8 Advanced Expressions and Functions

CASIO
Essential Materials

Introduction

These teaching materials were created with the hope of conveying to many teachers and students the appeal of scientific calculators.

(1) Change awareness (emphasizing the thinking process) and boost efficiency in learning mathematics

- By reducing the time spent on manual calculations, we can have learning with a focus on the thinking process that is more efficient.
- This reduces the aversion to mathematics caused by complicated calculations, and allows students to experience the joy of thinking, which is the essence of mathematics.

(2) Diversification of learning materials and problem-solving methods

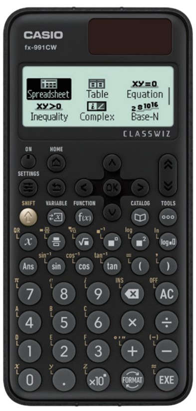
- Making it possible to do difficult calculations manually allows for diversity in learning materials and problem-solving methods.

(3) Promoting understanding of mathematical concepts

- By using the various functions of the scientific calculator in creative ways, students are able to deepen their understanding of mathematical concepts through calculations and discussions from different perspectives than before.
- This allows for exploratory learning through easy trial and error of questions.
- Listing and graphing of numerical values by means of tables allows students to discover laws and to understand visually.

Features of this book

- As well as providing first-time scientific calculator users with opportunities to learn basic scientific calculator functions from the ground up, the book also has material to show people who already use scientific calculators the appeal of scientific calculators described above.
- You can also learn about functions and techniques that are not available on conventional Casio models or other brands of scientific calculators.
- This book covers many units of high school mathematics, allowing students to learn how to use the scientific calculator as they study each topic.
- This book can be used in a variety of situations, from classroom activities to independent study and homework by students.



Better Mathematics Learning with Scientific Calculator

Structure

Quadratic functions

TARGET To understand the concept of functions.

STUDY GUIDE

Functions

Definition of a function

For 2 variables, x and y, if the value of x is determined and if just 1 corresponding value of y is determined, then we say that **y is a function of x.**

Ex. Given $x=2$ ➡ Functions ($y=x+1$) ➡ We can determine $y=3$

How to describe functions

When y is a function of x, generally y is expressed as $f(x)$ or $g(x)$. Furthermore, in the function $y=f(x)$, then when $x=a$, the value of y is expressed as $f(a)$.

Ex. In the function $f(x)=3x+3$, when $x=1$, then when $f(1)=3\times1+3=6$ and $x=5$, then we get $f(5)=3\times5+3=18$.

Linear functions & Quadratic functions

When y is expressed as a linear equation of x, y is called the **linear function** of x, which is expressed as follows.

$$\boxed{y=ax+b \quad (a\neq0)}$$

When y is expressed as a quadratic equation of x, y is called the **quadratic function** of x, which is expressed as follows.

$$\boxed{y=ax^2+bx+c \quad (a\neq0)}$$

TARGET

Students can identify the objective to learn in each section.

STUDY GUIDE

Mathematical theorems and concepts are explained in detail. A scientific calculator is used to check and derive formulas according to the topic.

EXERCISE

1 For the quadratic function $f(x)=3x^2+2x+6$, find the various following values.

(1) $f(0)$

$f(0)=3\cdot0^2+2\cdot0+6=6$

<u>6</u>

(2) $f(1)$

$f(1)=3\cdot1^2+2\cdot1+6=11$

<u>11</u>

(3) $f(-1)$

$f(-1)=3\cdot(-1)^2+2\cdot(-1)+6=7$

<u>7</u>

(4) $f(109)$

$f(109)=3\cdot109^2+2\cdot109+6=35867$

<u>35867</u>

check

Use the FUNCTION function to calculate the value of $f(x)$.

Press ⓞ, select [Calculate], press OK

Press f(x), select [Define f(x)], press OK ③ ⓧ ⓧ² ⊕ ② ⓧ ⊕ ⑥ EXE

Calculate Statistics Distribution Spreadsheet Table Equation

f(x) g(x) Define f(x) Define g(x)

f(x)=3x²+2x+6

EXERCISE

Students learn basic examples based on the explanation in Study Guide.

check

Explains how to use the scientific calculator to solve problems and check answers.

PRACTICE

1 For the quadratic function $f(x)=-2x^2+4x+5$, find the various following values.

(1) $f(0)$

$f(0)=-2\cdot0^2+4\cdot0+5=5$

5

(2) $f(2)$

$f(2)=-2\cdot2^2+4\cdot2+5=5$

5

(3) $f(-3)$

$f(-3)=-2\cdot(-3)^2+4\cdot(-3)+5=-25$

-25

(4) $f(52)$

$f(52)=-2\cdot52^2+4\cdot52+5=-5195$

-5195

PRACTICE

Students can do practice problems similar to those in EXERCISE. They can also practice using the scientific calculator as they learned to in Check.

ADVANCED

3 We know that when an object is thrown vertically upward from a height of 0 m at a velocity of v m/s, the relation between the height y m of the object and the time x seconds since it was thrown is $y=-4.9x^2+vx$. So, when a ball is thrown vertically from a height of 0 m at a velocity of 9.8 m/s, find the time it takes to reach the highest point and the height of that highest point. Note that the air resistance can be ignored.

$$y=-4.9x^2+9.8x=-4.9(x^2-2x)=-4.9\{(x-1)^2-1^2\}=-4.9(x-1)^2+4.9$$

Therefore, the graph of this function is convex upward, the axis is $x=1$, and the vertex is $(1, 4.9)$.

Thus, the ball takes 1 second to reach its highest point, and the height of that highest point is 4.9 m.

1 second, 4.9 m

check

Press ⓞ, select [Equation], press OK

Select [Polynomial], press OK, select [ax²+ bx+ c], press OK

⊖ ④ ⊙ ⑨ EXE ⑨ ⊙ ⑧ EXE ⓪ EXE EXE

Calculate Statistics Distribution Spreadsheet Table Equation

Simul Equation Polynomial Solver

ax²+bx+c ax³+bx²+cx+d ax⁴+bx³+cx²+dx+e

EXE EXE EXE **Press ⇧ ⓧ, scan the QR code to display a graph.**

Max of y=ax²+bx+c x= 1

Max of y=ax²+bx+c y= 49/10

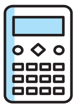
ADVANCED

Practical problems have been included in several topics. Solutions using a scientific calculator are also presented as necessary.

Other marks

Ex.	Simple examples on how to apply equations and theorems
explanation	Formulas and their supplementary explanations
proof	Proofs and checks of mathematical formulas
EXTRA Info.	Knowledge and information on formulas and other supplementary information in other units
OTHER METHODS	Alternative solutions and different verification methods for previously presented problems

Calculator mark



Where to use the scientific calculator

Colors of fonts in the teaching materials

- In STUDY GUIDE, important mathematical terms and formulas are printed in blue.
- In PRACTICE and ADVANCED the answers are printed in red.
  (Separate data is also available without the red parts, so it can be used for exercises.)

Applicable models

The applicable model is fx-991CW.
(Instructions on how to do input are for the fx-991CW, but in many cases similar calculations can be done on other models.)

Related Links

- Information and educational materials relevant to scientific calculators can be viewed on the following site.
  https://edu.casio.com
- The following video can be viewed to learn about the multiple functions of scientific calculators.
  https://www.youtube.com/playlist?list=PLRgxo9AwbIZLurUCZnrbr4cLfZdqY6aZA

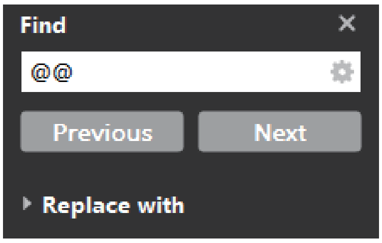
How to use PDF data

About types of data

- Data for all unit editions and data for each unit are available.
- For the above data, the PRACTICE and ADVANCED data without the answers in red is also available.

How to find where the scientific calculator is used

(1) Open a search window in the PDF Viewer.
(2) Type in "@@" as a search term.
(3) You can sequentially check where the calculator marks appear in the data.



How to search for a unit and section

(1) Search for units of data in all unit editions
- The data in all unit editions has a unit table of contents.
- Selecting a unit in the table of contents lets you jump to the first page of that unit.
- There is a bookmark on the first page of each unit, so you can jump from there also.

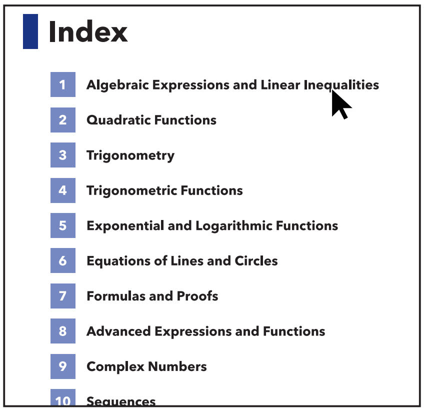
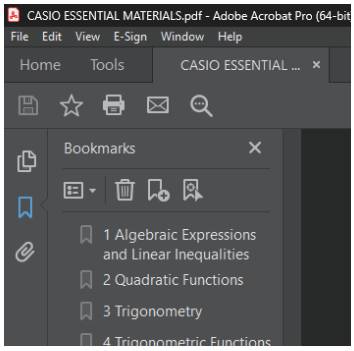


Table of contents of unit



Bookmark of unit

(2) Search for sections
- There are tables of contents for sections on the first page of units.
- Selecting a section in the table of contents takes you to the first page of that section.

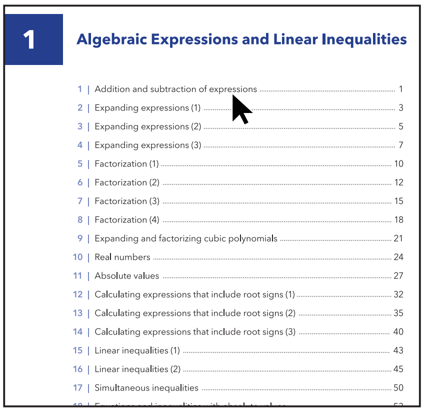


Table of contents of section

Parametrization of a curve

TARGET To understand the general forms of the curves expressed by parameters.

STUDY GUIDE

Parametrization

When the coordinates of a point (x, y) on a curve are expressed as the function of a single variable t as in $x=f(t)$ and $y=g(t)$, we say it is the **parametrization** of that curve, and we say t is the **parameter**. Furthermore, **by eliminating the parameters** from $x=f(t)$ and $y=g(t)$, we can find the equations of the graph for x and y.

Equation of a curve

$$\begin{cases} x = f(t) \\ y = g(t) \end{cases} \xrightarrow[\text{Eliminate } t]{} \text{Equation for } x \text{ and } y$$

EXERCISE

◆ Given a parameter t, find the shape of the curve expressed by the following equation.

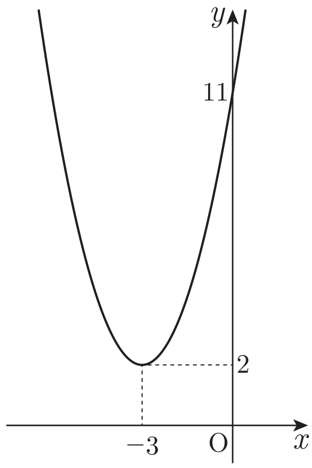
(1) $x = t-3$, $y = t^2+2$

From $x = t-3$, we can get $t = x+3$.

This is substituted into $y = t^2+2$, to give us $y = (x+3)^2+2 = x^2+6x+11$.

Therefore, the curve we find is a parabola $y = x^2+6x+11$.

Parabola $y = x^2+6x+11$



(2) $x = 3^t+3^{-t}$, $y = 3^t-3^{-t}$

Squaring both sides of these 2 equations gives us

$x^2 = 3^{2t}+2+3^{-2t}, y^2 = 3^{2t}-2+3^{-2t}$.

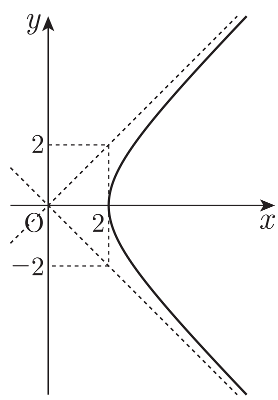
Therefore, $x^2-y^2 = 4$.

Here $3^t > 0, 3^{-t} > 0$, so,

from the relation of the arithmetic mean to the geometric mean, we get

$x = 3^t+3^{-t} \geq 2\sqrt{3^t \cdot 3^{-t}} = 2$.

Therefore, the curve we find is a hyperbola $x^2-y^2 = 4\,(x \geq 2)$.

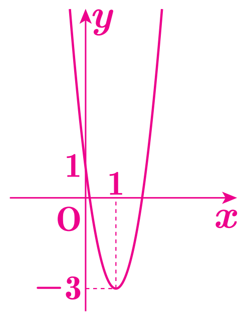


Hyperbola $x^2-y^2 = 4\,(x \geq 2)$

PRACTICE

◆ Given a parameter t, find the shape of the curve expressed by the following equation.

(1) $x=\frac{1}{2}t+1, y=t^2-3$



From $x=\frac{1}{2}t+1$, we get $t=2(x-1)$.

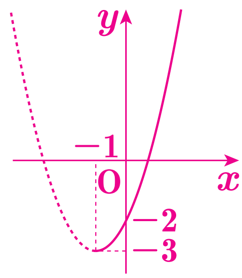
This is substituted into $y=t^2-3$,

to give us $y=\{2(x-1)\}^2-3=4x^2-8x+1$.

Therefore, the curve we find is a parabola $y=4x^2-8x+1$.

Parabola $y=4x^2-8x+1$

(2) $x=2\sqrt{t}-1, y=4t-3$



From $x=2\sqrt{t}-1$, we get $\sqrt{t}=\frac{1}{2}(x+1), t=\frac{1}{4}(x+1)^2$.

This is substituted into $y=4t-3$,

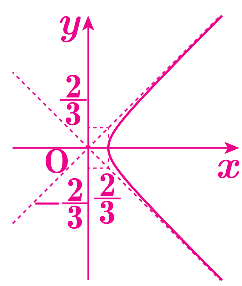
to give us $y=4\cdot\frac{1}{4}(x+1)^2-3=x^2+2x-2$.

Then, since $\sqrt{t}\geq 0$, we get $x\geq -1$.

Therefore, the curve we find is a parabola $y=x^2+2x-2\,(x\geq -1)$.

Parabola $y=x^2+2x-2\,(x\geq -1)$

(3) $x=\frac{1}{3}(2^t+2^{-t}), y=\frac{1}{3}(2^t-2^{-t})$



Squaring both sides of these 2 equations

gives us $x^2=\frac{1}{9}(2^{2t}+2+2^{-2t}), y^2=\frac{1}{9}(2^{2t}-2+2^{-2t})$.

Therefore, $x^2-y^2=\frac{4}{9}$

Here $2^t>0, 2^{-t}>0$, so,

from the relation of the arithmetic mean to the geometric

mean, we get $x=\frac{1}{3}(2^t+2^{-t})\geq\frac{2}{3}\sqrt{2^t\cdot 2^{-t}}=\frac{2}{3}$.

Therefore, the curve we find is a hyperbola $x^2-y^2=\frac{4}{9}\left(x\geq\frac{2}{3}\right)$.

Hyperbola $x^2-y^2=\frac{4}{9}\left(x\geq\frac{2}{3}\right)$

Parametrization by using trigonometric functions

TARGET To understand the general forms of the curves expressed by parametrization using trigonometric functions.

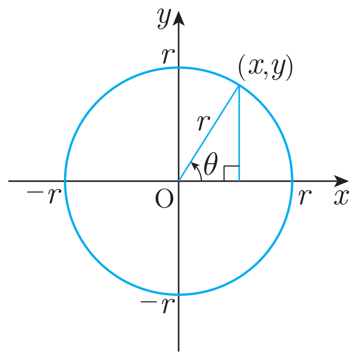
STUDY GUIDE

Parametrization of a quadratic curve

Quadratic curves, such as ellipses and hyperbolas, are generally parametrized by using **trigonometric functions**. The formulas for these curves are expressed using $\sin\theta$ or $\cos\theta$ and $\tan\theta$ or $\dfrac{1}{\cos\theta}$ variously for x and y, which can be solved by eliminating θ by using the formulas $\sin^2\theta+\cos^2\theta=1$ and $1+\tan^2\theta=\dfrac{1}{\cos^2\theta}$.

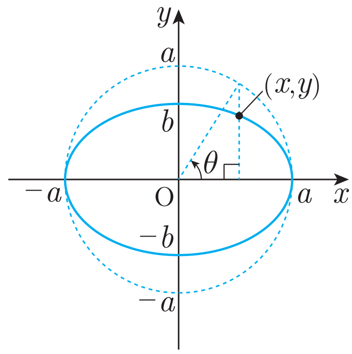
Circle $x^2+y^2=r^2$

$$\boxed{x=r\cos\theta,\ y=r\sin\theta}$$



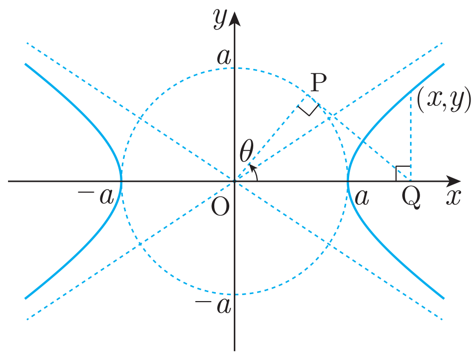
Ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$

$$\boxed{x=a\cos\theta,\ y=b\sin\theta}$$



Hyperbola $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$

$$\boxed{x=\frac{a}{\cos\theta},\ y=b\tan\theta}$$



explanation

For $\triangle$OPQ in the figure on the right, from $\dfrac{\mathrm{OP}}{\mathrm{OQ}}=\cos\theta$ we get

$\dfrac{a}{x}=\cos\theta, x=\dfrac{a}{\cos\theta}$.

By substituting this into $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$, we can solve for y, to get $y=b\tan\theta$.

EXERCISE

◆ Given a parameter θ, find the shape of the curve expressed by the following equation.

(1) $x=\sqrt{3}\cos\theta-1$, $y=\sqrt{3}\sin\theta+2$

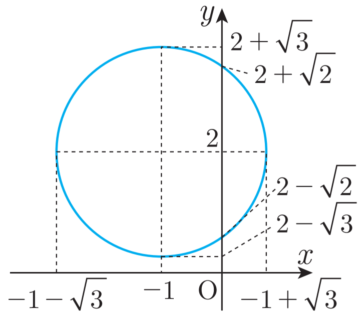
From $x=\sqrt{3}\cos\theta-1$, we can get $\sqrt{3}\cos\theta=x+1$.

Square both sides, $3\cos^2\theta=(x+1)^2$ ...(i)

In the same way, from $y=\sqrt{3}\sin\theta+2$, we get $3\sin^2\theta=(y-2)^2$ ...(ii)

From (i) and (ii), we get $(x+1)^2+(y-2)^2=3(\cos^2\theta+\sin^2\theta)=3$.

Therefore, the curve we find is a circle $(x+1)^2+(y-2)^2=3$.



Circle $(x+1)^2+(y-2)^2=3$

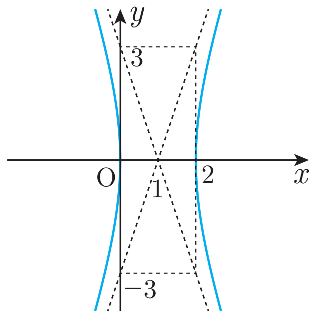
(2) $x=\dfrac{1}{\cos\theta}+1$, $y=3\tan\theta$

From $x=\dfrac{1}{\cos\theta}+1$, we can get $\dfrac{1}{\cos\theta}=x-1$.

Also, from $y=3\tan\theta$, we get $\tan\theta=\dfrac{y}{3}$.

This is substituted into $1+\tan^2\theta=\dfrac{1}{\cos^2\theta}$, to give us $1+\dfrac{y^2}{9}=(x-1)^2$.

Therefore, the curve we find is a hyperbola $(x-1)^2-\dfrac{y^2}{9}=1$.



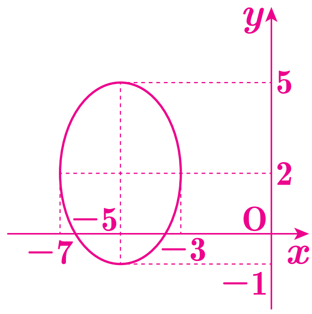
Hyperbola $(x-1)^2-\dfrac{y^2}{9}=1$

PRACTICE

◆ Given a parameter θ, find the shape of the curve expressed by the following equation.

(1) $x = 2\cos\theta - 5, y = 3\sin\theta + 2$

From $x = 2\cos\theta - 5$, we get $\cos\theta = \dfrac{x+5}{2}$.



Also, from $y = 3\sin\theta + 2$, we get $\sin\theta = \dfrac{y-2}{3}$.

These are substituted into $\sin^2\theta + \cos^2\theta = 1$,

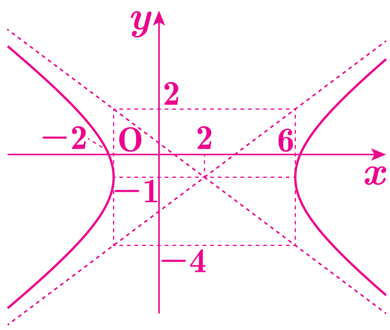
to give us $\left(\dfrac{y-2}{3}\right)^2 + \left(\dfrac{x+5}{2}\right)^2 = 1$.

Therefore, the curve we find is an ellipse $\dfrac{(x+5)^2}{4} + \dfrac{(y-2)^2}{9} = 1$.

Ellipse $\dfrac{(x+5)^2}{4} + \dfrac{(y-2)^2}{9} = 1$

(2) $x = \dfrac{4}{\cos\theta} + 2, y = 3\tan\theta - 1$

From $x = \dfrac{4}{\cos\theta} + 2$, we get $\dfrac{1}{\cos\theta} = \dfrac{x-2}{4}$.



Also, from $y = 3\tan\theta - 1$, we get $\tan\theta = \dfrac{y+1}{3}$.

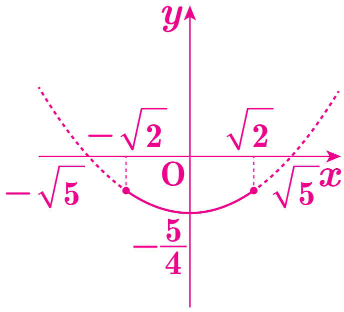
These are substituted into $1 + \tan^2\theta = \dfrac{1}{\cos^2\theta}$,

to give us $1 + \left(\dfrac{y+1}{3}\right)^2 = \left(\dfrac{x-2}{4}\right)^2$.

Therefore, the curve we find is a hyperbola $\dfrac{(x-2)^2}{16} - \dfrac{(y+1)^2}{9} = 1$.

Hyperbola $\dfrac{(x-2)^2}{16} - \dfrac{(y+1)^2}{9} = 1$

(3) $x=\sin\theta+\cos\theta, y=\frac{1}{4}\sin 2\theta-1$



From $y=\frac{1}{4}\sin 2\theta-1$,

we get $y=\frac{1}{4}\cdot 2\sin\theta\cos\theta-1=\frac{1}{2}\sin\theta\cos\theta-1$.

From this, we get $\sin\theta\cos\theta=2(y+1)$**, so**

$$\begin{aligned} x^2 &= (\sin\theta+\cos\theta)^2 \\ &= \sin^2\theta+2\sin\theta\cos\theta+\cos^2\theta \\ &= 2\sin\theta\cos\theta+1 \\ &= 4(y+1)+1 \\ &= 4y+5 \end{aligned}$$

Therefore, we get $y=\frac{1}{4}x^2-\frac{5}{4}$.

Now, from $x=\sin\theta+\cos\theta=\sqrt{2}\sin\left(\theta+\frac{\pi}{4}\right)$**, we have** $-\sqrt{2}\le x\le\sqrt{2}$.

Therefore, the curve we find is a parabola $y=\frac{1}{4}x^2-\frac{5}{4}(-\sqrt{2}\le x\le\sqrt{2})$.

Parabola $y=\frac{1}{4}x^2-\frac{5}{4}(-\sqrt{2}\le x\le\sqrt{2})$

EXTRA Info.

Diagrams of curves expressed by parameters

Use the Spreadsheet and QR code functions.

EXERCISE

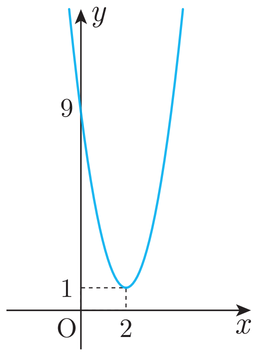
1 Given a parameter t, find the shape of the curve expressed by the equations $x=t+2, y=2t^2+1$.

From $x=t+2$, we get $t=x-2$.

This is substituted into $y=2t^2+1$, to give us $y=2(x-2)^2+1=2x^2-8x+9$.

Therefore, the curve we find is a parabola $y=2x^2-8x+9$.

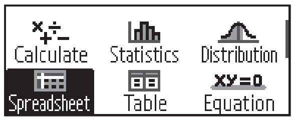
Parabola $y=2x^2-8x+9$



check

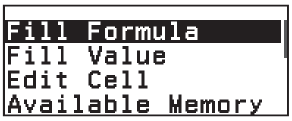
By using the Spreadsheet and QR code functions, we can show the curve expressed by the parameters.

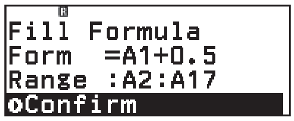
Press (⌂), select [Spreadsheet], press (OK), then clear the previous data by pressing (⏻)

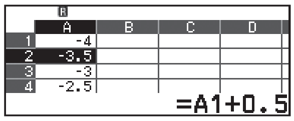


Input -4 in cell A1, then use [Fill Formula] to fill cells A2 to A17 with values from -3.5 to 4 in increments of 0.5.

(−) (4) (EXE) (⋯) (OK) (⇧) (4) (1) (+) (0) (.) (5) (EXE) (>) (>) (>) (>) (>) (>) (⌫) (1) (7) (EXE) (EXE)

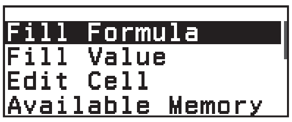


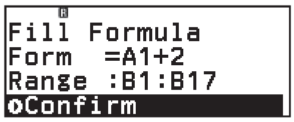


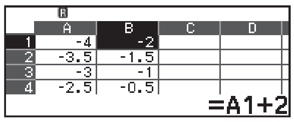


Let the values in column A be the values of t, then use [Fill Formula] so the values of the x coordinate, corresponding to each value of t, appear in column B.

(∧) (>) (⋯) (OK) (⇧) (4) (1) (+) (2) (EXE) (>) (>) (>) (>) (>) (>) (7) (EXE) (EXE)

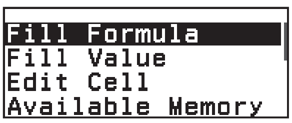


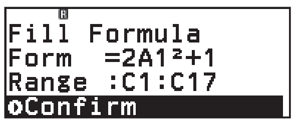


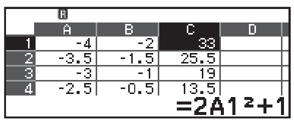


Let the values in column A be the values of t, then use [Fill Formula] so the values of the y coordinates, corresponding to each value of t, appear in column C.

(>) (⋯) (OK) (2) (⇧) (4) (1) (■²) (+) (1) (EXE) (>) (>) (>) (>) (>) (>) (7) (EXE) (EXE)





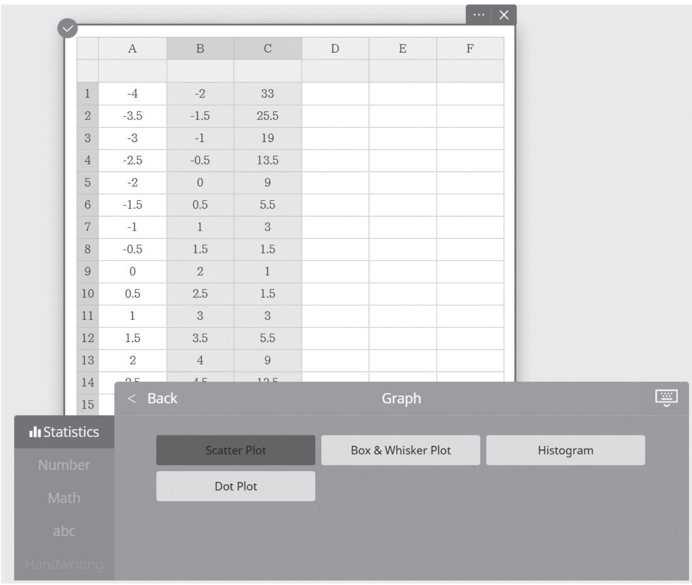


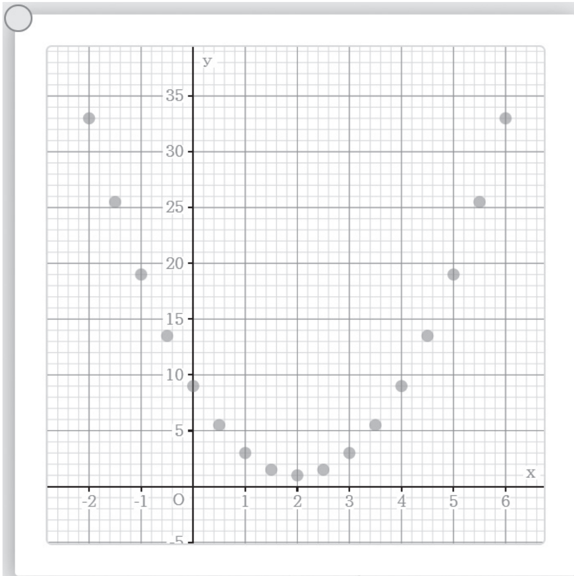
Press (⇧) (x), scan the QR code to display the spreadsheet.

In the spreadsheet that is displayed, select column B and column C, so the 2 columns become gray.

Now, in the [Statistics] tab of the displayed pop-up, select [Graph]→[Scatter Plot] in this order, to display a scatter plot.

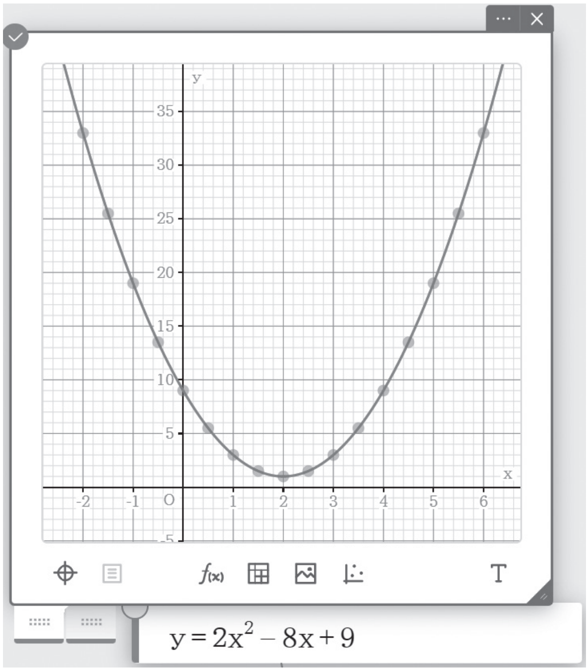
This scatter plot is a plot of the points that correlate to the shape expressed by the equations $x = t + 2, y = 2t^2 + 1$.

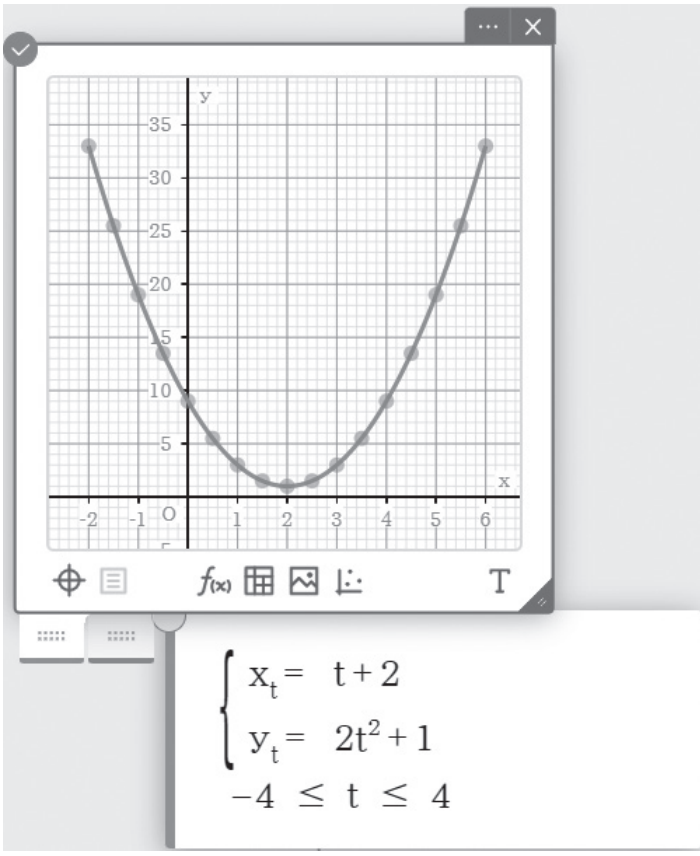




As shown in the figure at the bottom left, when we display a graph of $y = 2x^2 - 8x + 9$ on this scatter plot, we can see that it passes through all the plotted points.

Furthermore, as shown in the figure at the bottom right, we can display the graph on the scatter plot while also showing the parameters. In this case too, we can see that it passes through all the plotted points.





2 Given a parameter θ, find the shape of the curve expressed by the equations $x = 2\cos\theta + 1, y = 3\sin\theta - 1$.

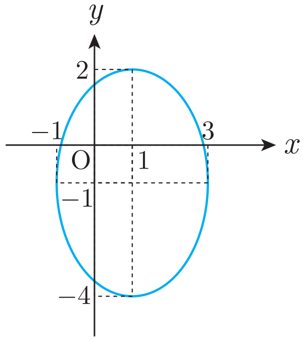
From $x = 2\cos\theta + 1$, we get $\cos\theta = \frac{x-1}{2}$.

Also, from $y = 3\sin\theta - 1$, we get $\sin\theta = \frac{y+1}{3}$.

These are substituted into $\sin^2\theta + \cos^2\theta = 1$,

to give us $\left(\frac{y+1}{3}\right)^2 + \left(\frac{x-1}{2}\right)^2 = 1$.

Therefore, the curve we find is an ellipse $\frac{(x-1)^2}{4} + \frac{(y+1)^2}{9} = 1$.

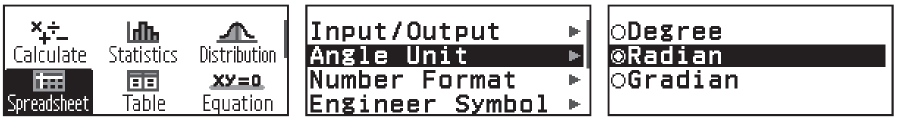


Ellipse $\frac{(x-1)^2}{4} + \frac{(y+1)^2}{9} = 1$

check

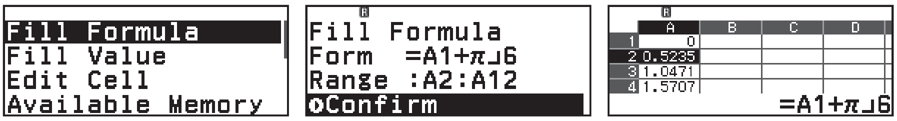
Press ⌂, select [Spreadsheet], press OK, then clear the previous data by pressing ⏻

Press ☰, select [Calc Settings], press OK, select [Angle Unit], press OK, select [Radian], press OK, press AC



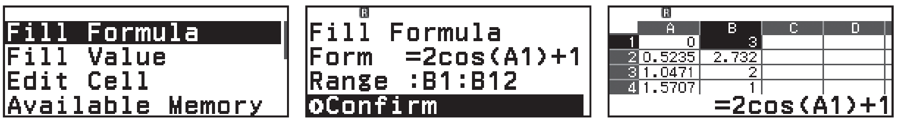
Input 0 in cell A1, then use [Fill Formula] to fill cells A2 to A12 with values from $\frac{\pi}{6}$ to $\frac{11}{6}\pi$ in increments of $\frac{\pi}{6}$.

0 EXE ⋯ OK ⇧ 4 1 + ⇧ 7 ▤ 6 EXE > > > > > 1 EXE EXE



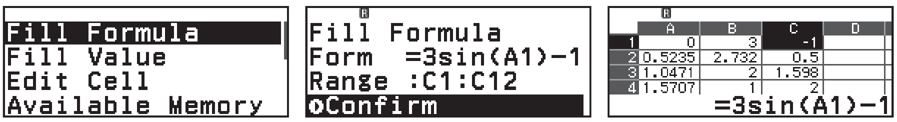
Let the values in column A be the values of θ, then use [Fill Formula] so the values of the x coordinate, corresponding to each value of θ, appear in column B.

∧ > ⋯ OK 2 cos ⇧ 4 1) + 1 EXE > > > > > > 2 EXE EXE



Let the values in column A be the values of θ, then use [Fill Formula] so the values of the y coordinates, corresponding to each value of θ, appear in column C.

> ⋯ OK 3 sin ⇧ 4 1) − 1 EXE > > > > > > 2 EXE EXE

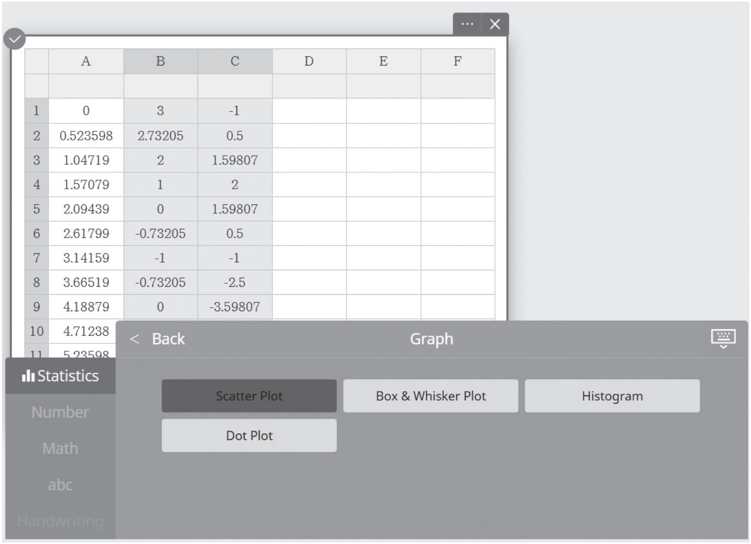


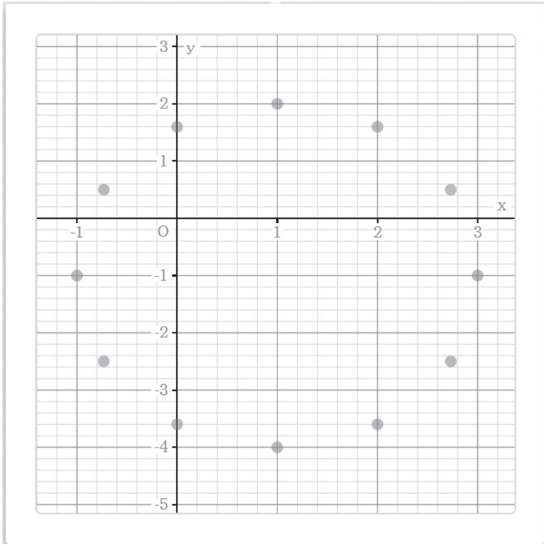
Press ⇧ ⓧ, scan the QR code to display the spreadsheet.

In the spreadsheet that is displayed, select column B and column C, so the 2 columns become gray.

Now, in the [Statistics] tab of the displayed pop-up, select [Graph]→[Scatter Plot] in this order, to display a scatter plot.

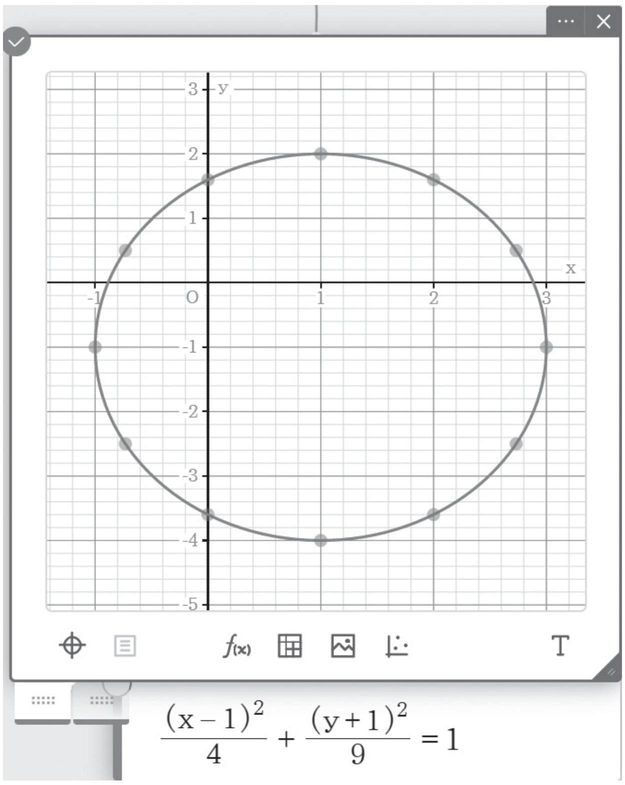
This scatter plot is a plot of the points that correlate to the shape expressed by the equations $x = 2\cos\theta + 1, y = 3\sin\theta - 1$.

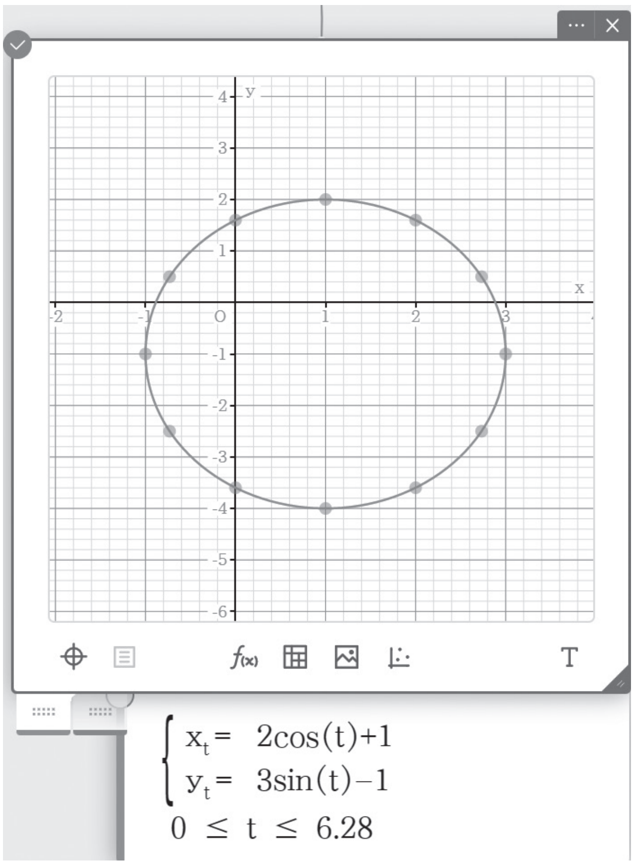




As shown in the figure at the bottom left, when we display the ellipse $\dfrac{(x-1)^2}{4} + \dfrac{(y+1)^2}{9} = 1$ on this scatter plot, we can see that it passes through all the plotted points.

Furthermore, as shown in the figure at the bottom right, we can display the curve on the scatter plot while also showing the parameters. In this case too, we can see that it passes through all the plotted points.

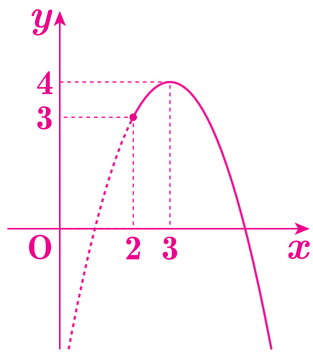




PRACTICE

1 Given a parameter t, find the shape of the curve expressed by the equations $x=\sqrt{t}+2, y=2\sqrt{t}-t+3$.

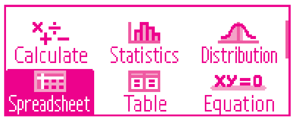
From $x=\sqrt{t}+2$, we get $\sqrt{t}=x-2, t=(x-2)^2$.
This is substituted into $y=2\sqrt{t}-t+3$,
to give us $y=2(x-2)-(x-2)^2+3=-x^2+6x-5$.
Then, since $\sqrt{t}\geq 0$, we get $x\geq 2$
Therefore, the curve we find is a parabola
$y=-x^2+6x-5\ (x\geq 2)$.



Parabola $y=-x^2+6x-5\ (x\geq 2)$

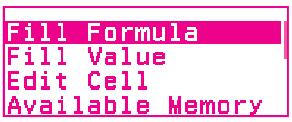
check

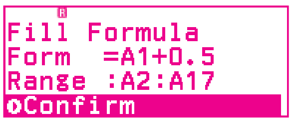
By using the Spreadsheet and QR code functions, we can show the curve expressed by the parameters.
Press ⌂, select [Spreadsheet], press OK, then clear the previous data by pressing ⏻

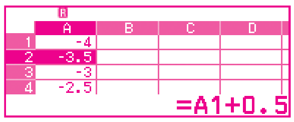


Input -4 in cell A1, then use [Fill Formula] to fill cells A2 to A17 with values from -3.5 to 4 in increments of 0.5.

(−) 4 EXE   ⋯ OK ⇧ 4 1 + 0 . 5 EXE   > > > > > > ⌫ 1 7 EXE   EXE

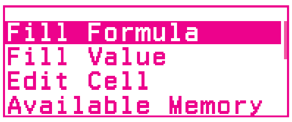


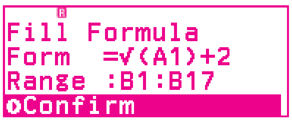


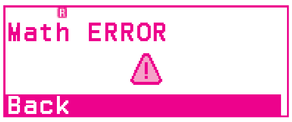


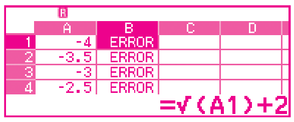
Let the values in column A be the values of t, then use [Fill Formula] so the values of the x coordinate, corresponding to each value of t, appear in column B.

∧ >   ⋯ OK √ ⇧ 4 1) + 2 EXE   > > > > > > 7 EXE   EXE   EXE



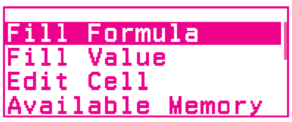


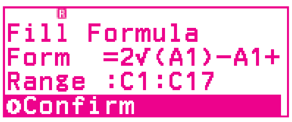


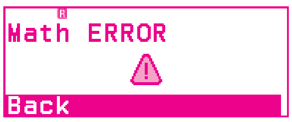


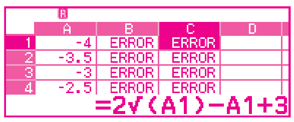
Let the values in column A be the values of t, then use [Fill Formula] so the values of the y coordinates, corresponding to each value of t, appear in column C.

>   ⋯ OK 2 √ ⇧ 4 1) − ⇧ 4 1 + 3 EXE   > > > > > > 7 EXE   EXE   EXE



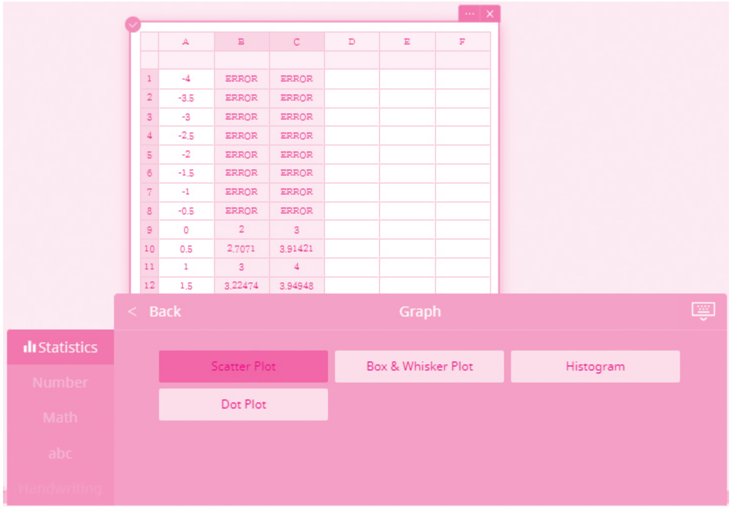


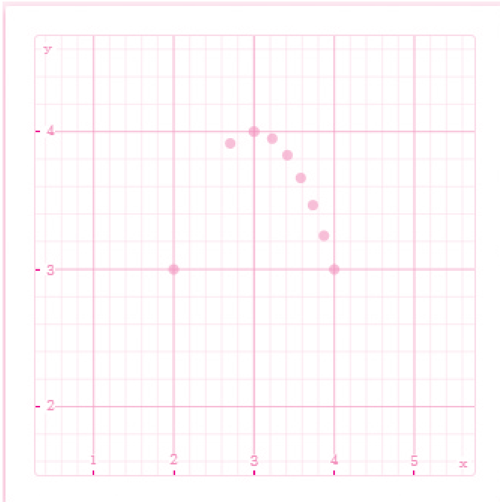




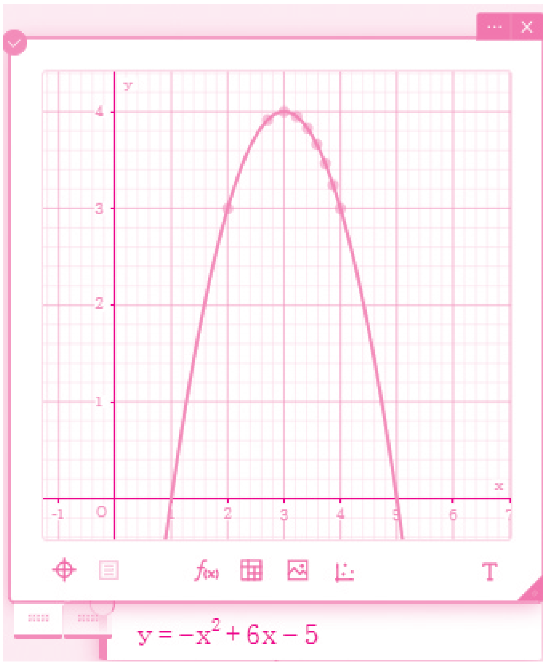
Press ⇧ x, scan the QR code to display the spreadsheet.
*** ERROR appears in this case because there is a negative value under the √ sign.**
Conversely, this shows that the range of values for t is greater than or equal to 0.

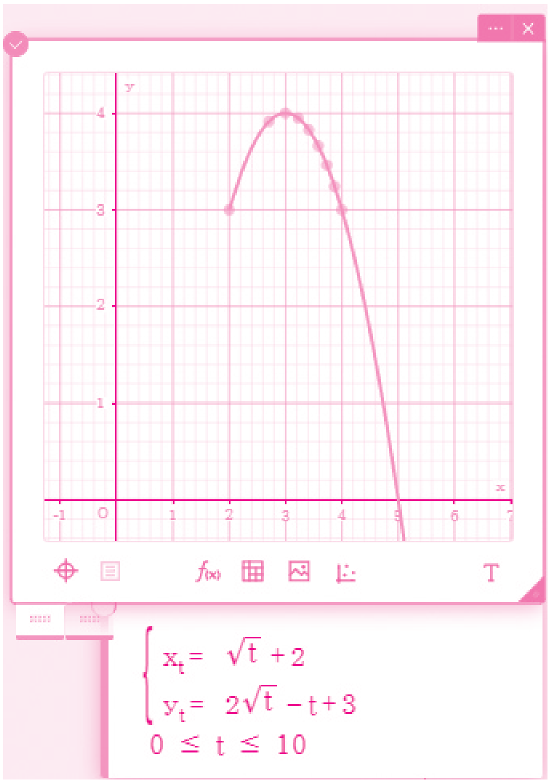
In the spreadsheet that is displayed, select column B and column C, so the 2 columns become gray.
Now, in the [Statistics] tab of the displayed pop-up, select [Graph]→[Scatter Plot] in this order, to display a scatter plot.
This scatter plot is a plot of the points that correlate to the shape expressed by the equations $x=\sqrt{t}+2$ and $y=2\sqrt{t}-t+3$.





As shown in the figure at the bottom left, when we display a graph of $y=-x^2+6x-5$ on this scatter plot, we can see that it passes through all the plotted points.
Furthermore, as shown in the figure at the bottom right, we can display the graph on the scatter plot while also showing the parameters. In this case too, we can see that it passes through all the plotted points.





2 Given a parameter θ, find the shape of the curve expressed by the equations $x=\sin\theta, y=\cos 2\theta$.

From $y=\cos 2\theta$, we get

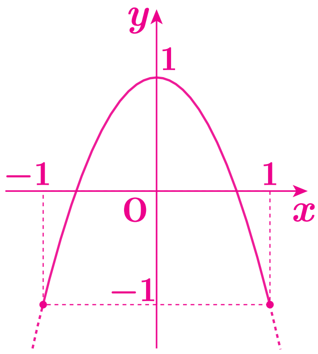
$y=\cos^2\theta-\sin^2\theta=(1-\sin^2\theta)-\sin^2\theta=1-2\sin^2\theta$.

From this, we get $y=1-2x^2$.

Now, from $x=\sin\theta$, we get $-1\leq x\leq 1$.

Therefore, the curve we find is a parabola

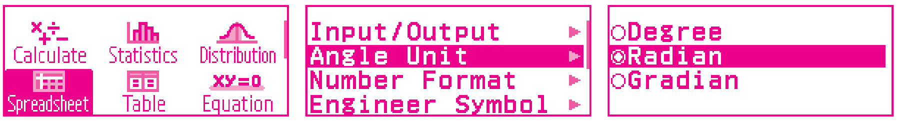
$y=1-2x^2\ (-1\leq x\leq 1)$.



Parabola $y=1-2x^2\ (-1\leq x\leq 1)$

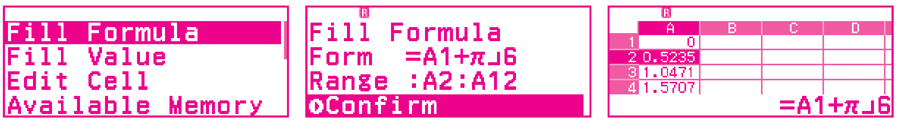
check

Press ⌂, select [Spreadsheet], press OK, then clear the previous data by pressing ⏻
Press ☰, select [Calc Settings], press OK, select [Angle Unit], press OK, select [Radian], press OK, press AC



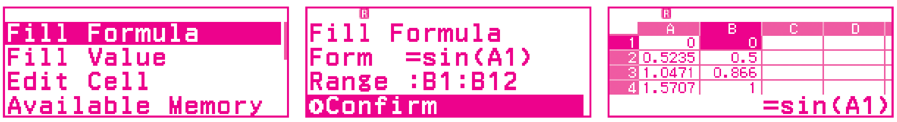
Input 0 in cell A1, then use [Fill Formula] to fill cells A2 to A12 with values from $\frac{\pi}{6}$ to $\frac{11}{6}\pi$ in increments of $\frac{\pi}{6}$.

0 EXE ⋯ OK ⇧ 4 1 + ⇧ 7 $\frac{\square}{\square}$ 6 EXE > > > > > 1 EXE EXE



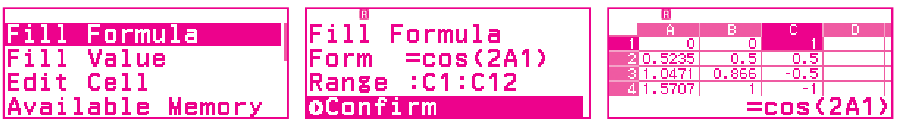
Let the values in column A be the values of θ, then use [Fill Formula] so the values of the x coordinate, corresponding to each value of θ, appear in column B.

^ > ⋯ OK sin ⇧ 4 1) EXE > > > > > > 2 EXE EXE

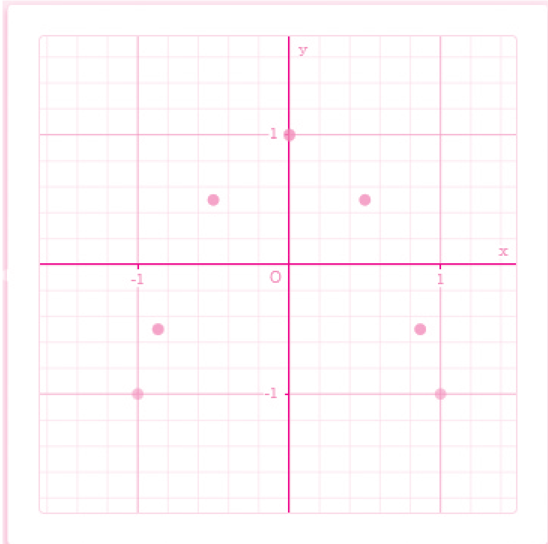


Let the values in column A be the values of θ, then use [Fill Formula] so the values of the y coordinates, corresponding to each value of θ, appear in column C.

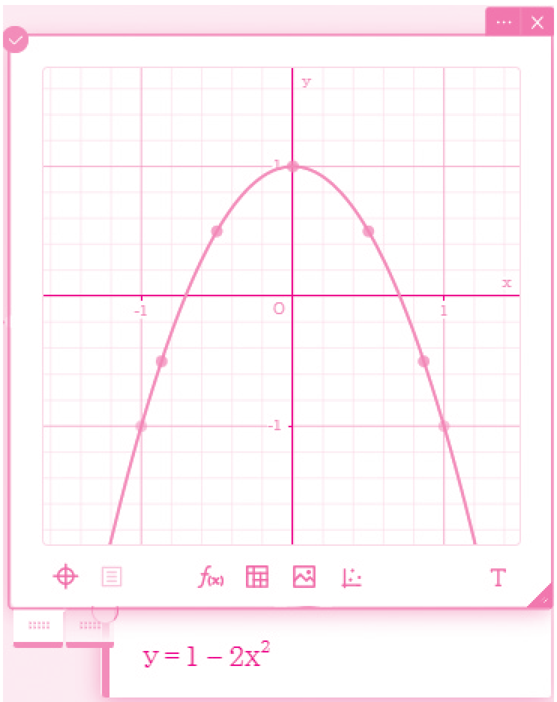
> ⋯ OK cos 2 ⇧ 4 1) EXE > > > > > > 2 EXE EXE

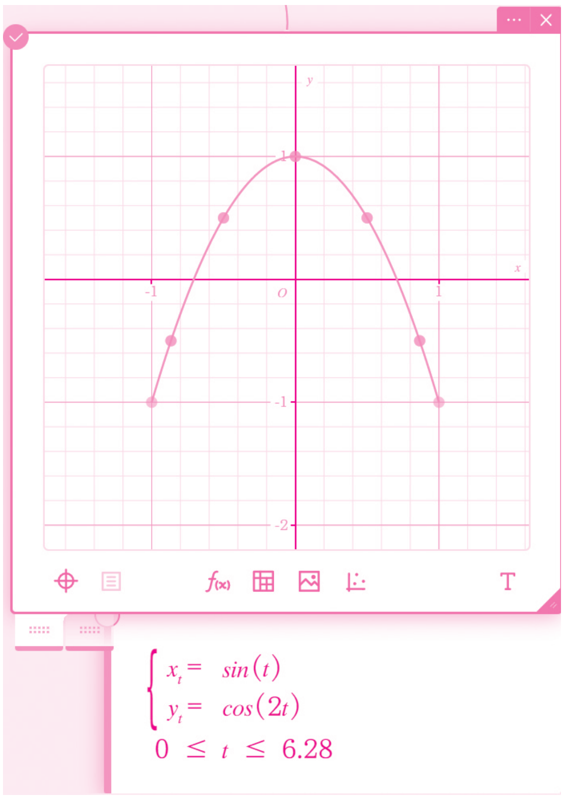


Press ⇧ ⓧ, scan the QR code to display the spreadsheet.
In the spreadsheet that is displayed, select column B and column C, so the 2 columns become gray.
Now, in the [Statistics] tab of the displayed pop-up, select [Graph]→[Scatter Plot] in this order, to display a scatter plot.
This scatter plot is a plot of the points that correlate to the shape expressed by the equations $x = \sin\theta$ and $y = \cos 2\theta$.



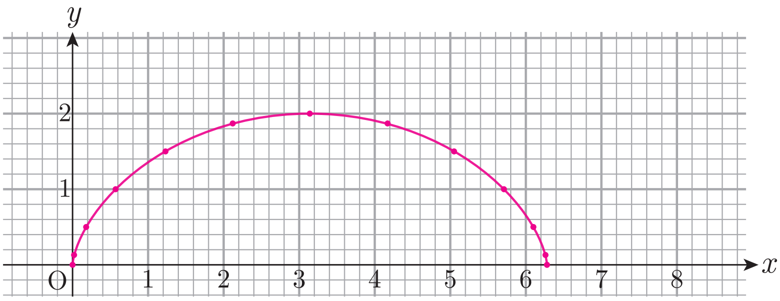
As shown in the figure at the bottom left, when we display a graph of $y = 1 - 2x^2$ on this scatter plot, we can see that it passes through all the plotted points.
Furthermore, as shown in the figure at the bottom right, we can display the curve on the scatter plot while also showing the parameters. In this case too, we can see that it passes through all the plotted points.



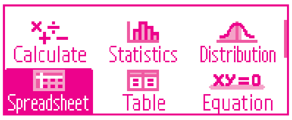


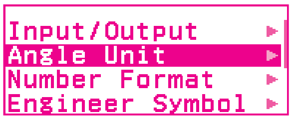
[3] Given θ as the parameter, consider the curve expressed by the equations $x=\theta-\sin\theta, y=1-\cos\theta$. Fill in the following table and plot the points that correspond to the various values of θ on the coordinate plane, and from that draw a sketch of the curve given $0 \leq \theta < 2\pi$.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2}{3}\pi$	$\frac{5}{6}\pi$	π	$\frac{7}{6}\pi$	$\frac{4}{3}\pi$	$\frac{3}{2}\pi$	$\frac{5}{3}\pi$	$\frac{11}{6}\pi$	2π
x	**0**	**0.02**	**0.18**	**0.57**	**1.23**	**2.12**	**3.14**	**4.17**	**5.05**	**5.71**	**6.10**	**6.26**	**6.28**
y	**0**	**0.13**	**0.5**	**1**	**1.5**	**1.87**	**2**	**1.87**	**1.5**	**1**	**0.5**	**0.13**	**0**



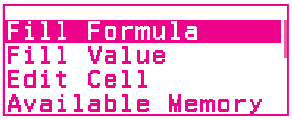
Press [HOME], select [Spreadsheet], press [OK], then clear the previous data by pressing [ON]
Press [SETTINGS], select [Calc Settings], press [OK], select [Angle Unit], press [OK], select [Radian], press [OK], press [AC]

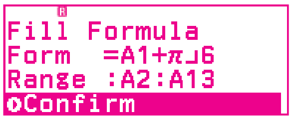


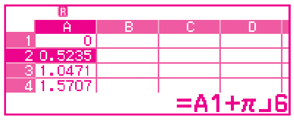


Input 0 in cell $A1$, then use [Fill Formula] to fill cells $A2$ to $A13$ with values from $\frac{\pi}{6}$ to 2π in increments of $\frac{\pi}{6}$.

[0] [EXE]   [⋯] [OK] [⇧] [4] [1] [+] [⇧] [7] [▭/▭] [6] [EXE]   [>] [>] [>] [>] [>] [>] [⌫] [1] [3] [EXE] [EXE]

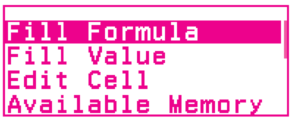


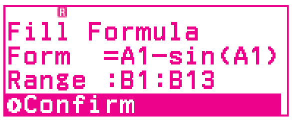


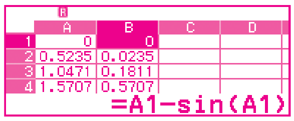


Let the values in column A be the values of θ, then use [Fill Formula] so the values of the x coordinate, corresponding to each value of θ, appear in column B.

[∧] [>]   [⋯] [OK] [⇧] [4] [1] [−] [sin] [⇧] [4] [1] [)] [EXE]   [>] [>] [>] [>] [>] [>] [3] [EXE]   [EXE]

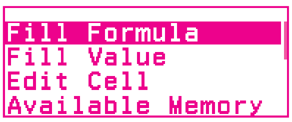


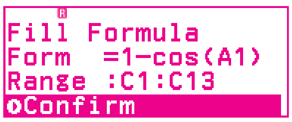


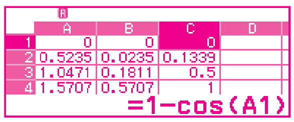


Let the values in column A be the values of θ, then use [Fill Formula] so the values of the y coordinates, corresponding to each value of θ, appear in column C.

[>]   [⋯] [OK] [1] [−] [cos] [⇧] [4] [1] [)] [EXE]   [>] [>] [>] [>] [>] [>] [3] [EXE]   [EXE]



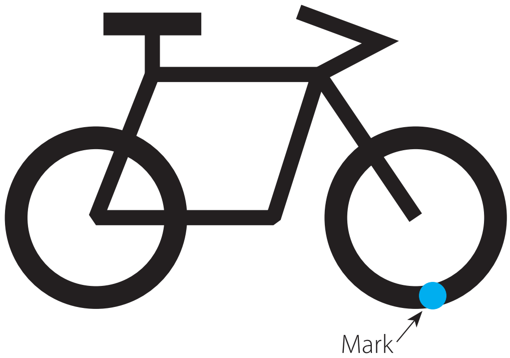




EXTRA Info.

Cylcoid

If a fixed point P on the circumference is rotated without slipping along a fixed straight line, the curve that the point P describes is called a **cycloid**.



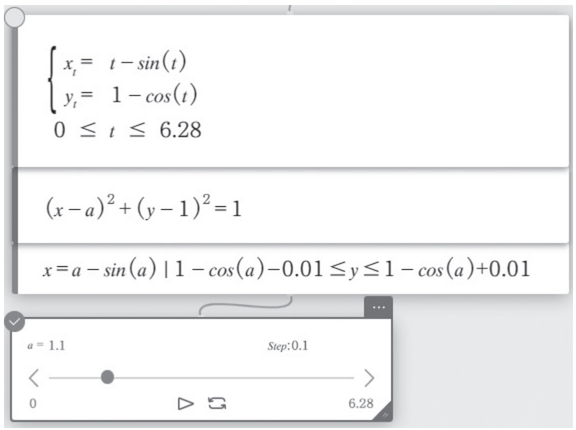
When the radius of the circle is a, then the parameters of the cycloid are expressed as follows.

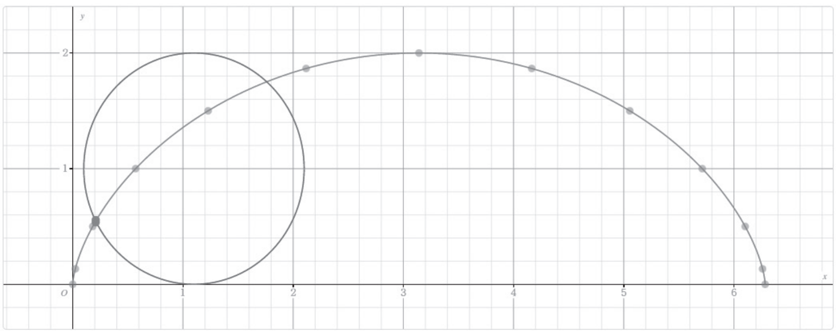
$x = a(\theta - \sin\theta), y = a(1 - \cos\theta)$

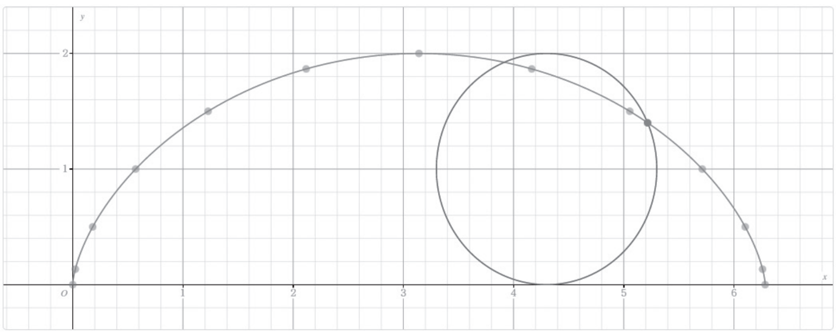
For example, if we paint a mark on the wheel of a bicycle, then as we ride it, the curve that the mark describes is a cycloid.
Furthermore, the curve that we drew in PRACTICE $\boxed{3}$ is a cycloid when a=1.
Note that the cycloid cannot be expressed in an equation of x and y by eliminating θ.

Use the Spreadsheet and QR code functions to draw a cycloid for which a=1.
Input values into Spreadsheet according to the answer in PRACTICE $\boxed{3}$, then press ⬆ ⓧ and scan the QR code to display the spreadsheet.

In the spreadsheet that is displayed, select column B and column C, so the 2 columns become gray.
Now, in the [Statistics] tab of the displayed pop-up, select [Graph]→[Scatter Plot] in this order, to display a scatter plot.
This scatter plot is a plot of the points that correlate to the shape expressed by the equations $x = \theta - \sin\theta, y = 1 - \cos\theta$.
Input the curve, as on the right, into the scatter plot, then set variable a.
After that, you can confirm that the fixed point on the circumference draws a curve when the play button (▷) is pressed, according to the setting of variable a.





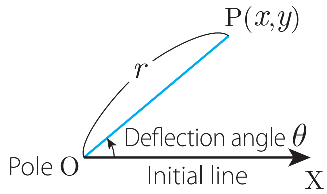


Polar and cartesian coordinates

TARGET To understand the relationship of polar coordinates and cartesian coordinates.

STUDY GUIDE

Defining polar coordinates



When a fixed point O and a half-line OX are defined on a plane, the position of an arbitrary point P on the plane is determined by the length r of OP and the angle θ between the half-lines OP and OX. The pair of these 2 numbers (r, θ) is called the **polar coordinate** of point P, and fixed point O, as a reference, is called the **pole**, the half-line OX is called the **initial line**, and angle θ is called the **deflection angle**.

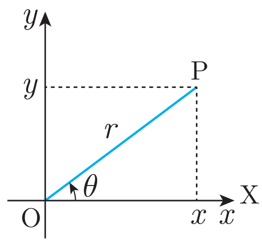
The polar coordinates of pole O are defined as $(0, \theta)$ (where θ is an arbitrary number). Since (r, θ) and $(r, \theta+2n\pi)$ (n is an integer) represent the same point, the polar coordinates of the point P are determined generally as the range of values of the deflection angle θ being $0 \leq \theta < 2\pi$.

Polar and cartesian coordinates

As shown in the figure on the right, when the pole of the polar coordinates is taken as the origin of the cartesian coordinates, and the initial line taken to be the positive direction on the x axis, then we can derive the following relation between cartesian coordinates (x, y) and polar coordinates (r, θ) for point P.

(1) $\quad x = r\cos\theta, y = r\sin\theta$

(2) $\quad r = \sqrt{x^2 + y^2}, \cos\theta = \dfrac{x}{r}, \sin\theta = \dfrac{y}{r} \quad (r \neq 0)$



Distance between 2 points and area of a triangle

The same as for cartesian coordinates, the distance between 2 points and the area of a triangle can be obtained from r and θ of polar coordinates.

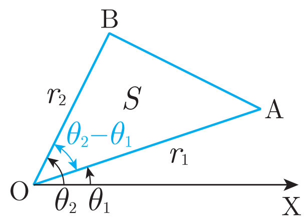
For $\triangle\mathrm{OAB}$ in the figure on the right, the cosine formula and the formula for the area of a triangle, lead us to the following formula.

Distance between 2 points A and B

$$\mathrm{AB} = \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_2 - \theta_1)}$$

Area of $\triangle\mathrm{OAB}$

$$S = \frac{1}{2}r_1r_2\,|\sin(\theta_2 - \theta_1)|$$



EXERCISE

1 For each of the following points, points A and B in polar coordinates are represented by the cartesian coordinates (x, y), and points C and D in the cartesian coordinates are represented by the polar coordinates (r, θ). Provided that $0 \leq \theta < 2\pi$.

$$\mathrm{A}\left(2, \frac{2}{3}\pi\right),\ \mathrm{B}\left(4, -\frac{\pi}{6}\right),\ \mathrm{C}(-2, -2),\ \mathrm{D}(\sqrt{3}, -1)$$

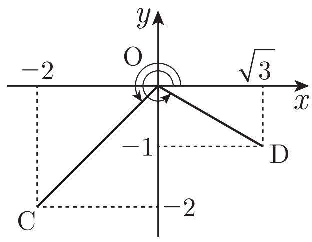
We can express point A in cartesian coordinates.

From $x = 2\cos\frac{2}{3}\pi = 2\cdot\left(-\frac{1}{2}\right) = -1, y = 2\sin\frac{2}{3}\pi = 2\cdot\frac{\sqrt{3}}{2} = \sqrt{3}$, we can get $\mathrm{A}(-1, \sqrt{3})$.

We can express point B in cartesian coordinates.

From $x = 4\cos\left(-\frac{\pi}{6}\right) = 4\cdot\frac{\sqrt{3}}{2} = 2\sqrt{3}, y = 4\sin\left(-\frac{\pi}{6}\right) = 4\cdot\left(-\frac{1}{2}\right) = -2$, we can get $\mathrm{B}(2\sqrt{3}, -2)$.

We can express point C in polar coordinates.



$r = \sqrt{(-2)^2 + (-2)^2} = \sqrt{8} = 2\sqrt{2}$, so from the figure to the right, $\theta = \frac{5}{4}\pi$,

and we thus get $\mathrm{C}\left(2\sqrt{2}, \frac{5}{4}\pi\right)$.

We can express point D in polar coordinates.

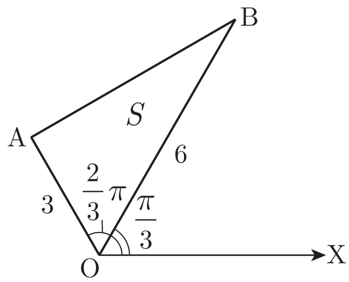
$r = \sqrt{(\sqrt{3})^2 + (-1)^2} = \sqrt{4} = 2$, so from the figure to the right, $\theta = \frac{11}{6}\pi$,

and we thus get $\mathrm{D}\left(2, \frac{11}{6}\pi\right)$.

$$\mathbf{A(-1, \sqrt{3}), B(2\sqrt{3}, -2), C\left(2\sqrt{2}, \frac{5}{4}\pi\right), D\left(2, \frac{11}{6}\pi\right)}$$

2 Solve the following problems with regards to 2 points $\mathrm{A}\left(3, \frac{2}{3}\pi\right), \mathrm{B}\left(6, \frac{\pi}{3}\right)$ and a pole O expressed in polar coordinates.

(1) Find the length of segment AB.



$\angle\mathrm{AOB} = \frac{2}{3}\pi - \frac{\pi}{3} = \frac{\pi}{3}$

For $\triangle\mathrm{OAB}$, from the cosine formula, we get

$\mathrm{AB}^2 = 3^2 + 6^2 - 2\cdot 3\cdot 6\cos\frac{\pi}{3} = 27$.

Since $\mathrm{AB} > 0$, we get $\mathrm{AB} = \sqrt{27} = 3\sqrt{3}$.

$$\mathbf{3\sqrt{3}}$$

(2) Find the area S of $\triangle$OAB.

$$S = \frac{1}{2}\cdot 3\cdot 6\sin\frac{\pi}{3} = 9\cdot\frac{\sqrt{3}}{2} = \frac{9\sqrt{3}}{2}$$

$$\mathbf{\frac{9\sqrt{3}}{2}}$$

PRACTICE

1. For each of the following points, points A and B in polar coordinates are represented by the cartesian coordinates (x, y), and points C and D in the cartesian coordinates are represented by the polar coordinates (r, θ). Provided that $0 \leq \theta < 2\pi$.

$\mathrm{A}\left(3, \frac{\pi}{2}\right),\ \mathrm{B}\left(2, -\frac{5}{4}\pi\right),\ \mathrm{C}(-2, 0),\ \mathrm{D}(1, \sqrt{3})$

Point A $x = 3\cos\frac{\pi}{2} = 3 \cdot 0 = 0, y = 3\sin\frac{\pi}{2} = 3 \cdot 1 = 3, \mathrm{A}(0, 3)$

Point B

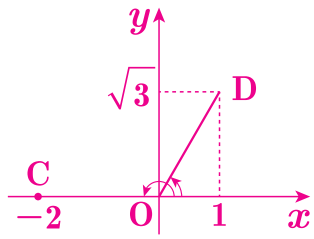
$$x = 2\cos\left(-\frac{5}{4}\pi\right) = 2 \cdot \left(-\frac{\sqrt{2}}{2}\right) = -\sqrt{2}, y = 2\sin\left(-\frac{5}{4}\pi\right) = 2 \cdot \frac{\sqrt{2}}{2} = \sqrt{2}, \mathrm{B}(-\sqrt{2}, \sqrt{2})$$

Point C $r = \sqrt{(-2)^2 + 0^2} = \sqrt{4} = 2$

From the figure on the right, we get $\theta = \pi, \mathrm{C}(2, \pi)$.

Point D $r = \sqrt{1^2 + (\sqrt{3})^2} = \sqrt{4} = 2$

From the figure on the right, we get $\theta = \frac{\pi}{3}, \mathrm{D}\left(2, \frac{\pi}{3}\right)$.

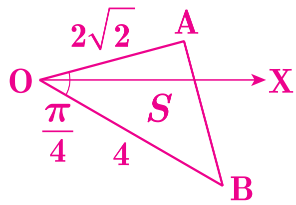


$$\mathrm{A}(0, 3), \mathrm{B}(-\sqrt{2}, \sqrt{2}), \mathrm{C}(2, \pi), \mathrm{D}\left(2, \frac{\pi}{3}\right)$$

2. Solve the following problems with regards to 2 points $\mathrm{A}\left(2\sqrt{2}, \frac{\pi}{12}\right), \mathrm{B}\left(4, -\frac{\pi}{6}\right)$ and a pole O expressed in polar coordinates.

(1) Find the length of segment AB.

$\angle \mathrm{AOB} = \frac{\pi}{12} - \left(-\frac{\pi}{6}\right) = \frac{3}{12}\pi = \frac{\pi}{4}$



For $\triangle \mathrm{OAB}$, from the cosine formula, we get

$\mathrm{AB}^2 = (2\sqrt{2})^2 + 4^2 - 2 \cdot 2\sqrt{2} \cdot 4\cos\frac{\pi}{4} = 8$.

Since $\mathrm{AB} > 0$, we get $\mathrm{AB} = \sqrt{8} = 2\sqrt{2}$.

$2\sqrt{2}$

(2) Find the area S of $\triangle\mathrm{OAB}$.

$S = \frac{1}{2} \cdot 2\sqrt{2} \cdot 4\sin\frac{\pi}{4} = \sqrt{2} \cdot 4 \cdot \frac{1}{\sqrt{2}} = 4$

4

Polar equations

TARGET To understand the general forms of the curves expressed by polar equations.

STUDY GUIDE

Polar equation for a curve

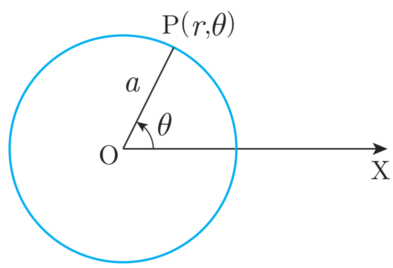
This equation is called the **polar equation** for a curve when F $(r, \theta)=0$ or $r=f(\theta)$ for the polar coordinate (r, θ) holds for any point P on the curve.

Polar equations for a circle

(1) A circle having a radius a centered on a pole O

From $\mathrm{OP}=a$, we get $r=a$.

$$\boxed{r = a}$$



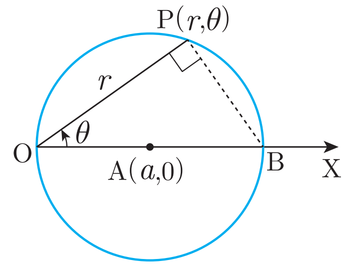
(2) A circle having a radius a centered on a point A $(a, 0)$

For $\triangle\mathrm{OPB}$ in the figure on the right, from $\dfrac{\mathrm{OP}}{\mathrm{OB}} = \cos\theta$ we get

$\dfrac{\mathrm{OP}}{2\mathrm{OA}} = \cos\theta, \dfrac{r}{2a} = \cos\theta$.

Therefore, we get $r = 2a\cos\theta$.

$$\boxed{r = 2a\cos\theta}$$



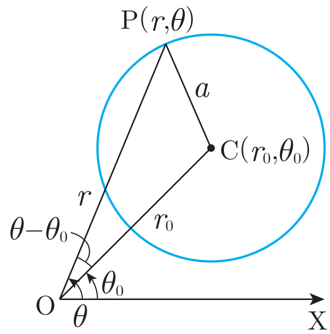
(3) A circle having a radius a centered on a point $\mathrm{C}(r_0, \theta_0)$

For $\triangle\mathrm{OPC}$ in the figure on the right,

from the cosine formula, we get $\mathrm{CP}^2 = \mathrm{OP}^2 + \mathrm{OC}^2 - 2\,\mathrm{OP}\cdot\mathrm{OC}\cos\angle\mathrm{POC}$.

Therefore, we get $a^2 = r^2 + r_0^{\,2} - 2rr_0\cos(\theta-\theta_0)$.

$$\boxed{r^2 + r_0^{\,2} - 2rr_0\cos(\theta-\theta_0) = a^2}$$



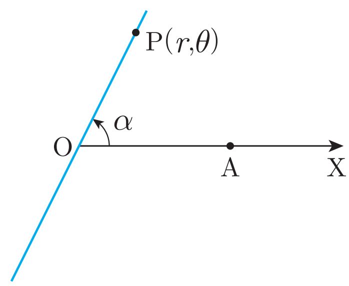
Polar equation for a line

(1) A line passing through pole O and forming an angle α with the initial line

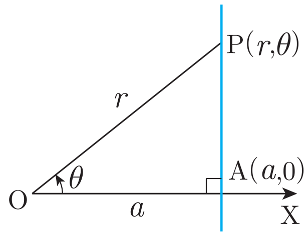
As shown in the figure on the right, given point A on the line OX,

from $\angle\mathrm{POA}=\alpha$, we get $\theta=\alpha$.

$$\boxed{\theta = \alpha}$$



(2) A line passing through point A $(a, 0)$ and perpendicular to the initial line

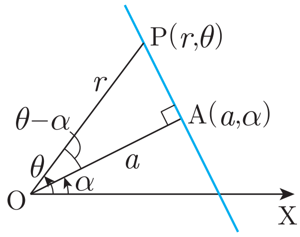


For $\triangle$OPA in the figure on the right, from $\dfrac{\mathrm{OA}}{\mathrm{OP}} = \cos\theta$ we get $\dfrac{a}{r} = \cos\theta$.

Therefore, we get $a = r\cos\theta$.

$$\boxed{\boldsymbol{r\cos\theta = a}}$$

(3) A line passing through point A (a, α) and perpendicular to line OA



For $\triangle$OPA in the figure on the right, from $\dfrac{\mathrm{OA}}{\mathrm{OP}} = \cos(\theta-\alpha)$ we get $\dfrac{a}{r} = \cos(\theta-\alpha)$.

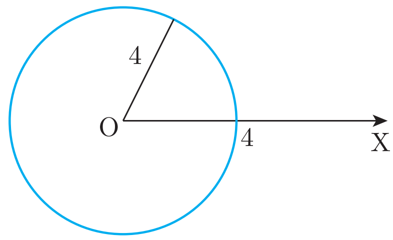
Therefore, we get $a = r\cos(\theta-\alpha)$.

$$\boxed{\boldsymbol{r\cos(\theta-\alpha) = a \quad (a>0)}}$$

EXERCISE

1 Solve whether the shape expressed by the following polar equation is a straight line or a curve.

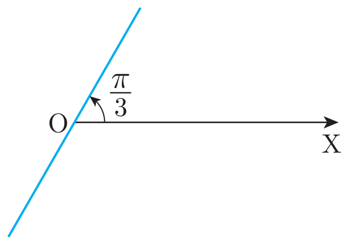
(1) $r=4$



Since the distance from the pole O is constant regardless of the value of the deflection angle, it becomes a circle of radius 4 centered on the pole O.

A circle having a radius of 4 centered on a pole O

(2) $\theta = \dfrac{\pi}{3}$

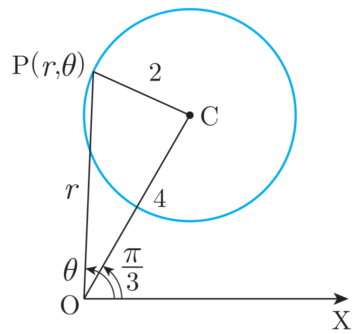


Since the deflection angle is always $\dfrac{\pi}{3}$, the angle formed by passing through the pole O and initial line OX is a straight line $\dfrac{\pi}{3}$.

A straight line forming an angle $\dfrac{\pi}{3}$ with a line passing through the pole O and initial line OX

2 Find the polar equations of the shapes shown below by using polar coordinates.

(1) A circle having a radius of 2 centered on a point $\mathrm{C}\left(4, \dfrac{\pi}{3}\right)$



Let (r, θ) be the coordinates of an arbitrary point P on the circumference.

For $\triangle$OPC, from the cosine formula, we get $2^2 = r^2 + 4^2 - 2r\cdot 4\cos\left(\theta - \dfrac{\pi}{3}\right)$.

This is arranged into $r^2 - 8r\cos\left(\theta - \dfrac{\pi}{3}\right) + 12 = 0$.

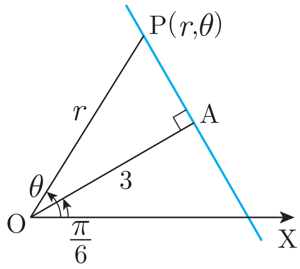
$$\boldsymbol{r^2 - 8r\cos\left(\theta - \frac{\pi}{3}\right) + 12 = 0}$$

(2) A line passing through a point $A\left(3, \frac{\pi}{6}\right)$ and perpendicular to line OA

Let (r, θ) be the coordinates of an arbitrary point P on the line.

The figure on the right gives us $\angle POA = \left|\theta - \frac{\pi}{6}\right|$, so we get $\frac{OA}{OP} = \cos\left(\theta - \frac{\pi}{6}\right)$.

From $OP = r$ and $OA = 3$, we get $r\cos\left(\theta - \frac{\pi}{6}\right) = 3$.



$$r\cos\left(\theta - \frac{\pi}{6}\right) = 3$$

PRACTICE

◆ Find the polar equations of the shapes shown below by using polar coordinates.

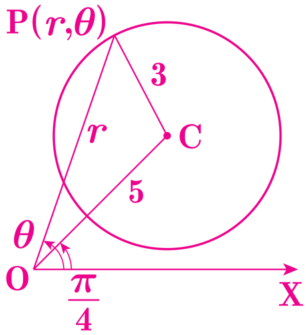
(1) A circle having a radius of 3 centered on a point $C\left(5, \frac{\pi}{4}\right)$

Let (r, θ) be the coordinates of an arbitrary point P on the circumference.

For $\triangle OPC$, from the cosine formula, we get

$3^2 = r^2 + 5^2 - 2r \cdot 5\cos\left(\theta - \frac{\pi}{4}\right)$.

This is arranged into $r^2 - 10r\cos\left(\theta - \frac{\pi}{4}\right) + 16 = 0$.



$$r^2 - 10r\cos\left(\theta - \frac{\pi}{4}\right) + 16 = 0$$

(2) A line passing through a point $A\left(4, \frac{2}{3}\pi\right)$ and perpendicular to line OA

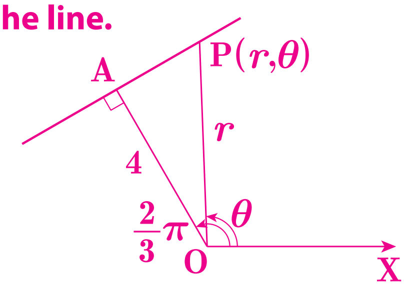
Let (r, θ) be the coordinates of an arbitrary point P on the line.

The figure on the right gives us $\angle POA = \left|\frac{2}{3}\pi - \theta\right|$,

so we get $\frac{OA}{OP} = \cos\left(\frac{2}{3}\pi - \theta\right)$.

From $OP = r$ and $OA = 4$, we get $r\cos\left(\frac{2}{3}\pi - \theta\right) = 4$,

specifically, $r\cos\left(\theta - \frac{2}{3}\pi\right) = 4$.



$$r\cos\left(\theta - \frac{2}{3}\pi\right) = 4$$

Equations using polar equations and cartesian coordinates

TARGET To understand the relation between equations using polar equations and cartesian coordinates.

STUDY GUIDE

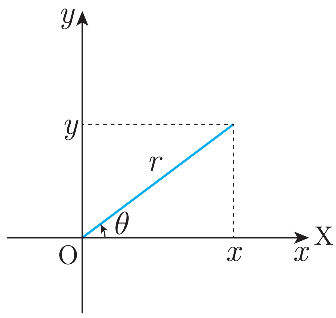
Relation between equations using polar equations and cartesian coordinates

Methods for expressing curves are those using cartesian coordinates and those using polar equations. They can be transformed into each other using the following relations for polar coordinates (r, θ) and cartesian coordinates (x, y).

Relation between polar coordinates (r, θ) and cartesian coordinates (x, y)

$$r = \sqrt{x^2 + y^2}$$

$$\begin{cases} \cos\theta = \dfrac{x}{r} & (x = r\cos\theta) \\ \sin\theta = \dfrac{y}{r} & (y = r\sin\theta) \end{cases}$$



EXERCISE

1 Use formulas of cartesian coordinates to express the curves and lines shown using the following polar equations.

(1) $r = 4\cos\theta$

From $r = 4\cos\theta$, we can get $r^2 = 4r\cos\theta$.

Substituting $r^2 = x^2 + y^2, r\cos\theta = x$ into this, gives us $x^2 + y^2 = 4x, x^2 - 4x + 4 + y^2 = 4, (x-2)^2 + y^2 = 4$.

Therefore, we get a circle $(x-2)^2 + y^2 = 4$.

$$\boldsymbol{(x-2)^2 + y^2 = 4}$$

(2) $r\cos\left(\theta - \dfrac{\pi}{6}\right) = 1$

Use the cosine addition theorem on $r\cos\left(\theta - \dfrac{\pi}{6}\right) = 1$.

$$r\left(\cos\theta\cos\frac{\pi}{6} + \sin\theta\sin\frac{\pi}{6}\right) = 1, r\left(\cos\theta\cdot\frac{\sqrt{3}}{2} + \sin\theta\cdot\frac{1}{2}\right) = 1, \frac{\sqrt{3}}{2}r\cos\theta + \frac{1}{2}r\sin\theta = 1$$

Substituting $r\cos\theta = x, r\sin\theta = y$ into this, gives us $\dfrac{\sqrt{3}}{2}x + \dfrac{1}{2}y = 1, y = -\sqrt{3}x + 2$.

Therefore, we get a line $y = -\sqrt{3}x + 2$.

$$\boldsymbol{y = -\sqrt{3}x + 2}$$

(3) $r\cos^2\theta = \dfrac{1}{2}\sin\theta$

From $r\cos^2\theta = \dfrac{1}{2}\sin\theta$, we can get $2(r\cos\theta)^2 = r\sin\theta$.

Substituting $r\cos\theta = x, r\sin\theta = y$ into this, gives us $2x^2 = y$.

Therefore, we get a parabola $y = 2x^2$.

$$\boldsymbol{y = 2x^2}$$

2 Express as polar equations the following equations shown using cartesian coordinates.

(1) $x - y = 2$

Substitute $x = r\cos\theta, y = r\sin\theta$ into $x - y = 2$ to get $r\cos\theta - r\sin\theta = 2, r(\sin\theta - \cos\theta) = -2$. …(i)

Now, from composite trigonometric functions, we get $\sin\theta - \cos\theta = \sqrt{2}\sin\left(\theta - \dfrac{\pi}{4}\right)$.

Therefore, the equation for (i) is $\sqrt{2}r\sin\left(\theta - \dfrac{\pi}{4}\right) = -2, r\sin\left(\theta - \dfrac{\pi}{4}\right) = -\sqrt{2}$.

$$\boldsymbol{r\sin\left(\theta - \frac{\pi}{4}\right) = -\sqrt{2}}$$

OTHER METHODS

Substitute $x = r\cos\theta, y = r\sin\theta$ into $x - y = 2$ to get $r\cos\theta - r\sin\theta = 2, r(\cos\theta - \sin\theta) = 2$. …(i)

Now, use the cosine addition theorem to get

$$\cos\theta - \sin\theta = \sqrt{2}\left(\cos\theta \cdot \frac{1}{\sqrt{2}} - \sin\theta \cdot \frac{1}{\sqrt{2}}\right) = \sqrt{2}\left(\cos\theta \cdot \cos\frac{\pi}{4} - \sin\theta \cdot \sin\frac{\pi}{4}\right) = \sqrt{2}\cos\left(\theta + \frac{\pi}{4}\right).$$

Therefore, the equation for (i) is $\sqrt{2}r\cos\left(\theta + \dfrac{\pi}{4}\right) = 2, r\cos\left(\theta + \dfrac{\pi}{4}\right) = \sqrt{2}$.

$$\boldsymbol{r\cos\left(\theta + \frac{\pi}{4}\right) = \sqrt{2}}$$

(2) $x^2 + (y-3)^2 = 9$

From $x^2 + (y-3)^2 = 9$, we can get $x^2 + y^2 - 6y = 0$.

Substituting $x^2 + y^2 = r^2, y = r\sin\theta$ into this, gives us $r^2 - 6r\sin\theta = 0, r(r - 6\sin\theta) = 0$.

Therefore, we get $r = 0$, $6\sin\theta$.

Now, $r = 0$ indicates the pole O, which is included in $r = 6\sin\theta$.

Therefore, the polar equation we find is $r = 6\sin\theta$.

$$\boldsymbol{r = 6\sin\theta}$$

PRACTICE

1 Use formulas of cartesian coordinates to express the curves and lines shown using the following polar equations.

(1) $r = 2\cos\theta$

From $r = 2\cos\theta$, we can get $r^2 = 2r\cos\theta$.

Substituting $r^2 = x^2 + y^2, r\cos\theta = x$ into this, gives us

$x^2 + y^2 = 2x, (x-1)^2 + y^2 = 1$.

Therefore, we get a circle $(x-1)^2 + y^2 = 1$.

$$\boldsymbol{(x-1)^2 + y^2 = 1}$$

(2) $r\sin\left(\theta-\frac{\pi}{3}\right)=2$

From $r\sin\left(\theta-\frac{\pi}{3}\right)=2$, we get

$r\left(\sin\theta\cos\frac{\pi}{3}-\cos\theta\sin\frac{\pi}{3}\right)=2, \frac{1}{2}r\sin\theta-\frac{\sqrt{3}}{2}r\cos\theta=2$.

Substituting $r\sin\theta=y, r\cos\theta=x$ into this, gives us

$\frac{1}{2}y-\frac{\sqrt{3}}{2}x=2, y=\sqrt{3}x+4$.

Therefore, we get a line $y=\sqrt{3}x+4$.

$$y=\sqrt{3}x+4$$

(3) $r^2\cos 2\theta=1$

From $r^2\cos 2\theta=1$, we can get $r^2(\cos^2\theta-\sin^2\theta)=1, (r\cos\theta)^2-(r\sin\theta)^2=1$.

Substituting $r\cos\theta=x, r\sin\theta=y$ into this, gives us $x^2-y^2=1$.

Therefore, we get a hyperbola $x^2-y^2=1$.

$$x^2-y^2=1$$

2 Express as polar equations the following equations shown using cartesian coordinates.

(1) $x-\sqrt{3}y=6$

Substitute $x=r\cos\theta, y=r\sin\theta$ into $x-\sqrt{3}y=6$ to get

$r\cos\theta-\sqrt{3}r\sin\theta=6, r(\cos\theta-\sqrt{3}\sin\theta)=6,$

$r\left(\frac{1}{2}\cos\theta-\frac{\sqrt{3}}{2}\sin\theta\right)=3, r\cos\left(\theta+\frac{\pi}{3}\right)=3$.

$$r\cos\left(\theta+\frac{\pi}{3}\right)=3$$

OTHER METHODS

$r\cos\theta-\sqrt{3}r\sin\theta=6, r(\sqrt{3}\sin\theta-\cos\theta)=-6$

From composite trigonometric functions, we get

$2r\sin\left(\theta-\frac{\pi}{6}\right)=-6, r\sin\left(\theta-\frac{\pi}{6}\right)=-3$.

$$r\sin\left(\theta-\frac{\pi}{6}\right)=-3$$

(2) $(x-1)^2+y^2=1$

From $(x-1)^2+y^2=1$, we can get $x^2+y^2-2x=0$.

Substituting $x^2+y^2=r^2, x=r\cos\theta$ into this, gives us

$r^2-2r\cos\theta=0, r(r-2\cos\theta)=0$.

Therefore, we get $r=0$, $2\cos\theta$.

Now, $r=0$ indicates the pole O, which is included in $r=2\cos\theta$.

Therefore, the polar equation we find is $r=2\cos\theta$.

$$r=2\cos\theta$$

Fractional function graphs

TARGET To understand the characteristics and general forms of fractional function graphs.

STUDY GUIDE

Fractional function graphs

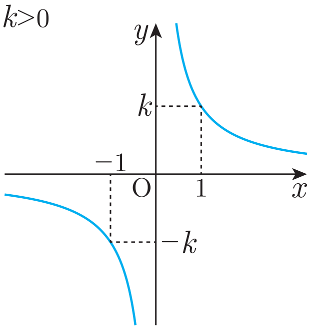
Fractional functions

When we express some function of y, such as $y=-\dfrac{2}{x}$ or $y=\dfrac{5x-4}{2x+1}$, as a fractional expression of x, we say y is a **fractional function** of x. However, the value of x whose denominator is 0 is not included in the domain.

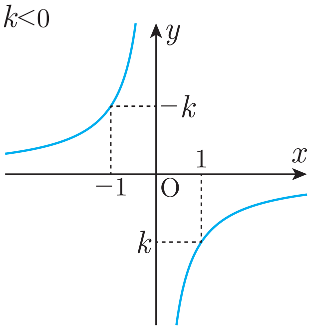
Graph of $y=\dfrac{k}{x}$

If it is an **equilateral hyperbola** (a hyperbola where the 2 asymptotes are at right angles to each other) where the x axis and y axis are asymptotes, then it is a **curve symmetrical about the origin O**.

$k>0$



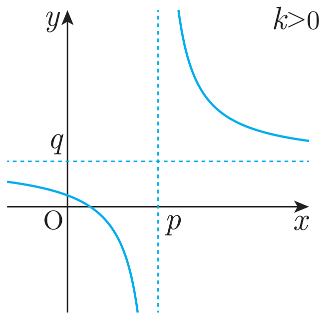
$k<0$



Graph of $y=\dfrac{k}{x-p}+q$

The graph of $y=\dfrac{k}{x}$ is an equilateral hyperbola that has been parallel translated by p along the x axis and by q along the y axis. This is a **symmetric curve in relation to point $(p,\ q)$**, and its asymptotes are $x=p$ and $y=q$.

In general, a graph of $y=f(x)$ that has been parallel translated by p along the x axis and by q along the y axis is $y-q=f(x-p)$, specifically it is a graph of $y=f(x-p)+q$.



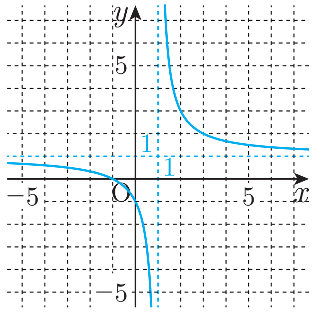
Graph of $y=\dfrac{ax+b}{cx+d}$

When $c\neq 0, ad\neq bc$, if we divide the numerator by the denominator, it changes to $y=\dfrac{k}{x-p}+q$, so we can find some form of a graph of $y=\dfrac{k}{x}$ by parallel translation.

EXERCISE

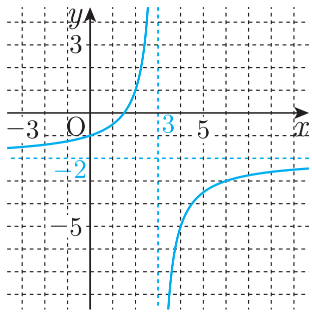
◆ Draw the following fractional function graphs.

(1) $y=\dfrac{2}{x-1}+1$



The graph of $y=\dfrac{2}{x}$ has been parallel translated by 1 along the x axis and y axis, and its asymptotes are $x=1$ and $y=1$.

(2) $y=\dfrac{2x-3}{3-x}$



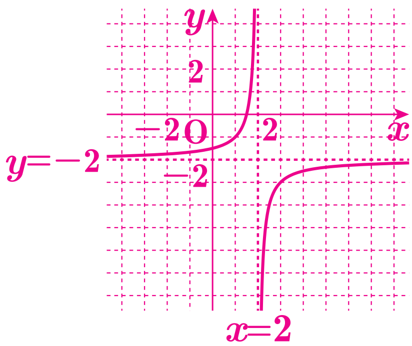
From $y=\dfrac{2x-3}{3-x}=\dfrac{-2(-x+3)+3}{-x+3}=-\dfrac{3}{x-3}-2$, the graph of $y=-\dfrac{3}{x}$ has been parallel translated by 3 along the x axis and by -2 along the y axis, so the asymptotes are $x=3$ and $y=-2$.

PRACTICE

◆ Draw the following fractional function graphs.

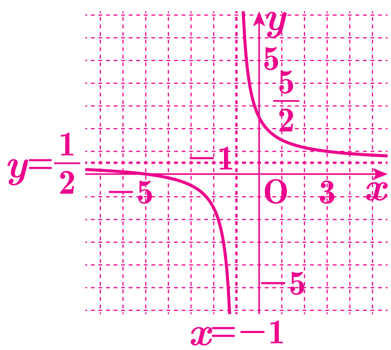
(1) $y=-\dfrac{1}{x-2}-2$

The graph of $y=-\dfrac{1}{x}$ has been parallel translated by 2 along the x axis and by -2 along the y axis, so the asymptotes are $x=2$ and $y=-2$.



(2) $y=\dfrac{x+5}{2x+2}$

From $y=\dfrac{x+5}{2x+2}=\dfrac{x+1+4}{2(x+1)}=\dfrac{2}{x+1}+\dfrac{1}{2}$, the graph of $y=\dfrac{2}{x}$ has been parallel translated by -1 along the x axis and by $\dfrac{1}{2}$ along the y axis, so its asymptotes are $x=-1$ and $y=\dfrac{1}{2}$.



Fractional function graphs and common points of lines

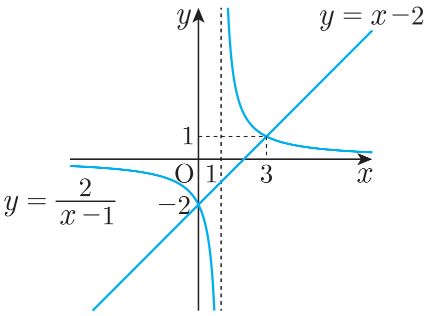
TARGET To understand how the coordinates of intersections of lines and hyperbolas correspond to real roots of simultaneous equations.

STUDY GUIDE

Solving simultaneous equations and coordinates of common points on 2 graphs

By solving simultaneous equations, it is possible to find the coordinates of the common points of 2 graphs, and then to use the graph to solve the **fractional equation** backwards.

Ex. Coordinates of the intersection of graph of function $y=\dfrac{2}{x-1}$ and line $y=x-2$



From the above graph, we see the coordinates of the intersections are $(0,-2)$ and $(3,1)$.

Correspond ⟷

Real root of simultaneous equation $\begin{cases} y=\dfrac{2}{x-1} \\ y=x-2 \end{cases}$

From $\dfrac{2}{x-1}=x-2$, we get $2=(x-2)(x-1), x^2-3x=0, x(x-3)=0$

Therefore, $x=0, 3$

When $x=0$, $y=0-2=-2$

When $x=3$, $y=3-2=1$

Therefore, $x=0$ and $y=-2$ or $x=3$ and $y=1$

EXERCISE

◆ Find the coordinates of the intersection of graph of function $y=\dfrac{3x-7}{x-2}$ and line $y=-x+5$.

From $\dfrac{3x-7}{x-2}=-x+5$, we get $3x-7=(-x+5)(x-2), 3x-7=-x^2+7x-10,$

$x^2-4x+3=0, (x-1)(x-3)=0$.

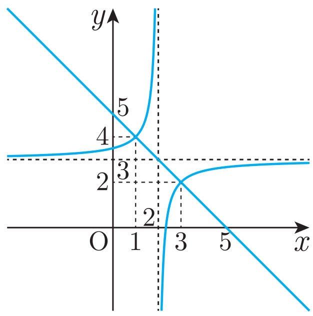
Therefore, $x=1, 3$.

When $x=1$, $y=-1+5=4$.

When $x=3$, $y=-3+5=2$.

Therefore, $x=1$ and $y=4$ or $x=3$ and $y=2$.

So, the coordinates of the intersections are $(1,4)$ and $(3,2)$.

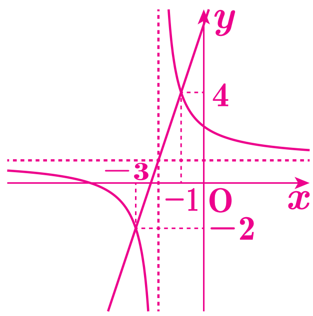


(1, 4), (3, 2)

PRACTICE

◆ Find the coordinates of the intersection of the functional graphs and lines below.

(1) $y=\dfrac{x+5}{x+2}$, $y=3x+7$



From $\dfrac{x+5}{x+2}=3x+7$**, we get** $x+5=(3x+7)(x+2)$,

$x+5=3x^2+13x+14, 3x^2+12x+9=0,$

$x^2+4x+3=0, (x+1)(x+3)=0.$

Therefore, $x=-1, -3$.

When $x=-1$, $y=3\times(-1)+7=4$.

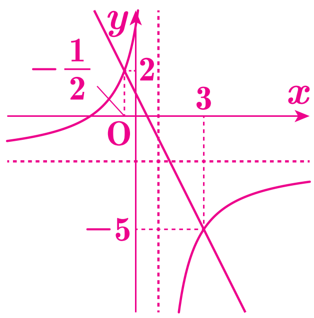
When $x=-3$, $y=3\times(-3)+7=-2$.

Therefore, $x=-1$ **and** $y=4$ **or** $x=-3$ **and** $y=-2$.

So, the coordinates of the intersections are $(-1, 4)$ **and** $(-3, -2)$.

$(-1, 4), (-3, -2)$

(2) $y=-\dfrac{2x+4}{x-1}$, $y=-2x+1$



From $-\dfrac{2x+4}{x-1}=-2x+1$**, we get** $2x+4=(2x-1)(x-1)$,

$2x+4=2x^2-3x+1, 2x^2-5x-3=0,$

$(2x+1)(x-3)=0.$

Therefore, $x=-\dfrac{1}{2}, 3$.

When $x=-\dfrac{1}{2}$**, then** $y=-2\times\left(-\dfrac{1}{2}\right)+1=2$.

When $x=3$, $y=-2\times3+1=-5$.

Therefore, $x=-\dfrac{1}{2}$ **and** $y=2$ **or** $x=3$, $y=-5$.

So, the coordinates of the intersections are $\left(-\dfrac{1}{2}, 2\right), (3, -5)$.

$\left(-\dfrac{1}{2}, 2\right), (3, -5)$

How to solve fractional inequalities

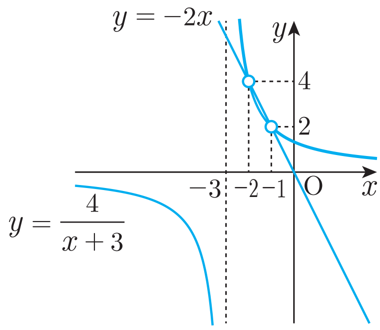
TARGET To understand how to solve fractional inequalities by using graphs.

STUDY GUIDE

How to solve fractional inequalities

We can find the solution of a **fractional inequality** by using the **positional relation of being higher or lower** in the graph, which corresponds to the **magnitude relation** of the inequality. Note that the value of x corresponding to the asymptote is not included in the solution.

Ex. Solve for $\dfrac{x-1}{x+3} < 2x+1$



From $\dfrac{x-1}{x+3} < 2x+1$, we get $-\dfrac{4}{x+3}+1 < 2x+1, \dfrac{4}{x+3} > -2x$. $\quad \ldots(\mathrm{i})$

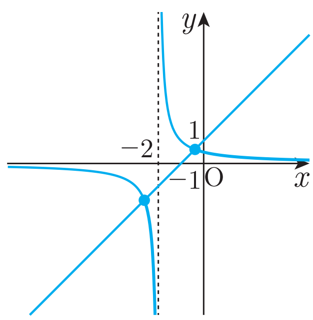
Here, given $y=\dfrac{4}{x+3}$ and $y=-2x$, whose graphs are in the figure on the right, by solving the equation $\dfrac{4}{x+3}=-2x$, we get $-2, -1$ the coordinates of the intersection of x.

The inequality (i) means that the graph of function $y=\dfrac{4}{x+3}$ is above line $y=-2x$.

Therefore, from the positional relation of the graph, we can see that the solution is $-3<x<-2, -1<x$.

EXERCISE

◆ Solve for inequality $\dfrac{1}{x+2} \leq x+1$.



Given $y=\dfrac{1}{x+2}$ and $y=x+1$, whose graphs are shown in the figure on the right.

The x coordinate for the intersection of these graphs comes from $\dfrac{1}{x+2}=x+1$, so we get $1=(x+1)(x+2), x^2+3x+1=0, x=\dfrac{-3\pm\sqrt{5}}{2}$.

To solve the inequality, find the range of the value of x that is under the line $y=x+1$ in the graph of the function $y=\dfrac{1}{x+2}$.

Therefore, we get $-\dfrac{3+\sqrt{5}}{2} \leq x < -2, \dfrac{-3+\sqrt{5}}{2} \leq x$.

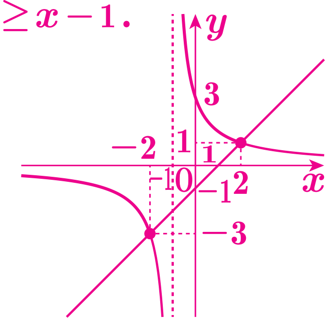
$$-\dfrac{3+\sqrt{5}}{2} \leq x < -2, \dfrac{-3+\sqrt{5}}{2} \leq x$$

PRACTICE

◆ Solve the following inequalities.

(1) $\dfrac{2x-1}{x+1} \leq -x+3$

From $\dfrac{2x-1}{x+1} \leq -x+3$, we get $-\dfrac{3}{x+1}+2 \leq -x+3$, $\dfrac{3}{x+1} \geq x-1$.



Now, given $y=\dfrac{3}{x+1}$ and $y=x-1$, whose graphs are shown in the figure on the right.

The x coordinate for the intersection of these graphs comes from $\dfrac{3}{x+1}=x-1$, so we get

$3=(x-1)(x+1), x^2=4, x=\pm 2$.

To solve the inequality, find the range of the value of x that is above the line $y=x-1$ in the graph of the function $y=\dfrac{3}{x+1}$.

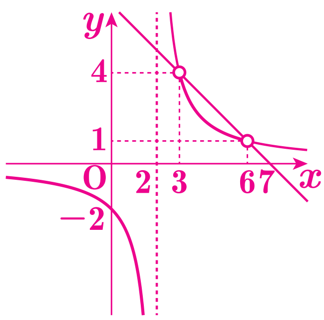
Therefore, we get $x \leq -2$, $-1 < x \leq 2$.

$x \leq -2$, $-1 < x \leq 2$

(2) $\dfrac{3x-2}{x-2} < -x+10$

From $\dfrac{3x-2}{x-2} < -x+10$, we get

$\dfrac{4}{x-2}+3 < -x+10$, $\dfrac{4}{x-2} < -x+7$.



Now, given $y=\dfrac{4}{x-2}$ and $y=-x+7$, whose graphs are shown in the figure on the right.

The x coordinate for the intersection of these graphs comes from $\dfrac{4}{x-2}=-x+7$, so we get $4=(-x+7)(x-2)$,

$x^2-9x+18=0, (x-3)(x-6)=0, x=3, 6$.

To solve the inequality, find the range of the value of x that is under the line $y=-x+7$ in the graph of the function $y=\dfrac{4}{x-2}$.

Therefore, we get $x<2$, $3<x<6$.

$x<2$, $3<x<6$

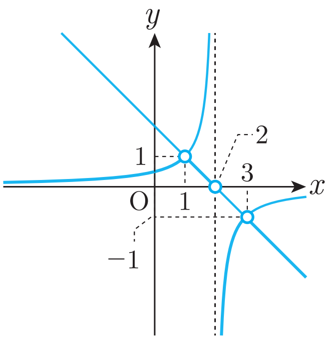
EXTRA Info.

Studying fractional inequalities

Use the Table, Equation, Inequality, and QR code functions.

EXERCISE

◆ Solve the inequality $-\dfrac{1}{x-2}>-x+2$.



Given $y=-\dfrac{1}{x-2}, y=-x+2$, we can see the graphs in the figure on the right.

The x coordinate for the intersection of these graphs comes

from $-\dfrac{1}{x-2}=-x+2$, so we get $-1=-(x-2)^2, x^2-4x+3=0,$
$(x-1)(x-3)=0, x=1,3$

To solve the inequality, find the range of the value of x that is above the line

$y=-x+2$ in the graph of the function $y=-\dfrac{1}{x-2}$.

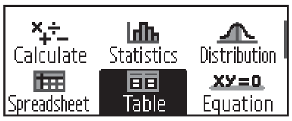
Therefore, we get $1<x<2$ and $3<x$.

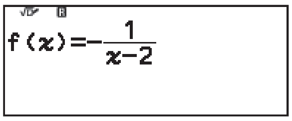
$1<x<2, 3<x$

check

Press ⌂, select [Table], press OK, then clear the previous data by pressing ⏻

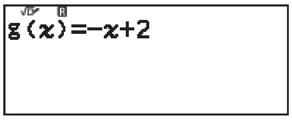
Press ⋯, select [Define f(x)/g(x)], press OK, select [Define f(x)], press OK, after inputting $f(x)=-\dfrac{1}{x-2}$, press EXE

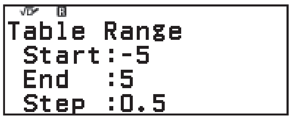




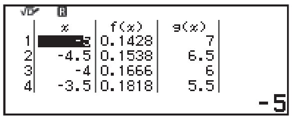
In the same way, input $g(x)=-x+2$.

Press ⋯, select [Table Range], press OK, after inputting [Start:−5, End:5, and Step:0.5], select [Execute], press EXE



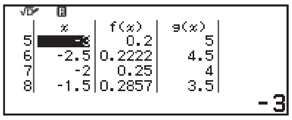


Press ⇧ x, scan the QR code to display a graph.



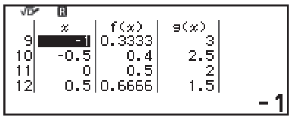
	x	f(x)	g(x)
1	-5	0.1428	7
2	-4.5	0.1538	6.5
3	-4	0.1666	6
4	-3.5	0.1818	5.5

-5



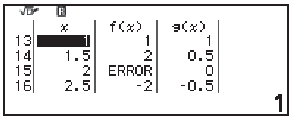
	x	f(x)	g(x)
5	-3	0.2	5
6	-2.5	0.2222	4.5
7	-2	0.25	4
8	-1.5	0.2857	3.5

-3



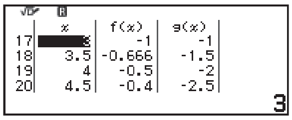
	x	f(x)	g(x)
9	-1	0.3333	3
10	-0.5	0.4	2.5
11	0	0.5	2
12	0.5	0.6666	1.5

-1



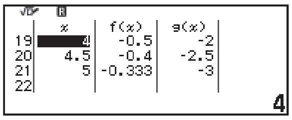
	x	f(x)	g(x)
13	1	1	1
14	1.5	2	0.5
15	2	ERROR	0
16	2.5	-2	-0.5

1



	x	f(x)	g(x)
17	3	-1	-1
18	3.5	-0.666	-1.5
19	4	-0.5	-2
20	4.5	-0.4	-2.5

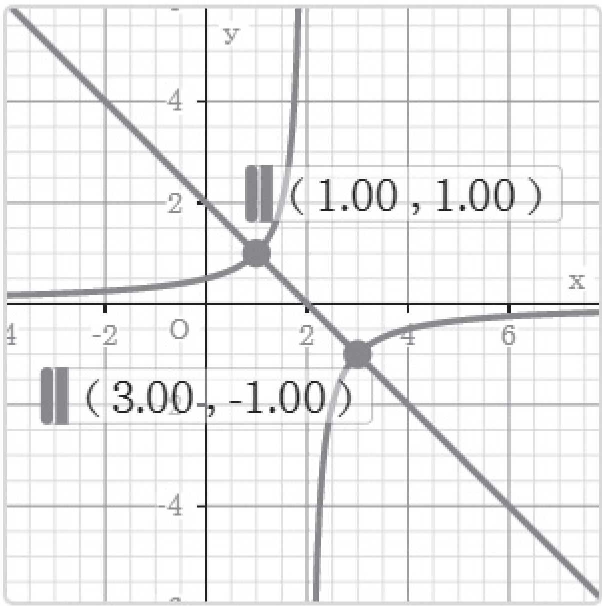
3



	x	f(x)	g(x)
19	4	-0.5	-2
20	4.5	-0.4	-2.5
21	5	-0.333	-3
22			

4

1/1



We can also consider the following.

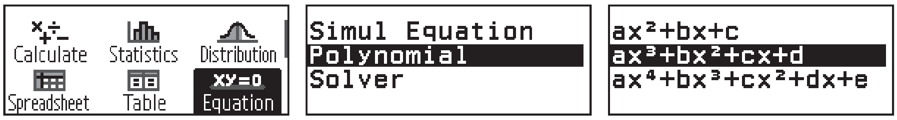
Multiply both sides of the given inequality $-\dfrac{1}{x-2} > -x+2$ by $(x-2)^2$

$$
\begin{aligned}
-(x-2) &> -(x-2)^3 \\
(x-2)\{(x-2)^2-1\} &> 0 \\
(x-2)(x^2-4x+3) &> 0 \\
x^3-6x^2+11x-6 &> 0 \quad ((x-1)(x-2)(x-3)>0)
\end{aligned}
$$

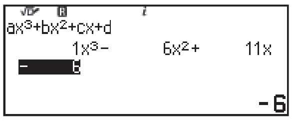
Now, use $\mathrm{Equation}$ to draw a graph to confirm the common points on the x axis, and then find the range of the solution to the inequality.

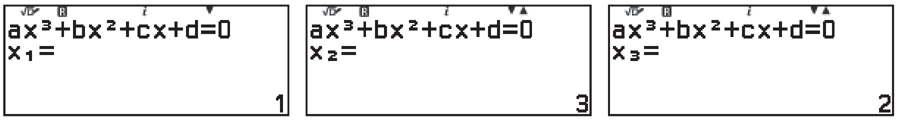
Press ⌂, select [Equation], press OK

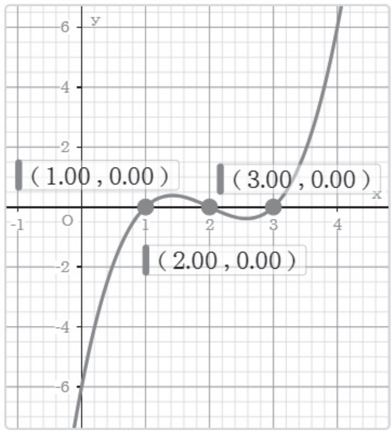
Select [Polynomial], press OK, select [$\mathrm{ax^3+bx^2+cx+d}$], press OK



1 EXE (−) 6 EXE 1 1 EXE (−) 6 EXE EXE EXE EXE







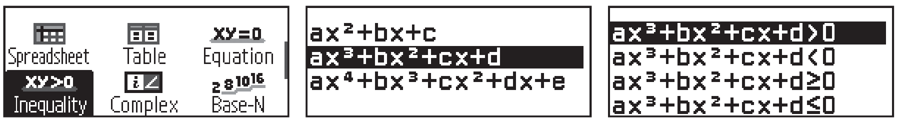
Press ⇧ x, scan the QR code to display a graph.

From the graph, we can find the solution to the inequality is $1<x<2$ and $3<x$.

You can also use $\mathrm{Inequality}$ to directly solve the cubic inequality and show the results in a diagram.

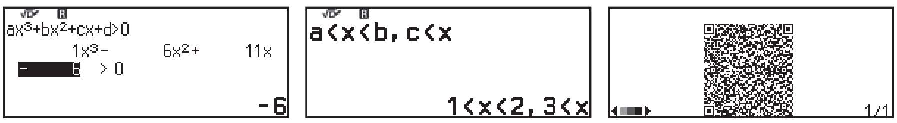
Press ⌂, select [Inequality], press OK

Select [$\mathrm{ax^3+bx^2+cx+d}$], press OK, select [$\mathrm{ax^3+bx^2+cx+d>0}$], press OK



1 EXE (−) 6 EXE 1 1 EXE (−) 6 EXE EXE

Press ⇧ x, scan the QR code to show the results in a diagram.

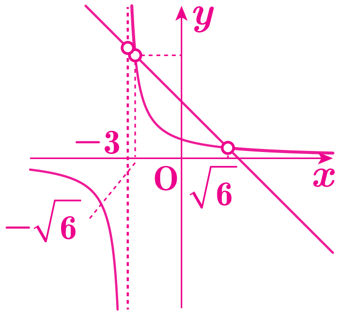


PRACTICE

◆ Solve the inequality $\dfrac{x}{x+3} < x-2$.

From $\dfrac{x}{x+3} < x-2$, we get $1-\dfrac{3}{x+3} < x-2$, $\dfrac{3}{x+3} > -x+3$

Now, given $y=\dfrac{3}{x+3}$, $y=-x+3$, we can see the graphs in the figure on the right.



The x coordinate for the intersection of these graphs comes from $\dfrac{3}{x+3}=-x+3$, so we get

$3=-(x-3)(x+3), x^2=6, x=\pm\sqrt{6}$

To solve the inequality, find the range of the value of x that is above the line $y=-x+3$ in the graph of the function $y=\dfrac{3}{x+3}$.

Therefore, we get $-3<x<-\sqrt{6}$, $\sqrt{6}<x$.

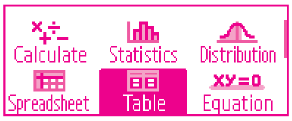
$-3<x<-\sqrt{6}$ and $\sqrt{6}<x$

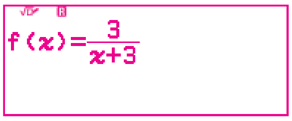
check

Press ⌂, select [Table], press OK, then clear the previous data by pressing ⏻

Press ⋯, select [Define f(x)/g(x)], press OK, select [Define f(x)], press OK

After inputting f(x)$=\dfrac{3}{x+3}$, press EXE

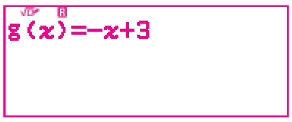


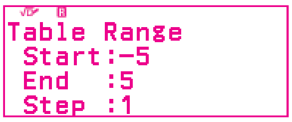


In the same way, input g(x)$=-x+3$.

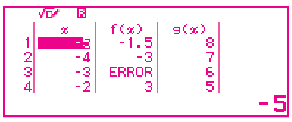
Press ⋯, select [Table Range], press OK

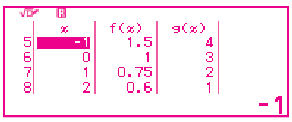
After inputting [Start:-5, End:5, and Step:1], select [Execute], press EXE

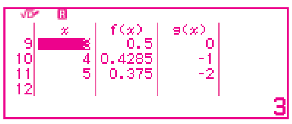


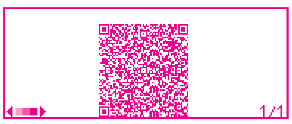


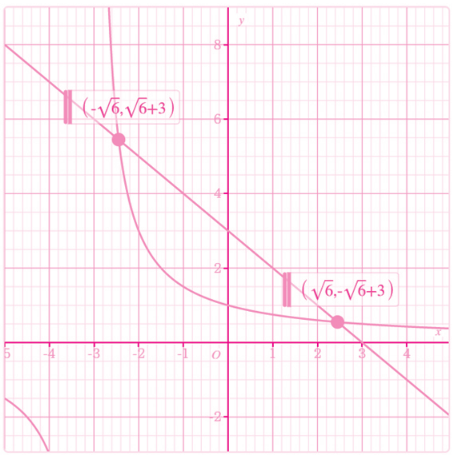
Press ⇧ x, scan the QR code to display a graph.











We can also consider the following.

Multiply both sides of the given inequality $\dfrac{x}{x+3} < x-2$ **by** $(x+3)^2$

$$x(x+3) < (x-2)(x+3)^2$$

$$(x+3)\{(x-2)(x+3)-x\} > 0$$

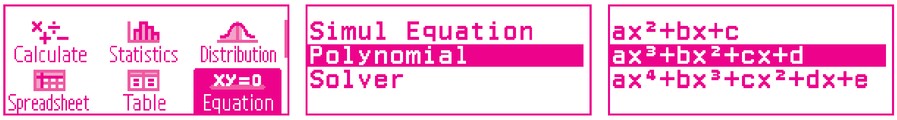
$$(x+3)(x^2-6) > 0$$

$$x^3+3x^2-6x-18 > 0 \quad ((x+3)(x+\sqrt{6})(x-\sqrt{6}) > 0)$$

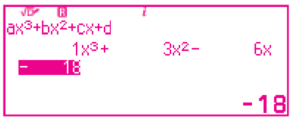
Now, use Equation **to draw a graph to confirm the common points on the** x **axis, and then find the range of the solution to the inequality.**

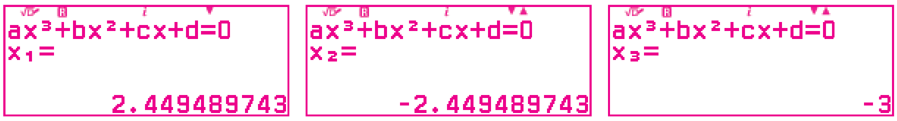
Press ⌂**, select** [Equation]**, press** OK

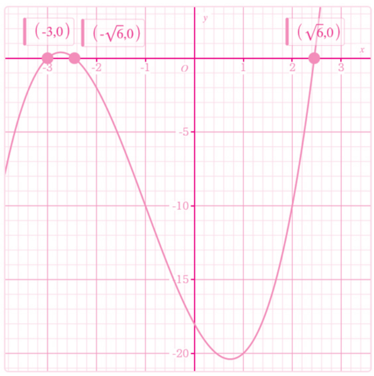
Select [Polynomial]**, press** OK**, select** [$\mathrm{ax^3+bx^2+cx+d}$]**, press** OK



1 EXE 3 EXE (−) 6 EXE (−) 1 8 EXE EXE EXE EXE







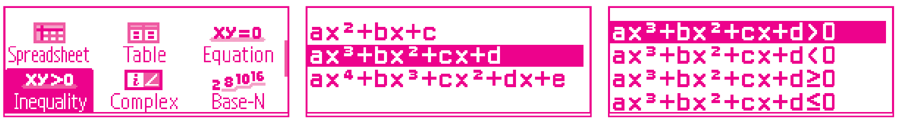
Press ⇧ (x)**, scan the** QR **code to display a graph.**

From the graph, we can find the solution to the inequality is $-3 < x < -\sqrt{6}$ **and** $\sqrt{6} < x$**.**

You can also use Inequality **to directly solve the cubic inequality and show the results in a diagram.**

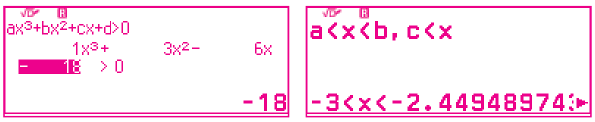
Press ⌂**, select** [Inequality]**, press** OK

Select [$\mathrm{ax^3+bx^2+cx+d}$]**, press** OK**, select** [$\mathrm{ax^3+bx^2+cx+d>0}$]**, press** OK



1 EXE 3 EXE (−) 6 EXE (−) 1 8 EXE EXE

Press ⇧ (x)**, scan the** QR **code to show the results in a diagram.**



Irrational function graphs

TARGET To understand the characteristics and general forms of irrational function graphs.

STUDY GUIDE

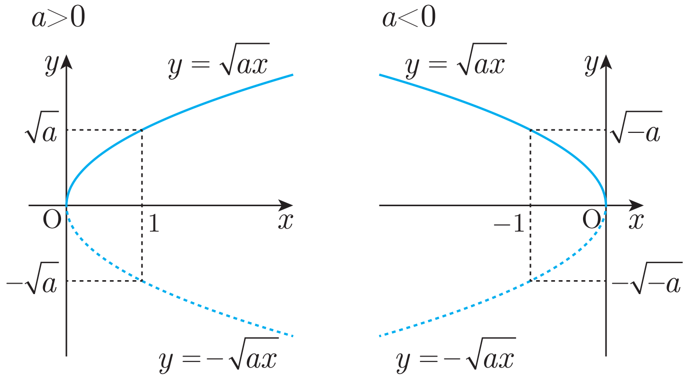
Irrational function graphs

Irrational functions

An expression that includes a letter under the root sign is called an **irrational expression**, and for a function of y, such that $y=\sqrt{x+1}$ or $y=4\sqrt{3x-1}$, when it appears in an irrational function of x, we say y is an **irrational function** of x. However, the range of values x that are negative under the root sign are not included in the domain.

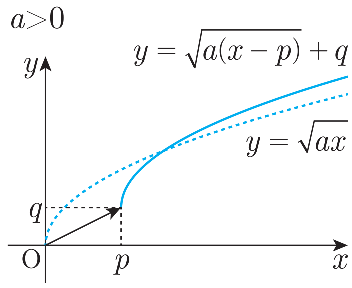
Graph of $y=\sqrt{ax}$

This is the part of a parabola $y^2=ax$, whose axis is the x axis and has its apex at the origin, where $y \geq 0$ (the part above the x axis).



Graph of $y=\sqrt{a(x-p)}+q$

The graph of $y=\sqrt{ax}$ was simply parallel translated by p along the x axis and by q along the y axis.



Graph of $y=\sqrt{ax+b}+q$

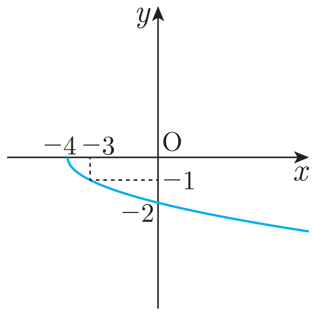
By enclosing with a, the coefficient of x, under the root sign, we change the expression to a $y=\sqrt{a\left(x+\frac{b}{a}\right)}+q$ format, so we can find how the graph of $y=\sqrt{ax}$ has parallel translated.

EXERCISE

◆ Draw the following irrational function graphs.

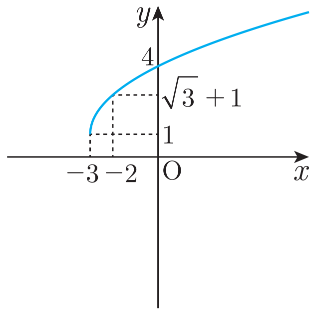
(1) $y=-\sqrt{x+4}$

From $y=-\sqrt{x+4}=-\sqrt{x-(-4)}$, the graph of $y=-\sqrt{x}$ was simply parallel translated by -4 along the x axis.



(2) $y=\sqrt{3x+9}+1$

From $y=\sqrt{3x+9}+1=\sqrt{3\{x-(-3)\}}+1$, the graph of $y=\sqrt{3x}$ was simply parallel translated by -3 along the x axis and by 1 along the y axis.

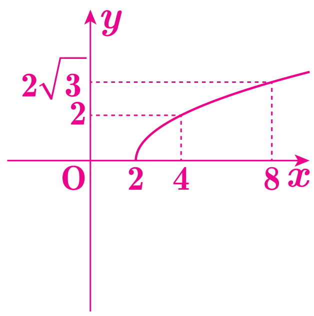


PRACTICE

◆ Draw the following irrational function graphs.

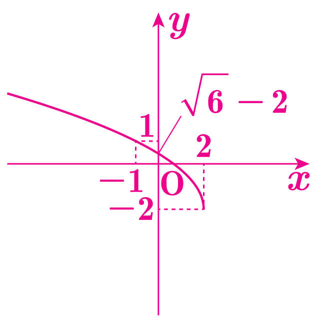
(1) $y=\sqrt{2x-4}$

From $y=\sqrt{2x-4}=\sqrt{2(x-2)}$, the graph of $y=\sqrt{2x}$ was simply parallel translated by 2 along the x axis.



(2) $y=\sqrt{-3x+6}-2$

From $y=\sqrt{-3x+6}-2=\sqrt{-3(x-2)}-2$, the graph of $y=\sqrt{-3x}$ was simply parallel translated by 2 along the x axis and by -2 along the y axis.



Irrational function graphs and common points of lines

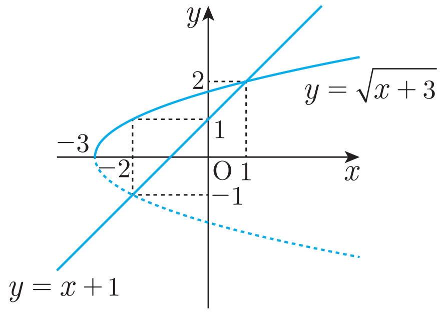
TARGET To understand how the coordinates of intersections of lines and irrational function graphs correspond to real roots of simultaneous equations.

STUDY GUIDE

Solving simultaneous equations and coordinates of common points on 2 graphs

By solving simultaneous equations, it is possible to find the coordinates of the common points of 2 graphs, and then to use the graph to solve the **irrational equation** backwards.

Ex. Graph of function $y=\sqrt{x+3}$ and coordinates of intersections of a line $y=x+1$.



From the above graph, we see the coordinates of the intersection are $(1, 2)$.

Correspond ⟷

Real root of simultaneous equation $\begin{cases} y=\sqrt{x+3} \\ y=x+1 \end{cases}$

Rearrange this by squaring both sides of $\sqrt{x+3}=x+1$.

$x+3=(x+1)^2, x+3=x^2+2x+1,$

$x^2+x-2=0, (x-1)(x+2)=0.$

Therefore, $x=1, -2$

Now, from $\sqrt{x+3}\geq 0$, we get $y\geq 0$, so from $x+1\geq 0$, we get $x\geq -1$

Therefore, $x=1$ and $y=2$

EXERCISE

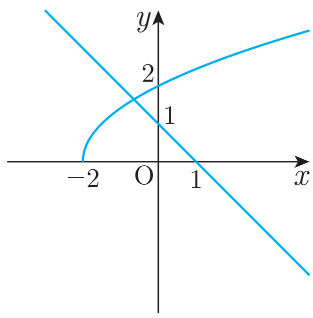
◆ Find the coordinates of the intersection of the graph of function $y=\sqrt{2x+4}$ and line $y=-x+1$.

Rearrange this by squaring both sides of $\sqrt{2x+4}=-x+1$.

$2x+4=(-x+1)^2, 2x+4=x^2-2x+1, x^2-4x-3=0$.

Therefore, $x=2\pm\sqrt{7}$.

Now, from $y=\sqrt{2x+4}=\sqrt{2(x+2)}$, the irrational function graph and line are shown in the figure on the right.



Therefore, $x=2+\sqrt{7}$ is not suitable, so we get

$x=2-\sqrt{7}, y=-x+1=-1+\sqrt{7}$.

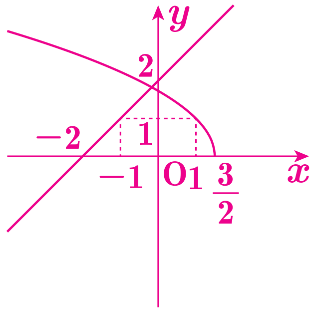
So, the coordinates of the intersections are $(2-\sqrt{7}, -1+\sqrt{7})$.

$\mathbf{(2-\sqrt{7}, -1+\sqrt{7})}$

PRACTICE

1 Find the coordinates of the intersection of the functional graph and line below.

(1) $y=\sqrt{3-2x}, y=x+2$



Rearrange this by squaring both sides of $\sqrt{3-2x}=x+2$.

$3-2x=(x+2)^2, 3-2x=x^2+4x+4, x^2+6x+1=0$.

Therefore, $x=-3\pm2\sqrt{2}$.

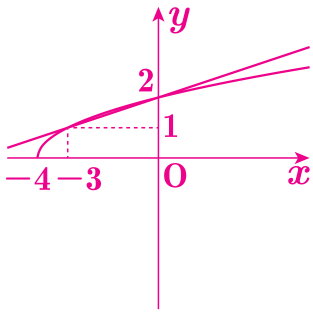
Now, from $y=\sqrt{3-2x}=\sqrt{-2\left(x-\frac{3}{2}\right)}$, the irrational function graph and line are shown in the figure on the right.

Therefore, $x=-3-2\sqrt{2}$ is not suitable, so we get $x=-3+2\sqrt{2}, y=-1+2\sqrt{2}$.

So, the coordinates of the intersections are $(-3+2\sqrt{2}, -1+2\sqrt{2})$.

$$(-3+2\sqrt{2}, -1+2\sqrt{2})$$

(2) $y=\sqrt{x+4}, y=\frac{1}{3}x+2$



Rearrange this by multiplying by 3 and squaring both sides of $\sqrt{x+4}=\frac{1}{3}x+2$.

$3\sqrt{x+4}=x+6, 9(x+4)=(x+6)^2$,

$9x+36=x^2+12x+36, x^2+3x=0, x(x+3)=0$.

Therefore, $x=0, -3$

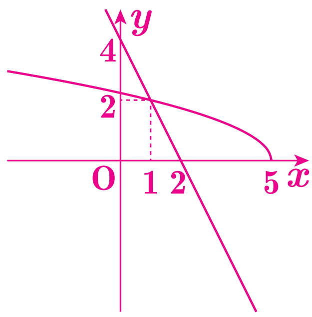
Now, the irrational function graph and line are shown in the figure on the right, and the 2 real roots solve the irrational equation.

Therefore, when $x=0$ then $y=2$, and when $x=-3$ then $y=1$.

So, the coordinates of the intersections are $(0, 2)$ and $(-3, 1)$.

$$(0, 2), (-3, 1)$$

2 Use the graph to solve the irrational equation $\sqrt{5-x}=-2x+4$.



The irrational function $y=\sqrt{5-x}$ and line $y=-2x+4$ are shown in the figure on the right expressed as a graph. The solution to the equation is $x=1$ because it corresponds to the x coordinate of the common points of the 2 graphs.

$$x=1$$

How to solve irrational inequalities

TARGET To understand how to solve irrational inequalities by using graphs.

STUDY GUIDE

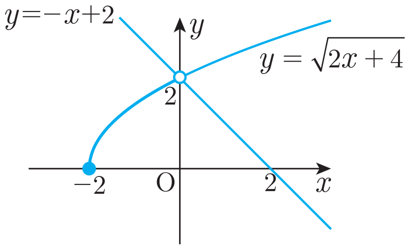
How to solve irrational inequalities

We can find the solution of an **irrational inequality** by using the **positional relation of being higher or lower** in the graph, which corresponds to the **magnitude relation** of the inequality. Note that a value of x that becomes a negative under the root sign is not a solution.

Ex. Solve for $\sqrt{2x+4} < -x+2$

Given $y=\sqrt{2x+4}=\sqrt{2(x+2)}$ and $y=-x+2$.

These graphs are shown in the figure on the right.



Furthermore, by solving the equation $\sqrt{2x+4}=-x+2$, we can find that the x coordinate of the intersection in the graph is $x=0$.

The inequality means that the graph of the irrational function $y=\sqrt{2x+4}$ is below line $y=-x+2$.

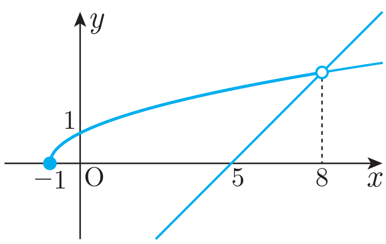
Therefore, from the positional relation of the graph, we can see that the solution is $-2 \leq x < 0$.

EXERCISE

◆Solve for inequality $x-\sqrt{x+1}<5$.

From $x-\sqrt{x+1}<5$, we can get $\sqrt{x+1}>x-5$.

Now, given $y=\sqrt{x+1}$ and $y=x-5$, whose graphs are shown in the figure on the right.



By solving $\sqrt{x+1}=x-5$, we can find the x coordinate of the intersection in these graphs.

$x+1=(x-5)^2, x+1=x^2-10x+25, x^2-11x+24=0,$
$(x-3)(x-8)=0, x=3, 8$.

From the figure on the right, we see that $x=3$ is not suitable, so we have $x=8$.

To solve the inequality, find the range of the value of x that is above the line $y=x-5$ in the graph of the irrational function $y=\sqrt{x+1}$.

Therefore, we get $-1 \leq x < 8$.

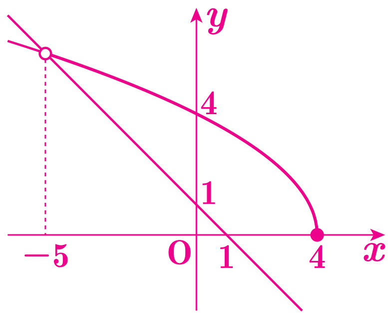
$\mathbf{-1 \leq x < 8}$

PRACTICE

◆ Solve the following inequalities.

(1) $2\sqrt{4-x}+x>1$

From $2\sqrt{4-x}+x>1$, we can get
$2\sqrt{4-x}>-x+1$.
Now, given $y=2\sqrt{4-x}$ and $y=-x+1$, whose graphs are shown in the figure on the right.
By solving $2\sqrt{4-x}=-x+1$, we can find the x coordinate of the intersection in these graphs.

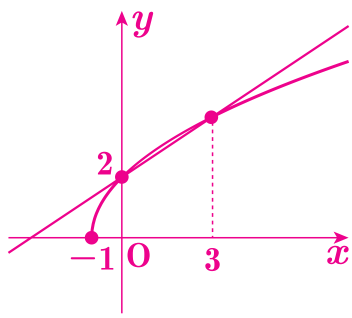


$4(4-x)=(-x+1)^2, 16-4x=x^2-2x+1,$
$x^2+2x-15=0, (x-3)(x+5)=0, x=3, -5.$
From the figure on the right, we see that $x=3$ is not suitable, so we have $x=-5$.
To solve the inequality, find the range of the value of x that is above the line $y=-x+1$ in the graph of the irrational function $y=2\sqrt{4-x}$.
Therefore, we get $-5<x\leq 4$.

$-5<x\leq 4$

(2) $\sqrt{4x+4}\leq\dfrac{2}{3}x+2$

Given $y=\sqrt{4x+4}=2\sqrt{x+1}, y=\dfrac{2}{3}x+2$, whose graphs are shown in the figure on the right.
By solving $2\sqrt{x+1}=\dfrac{2}{3}x+2$, we can find the x coordinate of the intersection in these graphs.



$3\sqrt{x+1}=x+3, 9(x+1)=(x+3)^2,$
$9x+9=x^2+6x+9, x^2-3x=0, x(x-3)=0, x=0, 3.$
To solve the inequality, find the range of the value of x that is below the line $y=\dfrac{2}{3}x+2$ in the graph of the irrational function $y=\sqrt{4x+4}$.
Therefore, we get $-1\leq x\leq 0, 3\leq x$.

$-1\leq x\leq 0, 3\leq x$

EXTRA Info.

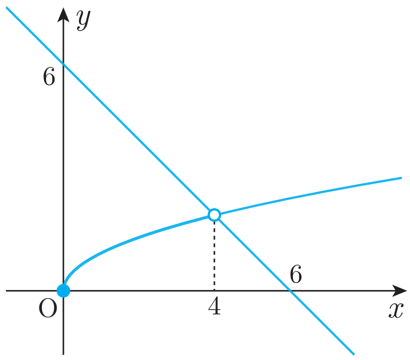
Studying irrational inequalities

Use the Table and QR code functions.

EXERCISE

◆ Solve the inequality $\sqrt{x} < -x+6$.

Given $y=\sqrt{x}$ and $y=-x+6$, we can see the graphs in the figure on the right.



By solving $\sqrt{x} = -x+6$, we can find the x coordinate of the intersection in these graphs.

$x=(-x+6)^2, x=x^2-12x+36, x^2-13x+36=0,$
$(x-4)(x-9)=0, x=4, 9$

From the figure on the right, we see that x=9 is not suitable, so we have x=4.

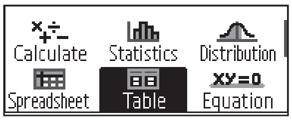
To solve the inequality, find the range of the value of x that is below the line $y=-x+6$ in the graph of the irrational function $y=\sqrt{x}$.

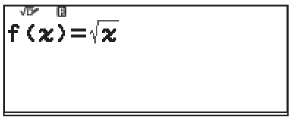
Therefore, we get $0 \le x < 4$. **$0 \le x < 4$**

check

Press ⌂, select [Table], press OK, then clear the previous data by pressing ⏻.

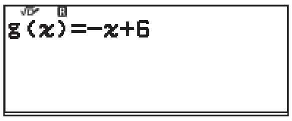
Press ⋯, select [Define f(x)/g(x)], press OK, select [Define f(x)], press OK, after inputting f(x)=$\sqrt{x}$, press EXE.

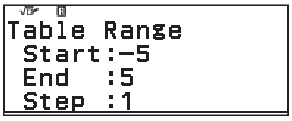




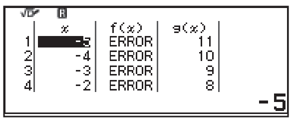
In the same way, input g(x)=$-x+6$.

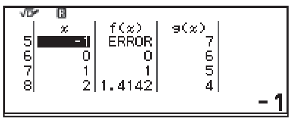
Press ⋯, select [Table Range], press OK, after inputting [Start:−5, End:5, and Step:1], select [Execute], press EXE.

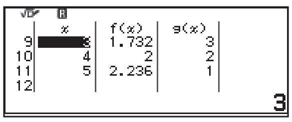


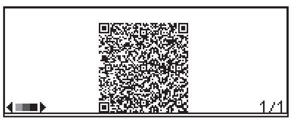


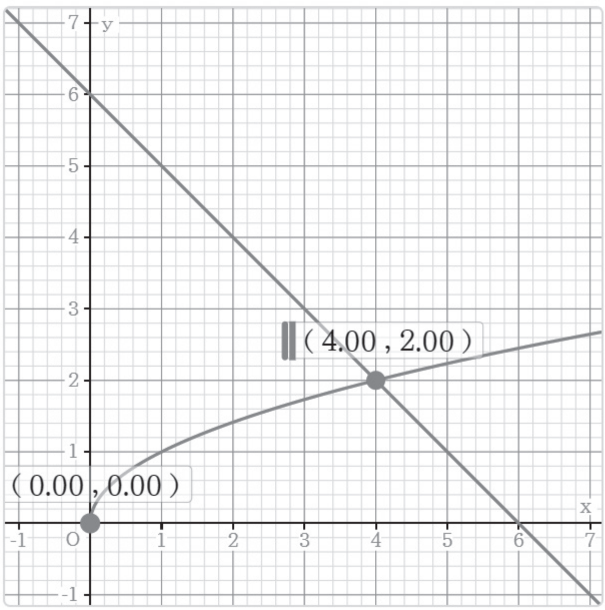
Press ⇧ ⓧ, scan the QR code to display a graph.







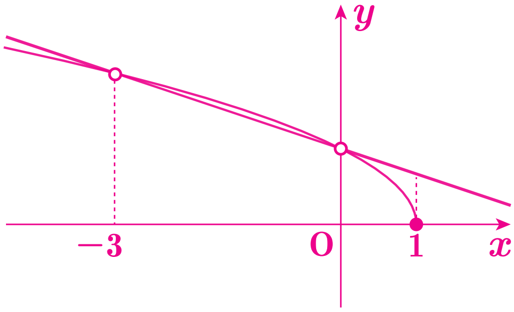




PRACTICE

◆ Solve the inequality $\sqrt{1-x}<-\frac{1}{3}x+1$.

Given $y=\sqrt{1-x}, y=-\frac{1}{3}x+1$, we can see the graphs in the figure on the right.



By solving $\sqrt{1-x}=-\frac{1}{3}x+1$, we can find the x coordinate of the intersection in these graphs.

$3\sqrt{1-x}=-x+3, 9(1-x)=(-x+3)^2, 9-9x=x^2-6x+9,$

$x^2+3x=0, x(x+3)=0, x=0,-3$

To solve the inequality, find the range of the value of x that is below the line $y=-\frac{1}{3}x+1$ in the graph of the irrational function $y=\sqrt{1-x}$.

Therefore, we get $x<-3$ and $0<x\leq 1$.

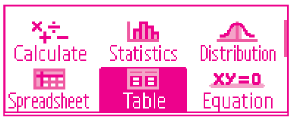
$x<-3, 0<x\leq 1$

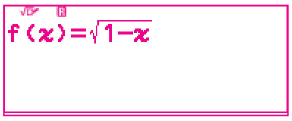
check

Press ⌂, select [Table], press OK, then clear the previous data by pressing ⏻

Press ⋯, select [Define f(x)/g(x)], press OK, select [Define f(x)], press OK

After inputting f(x)$=\sqrt{1-x}$, press EXE

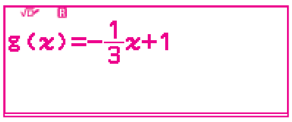


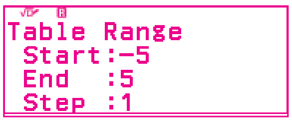


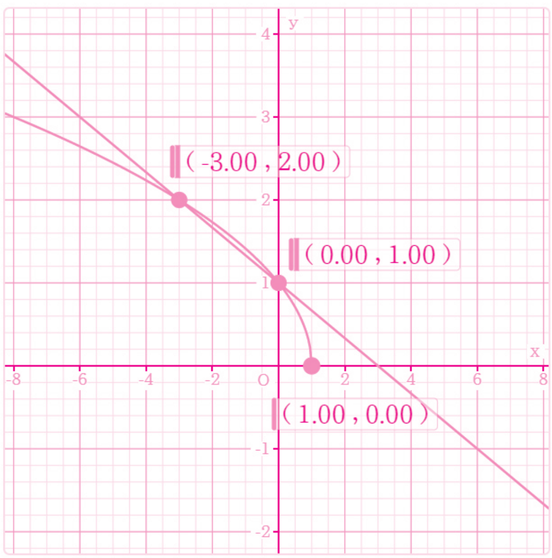
In the same way, input g(x)$=-\frac{1}{3}x+1$.

Press ⋯, select [Table Range], press OK

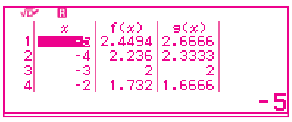
After inputting [Start:−5, End:5, and Step:1], select [Execute], press EXE

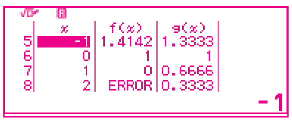


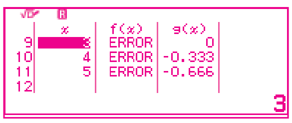


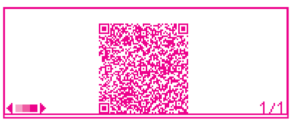


Press ⇧ x, scan the QR code to display a graph.









Inverse functions and composite functions

TARGET To understand the properties of inverse functions and composite functions.

STUDY GUIDE

Inverse functions

Definition of a function

In sets X and Y with real numbers as elements, let x be elements of X and y be elements of Y. In these sets, if you take an element x for which **only 1** element y corresponds to x, as determined based on a rule f, then y is said to **be a function of x**, and is expressed as $y = f(x)$.

1-to-1 functions

In the function $f(x)$, when different values of y always correspond to different values of x, the function $y=f(x)$ is said to be a **1-to-1 function**. In other words, we can say that the function $y=f(x)$ is a 1-to-1 function when the following is always true.

$x_1, x_2 (x_1 \neq x_2)$ that satisfies $f(x_1) = f(x_2)$ does not exist.
($x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$)

Inverse functions

When $y=f(x)$ is a 1-1 function, x is a function of y because once a value is determined for y, only 1 value can be determined for x. When this correspondence is expressed as $x=g(y)$ by transforming $y=f(x)$, this function g is called the **inverse function** of f, and is expressed as f^{-1} (**inverse**).

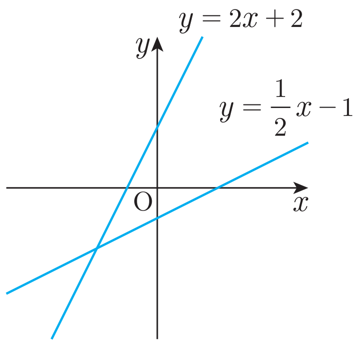
In other words, as $y = f(x) \Leftrightarrow x = f^{-1}(y)$, we say that $f^{-1}(x)$ is the **inverse function** of the function $f(x)$.

Ex. Find the inverse function of $y = \frac{1}{2}x - 1$ …(i), which is a function with 1-1 correspondence.

In the function of (i), once the value for y is determined, only 1 value can be determined for x.

From (i), we can express this correspondence as $x = 2y + 2$.

This is $x = f^{-1}(y)$, so by replacing the variables x and y, we get the inverse function $y = f^{-1}(x)$, from which we can derive $y = 2x + 2$.



How to find inverse functions

When a function $f(x)$ has 1-to-1 correspondence

(1) Solve $y = f(x)$ for x. $x = f^{-1}(y)$

(2) Replace variables x and y. $y = f^{-1}(x)$

Properties of inverse functions

(1) For $y = f(x)$ and $y = f^{-1}(x)$, the domain and range are replaced.

(2) The graph of $y = f(x)$ and the graph of $y = f^{-1}(x)$ are symmetric around the line $y=x$.

(3) When $f^{-1}(x) = g(x)$, then $g^{-1}(x) = f(x)$.
(The inverse function of an inverse function is the original function.)

Composite functions

When the range of $f(x)$ is included in the domain of $g(x)$ in 2 functions $f(x)$ and $g(x)$, we can consider a new function $g\big(f(x)\big)$ by substituting $f(x)$ into x of $g(x)$. This is called a **composite function** of $f(x)$ and $g(x)$, and is expressed as $(g \circ f)(x)$.

$$(g \circ f)(x) = g\big(f(x)\big) \quad \text{and generally} \quad (f \circ g)(x) \neq (g \circ f)(x)$$

explanation

When $f(x) = x^2 - 4, g(x) = 3x + 1$, we should consider $(f \circ g)(x)$ and $(g \circ f)(x)$.

$(f \circ g)(x) = f\big(g(x)\big) = (3x+1)^2 - 4 = 9x^2 + 6x - 3$.

$(g \circ f)(x) = g\big(f(x)\big) = 3(x^2 - 4) + 1 = 3x^2 - 11$.

Therefore, we get $(f \circ g)(x) \neq (g \circ f)(x)$.

Composite functions and inverse functions

From the definition of an inverse function, when $y=f(x)$, then $x = f^{-1}(y)$.

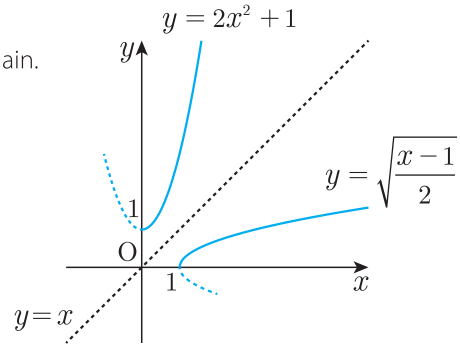
Therefore, we get $(f^{-1} \circ f)(x) = f^{-1}\big(f(x)\big) = f^{-1}(y) = x$.

Furthermore, we have $(f \circ f^{-1})(y) = f\big(f^{-1}(y)\big) = f(x) = y$, so we can derive the following.

$$(f^{-1} \circ f)(x) = x, (f \circ f^{-1})(y) = y$$

EXERCISE

1 Find the inverse function of the function $y = 2x^2 + 1 \quad (x \geq 0)$, and its domain.



From $y = 2x^2 + 1$, we can get $x^2 = \dfrac{y-1}{2}$.

Since $x \geq 0$, we get $x = \sqrt{\dfrac{y-1}{2}}$.

By replacing x and y, we get the inverse function $y = \sqrt{\dfrac{x-1}{2}}$.

Furthermore, the domain of the original function $y = 2x^2 + 1$ is $y \geq 1$, so the domain of the inverse function is $x \geq 1$.

$$\boldsymbol{y = \sqrt{\frac{x-1}{2}}, x \geq 1}$$

2 For $f(x) = 3x, g(x) = x + 2, h(x) = 2x^2 - 1$, find the following composite functions.

(1) $(f \circ g)^{-1}(x)$

$(f \circ g)(x) = f\big(g(x)\big) = f(x+2) = 3(x+2) = 3x + 6$.

Now, given $y = 3x + 6$, then $x = \dfrac{y-6}{3}$, so the inverse function is $y = \dfrac{1}{3}x - 2$.

Therefore, we get $(f \circ g)^{-1}(x) = \dfrac{1}{3}x - 2$.

$$\boldsymbol{(f \circ g)^{-1}(x) = \frac{1}{3}x - 2}$$

(2) $(g^{-1} \circ f^{-1})(x)$

Given $y = 3x$ then $x = \dfrac{1}{3}y$, which gives us $f^{-1}(x) = \dfrac{1}{3}x$.

Given $y = x + 2$ then $x = y - 2$, which gives us $g^{-1}(x) = x - 2$.

Therefore, we get $(g^{-1} \circ f^{-1})(x) = g^{-1}\big(f^{-1}(x)\big) = \dfrac{1}{3}x - 2$.

$$\boldsymbol{(g^{-1} \circ f^{-1})(x) = \frac{1}{3}x - 2}$$

(3) $\big((f \circ g) \circ h\big)(x)$

From (1), we get $(f \circ g)(x) = 3x + 6$.

Therefore, we get $\big((f \circ g) \circ h\big)(x) = (f \circ g)(2x^2 - 1) = 3(2x^2 - 1) + 6 = 6x^2 + 3$.

$$\boldsymbol{\big((f \circ g) \circ h\big)(x) = 6x^2 + 3}$$

(4) $\big(f \circ (g \circ h)\big)(x)$

$(g \circ h)(x) = g\big(h(x)\big) = g(2x^2 - 1) = (2x^2 - 1) + 2 = 2x^2 + 1$.

Therefore, we get $\big(f \circ (g \circ h)\big)(x) = f(2x^2 + 1) = 3(2x^2 + 1) = 6x^2 + 3$.

$$\boldsymbol{\big(f \circ (g \circ h)\big)(x) = 6x^2 + 3}$$

EXTRA Info.

The above items (1) and (2) and items (3) and (4) are examples of the following properties.

$$\boldsymbol{(f \circ g)^{-1}(x) = (g^{-1} \circ f^{-1})(x), \ \big((f \circ g) \circ h\big)(x) = \big(f \circ (g \circ h)\big)(x)}$$

PRACTICE

1 Find the inverse functions, and their domains, of the following functions.

(1) $y = 2x - 3 \ (0 \leq x < 2)$

From $y = 2x - 3$, we can get $x = \dfrac{y+3}{2}$. Therefore, the inverse function is $y = \dfrac{x+3}{2}$.

Furthermore, the domain of the original function is $-3 \leq y < 1$, so the domain of the inverse function is $-3 \leq x < 1$.

$$y = \frac{x+3}{2},\ -3 \leq x < 1$$

(2) $y = 3^{x-2}$

From $y = 3^{x-2}$, we can get $x - 2 = \log_3 y, x = \log_3 y + 2$.

Therefore, the inverse function is $y = \log_3 x + 2$.

Furthermore, the domain of the original function is $y > 0$, so the domain of the inverse function is $x > 0$.

$$y = \log_3 x + 2,\ x > 0$$

2 Find the composite functions $(f \circ g)(x), (g \circ f)(x)$ for each of the following functions $f(x)$ and $g(x)$.

(1) $f(x) = x(2x+1), g(x) = \dfrac{1}{2}x - 3$

$$(f \circ g)(x) = f\left(\frac{1}{2}x - 3\right) = \left(\frac{1}{2}x - 3\right)(x-5) = \frac{1}{2}x^2 - \frac{11}{2}x + 15$$

$$(g \circ f)(x) = g\left(x(2x+1)\right) = \frac{1}{2}x(2x+1) - 3 = x^2 + \frac{1}{2}x - 3$$

$$(f \circ g)(x) = \frac{1}{2}x^2 - \frac{11}{2}x + 15, (g \circ f)(x) = x^2 + \frac{1}{2}x - 3$$

(2) $f(x) = \log_{27} x, g(x) = 3^x$

$$(f \circ g)(x) = f(3^x) = \log_{27} 3^x = x \cdot \frac{\log_3 3}{\log_3 27} = \frac{1}{3}x$$

$$(g \circ f)(x) = g(\log_{27} x) = 3^{\log_{27} x}$$

Now, from $\log_{27} x = \dfrac{\log_3 x}{\log_3 27} = \dfrac{1}{3}\log_3 x = \log_3 x^{\frac{1}{3}}$, we get $3^{\log_{27} x} = 3^{\log_3 x^{\frac{1}{3}}} = x^{\frac{1}{3}}$.

Therefore, we get $(g \circ f)(x) = x^{\frac{1}{3}}$.

$$(f \circ g)(x) = \frac{1}{3}x, (g \circ f)(x) = x^{\frac{1}{3}}$$

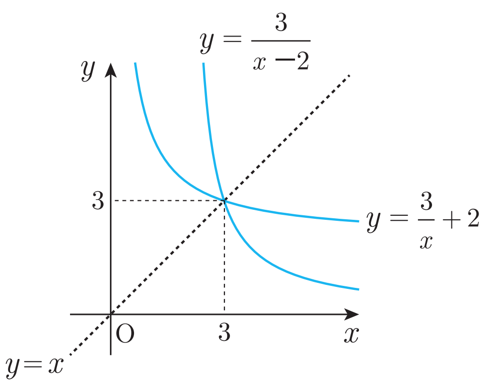
EXTRA Info.

Studying inverse functions and composite functions

Use the Table, Spreadsheet, and QR code functions.

EXERCISE

1 Find the inverse function of the function $y=\dfrac{3}{x}+2\ (x>0)$, and find its domain.



From $y=\dfrac{3}{x}+2$, we get $\dfrac{3}{x}=y-2$.

The range of the origin function is $y>2$, so we get $x=\dfrac{3}{y-2}$.

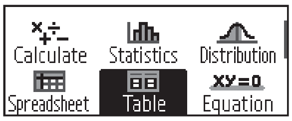
By replacing x and y, we get the inverse function $y=\dfrac{3}{x-2}\ (x>2)$.

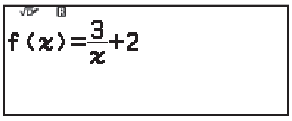
$$\boldsymbol{y=\frac{3}{x-2},\quad x>2}$$

check

Press ⌂, select [Table], press OK, then clear the previous data by pressing ⏻

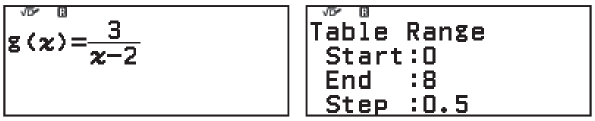
Press ⋯, select [Define f(x)/g(x)], press OK, select [Define f(x)], press OK, after inputting f(x)$=\dfrac{3}{x}+2$, press EXE





In the same way, input g(x)$=\dfrac{3}{x-2}$.

Press ⋯, select [Table Range], press OK, after inputting [Start:0, End:8, and Step:0.5], select [Execute], press EXE



	x	f(x)	g(x)
1	0	ERROR	-1.5
2	0.5	8	-2
3	1	5	-3
4	1.5	4	-6

	x	f(x)	g(x)
5	2	3.5	ERROR
6	2.5	3.2	6
7	3	3	3
8	3.5	2.8571	2

	x	f(x)	g(x)
9	4	2.75	1.5
10	4.5	2.6666	1.2
11	5	2.6	1
12	5.5	2.5454	0.8571

	x	f(x)	g(x)
13	6	2.5	0.75
14	6.5	2.4615	0.6666
15	7	2.4285	0.6
16	7.5	2.4	0.5454

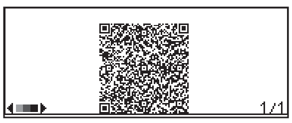
	x	f(x)	g(x)
15	7	2.4285	0.6
16	7.5	2.4	0.5454
17	8	2.375	0.5
18			

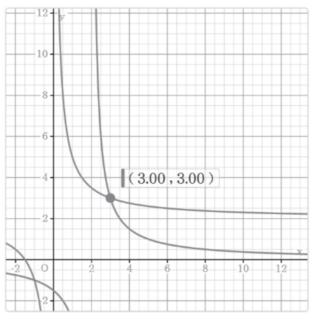
When $x=0.5$, then $f(0.5)=8$; this time, we get $g(8)=0.5$.

When $x=1$, then $f(1)=5$; this time, we get $g(5)=1$.

The same holds for any x, so $f(x)$ and $g(x)$ are inverse functions of each other.

Press ⇧ x, scan the QR code to display a graph.





2 For $f(x) = x + 2, g(x) = 2x - 1$, find the composite function $(g \circ f)(x)$.

$(g \circ f)(x) = g\big(f(x)\big) = g(x+2) = 2(x+2) - 1 = 2x + 3$

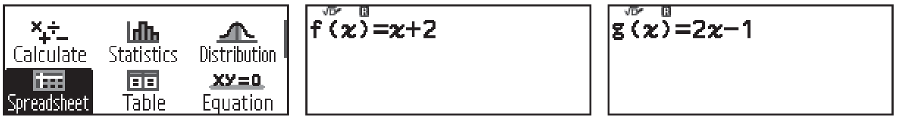
$(g \circ f)(x) = 2x + 3$

check

Press ⌂, select [Spreadsheet], press OK, then clear the previous data by pressing ⏻

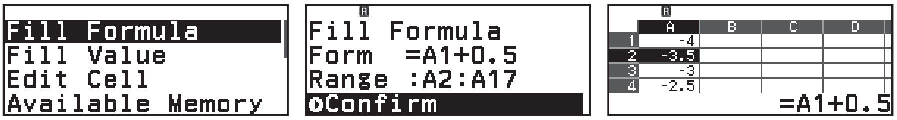
Press f(x), select [Define f(x)], press OK, press x ⊕ ② EXE

In the same way, press f(x), select [Define g(x)]press OK, press ② x ⊖ ① EXE



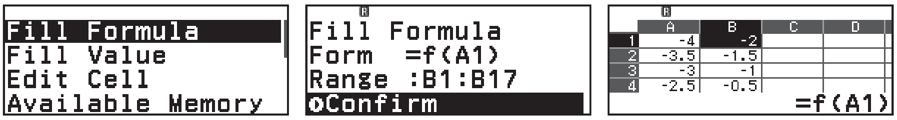
Input -4 in cell A1, then use [Fill Formula] to fill cells A2 to A17 with values from -3.5 to 4 in increments of 0.5.

⊖ ④ EXE ⋯ OK ⬆ ④ ① ⊕ ⓪ ⊙ ⑤ EXE ▷ ▷ ▷ ▷ ▷ ▷ ⌫ ① ⑦ EXE EXE



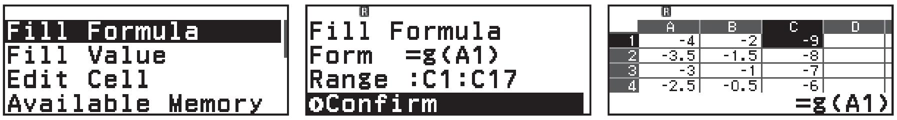
Let the values in column A be the values of x, then use [Fill Formula] so the values of f(x), corresponding to each value of x, appear in column B.

△ ▷ ⋯ OK f(x) OK ⬆ ④ ① ⟩ EXE ▷ ▷ ▷ ▷ ▷ ▷ ⑦ EXE EXE



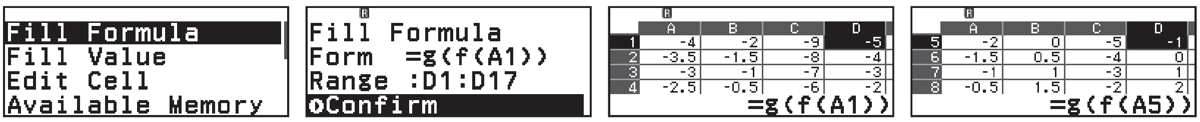
Let the values in column A be the values of x, then use [Fill Formula] so the values of g(x), corresponding to each value of x, appear in column C.

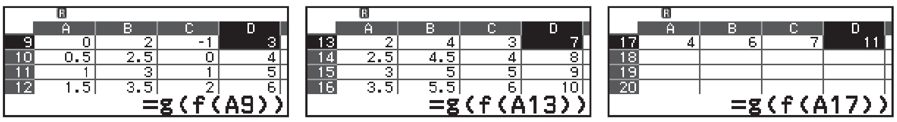
▷ ⋯ OK f(x) ▽ OK ⬆ ④ ① ⟩ EXE ▷ ▷ ▷ ▷ ▷ ▷ ⑦ EXE EXE



Let the values in column A be the values of x, then use [Fill Formula] so the values of $(g \circ f)(x)$, corresponding to each value of x, appear in column D.

▷ ⋯ OK f(x) ▽ OK f(x) OK ⬆ ④ ① ⟩ ⟩ EXE ▷ ▷ ▷ ▷ ▷ ▷ ⑦ EXE EXE





In the figure above $f(-4) = -2$ and $g(-2) = -5$, so we get $(g \circ f)(-4) = -5$.

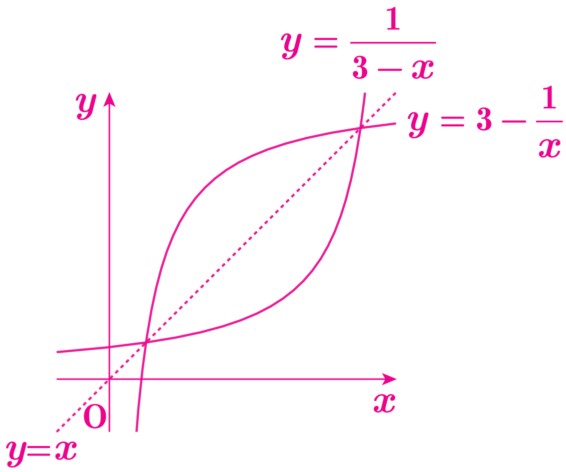
PRACTICE

1 Find the inverse function of the function $y = 3 - \frac{1}{x}$ ($x>0$), and find its domain.

From $y = 3 - \frac{1}{x}$**, we get** $\frac{1}{x} = 3 - y$**.**

The range of the origin function is $y<3$**, so we get** $x = \frac{1}{3-y}$**.**

By replacing x **and** y**, we get the inverse function** $y = \frac{1}{3-x}$ **(**$x<3$**).**



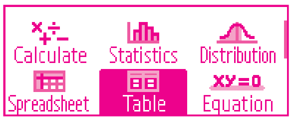
$$y = \frac{1}{3-x}\ (x<3)$$

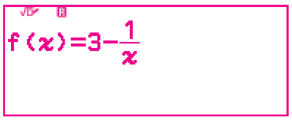
check

Press ⌂**, select** [Table]**, press** OK**, then clear the previous data by pressing** ⏻

Press ⋯**, select** [Define f(x)/g(x)]**, press** OK**, select** [Define f(x)]**, press** OK

After inputting f(x)$=3-\frac{1}{x}$**, press** EXE

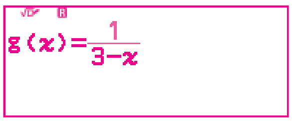


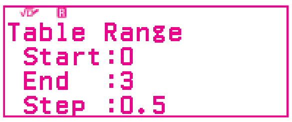


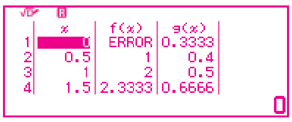
In the same way, input g(x)$=\frac{1}{3-x}$**.**

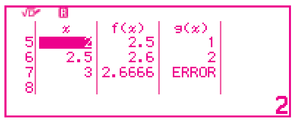
Press ⋯**, select** [Table Range]**, press** OK

After inputting [Start:0, End:3, and Step:0.5]**, select** [Execute]**, press** EXE



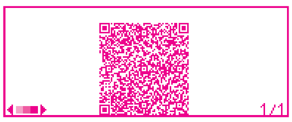


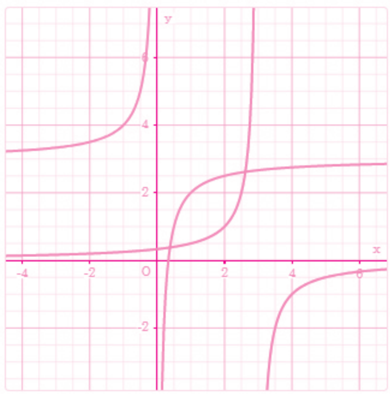




When x=0.5**, then** f(0.5)=1**; this time, we get** g(1)=0.5**.**
When x=1**, then** f(1)=2**; this time, we get** g(2)=1**.**
The same holds for any x**, so** $f(x)$ **and** $g(x)$ **are inverse functions of each other.**

Press ⇧ x**, scan the QR code to display a graph.**





2 For $f(x)=1-x, g(x)=x^2-3$, find the composite function $(g\circ f)(x)$.

$$(g\circ f)(x)=g\left(f(x)\right)=g(1-x)=(1-x)^2-3=x^2-2x-2$$

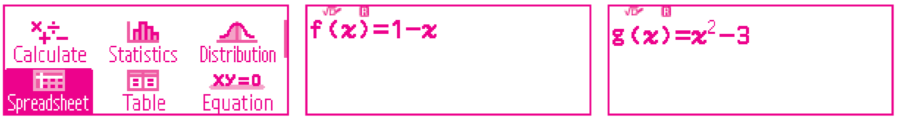
$$(g\circ f)(x)=x^2-2x-2$$

check

Press ⌂, select [Spreadsheet], press OK, then clear the previous data by pressing ⏻

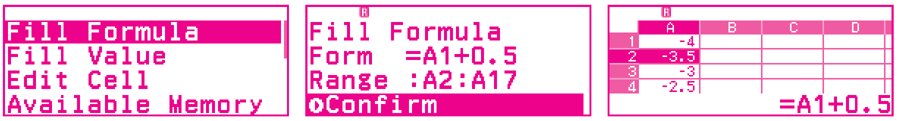
Press f(x), select [Define f(x)], press OK, press 1 ⊖ x EXE

In the same way, press f(x), select [Define g(x)], press OK, press x ■² ⊖ 3 EXE



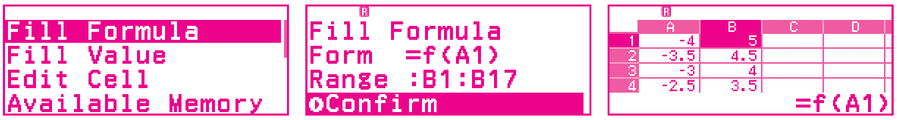
Input -4 in cell A1, then use [Fill Formula] to fill cells A2 to A17 with values from -3.5 to 4 in increments of 0.5.

⊖ 4 EXE   ⋯ OK ⇧ 4 1 ⊕ 0 . 5 EXE   > > > > > > ⌫ 1 7 EXE   EXE



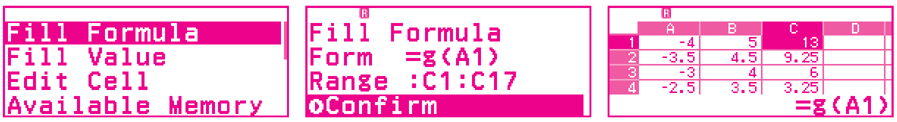
Let the values in column A be the values of x, then use [Fill Formula] so the values of f(x), corresponding to each value of x, appear in column B.

∧ >   ⋯ OK f(x) OK ⇧ 4 1) EXE   > > > > > > 7 EXE   EXE



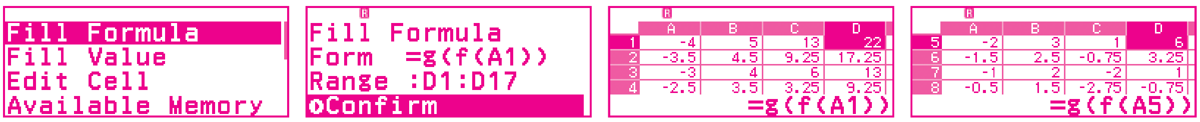
Let the values in column A be the values of x, then use [Fill Formula] so the values of g(x), corresponding to each value of x, appear in column C.

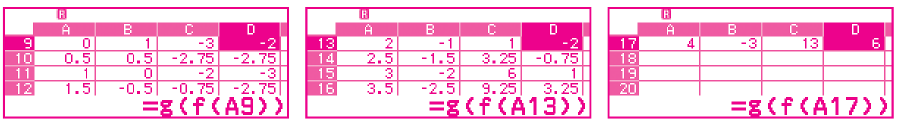
>   ⋯ OK f(x) ∨ OK ⇧ 4 1) EXE   > > > > > > 7 EXE   EXE



Let the values in column A be the values of x, then use [Fill Formula] so the values of $(g\circ f)(x)$, corresponding to each value of x, appear in column D.

>   ⋯ OK f(x) ∨ OK f(x) OK ⇧ 4 1)) EXE   > > > > > > 7 EXE   EXE





△

In the figure above $f(-3)=4$ and $g(4)=13$, so we get $(g\circ f)(-3)=13$.

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