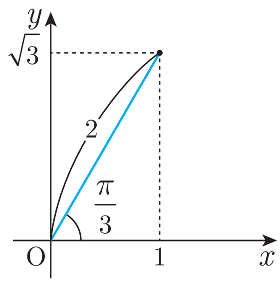
If $z=0$, then $r=0$, but the deflection angle is not determined.

Ex. Express $z=1+\sqrt{3}i$ in polar coordinates.

$z=1+\sqrt{3}i$ is $r=\sqrt{1^2+(\sqrt{3})^2}=\sqrt{4}=2$

Also, from $\cos\theta=\dfrac{1}{2}, \sin\theta=\dfrac{\sqrt{3}}{2}$, $0\leq\theta<2\pi$, so $\theta=\dfrac{\pi}{3}$

Therefore, $z=1+\sqrt{3}i=2\left(\dfrac{1}{2}+\dfrac{\sqrt{3}}{2}i\right)=2\left(\cos\dfrac{\pi}{3}+i\sin\dfrac{\pi}{3}\right)$



OTHER METHODS

$-2\pi\leq\theta<0$, so $\theta=-\dfrac{5\pi}{3}$

Therefore, $z=1+\sqrt{3}i=2\left\{\cos\left(-\dfrac{5\pi}{3}\right)+i\sin\left(-\dfrac{5\pi}{3}\right)\right\}$

The deflection angle is determined according to the conditions in the problem.

EXTRA Info.

Use the scientific calculator to change complex numbers to their polar forms.

EXERCISE

1 When the deflection angle θ is $(0 \leq \theta < 2\pi)$, express the following complex numbers in polar form.

(1) $1+i$

From $r=|1+i|=\sqrt{1^2+1^2}=\sqrt{2}, \cos\theta=\frac{1}{\sqrt{2}}, \sin\theta=\frac{1}{\sqrt{2}}$, $0 \leq \theta < 2\pi$, so $\theta=\frac{\pi}{4}$

Therefore, $1+i=\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$

$$\boldsymbol{\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)}$$

(2) $\sqrt{3}-i$

From $r=|\sqrt{3}-i|=\sqrt{(\sqrt{3})^2+(-1)^2}=\sqrt{4}=2, \cos\theta=\frac{\sqrt{3}}{2}, \sin\theta=-\frac{1}{2}$, $0 \leq \theta < 2\pi$, so $\theta=\frac{11\pi}{6}$

Therefore, $\sqrt{3}-i=2\left(\cos\frac{11\pi}{6}+i\sin\frac{11\pi}{6}\right)$

$$\boldsymbol{2\left(\cos\frac{11\pi}{6}+i\sin\frac{11\pi}{6}\right)}$$

(3) $\frac{-1+\sqrt{3}i}{4}$

Convert $\frac{-1+\sqrt{3}i}{4}=\frac{1}{4}(-1+\sqrt{3}i)$, then for simplicity, change $-1+\sqrt{3}i$ to polar form.

For $-1+\sqrt{3}i$, $r=|-1+\sqrt{3}i|=\sqrt{(-1)^2+(\sqrt{3})^2}=\sqrt{4}=2$

From $\cos\theta=-\frac{1}{2}, \sin\theta=\frac{\sqrt{3}}{2}$, $0 \leq \theta < 2\pi$, so $\theta=\frac{2\pi}{3}$

Therefore, $\frac{1}{4}(-1+\sqrt{3}i)=\frac{1}{4}\cdot 2\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)=\frac{1}{2}\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)$

$$\boldsymbol{\frac{1}{2}\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)}$$

(4) $-2i$

From $r=|-2i|=\sqrt{(-2)^2}=2, \cos\theta=0, \sin\theta=-\frac{2}{2}=-1$, $0 \leq \theta < 2\pi$, so $\theta=\frac{3\pi}{2}$

Therefore, $-2i=2\left(\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}\right)=2i\sin\frac{3\pi}{2}$

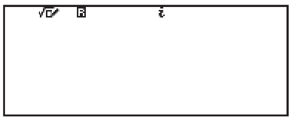
$$\boldsymbol{2i\sin\frac{3\pi}{2}}$$

check

Use the Complex function in the scientific calculator to display the polar form of complex numbers.

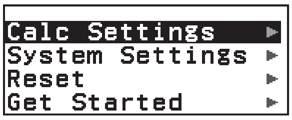
Press ⌂, select [Complex], press OK

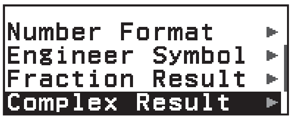
Change the i(a+bi display) at the top of the calculator screen to the $\angle$(polar form display).

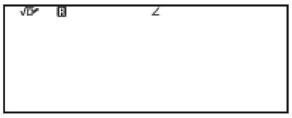


Press ☰, select [Calc Settings], press OK, select [Complex Result], press OK,

select [r$\angle\theta$], press OK, AC

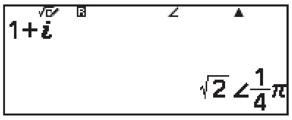






(1) Change $1+i$ to polar form.

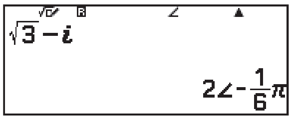
① ⊕ ⬆ ⑨ EXE



On the scientific calculator, the polar form $r(\cos\theta + i\sin\theta)$ is displayed as $r\angle\theta$.

(2) Change $\sqrt{3}-i$ to polar form.

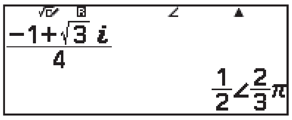
√ ③ ▷ ⊖ ⬆ ⑨ EXE



$2\angle-\dfrac{\pi}{6}$ is the same value as $2\angle\dfrac{11\pi}{6}$.

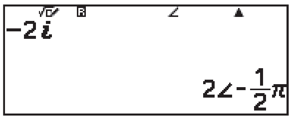
(3) Change $\dfrac{-1+\sqrt{3}i}{4}$ to polar form.

▭ ⊖ ① ⊕ √ ③ ▷ ⬆ ⑨ ▽ ④ EXE



(4) Change $-2i$ to polar form.

⊖ ② ⬆ ⑨ EXE



$2\angle-\dfrac{\pi}{2}$ is the same value as $2\angle\dfrac{3\pi}{2}$.

2 Show the polar form of the following complex numbers in the $a+bi$ form.

(1) $\sqrt{7}\left(\cos\dfrac{\pi}{3}+i\sin\dfrac{\pi}{3}\right)$

$=\sqrt{7}\left(\dfrac{1}{2}+\dfrac{\sqrt{3}}{2}i\right)=\dfrac{\sqrt{7}}{2}+\dfrac{\sqrt{21}}{2}i$

$$\underline{\dfrac{\sqrt{7}}{2}+\dfrac{\sqrt{21}}{2}i}$$

(2) $\sqrt{10}\left(\cos\dfrac{4\pi}{3}+i\sin\dfrac{4\pi}{3}\right)$

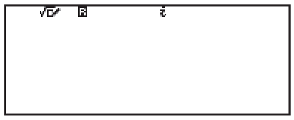
$=\sqrt{10}\left(-\dfrac{1}{2}-\dfrac{\sqrt{3}}{2}i\right)=-\dfrac{\sqrt{10}}{2}-\dfrac{\sqrt{30}}{2}i$

$$\underline{-\dfrac{\sqrt{10}}{2}-\dfrac{\sqrt{30}}{2}i}$$

check

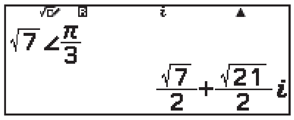
Change the ∠(polar form display) at the top of the calculator screen to the i(a+bi display).

[MENU] [OK] [▽] [▽] [▽] [▽] [▽] [OK] [OK] [AC]



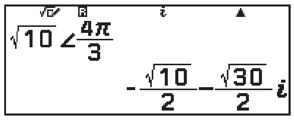
(1) Change $\sqrt{7}\left(\cos\dfrac{\pi}{3}+i\sin\dfrac{\pi}{3}\right)$ to the $a+bi$ form.

Press [√■] [7] [▷], [CATALOG], select [Complex], press [OK], select [∠], press [OK], [SHIFT] [7] [■/■] [3] [EXE]



(2) Change $\sqrt{10}\left(\cos\dfrac{4\pi}{3}+i\sin\dfrac{4\pi}{3}\right)$ to the $a+bi$ form.

[√■] [1] [0] [▷] [CATALOG] [OK] [▽] [OK] [4] [SHIFT] [7] [■/■] [3] [EXE]



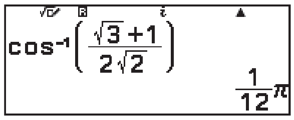
3 Find the absolute value r and the deflection angle $\theta(-\pi<\theta\leq\pi)$ of the following complex numbers. However, you can use a scientific calculator to find the deflection angle.

(1) $z=(1-i)(1+\sqrt{3}i)$

$$z=(1-i)(1+\sqrt{3}i)=(\sqrt{3}+1)+(\sqrt{3}-1)i$$
$$r=\sqrt{(\sqrt{3}+1)^2+(\sqrt{3}-1)^2}=\sqrt{8}=2\sqrt{2}$$

Furthermore, use the scientific calculator to find an angle θ that satisfies $\cos\theta=\dfrac{\sqrt{3}+1}{2\sqrt{2}}, \sin\theta=\dfrac{\sqrt{3}-1}{2\sqrt{2}}$.

[SHIFT] [cos] [▭/▭] [√▭] [3] [▷] [+] [1] [▽] [2] [√▭] [2] [▷] [▷] [)] [EXE]



Therefore, $-\pi<\theta\leq\pi$, so $\theta=\dfrac{\pi}{12}$

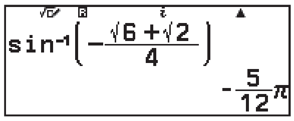
$$\boldsymbol{r=2\sqrt{2}, \theta=\frac{\pi}{12}}$$

(2) $z=\dfrac{\sqrt{3}-i}{\sqrt{2}+\sqrt{2}i}$

$$z=\frac{\sqrt{3}-i}{\sqrt{2}+\sqrt{2}i}=\frac{\sqrt{3}-i}{\sqrt{2}+\sqrt{2}i}\cdot\frac{\sqrt{2}-\sqrt{2}i}{\sqrt{2}-\sqrt{2}i}=\frac{\sqrt{6}-\sqrt{2}}{4}-\frac{\sqrt{6}+\sqrt{2}}{4}i=\frac{1}{4}\{(\sqrt{6}-\sqrt{2})-(\sqrt{6}+\sqrt{2})i\}$$
$$r=\frac{1}{4}|(\sqrt{6}-\sqrt{2})-(\sqrt{6}+\sqrt{2})i|=\frac{1}{4}\cdot\sqrt{(\sqrt{6}-\sqrt{2})^2+(\sqrt{6}+\sqrt{2})^2}=\frac{1}{4}\cdot\sqrt{16}=1$$

Furthermore, use the scientific calculator to find an angle θ that satisfies $\cos\theta=\dfrac{\sqrt{6}-\sqrt{2}}{4}, \sin\theta=-\dfrac{\sqrt{6}+\sqrt{2}}{4}$.

[SHIFT] [sin] [(−)] [▭/▭] [√▭] [6] [▷] [+] [√▭] [2] [▽] [4] [▷] [)] [EXE]



Therefore, $-\pi<\theta\leq\pi$, so $\theta=-\dfrac{5\pi}{12}$

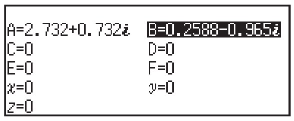
$$\boldsymbol{r=1, \theta=-\frac{5\pi}{12}}$$

check

Use the VARIABLE function in the scientific calculator to confirm the values of the absolute value r and the deflection angle θ.

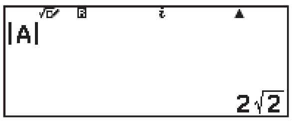
In the VARIABLE screen, input [A=$(1-i)(1+\sqrt{3}i)$ and B=$\dfrac{\sqrt{3}-i}{\sqrt{2}+\sqrt{2}i}$].

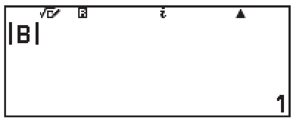
[VARIABLE] [(] [1] [−] [SHIFT] [9] [)] [(] [1] [+] [√▭] [3] [▷] [SHIFT] [9] [)] [EXE]
[▷] [▭/▭] [√▭] [3] [▷] [−] [SHIFT] [9] [▽] [√▭] [2] [▷] [+] [√▭] [2] [▷] [SHIFT] [9] [EXE] [↩]



Calculate the absolute value r.

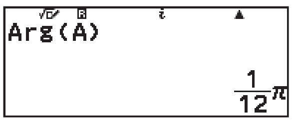
[CATALOG] [▽] [▽] [▽] [OK] [OK] [SHIFT] [4] [EXE]   [↩] [◁] [DEL] [SHIFT] [5] [EXE]

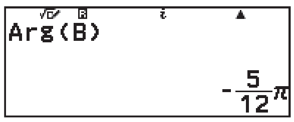




Calculate the value of the deflection angle θ.

Press [CATALOG], select [Complex], press [OK], select [Argument], press [OK], [SHIFT] [4] [)] [EXE]   [↩] [◁] [DEL] [SHIFT] [5] [EXE]





PRACTICE

1 When the deflection angle θ is $(0 \leq \theta < 2\pi)$, express the following complex numbers in polar form. However, you can use a scientific calculator to find the deflection angle in (4).

(1) $-\sqrt{3}-i$

From $r=|-\sqrt{3}-i|=\sqrt{(-\sqrt{3})^2+(-1)^2}=\sqrt{4}=2, \cos\theta=-\frac{\sqrt{3}}{2}, \sin\theta=-\frac{1}{2}$,

$0 \leq \theta < 2\pi$, **so** $\theta=\frac{7\pi}{6}$

Therefore, $-\sqrt{3}-i=2\left(\cos\frac{7\pi}{6}+i\sin\frac{7\pi}{6}\right)$

$$2\left(\cos\frac{7\pi}{6}+i\sin\frac{7\pi}{6}\right)$$

(2) $5-5i$

From

$r=|5-5i|=\sqrt{5^2+(-5)^2}=\sqrt{50}=5\sqrt{2}, \cos\theta=\frac{5}{5\sqrt{2}}=\frac{1}{\sqrt{2}}, \sin\theta=-\frac{5}{5\sqrt{2}}=-\frac{1}{\sqrt{2}}$,

$0 \leq \theta < 2\pi$, **so** $\theta=\frac{7\pi}{4}$

Therefore, $5-5i=5\sqrt{2}\left(\cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4}\right)$

$$5\sqrt{2}\left(\cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4}\right)$$

(3) $\frac{1+\sqrt{3}i}{6}$

Convert $\frac{1+\sqrt{3}i}{6}=\frac{1}{6}(1+\sqrt{3}i)$**, then for simplicity, change** $1+\sqrt{3}i$ **to polar form.**

For $1+\sqrt{3}i$**, so** $r=|1+\sqrt{3}i|=\sqrt{1^2+(\sqrt{3})^2}=\sqrt{4}=2$

From $\cos\theta=\frac{1}{2}, \sin\theta=\frac{\sqrt{3}}{2}$, $0 \leq \theta < 2\pi$, **so** $\theta=\frac{\pi}{3}$

Therefore, $\frac{1}{6}(1+\sqrt{3}i)=\frac{1}{6}\cdot 2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)=\frac{1}{3}\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$

$$\frac{1}{3}\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$$

(4) $3+4i$

$r=|3+4i|=\sqrt{3^2+4^2}=\sqrt{25}=5, \cos\theta=\frac{3}{5}, \sin\theta=\frac{4}{5}$

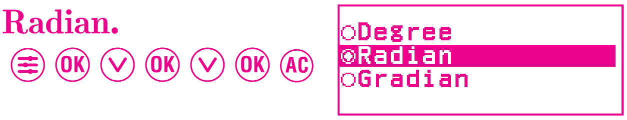
Therefore, $3+4i=5(\cos\theta+i\sin\theta)$, however θ is the angle that satisfies $\cos\theta=\frac{3}{5}, \sin\theta=\frac{4}{5}$.

Now, use the scientific calculator to calculate the value of θ and display it in Degree,

Therefore, $\theta \simeq 53.13°$

$5(\cos\theta+i\sin\theta)$ However, $\theta \simeq 53.13°$

After the calculation, return the setting to Radian.



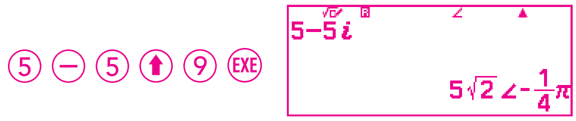
check

Change the i(a+bi display) at the top of the calculator screen to the $\angle$(polar form display).

(1) Change $-\sqrt{3}-i$ to polar form.

$2\angle-\frac{5\pi}{6}$ is the same value as $2\angle\frac{7\pi}{6}$.

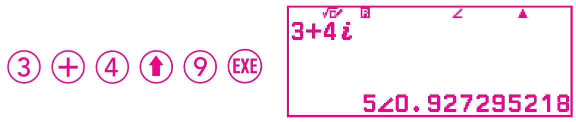
(2) Change $5-5i$ to polar form.



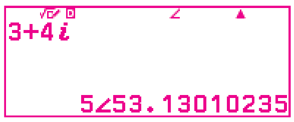
$5\sqrt{2}\angle-\frac{\pi}{4}$ is the same value as $5\sqrt{2}\angle\frac{7\pi}{4}$.

(3) Change $\frac{1+\sqrt{3}i}{6}$ to polar form.

(4) Change $3+4i$ to polar form.



0.927295218(rad) is the same value as 53.13...°.



2 Show the polar form of the following complex numbers in the $a+bi$ form.

(1) $\sqrt{11}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)$

$$=\sqrt{11}\left(-\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i\right)=-\frac{\sqrt{22}}{2}+\frac{\sqrt{22}}{2}i$$

$$-\frac{\sqrt{22}}{2}+\frac{\sqrt{22}}{2}i$$

(2) $\sqrt{17}\left(\cos\frac{11\pi}{6}+i\sin\frac{11\pi}{6}\right)$

$$=\sqrt{17}\left(\frac{\sqrt{3}}{2}-\frac{1}{2}i\right)=\frac{\sqrt{51}}{2}-\frac{\sqrt{17}}{2}i$$

$$\frac{\sqrt{51}}{2}-\frac{\sqrt{17}}{2}i$$

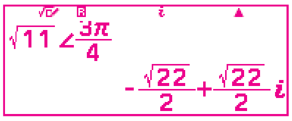
check

Change the ∠(polar form display) at the top of the calculator screen to the i(a+bi display).

[≡] [OK] [∨] [∨] [∨] [∨] [∨] [OK] [OK] [AC]

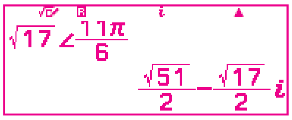
(1) Change $\sqrt{11}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)$ to the $a+bi$ form.

[√■] [1] [1] [>] [📖] [OK] [∨] [OK] [3] [⬆] [7] [▭/▭] [4] [EXE]



(2) Change $\sqrt{17}\left(\cos\frac{11\pi}{6}+i\sin\frac{11\pi}{6}\right)$ to the $a+bi$ form.

[√■] [1] [7] [>] [📖] [OK] [∨] [OK] [1] [1] [⬆] [7] [▭/▭] [6] [EXE]



3 Find the absolute value r and the deflection angle $\theta(-\pi<\theta\leq\pi)$ of the following complex numbers.

(1) $z=i(1-\sqrt{3}i)$

$z=i(1-\sqrt{3}i)=\sqrt{3}+i$

$r=\sqrt{(\sqrt{3})^2+1^2}=\sqrt{4}=2$

Also, by finding the angle θ that satisfies $\cos\theta=\dfrac{\sqrt{3}}{2}, \sin\theta=\dfrac{1}{2}$, we get

$-\pi<\theta\leq\pi$, so $\theta=\dfrac{\pi}{6}$

$$r=2, \theta=\frac{\pi}{6}$$

(2) $z=-3$

$r=\sqrt{(-3)^2}=3$

Furthermore, finding the angle θ that satisfies $\cos\theta=\dfrac{-3}{3}=-1, \sin\theta=0$, we get

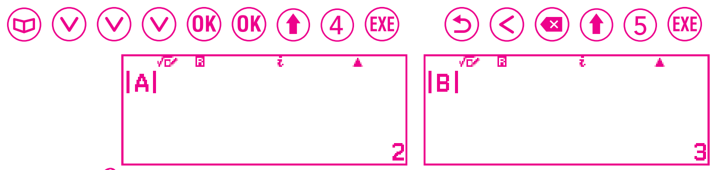
$-\pi<\theta\leq\pi$, so $\theta=\pi$

$$r=3, \theta=\pi$$

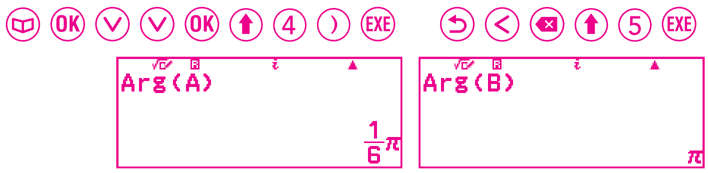
check

In the VARIABLE screen, input [A=$\sqrt{3}+i$ and B=-3].

Calculate the absolute value r.



Calculate the value of the deflection angle θ.



Multiplying & dividing polar forms

TARGET To understand about multiplying and dividing the polar forms of complex numbers.

STUDY GUIDE

Formulas for multiplying and dividing polar forms

When $z_1 = r_1(\cos\theta_1 + i\sin\theta_1), z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

(1) **Multiplying polar forms**

$$z_1 z_2 = r_1 r_2\{\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\}$$

$$|z_1 z_2| = |z_1||z_2| = r_1 r_2, \arg z_1 z_2 = \arg z_1 + \arg z_2$$

(2) **Dividing polarforms**

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\{\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)\}$$

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} = \frac{r_1}{r_2}, \arg\frac{z_1}{z_2} = \arg z_1 - \arg z_2$$

explanation

(1)
$$\begin{aligned} z_1 z_2 &= r_1 r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\ &= r_1 r_2(\cos\theta_1\cos\theta_2 + i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 + i^2\sin\theta_1\sin\theta_2) \\ &= r_1 r_2\{(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)\} \\ &= r_1 r_2\{\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\} \end{aligned}$$

(2)
$$\begin{aligned} \frac{z_1}{z_2} &= \frac{r_1}{r_2}\cdot\frac{\cos\theta_1 + i\sin\theta_1}{\cos\theta_2 + i\sin\theta_2} = \frac{r_1}{r_2}\cdot\frac{(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 - i\sin\theta_2)}{(\cos\theta_2 + i\sin\theta_2)(\cos\theta_2 - i\sin\theta_2)} \\ &= \frac{r_1}{r_2}\cdot\frac{\cos\theta_1\cos\theta_2 - i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 - i^2\sin\theta_1\sin\theta_2}{\cos\theta_2\cos\theta_2 - i\cos\theta_2\sin\theta_2 + i\sin\theta_2\cos\theta_2 - i^2\sin\theta_2\sin\theta_2} \\ &= \frac{r_1}{r_2}\cdot\frac{(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)}{\cos^2\theta_2 + \sin^2\theta_2} \\ &= \frac{r_1}{r_2}\cdot\frac{\{\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)\}}{1} \\ &= \frac{r_1}{r_2}\{\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)\} \end{aligned}$$

EXTRA Info.

Trigonometric addition theorem

Trigonometric addition theorem

$\sin(\theta_1+\theta_2) = \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2$

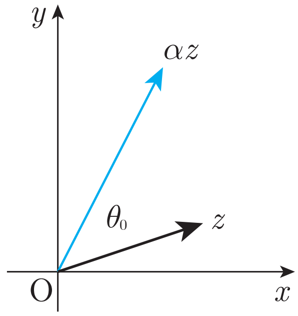
$\cos(\theta_1+\theta_2) = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2$

$\sin(\theta_1-\theta_2) = \sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2$

$\cos(\theta_1-\theta_2) = \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2$

Visualizing complex number multiplication

Given $\alpha = r_0(\cos\theta_0 + i\sin\theta_0)$, the point represented by αz is the **point z rotated around the origin by θ_0 and multiplied by r_0, the distance from the origin.**



Ex. Given $\alpha = \sqrt{2}\left(\cos\dfrac{\pi}{4} + i\sin\dfrac{\pi}{4}\right)$ and point $z(\sqrt{3},1)$ ($|z| = 2, \arg z = \dfrac{\pi}{6}$), then αz represents the point z rotated by $\dfrac{\pi}{4}$ around the origin and multiplied by $\sqrt{2}$ times the distance from the origin.

EXTRA Info.

Use the scientific calculator to multiply and divide polar forms.

EXERCISE

1 When $z_1 = \sqrt{5}\left(\cos\dfrac{\pi}{7} + i\sin\dfrac{\pi}{7}\right)$ and $z_2 = 3\left(\cos\dfrac{\pi}{5} + i\sin\dfrac{\pi}{5}\right)$, express the following complex numbers in polar form.

However, the deflection angle θ is $-\pi<\theta\le\pi$.

(1) $z_1 z_2$

$$= \sqrt{5}\cdot 3\left\{\cos\left(\frac{\pi}{7}+\frac{\pi}{5}\right) + i\sin\left(\frac{\pi}{7}+\frac{\pi}{5}\right)\right\} = 3\sqrt{5}\left(\cos\frac{12\pi}{35} + i\sin\frac{12\pi}{35}\right)$$

$$\mathbf{3\sqrt{5}\left(\cos\frac{12\pi}{35} + i\sin\frac{12\pi}{35}\right)}$$

(2) $\dfrac{z_1}{z_2}$

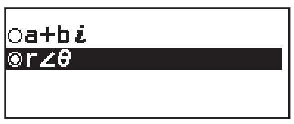
$$= \frac{\sqrt{5}}{3}\left\{\cos\left(\frac{\pi}{7}-\frac{\pi}{5}\right) + i\sin\left(\frac{\pi}{7}-\frac{\pi}{5}\right)\right\} = \frac{\sqrt{5}}{3}\left\{\cos\left(-\frac{2\pi}{35}\right) + i\sin\left(-\frac{2\pi}{35}\right)\right\}$$

$$\mathbf{\frac{\sqrt{5}}{3}\left\{\cos\left(-\frac{2\pi}{35}\right) + i\sin\left(-\frac{2\pi}{35}\right)\right\}}$$

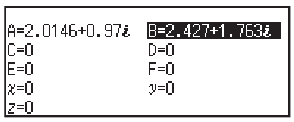
check

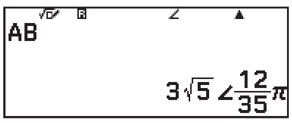
Press ⓐ, select [Complex], press OK

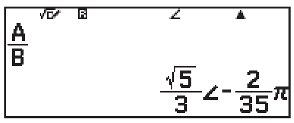
Change the i(a+bi display) at the top of the calculator screen to the ∠(polar form display).



In the VARIABLE screen, input [$A = \sqrt{5}\angle\dfrac{\pi}{7}, B = 3\angle\dfrac{\pi}{5}$].







[2] When $\theta = \dfrac{\pi}{18}$, find the value of $\dfrac{(\cos 8\theta + i\sin 8\theta)(\cos 3\theta + i\sin 3\theta)}{\cos 5\theta + i\sin 5\theta}$.

$$\frac{(\cos 8\theta + i\sin 8\theta)(\cos 3\theta + i\sin 3\theta)}{\cos 5\theta + i\sin 5\theta} = \frac{\cos(8\theta + 3\theta) + i\sin(8\theta + 3\theta)}{\cos 5\theta + i\sin 5\theta} = \frac{\cos 11\theta + i\sin 11\theta}{\cos 5\theta + i\sin 5\theta}$$

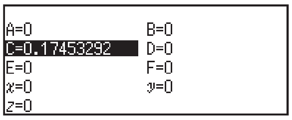
$$= \cos(11\theta - 5\theta) + i\sin(11\theta - 5\theta) = \cos 6\theta + i\sin 6\theta = \cos\left(6\cdot\frac{\pi}{18}\right) + i\sin\left(6\cdot\frac{\pi}{18}\right)$$

$$= \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$$

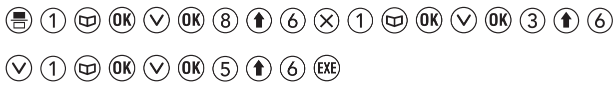
$$\mathbf{\frac{1}{2} + \frac{\sqrt{3}}{2}i}$$

check

In the VARIABLE screen, input $[C = \dfrac{\pi}{18}]$.



Calculate the value of $\dfrac{1\angle 8C \times 1\angle 3C}{1\angle 5C}$.

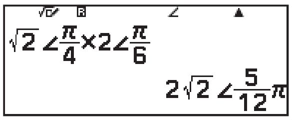


[3] Given $\alpha = \sqrt{2}\left(\cos\dfrac{\pi}{4} + i\sin\dfrac{\pi}{4}\right)$ and point $z(\sqrt{3}, 1)$, find the polar form of the complex number αz. Also, use the scientific calculator to display the polar form that you found in the $a+bi$ form.

$z = \sqrt{3} + i = 2\left(\dfrac{\sqrt{3}}{2} + \dfrac{1}{2}i\right) = 2\left(\cos\dfrac{\pi}{6} + i\sin\dfrac{\pi}{6}\right)$ is displayed, so

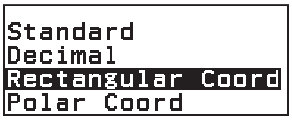
$$\alpha z = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\cdot 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = 2\sqrt{2}\left\{\cos\left(\frac{\pi}{4} + \frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{4} + \frac{\pi}{6}\right)\right\} = 2\sqrt{2}\left(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right)$$

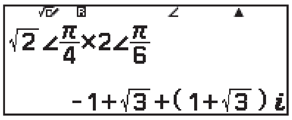
Calculate the values of $\sqrt{2}\angle\dfrac{\pi}{4} \times 2\angle\dfrac{\pi}{6}$ as a polar form.



Change the calculated value to the a+bi form.

Press FORMAT, select [Rectangular Coord], press OK





Therefore, the polar form is $2\sqrt{2}\left(\cos\dfrac{5\pi}{12} + i\sin\dfrac{5\pi}{12}\right)$ and the a+bi form is $-1 + \sqrt{3} + (1 + \sqrt{3})i$

$$\mathbf{2\sqrt{2}\left(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right), -1 + \sqrt{3} + (1 + \sqrt{3})i}$$

PRACTICE

1 When $z_1 = 4\left(\cos\frac{\pi}{9} + i\sin\frac{\pi}{9}\right)$ and $z_2 = \sqrt{7}\left(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}\right)$, express the following complex numbers in polar form.

However, the deflection angle θ is $-\pi<\theta\leq\pi$.

(1) z_1z_2

$$z_1z_2 = 4\cdot\sqrt{7}\left\{\cos\left(\frac{\pi}{9}+\frac{\pi}{8}\right)+i\sin\left(\frac{\pi}{9}+\frac{\pi}{8}\right)\right\} = 4\sqrt{7}\left(\cos\frac{17\pi}{72}+i\sin\frac{17\pi}{72}\right)$$

$$4\sqrt{7}\left(\cos\frac{17\pi}{72}+i\sin\frac{17\pi}{72}\right)$$

(2) $\frac{z_1}{z_2}$

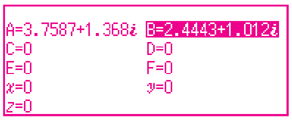
$$\frac{z_1}{z_2} = \frac{4}{\sqrt{7}}\left\{\cos\left(\frac{\pi}{9}-\frac{\pi}{8}\right)+i\sin\left(\frac{\pi}{9}-\frac{\pi}{8}\right)\right\} = \frac{4}{\sqrt{7}}\left\{\cos\left(-\frac{\pi}{72}\right)+i\sin\left(-\frac{\pi}{72}\right)\right\}$$

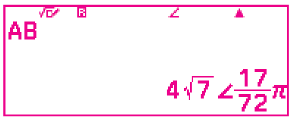
$$= \frac{4\sqrt{7}}{7}\left\{\cos\left(-\frac{\pi}{72}\right)+i\sin\left(-\frac{\pi}{72}\right)\right\}$$

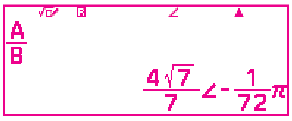
$$\frac{4\sqrt{7}}{7}\left\{\cos\left(-\frac{\pi}{72}\right)+i\sin\left(-\frac{\pi}{72}\right)\right\}$$

check

In the VARIABLE screen, input $[\mathrm{A} = 4\angle\frac{\pi}{9}, \mathrm{B} = \sqrt{7}\angle\frac{\pi}{8}]$.







2 When $\theta = \frac{\pi}{16}$, find the value of $\frac{(\cos 5\theta + i\sin 5\theta)(\cos 2\theta + i\sin 2\theta)}{\cos 3\theta + i\sin 3\theta}$.

$$\frac{(\cos 5\theta + i\sin 5\theta)(\cos 2\theta + i\sin 2\theta)}{\cos 3\theta + i\sin 3\theta} = \frac{\cos(5\theta + 2\theta) + i\sin(5\theta + 2\theta)}{\cos 3\theta + i\sin 3\theta}$$

$$= \frac{\cos 7\theta + i\sin 7\theta}{\cos 3\theta + i\sin 3\theta} = \cos(7\theta - 3\theta) + i\sin(7\theta - 3\theta)$$

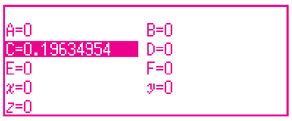
$$= \cos 4\theta + i\sin 4\theta = \cos\left(4\cdot\frac{\pi}{16}\right) + i\sin\left(4\cdot\frac{\pi}{16}\right)$$

$$= \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$$

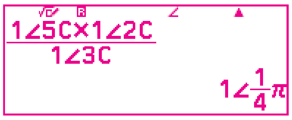
$$\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$$

check

In the VARIABLE screen, input $[\mathrm{C} = \frac{\pi}{16}]$.



Calculate the value of $\frac{1\angle 5\mathrm{C} \times 1\angle 2\mathrm{C}}{1\angle 3\mathrm{C}}$.



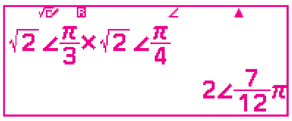
3 Given $\alpha = \sqrt{2}\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$ and point $z(1,1)$, find the polar form of the complex number αz. Also, use the scientific calculator to display the polar form that you found in the $a+bi$ form.

$z = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$ **is displayed, so**

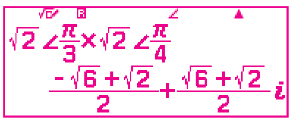
$$\alpha z = \sqrt{2}\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\cdot\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = 2\left\{\cos\left(\frac{\pi}{3} + \frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{3} + \frac{\pi}{4}\right)\right\}$$

$$= 2\left(\cos\frac{7\pi}{12} + i\sin\frac{7\pi}{12}\right)$$

Calculate the values of $\sqrt{2}\angle\frac{\pi}{3} \times \sqrt{2}\angle\frac{\pi}{4}$ **as a polar form.**



Change the calculated value to the a+bi form.



Therefore, the polar form is $2\left(\cos\frac{7\pi}{12} + i\sin\frac{7\pi}{12}\right)$ **and the a+bi form is** $\frac{-\sqrt{6}+\sqrt{2}}{2} + \frac{\sqrt{6}+\sqrt{2}}{2}i$

$$2\left(\cos\frac{7\pi}{12} + i\sin\frac{7\pi}{12}\right), \frac{-\sqrt{6}+\sqrt{2}}{2} + \frac{\sqrt{6}+\sqrt{2}}{2}i$$

De Moivre's theorem

TARGET To learn about De Moivre's theorem.

STUDY GUIDE

De Moivre's theorem, shown below, holds for any integer n and any real number θ.

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

explanation

(1) When $n=0$

We can derive that the left side $=(\cos\theta + i\sin\theta)^0 = 1$ and the right side $=\cos 0 + i\sin 0 = 1$.

(2) When n is a positive integer

(i) When $n=1$

We can derive that the left side $=(\cos\theta + i\sin\theta)^1 = \cos\theta + i\sin\theta$ and the right side $=\cos\theta + i\sin\theta$.

(ii) When $n=k(k\geq 1)$

We can derive that $(\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta$.

This means that

$$\begin{aligned}(\cos\theta + i\sin\theta)^{k+1} &= (\cos\theta + i\sin\theta)^k(\cos\theta + i\sin\theta) = (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta)\\ &= (\cos k\theta\cos\theta - \sin k\theta\sin\theta) + i(\sin k\theta\cos\theta + \cos k\theta\sin\theta)\\ &= \cos(k+1)\theta + i\sin(k+1)\theta\end{aligned}$$

We can show that this expression holds when $n=k+1$.

From (i) and (ii), this holds when n is a positive integer.

(3) When n is a negative integer

Given $n=-m$ (m is a positive integer),

$$(\cos\theta + i\sin\theta)^n = (\cos\theta + i\sin\theta)^{-m} = \frac{1}{(\cos\theta + i\sin\theta)^m} = \frac{1}{\cos m\theta + i\sin m\theta}$$

$$= \frac{1}{\cos m\theta + i\sin m\theta}\cdot\frac{\cos m\theta - i\sin m\theta}{\cos m\theta - i\sin m\theta} = \frac{\cos m\theta - i\sin m\theta}{(\cos m\theta)^2 + (\sin m\theta)^2}$$

$$= \cos m\theta - i\sin m\theta = \cos(-m)\theta + i\sin(-m)\theta = \cos n\theta + i\sin n\theta$$

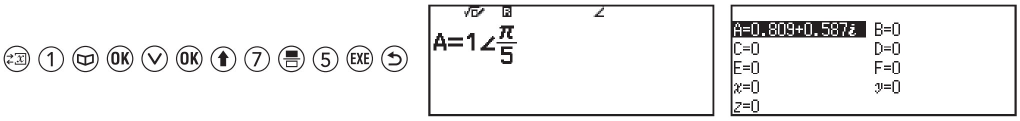
Therefore, this holds when n is a negative integer.

Ex. $(\cos\theta + i\sin\theta)^{13} = \cos 13\theta + i\sin 13\theta$

check

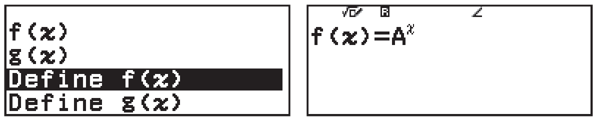
Confirm $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ is derived for $\theta = \frac{\pi}{5}$ when n=2, 3, 4, and 5.

In the VARIABLE screen, input [$\mathrm{A} = 1\angle\frac{\pi}{5}$].



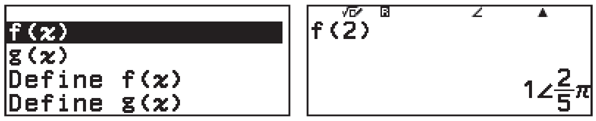
Register $\mathrm{f}(x)=\mathrm{A}^x$.

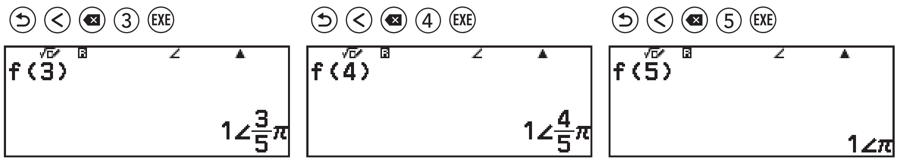
Press (f(x)), select [Define f(x)], press (OK), (⇧) (4) (■) (x) (EXE)



Confirm for the values f(2), f(3), f(4), and f(5).

Press (f(x)), select [f(x)], press (OK), (2) ()) (EXE)





From the above results, we can confirm that $f(x) = (\cos\theta + i\sin\theta)^x = \cos x\theta + i\sin x\theta$ is derived for $\theta = \frac{\pi}{5}$ when x=2, 3, 4, and 5.

EXTRA Info.

Use the scientific calculator to confirm the results of calculations using De Moivre's theorem.

EXERCISE

◆ Show the following complex numbers in the $a+bi$ form.

(1) $\left(\cos\frac{\pi}{12}+i\sin\frac{\pi}{12}\right)^4$

$$\left(\cos\frac{\pi}{12}+i\sin\frac{\pi}{12}\right)^4=\cos 4\cdot\frac{\pi}{12}+i\sin 4\cdot\frac{\pi}{12}=\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}=\frac{1}{2}+\frac{\sqrt{3}}{2}i$$

$\mathbf{\frac{1}{2}+\frac{\sqrt{3}}{2}i}$

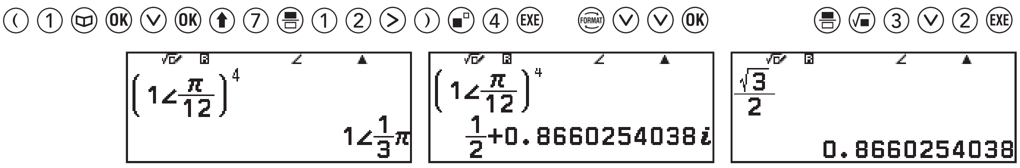
(2) $\left(\cos\frac{\pi}{8}+i\sin\frac{\pi}{8}\right)^{10}$

$$\left(\cos\frac{\pi}{8}+i\sin\frac{\pi}{8}\right)^{10}=\cos 10\cdot\frac{\pi}{8}+i\sin 10\cdot\frac{\pi}{8}=\cos\frac{5\pi}{4}+i\sin\frac{5\pi}{4}=-\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i$$

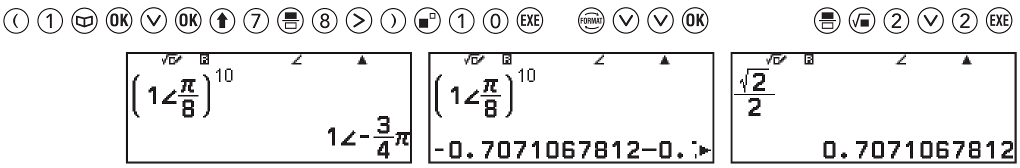
$\mathbf{-\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i}$

check

(1) Confirm that $\left(\cos\frac{\pi}{12}+i\sin\frac{\pi}{12}\right)^4 \Rightarrow \left(1\angle\frac{\pi}{12}\right)^4=1\angle\frac{\pi}{3}\Rightarrow\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}=\frac{1}{2}+\frac{\sqrt{3}}{2}i$.



(2) Confirm that $\left(\cos\frac{\pi}{8}+i\sin\frac{\pi}{8}\right)^{10} \Rightarrow \left(1\angle\frac{\pi}{8}\right)^{10}=1\angle-\frac{3\pi}{4}\Rightarrow\cos\left(-\frac{3\pi}{4}\right)+i\sin\left(-\frac{3\pi}{4}\right)=-\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i$.



PRACTICE

◆ Show the following complex numbers in the $a+bi$ form.

(1) $\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)^{13}$

$$\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)^{13}=\cos 13\cdot\frac{\pi}{6}+i\sin 13\cdot\frac{\pi}{6}=\cos\frac{13\pi}{6}+i\sin\frac{13\pi}{6}=\frac{\sqrt{3}}{2}+\frac{1}{2}i$$

$$\frac{\sqrt{3}}{2}+\frac{1}{2}i$$

(2) $\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)^{15}$

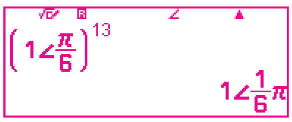
$$\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)^{15}=\cos 15\cdot\frac{\pi}{4}+i\sin 15\cdot\frac{\pi}{4}=\cos\frac{15\pi}{4}+i\sin\frac{15\pi}{4}=\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i$$

$$\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i$$

check

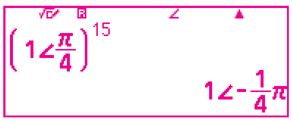
(1) Confirm that

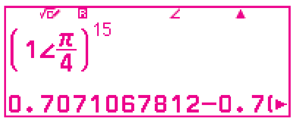
$$\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)^{13}\Rightarrow\left(1\angle\frac{\pi}{6}\right)^{13}=1\angle\frac{\pi}{6}\Rightarrow\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}=\frac{\sqrt{3}}{2}+\frac{1}{2}i.$$



(2) Confirm that $\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)^{15}\Rightarrow\left(1\angle\frac{\pi}{4}\right)^{15}=1\angle-\frac{\pi}{4}$

$$\Rightarrow\cos\left(-\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right)=\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i.$$





The n-th root of complex numbers

TARGET To understand the n-th root of complex numbers.

STUDY GUIDE

For a complex number α and natural number n, the complex number z that satisfies the equation $z^n = \alpha$ is called the n-th root of α.

Specifically, when $\alpha=1$, z is called the **n-th root of 1.**

EXTRA Info.

Use the scientific calculator to calculate the value of the n-th root of 1.

EXERCISE

◆ Find the 6th root of 1.

Since the solution to the 6th root of 1 $\Leftrightarrow$ $z^6 = 1$, find a complex number solution z that satisfies $z^6 = 1$.

Let the complex number z that satisfies the point of the question be $z = r(\cos\theta + i\sin\theta)$ $(r>0, 0\leq\theta<2\pi)$, such that

$z^6 = \{r(\cos\theta + i\sin\theta)\}^6 = r^6(\cos\theta + i\sin\theta)^6 = r^6(\cos 6\theta + i\sin 6\theta)$

And, it is expressed as $1 = 1(\cos 0 + i\sin 0)$, so $r^6(\cos 6\theta + i\sin 6\theta) = 1(\cos 0 + i\sin 0)$

By comparing the absolute value and the deflection angle of both sides, from $r^6 = 1$ and $r>0$, we get $r=1$

From $6\theta=2\pi\times k$ (k is an integer), we get $\theta = \frac{\pi}{3}\times k$ Since $0\leq\theta<2\pi$, that is to say $0\leq\frac{\pi}{3}\times k<2\pi$, then the values of k that satisfy this are $k=0, 1, 2, 3, 4$, and 5

Therefore, the solution to $z^6 = 1$ is $z = 1\cdot\left\{\cos\left(\frac{\pi}{3}\times k\right) + i\sin\left(\frac{\pi}{3}\times k\right)\right\}$ $(k=0, 1, 2, 3, 4, 5)$

When $k=0$, $z = 1\cdot(\cos 0 + i\sin 0) = 1$

When $k=1$, $z = 1\cdot\left\{\cos\left(\frac{\pi}{3}\times 1\right) + i\sin\left(\frac{\pi}{3}\times 1\right)\right\} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$

When $k=2$, $z = 1\cdot\left\{\cos\left(\frac{\pi}{3}\times 2\right) + i\sin\left(\frac{\pi}{3}\times 2\right)\right\} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$

When $k=3$, $z = 1\cdot\left\{\cos\left(\frac{\pi}{3}\times 3\right) + i\sin\left(\frac{\pi}{3}\times 3\right)\right\} = -1$

When $k=4$, $z = 1\cdot\left\{\cos\left(\frac{\pi}{3}\times 4\right) + i\sin\left(\frac{\pi}{3}\times 4\right)\right\} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$

When $k=5$, $z = 1\cdot\left\{\cos\left(\frac{\pi}{3}\times 5\right) + i\sin\left(\frac{\pi}{3}\times 5\right)\right\} = \frac{1}{2} - \frac{\sqrt{3}}{2}i$

$$\boldsymbol{z = 1, \frac{1}{2} + \frac{\sqrt{3}}{2}i, -\frac{1}{2} + \frac{\sqrt{3}}{2}i, -1, -\frac{1}{2} - \frac{\sqrt{3}}{2}i, \frac{1}{2} - \frac{\sqrt{3}}{2}i}$$

check

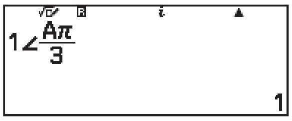
The solution of $z^6=1$, which is $z=1\cdot\left\{\cos\left(\frac{\pi}{3}\times k\right)+i\sin\left(\frac{\pi}{3}\times k\right)\right\}$, becomes $1\angle\frac{\mathrm{A}\pi}{3}$ $(\mathrm{A}=0,1,2,3,4,5)$ in the scientific calculator.

Change the $\angle$(polar form display) at the top of the calculator screen to the i (a+bi display).

[MENU] [OK] [∨] [∨] [∨] [∨] [∨] [OK] [OK] [AC]

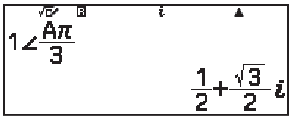
After inputting $1\angle\frac{\mathrm{A}\pi}{3}$, in the VARIABLE screen, input [A=0].

[1] [CATALOG] [OK] [∨] [OK] [SHIFT] [4] [SHIFT] [7] [FRACTION] [3] [VARIABLE] [0] [EXE] [BACK] [EXE]



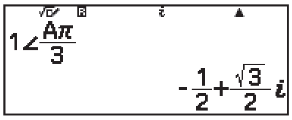
Input [A=1].

[VARIABLE] [1] [EXE] [BACK] [EXE]



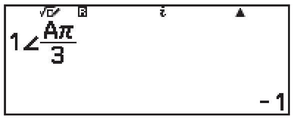
Input [A=2].

[VARIABLE] [2] [EXE] [BACK] [EXE]



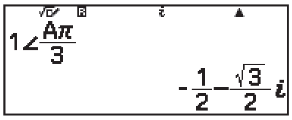
Input [A=3].

[VARIABLE] [3] [EXE] [BACK] [EXE]



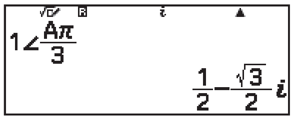
Input [A=4].

[VARIABLE] [4] [EXE] [BACK] [EXE]



Input [A=5].

[VARIABLE] [5] [EXE] [BACK] [EXE]

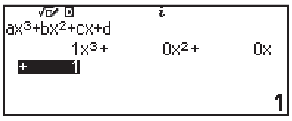


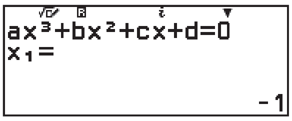
OTHER METHODS

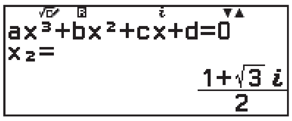
Since $z^6=1\Leftrightarrow z^6-1=0\Leftrightarrow(z^3+1)(z^3-1)=0$, simply find the solutions to $z^3+1=0$ and $z^3-1=0$.

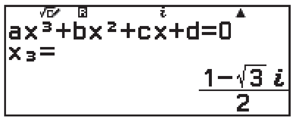
Press [HOME], select [Equation], press [OK], select [Polynomial], press [OK], select [ax³+bx²+cx+d], press [OK]

[1] [EXE] [0] [EXE] [0] [EXE] [1] [EXE] [EXE] [EXE] [EXE]

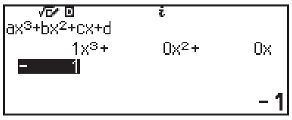


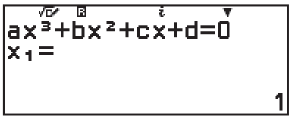


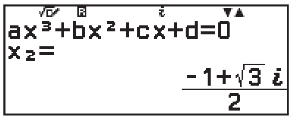


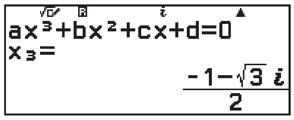


[EXE] [EXE] [>] [>] [>] [(−)] [1] [EXE] [EXE] [EXE] [EXE]









PRACTICE

1 Find the 8th root of 1.

Since the solution to the 8th root of $1 \Leftrightarrow z^8 = 1$, find a complex number solution z that satisfies $z^8 = 1$.

Let the complex number z that satisfies the point of the question be $z = r(\cos\theta + i\sin\theta)$ $(r>0, 0\leq\theta<2\pi)$, such that

$$z^8 = \{r(\cos\theta + i\sin\theta)\}^8 = r^8(\cos\theta + i\sin\theta)^8 = r^8(\cos 8\theta + i\sin 8\theta)$$

And, it is expressed as $1 = 1(\cos 0 + i\sin 0)$, so $r^8(\cos 8\theta + i\sin 8\theta) = 1(\cos 0 + i\sin 0)$

By comparing the absolute value and the deflection angle of both sides, from $r^8 = 1$ and $r>0$, we get $r=1$

From $8\theta = 2\pi \times k$ (k is an integer), we get $\theta = \frac{\pi}{4} \times k$ Since $0\leq\theta<2\pi$, that is to say $0\leq\frac{\pi}{4}\times k<2\pi$, then the values of k that satisfy this are $k=0, 1, 2, 3, 4, 5, 6$, and 7

Therefore, the solution to $z^8 = 1$ is

$$z = 1\cdot\left\{\cos\left(\frac{\pi}{4}\times k\right) + i\sin\left(\frac{\pi}{4}\times k\right)\right\} \ (k=0, 1, 2, 3, 4, 5, 6, 7)$$

When $k=0$, $z = 1\cdot(\cos 0 + i\sin 0) = 1$

When $k=1$, $z = 1\cdot\left\{\cos\left(\frac{\pi}{4}\times 1\right) + i\sin\left(\frac{\pi}{4}\times 1\right)\right\} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$

When $k=2$, $z = 1\cdot\left\{\cos\left(\frac{\pi}{4}\times 2\right) + i\sin\left(\frac{\pi}{4}\times 2\right)\right\} = i$

When $k=3$, $z = 1\cdot\left\{\cos\left(\frac{\pi}{4}\times 3\right) + i\sin\left(\frac{\pi}{4}\times 3\right)\right\} = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$

When $k=4$, $z = 1\cdot\left\{\cos\left(\frac{\pi}{4}\times 4\right) + i\sin\left(\frac{\pi}{4}\times 4\right)\right\} = -1$

When $k=5$, $z = 1\cdot\left\{\cos\left(\frac{\pi}{4}\times 5\right) + i\sin\left(\frac{\pi}{4}\times 5\right)\right\} = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$

When $k=6$, $z = 1\cdot\left\{\cos\left(\frac{\pi}{4}\times 6\right) + i\sin\left(\frac{\pi}{4}\times 6\right)\right\} = -i$

When $k=7$, $z = 1\cdot\left\{\cos\left(\frac{\pi}{4}\times 7\right) + i\sin\left(\frac{\pi}{4}\times 7\right)\right\} = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$

$$z = 1, \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i, i, -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i,$$
$$-1, -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i, -i, \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$$

check

The solution of $z^8=1$, which is $z=1\cdot\left\{\cos\left(\frac{\pi}{4}\times k\right)+i\sin\left(\frac{\pi}{4}\times k\right)\right\}$, becomes

$1\angle\frac{A\pi}{4}$ $(A=0, 1, 2, 3, 4, 5, 6, 7)$ in the scientific calculator.

After inputting $1\angle\frac{A\pi}{4}$, in the VARIABLE screen, input $[A=1$ to $7]$ respectively.

1 [∠] OK ∨ OK ⇧ 4 ⇧ 7 [÷] 4   [VAR] 0 EXE   ↺ EXE

$1\angle\frac{A\pi}{4}$ → 1

↺ [VAR] 1 EXE ↺ EXE → $\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i$

[VAR] 2 EXE ↺ EXE → i

[VAR] 3 EXE ↺ EXE → $-\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i$

[VAR] 4 EXE ↺ EXE → -1

[VAR] 5 EXE ↺ EXE → $-\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i$

[VAR] 6 EXE ↺ EXE → $-i$

[VAR] 7 EXE ↺ EXE → $\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i$

OTHER METHODS

Since $z^8=1 \Leftrightarrow z^8-1=0 \Leftrightarrow (z^4+1)(z^4-1)=0$, simply find the solutions to $z^4+1=0$ and $z^4-1=0$.

Press [HOME], select [Equation], press OK, select [Polynomial], press OK, select [$ax^4+bx^3+cx^2+dx+e$], press OK

1 EXE 0 EXE 0 EXE 0 EXE 1 EXE   EXE EXE EXE EXE

$ax^4+bx^3+\cdots+e=0$

$x_1=\frac{\sqrt{2}+\sqrt{2}i}{2}$ | $x_2=\frac{\sqrt{2}-\sqrt{2}i}{2}$ | $x_3=\frac{-\sqrt{2}+\sqrt{2}i}{2}$ | $x_4=\frac{-\sqrt{2}-\sqrt{2}i}{2}$

EXE   > > > > (−) 1 EXE   EXE EXE EXE EXE

$ax^4+bx^3+\cdots+e=0$

$x_1=1$ | $x_2=-1$ | $x_3=i$ | $x_4=-i$

2 Solve the equation $z^3 = 8i$.

Let the complex number z that satisfies the point of the question be

$z = r(\cos\theta + i\sin\theta)$ $(r>0, 0\le\theta<2\pi)$, **such that**

$$z^3 = \{r(\cos\theta + i\sin\theta)\}^3 = r^3(\cos\theta + i\sin\theta)^3 = r^3(\cos 3\theta + i\sin 3\theta)$$

And, it is expressed as $8i = 2^3\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$, **so**

$$r^3(\cos 3\theta + i\sin 3\theta) = 2^3\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$$

By comparing the absolute value and the deflection angle of both sides, from

$r^3 = 2^3$ **and** $r>0$, **we get** $r=2$

From $3\theta = \frac{\pi}{2} + 2\pi\times k$ (k **is an integer**), **we get** $\theta = \frac{\pi}{6} + \frac{2\pi}{3}\times k$

Since $0\le\theta<2\pi$, **that is to say** $0\le\frac{\pi}{6} + \frac{2\pi}{3}\times k < 2\pi$, **then the values of** k **that satisfy this are** $k=0$, 1, **and** 2

Therefore, the solution to $z^3 = 8i$ **is**

$$z = 2\cdot\left\{\cos\left(\frac{\pi}{6} + \frac{2\pi}{3}\times k\right) + i\sin\left(\frac{\pi}{6} + \frac{2\pi}{3}\times k\right)\right\} (k = 0, 1, 2)$$

When $k=0$, $z = 2\cdot\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = 2\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = \sqrt{3} + i$

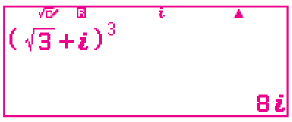
When $k=1$,

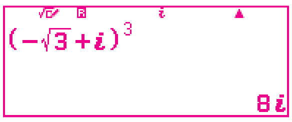
$$z = 2\cdot\left\{\cos\left(\frac{\pi}{6} + \frac{2\pi}{3}\right) + i\sin\left(\frac{\pi}{6} + \frac{2\pi}{3}\right)\right\} = 2\cdot\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right) = 2\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$$
$$= -\sqrt{3} + i$$

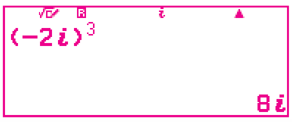
When $k=2$,

$$z = 2\cdot\left\{\cos\left(\frac{\pi}{6} + \frac{4\pi}{3}\right) + i\sin\left(\frac{\pi}{6} + \frac{4\pi}{3}\right)\right\} = 2\cdot\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right) = 2(-i) = -2i$$

$$z = \sqrt{3} + i, -\sqrt{3} + i, -2i$$







Plane figures and complex numbers

TARGET To apply complex number planes to plane figures.

STUDY GUIDE

Formulas for multiplying and dividing polar forms

The following relational expressions hold for 2 points, $\mathrm{A}(\alpha)$ and $\mathrm{B}(\beta)$, on the complex number plane.

(i) **Parallel translation of A (α) by just B (β) is to point $w \Rightarrow w=\alpha+\beta$**

(ii) **Distance between 2 points A (α) and B (β) $\Rightarrow |\beta-\alpha|$**

(iii) **For 2 points A (α) and B (β), the segment AB m:n, so**

Internal dividing point $\Rightarrow \dfrac{n\alpha+m\beta}{m+n}$

External dividing point $\Rightarrow \dfrac{-n\alpha+m\beta}{m-n}$

(iv) (1) **Rotation of A (α) around the origin O by just θ is point w**

$\Rightarrow w=(\cos\theta+i\sin\theta)\alpha$

(2) **Rotation of A (α) around B (β) by just θ is point w**

$\Rightarrow w-\beta=(\cos\theta+i\sin\theta)(\alpha-\beta)$

In addition, the following relational expression holds for a point $\mathrm{C}(\gamma)$ obtained by rotating the point $\mathrm{B}(\beta)$ by θ around the point $\mathrm{A}(\alpha)$ and multiplying the distance from point A by k $(k>0)$.

(v) **From $\gamma-\alpha=k(\cos\theta+i\sin\theta)(\beta-\alpha)$,**

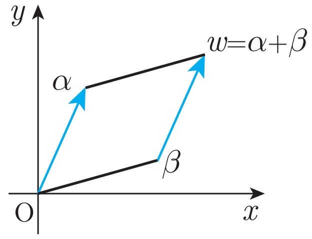
$$\frac{\gamma-\alpha}{\beta-\alpha}=k(\cos\theta+i\sin\theta)$$

$$\Rightarrow \theta=\angle\mathrm{BAC}=\arg\frac{\gamma-\alpha}{\beta-\alpha}$$

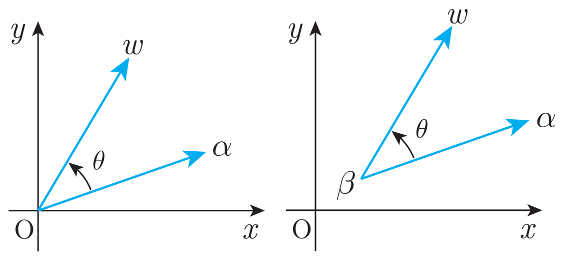
($\angle$BAC is the angle measured from the half-line AB to the half-line AC)

explanation

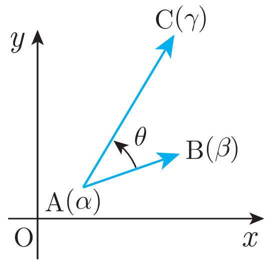
(i)



(iv)



(v)



EXTRA Info.

Use the scientific calculator to consider the relations between complex numbers and plane figures.

EXERCISE

1 Given 3 points $z=4-3i, \alpha=5+2i, \beta=1+6i$, on the complex number plane, solve each of the following problems.

(1) Find the point w, which was parallel translated to point z by just α.

$w=z+\alpha=(4-3i)+(5+2i)=9-i$

$\mathbf{9-i}$

(2) Find the distance between 2 points α and β.

$|\beta-\alpha|=|(1+6i)-(5+2i)|=|-4+4i|=\sqrt{(-4)^2+4^2}=\sqrt{32}=4\sqrt{2}$

$\mathbf{4\sqrt{2}}$

(3) For the 2 points A (α) and B (β), find the point Q that externally divides and the point P that internally divides line AB by 3:2.

Internal dividing point P is $\dfrac{2\alpha+3\beta}{3+2}=\dfrac{2}{5}\alpha+\dfrac{3}{5}\beta=\dfrac{2}{5}(5+2i)+\dfrac{3}{5}(1+6i)=\dfrac{13}{5}+\dfrac{22}{5}i$

External dividing point Q is $\dfrac{-2\alpha+3\beta}{3-2}=-2\alpha+3\beta=-2(5+2i)+3(1+6i)=-7+14i$

$\mathbf{P\cdots\dfrac{13}{5}+\dfrac{22}{5}i, Q\cdots-7+14i}$

(4) Find the point w by rotating point z by just $\dfrac{\pi}{4}$ around the origin O.

$$w=(\cos\theta+i\sin\theta)z=\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)(4-3i)=\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i\right)(4-3i)=\frac{7\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i$$

$\mathbf{\dfrac{7\sqrt{2}}{2}+\dfrac{\sqrt{2}}{2}i}$

(5) Find the point w by rotating the point z around the point z_0 $(-3-7i)$ by just $\dfrac{\pi}{6}$.

From $w-z_0=(\cos\theta+i\sin\theta)(z-z_0)$, we get $w=z_0+(\cos\theta+i\sin\theta)(z-z_0)$

Therefore, $w=(-3-7i)+\left(\cos\dfrac{\pi}{6}+i\sin\dfrac{\pi}{6}\right)\{(4-3i)-(-3-7i)\}=(-3-7i)+\left(\dfrac{\sqrt{3}}{2}+\dfrac{1}{2}i\right)(7+4i)$

$=(-3-7i)+\left(\dfrac{7\sqrt{3}-4}{2}+\dfrac{7+4\sqrt{3}}{2}i\right)=\dfrac{7\sqrt{3}-10}{2}+\dfrac{4\sqrt{3}-7}{2}i$

$\mathbf{\dfrac{7\sqrt{3}-10}{2}+\dfrac{4\sqrt{3}-7}{2}i}$

2 Given 3 points $A(\alpha)=2+2i, B(\beta)=3+4i, C(\gamma)=5+3i$ on the complex number plane, find the measure of $\angle$ BAC.

$$\angle BAC=\arg\frac{\gamma-\alpha}{\beta-\alpha}=\arg\frac{(5+3i)-(2+2i)}{(3+4i)-(2+2i)}=\arg\frac{3+i}{1+2i}=\arg\frac{3+i}{1+2i}\cdot\frac{1-2i}{1-2i}=\arg\frac{5-5i}{5}=\arg(1-i)$$

From $1-i=\sqrt{2}\left(\dfrac{1}{\sqrt{2}}-\dfrac{1}{\sqrt{2}}i\right)=\sqrt{2}\left\{\cos\left(-\dfrac{\pi}{4}\right)+i\sin\left(-\dfrac{\pi}{4}\right)\right\}$, we get $\arg(1-i)=-\dfrac{\pi}{4}$

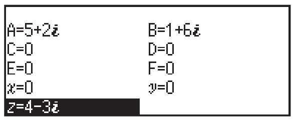
Therefore, the measure of $\angle$ BAC is $\dfrac{\pi}{4}$

$\mathbf{\dfrac{\pi}{4}}$

check

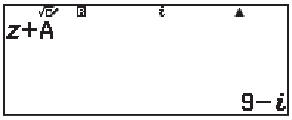
1 In the VARIABLE screen, input [$\mathrm{A}=5+2i$, $\mathrm{B}=1+6i$, and $z=4-3i$].

(VARIABLE) ⑤ ⊕ ② ⇧ ⑨ (EXE) ＞ ① ⊕ ⑥ ⇧ ⑨ (EXE) ∨ ∨ ∨ ∨ ④ ⊖ ③ ⇧ ⑨ (EXE)



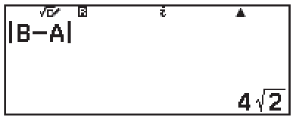
(1) Calculate the value of $z+\mathrm{A}$.

(BACK) ⇧ (×10^x) ⊕ ⇧ ④ (EXE)



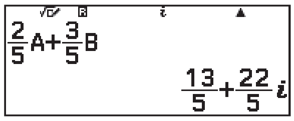
(2) Calculate the value of $|\mathrm{B}-\mathrm{A}|$.

(CATALOG) ∨ ∨ ∨ (OK) (OK) ⇧ ⑤ ⊖ ⇧ ④ (EXE)



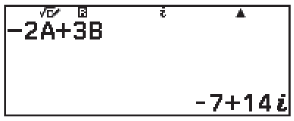
(3) Calculate the value of $\frac{2}{5}\mathrm{A}+\frac{3}{5}\mathrm{B}$.

② (FRACTION) ⑤ ＞ ⇧ ④ ⊕ ③ (FRACTION) ⑤ ＞ ⇧ ⑤ (EXE)



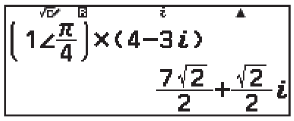
Calculate the value of $-2\mathrm{A}+3\mathrm{B}$.

⊖ ② ⇧ ④ ⊕ ③ ⇧ ⑤ (EXE)



(4) Calculate the value of $\left(1\angle\frac{\pi}{4}\right)\times(4-3i)$.

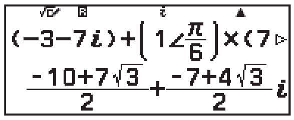
(① (CATALOG) (OK) ∨ (OK) ⇧ ⑦ (FRACTION) ④ ＞) ⊗ (④ ⊖ ③ ⇧ ⑨) (EXE)



(5) Calculate the value of $(-3-7i)+\left(1\angle\frac{\pi}{6}\right)\times(7+4i)$.

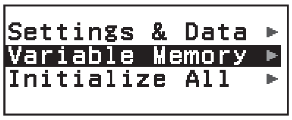
(⊖ ③ ⊖ ⑦ ⇧ ⑨)

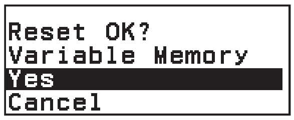
⊕ (① (CATALOG) (OK) ∨ (OK) ⇧ ⑦ (FRACTION) ⑥ ＞) ⊗ (⑦ ⊕ ④ ⇧ ⑨) (EXE)



2 Delete all the VARIABLE memory.

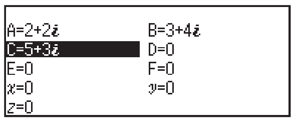
Press (MENU), select [Reset], press (OK), select [Variable Memory], press (OK), select [Yes], press (OK)





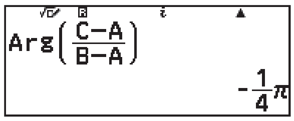
In the VARIABLE screen, input [$\mathrm{A}=2+2i$, $\mathrm{B}=3+4i$, and $\mathrm{C}=5+3i$].

(VARIABLE) ② ⊕ ② ⇧ ⑨ (EXE) ＞ ③ ⊕ ④ ⇧ ⑨ (EXE) ＞ ⑤ ⊕ ③ ⇧ ⑨ (EXE) (BACK)



Calculate the value of $\arg\frac{\mathrm{C}-\mathrm{A}}{\mathrm{B}-\mathrm{A}}$.

(CATALOG) (OK) ∨ ∨ (OK) (FRACTION) ⇧ ⑥ ⊖ ⇧ ④ ∨ ⇧ ⑤ ⊖ ⇧ ④ ＞) (EXE)



PRACTICE

1 Given 3 points $z=2-2i, \alpha=1+3i, \beta=-2+i$ on the complex number plane, solve each of the following problems.

(1) Find the point w, which was parallel translated to point z by just α.

$$w=z+\alpha=(2-2i)+(1+3i)=3+i$$

$3+i$

(2) Find the distance between 2 points α and β.

$$|\beta-\alpha|=|(-2+i)-(1+3i)|=|-3-2i|=\sqrt{(-3)^2+(-2)^2}=\sqrt{13}$$

$\sqrt{13}$

(3) For the 2 points A (α) and B (β), find the point Q that externally divides and the point P that internally divides line AB by 4:3.

Internal dividing point P is

$$\frac{3\alpha+4\beta}{4+3}=\frac{3}{7}\alpha+\frac{4}{7}\beta=\frac{3}{7}(1+3i)+\frac{4}{7}(-2+i)=-\frac{5}{7}+\frac{13}{7}i$$

External dividing point Q is

$$\frac{-3\alpha+4\beta}{4-3}=-3\alpha+4\beta=-3(1+3i)+4(-2+i)=-11-5i$$

$\mathrm{P}\cdots-\frac{5}{7}+\frac{13}{7}i, \mathrm{Q}\cdots-11-5i$

(4) Find the point w by rotating point z by just $\frac{\pi}{3}$ around the origin O.

$$w=(\cos\theta+i\sin\theta)z=\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)(2-2i)=\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)(2-2i)$$
$$=(\sqrt{3}+1)+(\sqrt{3}-1)i$$

$(\sqrt{3}+1)+(\sqrt{3}-1)i$

(5) Find the point w by rotating the point z around the point z_0 $(-3+4i)$ by just $\frac{\pi}{4}$.

From $w-z_0=(\cos\theta+i\sin\theta)(z-z_0)$**, we get** $w=z_0+(\cos\theta+i\sin\theta)(z-z_0)$

Therefore, $w=(-3+4i)+\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)\{(2-2i)-(-3+4i)\}$

$$=(-3+4i)+\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i\right)(5-6i)$$
$$=(-3+4i)+\left(\frac{11\sqrt{2}}{2}+\frac{-\sqrt{2}}{2}i\right)=\frac{11\sqrt{2}-6}{2}+\frac{8-\sqrt{2}}{2}i$$

$\frac{11\sqrt{2}-6}{2}+\frac{8-\sqrt{2}}{2}i$

2 Given 3 points $\alpha = 1 + i, \beta = \sqrt{3} + 1 + 2i, \gamma = 1 + 3i$ on the complex number plane, find the measure of $\angle \mathrm{BAC}$.

$$\frac{\gamma - \alpha}{\beta - \alpha} = \frac{(1+3i)-(1+i)}{(\sqrt{3}+1+2i)-(1+i)} = \frac{2i}{\sqrt{3}+i} = \frac{2i}{\sqrt{3}+i} \cdot \frac{\sqrt{3}-i}{\sqrt{3}-i} = \frac{2+2\sqrt{3}i}{4}$$

$$= \frac{1+\sqrt{3}i}{2} = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$$

$$\angle \mathrm{BAC} = \arg\frac{\gamma-\alpha}{\beta-\alpha} = \arg\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = \frac{\pi}{3}$$

Therefore, the measure of $\angle \mathrm{BAC}$ is $\frac{\pi}{3}$

$\frac{\pi}{3}$

check

In the VARIABLE screen, input [A=1+3i,B=−2+i, and z=2−2i].

⇄ 1 + 3 ⇧ 9 EXE > − 2 + ⇧ 9 EXE ∨ ∨ ∨ ∨ 2 − 2 ⇧ 9 EXE ↺

A=1+3i B=-2+i
C=0 D=0
E=0 F=0
x=0 y=0
z=2-2i

(1) **Calculate the value of z+A.**

⇧ ×10ˣ + ⇧ 4 EXE

z+A
3+i

(2) **Calculate the value of $|\mathrm{B} - \mathrm{A}|$.**

▭ ∨ ∨ ∨ OK OK ⇧ 5 − ⇧ 4 EXE

|B−A|
$\sqrt{13}$

(3) **Calculate the value of $\frac{3}{7}\mathrm{A} + \frac{4}{7}\mathrm{B}$.**

3 ▭ 7 > ⇧ 4 + 4 ▭ 7 > ⇧ 5 EXE

$\frac{3}{7}$A+$\frac{4}{7}$B
$-\frac{5}{7}+\frac{13}{7}i$

Calculate the value of $-3\mathrm{A} + 4\mathrm{B}$.

− 3 ⇧ 4 + 4 ⇧ 5 EXE

−3A+4B
-11−5i

(4) **Calculate the value of $\left(1\angle\frac{\pi}{3}\right) \times (2-2i)$.**

(1 ▭ OK ∨ OK ⇧ 7 ▭ 3 >) × (2 − 2 ⇧ 9) EXE

$\left(1\angle\frac{\pi}{3}\right)$×(2−2$i$)
$1+\sqrt{3}+(-1+\sqrt{3})i$

(5) **Calculate the value of $(-3+4i) + \left(1\angle\frac{\pi}{4}\right) \times (5-6i)$.**

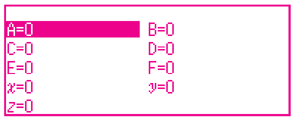
(− 3 + 4 ⇧ 9)
+ (1 ▭ OK ∨ OK ⇧ 7 ▭ 4 >) × (5 − 6 ⇧ 9) EXE

(−3+4i)+$\left(1\angle\frac{\pi}{4}\right)$×(5▹
$\frac{-6+11\sqrt{2}}{2}+\frac{8-\sqrt{2}}{2}i$

check

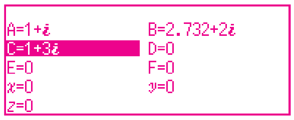
Delete all the VARIABLE memory.

(MENU) (∨) (∨) (OK) (∨) (OK) (OK)



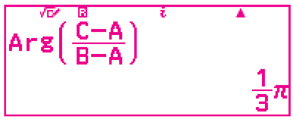
In the VARIABLE screen, input $[\mathrm{A}=1+i, \mathrm{B}=\sqrt{3}+1+2i, \mathrm{C}=1+3i]$.

(VAR) (1) (+) (SHIFT) (9) (EXE) (>) (√■) (3) (>) (+) (1) (+) (2) (SHIFT) (9) (EXE) (>) (1) (+) (3) (SHIFT) (9) (EXE) (BACK)



Calculate the value of $\arg\dfrac{\mathrm{C}-\mathrm{A}}{\mathrm{B}-\mathrm{A}}$.

(CATALOG) (OK) (∨) (∨) (OK) (FRAC) (SHIFT) (6) (−) (SHIFT) (4) (∨) (SHIFT) (5) (−) (SHIFT) (4) (>) ( )) (EXE)



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