7 Formulas and Proofs

CASIO
Essential Materials

Introduction

These teaching materials were created with the hope of conveying to many teachers and students the appeal of scientific calculators.

(1) Change awareness (emphasizing the thinking process) and boost efficiency in learning mathematics

- By reducing the time spent on manual calculations, we can have learning with a focus on the thinking process that is more efficient.
- This reduces the aversion to mathematics caused by complicated calculations, and allows students to experience the joy of thinking, which is the essence of mathematics.

(2) Diversification of learning materials and problem-solving methods

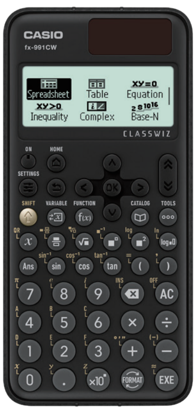
- Making it possible to do difficult calculations manually allows for diversity in learning materials and problem-solving methods.

(3) Promoting understanding of mathematical concepts

- By using the various functions of the scientific calculator in creative ways, students are able to deepen their understanding of mathematical concepts through calculations and discussions from different perspectives than before.
- This allows for exploratory learning through easy trial and error of questions.
- Listing and graphing of numerical values by means of tables allows students to discover laws and to understand visually.

Features of this book

- As well as providing first-time scientific calculator users with opportunities to learn basic scientific calculator functions from the ground up, the book also has material to show people who already use scientific calculators the appeal of scientific calculators described above.
- You can also learn about functions and techniques that are not available on conventional Casio models or other brands of scientific calculators.
- This book covers many units of high school mathematics, allowing students to learn how to use the scientific calculator as they study each topic.
- This book can be used in a variety of situations, from classroom activities to independent study and homework by students.



Better Mathematics Learning with Scientific Calculator

Structure

Quadratic functions

TARGET To understand the concept of functions.

STUDY GUIDE

Functions

Definition of a function

For 2 variables, x and y, if the value of x is determined and if just 1 corresponding value of y is determined, then we say that **y is a function of x**.

Ex. Given $x=2$ ➡ Functions ($y=x+1$) ➡ We can determine $y=3$

How to describe functions

When y is a function of x, generally y is expressed as $f(x)$ or $g(x)$. Furthermore, in the function $y=f(x)$, then when $x=a$, the value of y is expressed as $f(a)$.

Ex. In the function $f(x)=3x+3$, when $x=1$, then when $f(1)=3\times1+3=6$ and $x=5$, then we get $f(5)=3\times5+3=18$.

Linear functions & Quadratic functions

When y is expressed as a linear equation of x, y is called the **linear function** of x, which is expressed as follows.

$$\boxed{y=ax+b\ \ (a\neq0)}$$

When y is expressed as a quadratic equation of x, y is called the **quadratic function** of x, which is expressed as follows.

$$\boxed{y=ax^2+bx+c\ \ (a\neq0)}$$

TARGET

Students can identify the objective to learn in each section.

STUDY GUIDE

Mathematical theorems and concepts are explained in detail. A scientific calculator is used to check and derive formulas according to the topic.

EXERCISE

1 For the quadratic function $f(x)=3x^2+2x+6$, find the various following values.

(1) $f(0)$

$f(0)=3\cdot0^2+2\cdot0+6=6$

6

(2) $f(1)$

$f(1)=3\cdot1^2+2\cdot1+6=11$

11

(3) $f(-1)$

$f(-1)=3\cdot(-1)^2+2\cdot(-1)+6=7$

7

(4) $f(109)$

$f(109)=3\cdot109^2+2\cdot109+6=35867$

35867

check

Use the FUNCTION function to calculate the value of $f(x)$.

Press ⌂, select [Calculate], press OK

Press f(x), select [Define f(x)], press OK ③ x x² ⊕ ② x ⊕ ⑥ EXE

Calculate Statistics Distribution Spreadsheet Table Equation | f(x) g(x) Define f(x) Define g(x) | f(x)=3x²+2x+6

EXERCISE

Students learn basic examples based on the explanation in Study Guide.

check

Explains how to use the scientific calculator to solve problems and check answers.

PRACTICE

1 For the quadratic function $f(x)=-2x^2+4x+5$, find the various following values.

(1) $f(0)$

$f(0)=-2\cdot0^2+4\cdot0+5=5$

5

(2) $f(2)$

$f(2)=-2\cdot2^2+4\cdot2+5=5$

5

(3) $f(-3)$

$f(-3)=-2\cdot(-3)^2+4\cdot(-3)+5=-25$

-25

(4) $f(52)$

$f(52)=-2\cdot52^2+4\cdot52+5=-5195$

-5195

PRACTICE

Students can do practice problems similar to those in EXERCISE. They can also practice using the scientific calculator as they learned to in Check.

ADVANCED

3 We know that when an object is thrown vertically upward from a height of 0 m at a velocity of v m/s, the relation between the height y m of the object and the time x seconds since it was thrown is $y=-4.9x^2+vx$. So, when a ball is thrown vertically from a height of 0 m at a velocity of 9.8 m/s, find the time it takes to reach the highest point and the height of that highest point. Note that the air resistance can be ignored.

$y=-4.9x^2+9.8x=-4.9(x^2-2x)=-4.9\{(x-1)^2-1^2\}=-4.9(x-1)^2+4.9$

Therefore, the graph of this function is convex upward, the axis is $x=1$, and the vertex is $(1, 4.9)$.

Thus, the ball takes 1 second to reach its highest point, and the height of that highest point is 4.9 m.

1 second, 4.9 m

check

Press ⌂, select [Equation], press OK

Select [Polynomial], press OK, select [ax²+bx+c], press OK

⊖ ④ ⊙ ⑨ EXE ⑨ ⊙ ⑧ EXE ⓪ EXE EXE

Calculate Statistics Distribution Spreadsheet Table Equation | Simul Equation Polynomial Solver | ax²+bx+c ax³+bx²+cx+d ax⁴+bx³+cx²+dx+e | ax²+bx+c -4.9x²+ 9.8x 0

EXE EXE EXE **Press ⇧ x, scan the QR code to display a graph.**

Max of y=ax²+bx+c x= 1 | Max of y=ax²+bx+c y= 49/10 | 1/1

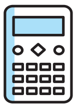
ADVANCED

Practical problems have been included in several topics. Solutions using a scientific calculator are also presented as necessary.

Other marks

Ex.	Simple examples on how to apply equations and theorems
explanation	Formulas and their supplementary explanations
proof	Proofs and checks of mathematical formulas
EXTRA Info.	Knowledge and information on formulas and other supplementary information in other units
OTHER METHODS	Alternative solutions and different verification methods for previously presented problems

Calculator mark



Where to use the scientific calculator

Colors of fonts in the teaching materials

- In STUDY GUIDE, important mathematical terms and formulas are printed in blue.
- In PRACTICE and ADVANCED the answers are printed in red.
  (Separate data is also available without the red parts, so it can be used for exercises.)

Applicable models

The applicable model is fx-991CW.
(Instructions on how to do input are for the fx-991CW, but in many cases similar calculations can be done on other models.)

Related Links

- Information and educational materials relevant to scientific calculators can be viewed on the following site.
  https://edu.casio.com
- The following video can be viewed to learn about the multiple functions of scientific calculators.
  https://www.youtube.com/playlist?list=PLRgxo9AwbIZLurUCZnrbr4cLfZdqY6aZA

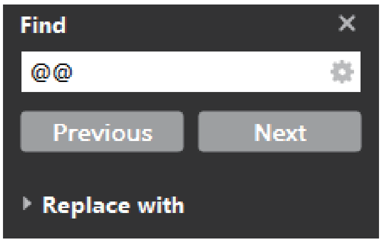
How to use PDF data

About types of data

- Data for all unit editions and data for each unit are available.
- For the above data, the PRACTICE and ADVANCED data without the answers in red is also available.

How to find where the scientific calculator is used

(1) Open a search window in the PDF Viewer.
(2) Type in "@@" as a search term.
(3) You can sequentially check where the calculator marks appear in the data.



How to search for a unit and section

(1) Search for units of data in all unit editions

- The data in all unit editions has a unit table of contents.
- Selecting a unit in the table of contents lets you jump to the first page of that unit.
- There is a bookmark on the first page of each unit, so you can jump from there also.

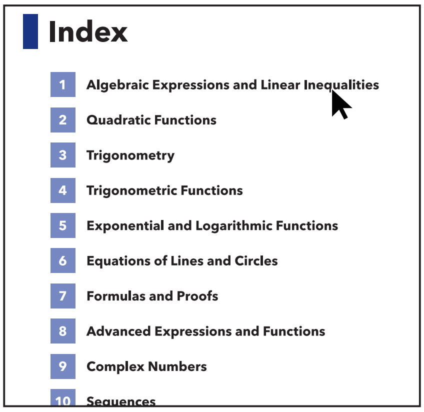
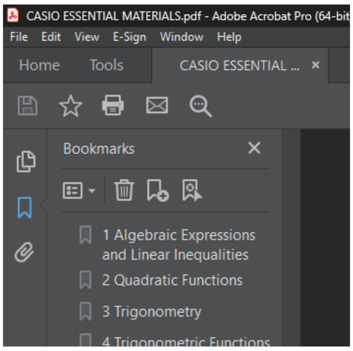


Table of contents of unit



Bookmark of unit

(2) Search for sections

- There are tables of contents for sections on the first page of units.
- Selecting a section in the table of contents takes you to the first page of that section.

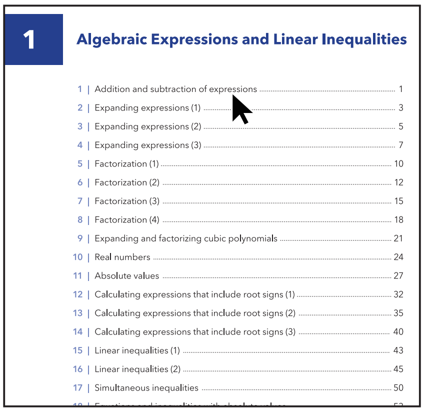


Table of contents of section

Formulas for expanding cubic polynomials

TARGET To understand formulas for expanding cubic polynomials.

STUDY GUIDE

Formulas for expanding cubic polynomials

The following formulas are often used to expand cubic polynomials.

$$(1)\quad (a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$
$$(2)\quad (a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$$
$$(3)\quad (a+b)(a^2 - ab + b^2) = a^3 + b^3$$
$$(4)\quad (a-b)(a^2 + ab + b^2) = a^3 - b^3$$

EXERCISE

◆ Expand the following equations.

(1) $(2x+1)^3 = (2x)^3 + 3\cdot(2x)^2\cdot 1 + 3\cdot 2x\cdot 1^2 + 1^3$
$= 8x^3 + 12x^2 + 6x + 1$

$8x^3 + 12x^2 + 6x + 1$

(2) $(x-1)^3 = x^3 - 3\cdot x^2\cdot 1 + 3\cdot x\cdot 1^2 - 1^3$
$= x^3 - 3x^2 + 3x - 1$

$x^3 - 3x^2 + 3x - 1$

(3) $(x+1)(x^2 - x + 1) = x^3 + 1^3$
$= x^3 + 1$

$x^3 + 1$

(4) $(x-2)(x^2 + 2x + 4) = x^3 - 2^3$
$= x^3 - 8$

$x^3 - 8$

PRACTICE

◆ Expand the following equations.

(1) $(3x+5y)^3$

$= (3x)^3 + 3\cdot(3x)^2\cdot 5y + 3\cdot 3x\cdot(5y)^2 + (5y)^3$
$= 27x^3 + 135x^2y + 225xy^2 + 125y^3$

$27x^3 + 135x^2y + 225xy^2 + 125y^3$

(2) $(x-2y^2)^3$

$= x^3 - 3\cdot x^2\cdot 2y^2 + 3\cdot x\cdot(2y^2)^2 - (2y^2)^3$
$= x^3 - 6x^2y^2 + 12xy^4 - 8y^6$

$x^3 - 6x^2y^2 + 12xy^4 - 8y^6$

(3) $(3x+2y)(9x^2 - 6xy + 4y^2)$

$= (3x)^3 + (2y)^3$
$= 27x^3 + 8y^3$

$27x^3 + 8y^3$

(4) $(2a-5b)(4a^2 + 10ab + 25b^2)$

$= (2a)^3 - (5b)^3$
$= 8a^3 - 125b^3$

$8a^3 - 125b^3$

Formulas for factorizing cubic polynomials

TARGET To understand formulas for factorizing cubic polynomials.

STUDY GUIDE

Formulas for factorizing cubic polynomials

The following formulas are often used to factorize cubic polynomials.

(1) $a^3+3a^2b+3ab^2+b^3=(a+b)^3$

(2) $a^3-3a^2b+3ab^2-b^3=(a-b)^3$

(3) $a^3+b^3=(a+b)(a^2-ab+b^2)$

(4) $a^3-b^3=(a-b)(a^2+ab+b^2)$

EXERCISE

◆ Factorize the following equations.

(1) $x^3+6x^2+12x+8=x^3+3\cdot x^2\cdot 2+3\cdot x\cdot 2^2+2^3$
$=(x+2)^3$

$(x+2)^3$

(2) $x^3-9x^2y+27xy^2-27y^3$
$=x^3-3\cdot x^2\cdot 3y+3\cdot x\cdot(3y)^2-(3y)^3$
$=(x-3y)^3$

$(x-3y)^3$

(3) $a^3+216=a^3+6^3$
$=(a+6)(a^2-6a+36)$

$(a+6)(a^2-6a+36)$

(4) $a^3-8b^3=a^3-(2b)^3$
$=(a-2b)(a^2+2ab+4b^2)$

$(a-2b)(a^2+2ab+4b^2)$

PRACTICE

◆ Factorize the following equations.

(1) $8x^3+36x^2y+54xy^2+27y^3$

$=(2x)^3+3\cdot(2x)^2\cdot 3y+3\cdot 2x\cdot(3y)^2+(3y)^3$
$=(2x+3y)^3$

$(2x+3y)^3$

(2) $x^3-12x^2y+48xy^2-64y^3$

$=x^3-3\cdot x^2\cdot 4y+3\cdot x\cdot(4y)^2-(4y)^3$
$=(x-4y)^3$

$(x-4y)^3$

(3) a^3+125b^3

$=a^3+(5b)^3$
$=(a+5b)(a^2-5ab+25b^2)$

$(a+5b)(a^2-5ab+25b^2)$

(4) $27x^3-64y^3$

$=(3x)^3-(4y)^3$
$=(3x-4y)(9x^2+12xy+16y^2)$

$(3x-4y)(9x^2+12xy+16y^2)$

Binomial theorem

TARGET To understand how to use the binomial theorem to expand equations.

STUDY GUIDE

Pascal's triangle

Consider the formula to expand $(a+b)^n$.

$$
\begin{aligned}
(a+b)^2 &= a^2+2ab+b^2 \\
(a+b)^3 &= a^3+3a^2b+3ab^2+b^3 \\
(a+b)^4 &= (a+b)(a+b)^3 \\
&= (a+b)(a^3+3a^2b+3ab^2+b^3) \\
&= a^4+3a^3b+3a^2b^2+ab^3+a^3b+3a^2b^2+3ab^3+b^4 \\
&= a^4+4a^3b+6a^2b^2+4ab^3+b^4
\end{aligned}
$$

$$
\begin{array}{l ccccccccccccc}
 & & & & & & 1 & & 1 & & & & & \\
(a+b)^2 & & & & & 1 & & 2 & & 1 & & & & \\
(a+b)^3 & & & & 1 & & 3 & & 3 & & 1 & & & \\
(a+b)^4 & & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
(a+b)^5 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
(a+b)^6 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}
$$

Here, just the coefficients of each term are extracted and arranged in order to consider the regularity.

(1) The number at both ends of each row is 1.

(2) After the 2nd row, the numbers, except at both ends, are equal to the sum of the numbers to the upper left and upper right.

(3) The numbers in each row are symmetrical.

This sequence of numbers in this regular triangular shape is called **Pascal's triangle**.

Binomial theorem

The binomial theorem is a method of expanding $(a+b)^n$ as a sum by multiplying a chosen term, either a or b, from an n number of $(a+b)$.

Such that, a chooses $n-r$ and b chooses r, which when multiplied are $a^{n-r}b^r$, the method to choose is ${}_n\mathrm{C}_r$.

Therefore, when we expand $(a+b)^n$, the coefficient of the term $a^{n-r}b^r$ is ${}_n\mathrm{C}_r$, from which we can derive the following formula. This is called the **binomial theorem**.

$$
\boxed{(a+b)^n = \underline{{}_n\mathrm{C}_0}\,a^n + \underline{{}_n\mathrm{C}_1}\,a^{n-1}b + \underline{{}_n\mathrm{C}_2}\,a^{n-2}b^2 + \cdots + \underline{{}_n\mathrm{C}_r}\,a^{n-r}b^r + \cdots + \underline{{}_n\mathrm{C}_n}\,b^n}
$$

EXTRA Info.

The method to choose r elements from various n elements is ${}_n\mathrm{C}_r$.

$$
{}_n\mathrm{C}_r = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}
$$

Also, this is ${}_n\mathrm{C}_1 = n, {}_n\mathrm{C}_n = 1$, specifically stated as ${}_n\mathrm{C}_0 = 1$.

The values of the underlined elements ${}_n\mathrm{C}_0, {}_n\mathrm{C}_1, {}_n\mathrm{C}_2, \cdots, {}_n\mathrm{C}_r, \cdots, {}_n\mathrm{C}_n$ are the same as the nth row in Pascal's triangle.

EXERCISE

1 Use the binomial theorem to find the expansion of $(a+b)^6$.

$$(a+b)^6 = {}_6C_0a^6 + {}_6C_1a^5b + {}_6C_2a^4b^2 + {}_6C_3a^3b^3 + {}_6C_4a^2b^4 + {}_6C_5ab^5 + {}_6C_6b^6$$
$$= a^6 + 6a^5b + \frac{6\cdot5}{2\cdot1}a^4b^2 + \frac{6\cdot5\cdot4}{3\cdot2\cdot1}a^3b^3 + \frac{6\cdot5}{2\cdot1}a^2b^4 + 6ab^5 + b^6$$
$$= \underset{1}{\underline{a^6}} + \underset{6}{\underline{6a^5b}} + \underset{15}{\underline{15a^4b^2}} + \underset{20}{\underline{20a^3b^3}} + \underset{15}{\underline{15a^2b^4}} + \underset{6}{\underline{6ab^5}} + \underset{1}{\underline{b^6}}$$

Can also be found in Pascal's triangle.

$\underline{\boldsymbol{a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6}}$

2 Find the coefficient of the term x^3 in the expansion of $(x+2)^6$.

The general term for $(x+2)^6$ is ${}_6C_rx^{6-r}2^r = {}_6C_r2^rx^{6-r}$.

The term for x^3 is $6-r=3$, when $r=3$.

The coefficient to be found is ${}_6C_32^3 = \frac{6\cdot5\cdot4}{3\cdot2\cdot1}\cdot2^3 = 160$.

160

PRACTICE

1 Use the binomial theorem to find the expansion of $(a+b)^7$.

$$(a+b)^7 = {}_7C_0a^7 + {}_7C_1a^6b + {}_7C_2a^5b^2 + {}_7C_3a^4b^3 + {}_7C_4a^3b^4 + {}_7C_5a^2b^5 + {}_7C_6ab^6 + {}_7C_7b^7$$
$$= a^7 + 7a^6b + \frac{7\cdot6}{2\cdot1}a^5b^2 + \frac{7\cdot6\cdot5}{3\cdot2\cdot1}a^4b^3 + \frac{7\cdot6\cdot5}{3\cdot2\cdot1}a^3b^4 + \frac{7\cdot6}{2\cdot1}a^2b^5 + 7ab^6 + b^7$$
$$= a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7$$

$a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7$

2 Find the coefficient of the term x^4 in the expansion of $(x-3)^7$.

The general term for $(x-3)^7$ is ${}_7C_rx^{7-r}(-3)^r = {}_7C_r(-3)^rx^{7-r}$.

The term for x^4 is $7-r=4$, when $r=3$.

The coefficient to be found is ${}_7C_3(-3)^3 = \frac{7\cdot6\cdot5}{3\cdot2\cdot1}\cdot(-3)^3 = -945$.

-945

Use the binomial theorem to expand equations

TARGET Use the binomial theorem to expand various equations.

STUDY GUIDE

Binomial theorem

$$(a+b)^n = {}_n\mathrm{C}_0a^n + {}_n\mathrm{C}_1a^{n-1}b + {}_n\mathrm{C}_2a^{n-2}b^2 + \cdots + {}_n\mathrm{C}_ra^{n-r}b^r + \cdots + {}_n\mathrm{C}_nb^n$$

In the expansion of $(a+b)^n$, the $(r+1)$th term, represented as ${}_n\mathrm{C}_ra^{n-r}b^r$, is called the **general term**.

The coefficients of each term ${}_n\mathrm{C}_0, {}_n\mathrm{C}_1, {}_n\mathrm{C}_2, \cdots, {}_n\mathrm{C}_{n-1}, {}_n\mathrm{C}_n$ are called **binomial coefficients**.

EXERCISE

1 Use the binomial theorem to find the expansion of $(x+3)^4$.

$$\begin{aligned}(x+3)^4 &= {}_4\mathrm{C}_0x^4 + {}_4\mathrm{C}_1x^3\cdot 3 + {}_4\mathrm{C}_2x^2\cdot 3^2 + {}_4\mathrm{C}_3x\cdot 3^3 + {}_4\mathrm{C}_4 3^4 \\ &= x^4 + 4x^3\cdot 3 + \frac{4\cdot 3}{2\cdot 1}x^2\cdot 3^2 + 4x\cdot 3^3 + 3^4 \\ &= x^4 + 12x^3 + 54x^2 + 108x + 81\end{aligned}$$

$x^4 + 12x^3 + 54x^2 + 108x + 81$

2 Find the coefficient of the term x^3y^2 in the expansion of $(x-4y)^5$.

The general term for $(x-4y)^5$ is ${}_5\mathrm{C}_rx^{5-r}(-4y)^r = {}_5\mathrm{C}_r(-4)^rx^{5-r}y^r$.

The term for x^3y^2 is $5-r=3$, when $r=2$.

The coefficient to be found is ${}_5\mathrm{C}_2(-4)^2 = \frac{5\cdot 4}{2\cdot 1}\cdot(-4)^2 = 160$.

160

PRACTICE

1 Use the binomial theorem to find the expansion of the following equations.

(1) $(2a+b)^5$

$$(2a+b)^5 = {}_5C_0(2a)^5 + {}_5C_1(2a)^4b + {}_5C_2(2a)^3b^2 + {}_5C_3(2a)^2b^3 + {}_5C_4(2a)b^4 + {}_5C_5b^5$$
$$= 32a^5 + 5\cdot 16a^4b + \frac{5\cdot 4}{2\cdot 1}\cdot 8a^3b^2 + \frac{5\cdot 4}{2\cdot 1}\cdot 4a^2b^3 + 5\cdot 2ab^4 + b^5$$
$$= 32a^5 + 80a^4b + 80a^3b^2 + 40a^2b^3 + 10ab^4 + b^5$$

$32a^5 + 80a^4b + 80a^3b^2 + 40a^2b^3 + 10ab^4 + b^5$

(2) $(3x-2y)^4$

$$(3x-2y)^4 = {}_4C_0(3x)^4 + {}_4C_1(3x)^3\cdot(-2y) + {}_4C_2(3x)^2\cdot(-2y)^2 + {}_4C_3(3x)\cdot(-2y)^3 + {}_4C_4(-2y)^4$$
$$= 81x^4 + 4\cdot 27x^3\cdot(-2y) + \frac{4\cdot 3}{2\cdot 1}\cdot 9x^2\cdot 4y^2 + 4\cdot 3x\cdot(-8y^3) + 16y^4$$
$$= 81x^4 - 216x^3y + 216x^2y^2 - 96xy^3 + 16y^4$$

$81x^4 - 216x^3y + 216x^2y^2 - 96xy^3 + 16y^4$

2 Find the coefficient of the term specified in [ ] in the expansion of the following equation.

(1) $(x-3y)^5$ $[x^4y]$

The general term for $(x-3y)^5$ is ${}_5C_r x^{5-r}(-3y)^r = {}_5C_r(-3)^r x^{5-r}y^r$.
The term for x^4y is $5-r=4$, when $r=1$.

The coefficient to be found is ${}_5C_1(-3)^1 = 5\cdot(-3) = -15$.

-15

(2) $(x^2-3)^6$ $[x^8]$

The general term for $(x^2-3)^6$ is ${}_6C_r(x^2)^{6-r}(-3)^r = {}_6C_r(-3)^r x^{12-2r}$.
The term for x^8 is $12-2r=8$, when $r=2$.

The coefficient to be found is ${}_6C_2(-3)^2 = \frac{6\cdot 5}{2\cdot 1}\cdot(-3)^2 = 135$.

135

EXTRA Info.

Use the scientific calculator to calculate the coefficient by using the binomial theorem.

In this section, we study how to use the scientific calculator to calculate coefficients by using the binomial theorem to effectively expand equations.

This manual combines traditional study of mathematics with solving problems on the scientific calculator for in-depth learning from multiple perspectives.

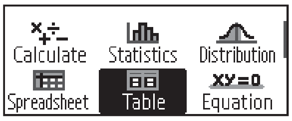
EXERCISE

1 Expand $(a+b)^5$ by using the scientific calculator.

General term of expansion: ${}_5\mathrm{C}_x a^{5-x} b^x$ $(x=0, 1, 2, 3, 4, 5)$

Use the Table function in the scientific calculator to find the value of the coefficient ${}_5\mathrm{C}_x$ as shown below.

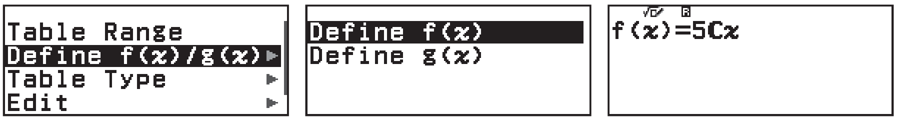
Press ⌂, select [Table], press OK, then clear the previous data by pressing ⏻



Press ⋯, select [Define $\mathrm{f}(x)/\mathrm{g}(x)$], press OK, select [Define $\mathrm{f}(x)$], press OK

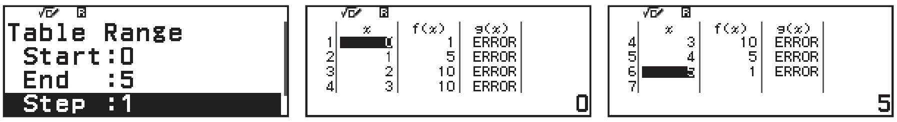
After inputting $\mathrm{f}(x)={}_5\mathrm{C}_x$, press EXE (*)

* How to input C (combination operation symbol): Press ⊡, select [Probability], press OK, select [Combination], press OK



Press ⋯, select [Table Range], press OK

After inputting [Start:0, End:5, and Step:1], select [Execute], press EXE



From the table, $(a+b)^5 = 1\cdot a^5b^0 + 5\cdot a^4b^1 + 10\cdot a^3b^2 + 10\cdot a^2b^3 + 5\cdot a^1b^4 + 1\cdot a^0b^5$

$= a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$

$a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$

2 Use the scientific calculator to calculate the numbers that go in the □ squares in the following equation.

$(3a-2b)^4 = \square\, a^4 + \square\, a^3b + \square\, a^2b^2 + \square\, ab^3 + \square\, b^4$

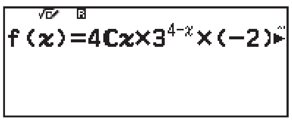
General term of expansion: ${}_4\mathrm{C}_x(3a)^{4-x}(-2b)^x = {}_4\mathrm{C}_x 3^{4-x}(-2)^x \cdot a^{4-x}b^x$ $(x=0,1,2,3,4)$

Use the Table function in the scientific calculator to find the value of the coefficient ${}_4\mathrm{C}_x 3^{4-x}(-2)^x$ as shown below.

Press ⌂, select [Table], press OK, then clear the previous data by pressing ⏻

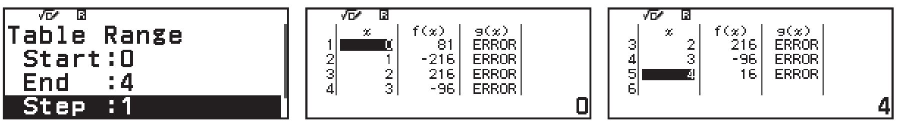
Press ⋯, select [Define $\mathrm{f}(x)/\mathrm{g}(x)$], press OK, select [Define $\mathrm{f}(x)$], press OK

After inputting $\mathrm{f}(x)={}_4\mathrm{C}_x 3^{4-x}(-2)^x$, press EXE



Press ⋯, select [Table Range], press OK

After inputting [Start:0, End:4, and Step:1], select [Execute], press EXE



From the table, $(3a-2b)^4 = 81a^4 - 216a^3b + 216a^2b^2 - 96ab^3 + 16b^4$

81, −216, 216, −96, 16

PRACTICE

1 Expand $(a+b)^8$ by using the scientific calculator.

General term of expansion: ${}_8C_x a^{8-x} b^x$ $(x=0, 1, 2, 3, 4, 5, 6, 7, 8)$

Use the scientific calculator to calculate the coefficient ${}_8C_x$**, which comes before the letters in** $a^{8-x}b^x$**.**

Press ⌂**, select** [Table]**, press** OK**, then clear the previous data by pressing** ⏻

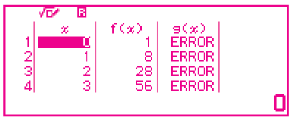
Press ⋯**, select** [Define f(x)/g(x)]**, press** OK**,**

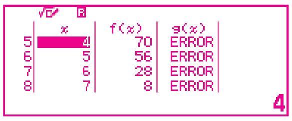
select [Define f(x)]**, press** OK

After inputting $f(x)={}_8C_x$**, press** EXE

Press ⋯**, select** [Table Range]**, press** OK

After inputting [Start:0,End:8, **and** Step:1]**, select** [Execute]**, press** EXE





From the table,

$(a+b)^8 = a^8+8a^7b+28a^6b^2+56a^5b^3+70a^4b^4+56a^3b^5+28a^2b^6+8ab^7+b^8$

$$a^8+8a^7b+28a^6b^2+56a^5b^3+70a^4b^4+56a^3b^5+28a^2b^6+8ab^7+b^8$$

2 Use the scientific calculator to calculate the numbers that go in the □ squares in the following equation.

$(2a-5b)^3 = \square a^3 + \square a^2b + \square ab^2 + \square b^3$

General term of expansion: ${}_3C_x(2a)^{3-x}(-5b)^x = {}_3C_x 2^{3-x}(-5)^x \cdot a^{3-x}b^x$ $(x=0, 1, 2, 3)$

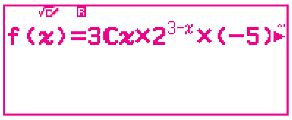
Use the scientific calculator to calculate the coefficient ${}_3C_x 2^{3-x}(-5)^x$**, which comes before the letters in** $a^{3-x}b^x$**.**

Press ⌂**, select** [Table]**, press** OK**, then clear the previous data by pressing** ⏻

Press ⋯**, select** [Define f(x)/g(x)]**, press** OK**,**

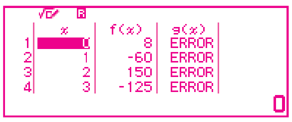
select [Define f(x)]**, press** OK

After inputting $f(x)={}_3C_x 2^{3-x}(-5)^x$**, press** EXE



Press ⋯**, select** [Table Range]**, press** OK

After inputting [Start:0, End:3, **and** Step:1]**, select** [Execute]**, press** EXE



From the table, $(2a-5b)^3 = 8a^3-60a^2b+150ab^2-125b^3$

$$8,\ -60,\ 150,\ -125$$

Multiplication and division of fractional expressions

TARGET To understand how to multiply and divide fractional expressions.

STUDY GUIDE

Fractional expressions

Monomials and polynomials are both called **integer polynomials**.

For the integer polynomials A and B, where $B\neq0$, an expression $\dfrac{A}{B}$ that includes a letter in the denominator is called a **fractional expression**. Fractional expressions can be handled the same way as calculating fractions.

(1) $\dfrac{AC}{BC}=\dfrac{A}{B}\ (C\neq0)$ (2) $\dfrac{A}{B}=\dfrac{AD}{BD}\ (D\neq0)$

Multiplication and division of fractional expressions

Multiplication $\dfrac{A}{B}\times\dfrac{C}{D}=\dfrac{AC}{BD}$ **Division** $\dfrac{A}{B}\div\dfrac{C}{D}=\dfrac{A}{B}\times\dfrac{D}{C}=\dfrac{AD}{BC}$

How to calculate

(1) Convert division to **multiplication**.

(2) **Factorize** the denominator and numerator.

(3) After factorizing, **reduce** if possible.

EXERCISE

◆ Calculate the following.

(1) $\dfrac{x-4}{x+3}\times\dfrac{x^2+x-6}{x^2+2x-24}=\dfrac{\cancel{x-4}}{\cancel{x+3}}\times\dfrac{(x-2)\cancel{(x+3)}}{\cancel{(x-4)}(x+6)}$

$=\dfrac{x-2}{x+6}$

$\dfrac{x-2}{x+6}$

(2) $\dfrac{x^2+2x}{x^2-4x-5}\div\dfrac{x^2+4x+4}{x^2-2x-3}=\dfrac{x^2+2x}{x^2-4x-5}\times\dfrac{x^2-2x-3}{x^2+4x+4}$

$=\dfrac{x\cancel{(x+2)}}{\cancel{(x+1)}(x-5)}\times\dfrac{\cancel{(x+1)}(x-3)}{(x+2)^{\cancel{2}}}$

$=\dfrac{x(x-3)}{(x-5)(x+2)}$

$\dfrac{x(x-3)}{(x-5)(x+2)}$

PRACTICE

◆ Calculate the following.

(1) $\dfrac{a^2+6a+9}{a^2-a-2}\times\dfrac{a^2-4}{a^2+2a-3}$

$$=\frac{(a+3)^{\cancel{2}}}{(a+1)\cancel{(a-2)}}\times\frac{(a+2)\cancel{(a-2)}}{(a-1)\cancel{(a+3)}}$$

$$=\frac{(a+3)(a+2)}{(a+1)(a-1)}$$

$$\frac{(a+3)(a+2)}{(a+1)(a-1)}$$

(2) $\dfrac{2x^2+xy-6y^2}{x^2-xy-6y^2}\times\dfrac{x^2-3xy}{2x^2-xy-3y^2}$

$$=\frac{\cancel{(2x-3y)}\cancel{(x+2y)}}{\cancel{(x+2y)}\cancel{(x-3y)}}\times\frac{x\cancel{(x-3y)}}{\cancel{(2x-3y)}(x+y)}$$

$$=\frac{x}{x+y}$$

$$\frac{x}{x+y}$$

(3) $\dfrac{x^2+4x-12}{x^2+x-30}\div\dfrac{x^2+7x-18}{x^2-x-20}$

$$=\frac{x^2+4x-12}{x^2+x-30}\times\frac{x^2-x-20}{x^2+7x-18}$$

$$=\frac{\cancel{(x-2)}\cancel{(x+6)}}{\cancel{(x-5)}\cancel{(x+6)}}\times\frac{(x+4)\cancel{(x-5)}}{\cancel{(x-2)}(x+9)}$$

$$=\frac{x+4}{x+9}$$

$$\frac{x+4}{x+9}$$

(4) $\dfrac{x+2}{x+5}\div\dfrac{x^2-7x-18}{x^2+x-12}\times\dfrac{x^2-8x-9}{x^2+5x-24}$

$$=\frac{x+2}{x+5}\times\frac{x^2+x-12}{x^2-7x-18}\times\frac{x^2-8x-9}{x^2+5x-24}$$

$$=\frac{\cancel{x+2}}{x+5}\times\frac{\cancel{(x-3)}(x+4)}{\cancel{(x+2)}\cancel{(x-9)}}\times\frac{(x+1)\cancel{(x-9)}}{\cancel{(x-3)}(x+8)}$$

$$=\frac{(x+4)(x+1)}{(x+5)(x+8)}$$

$$\frac{(x+4)(x+1)}{(x+5)(x+8)}$$

Addition and subtraction of fractional expressions

TARGET To understand how to add and subtract fractional expressions.

STUDY GUIDE

Reducing fractional expressions to a common denominator

Making the denominators of fractional expressions to be the same integer polynomial is called **reducing a fraction to a common denominator**.

Ex. Reduce $\dfrac{1}{x^2-1}$ and $\dfrac{1}{x^2+x}$ to a common denominator

$$\frac{1}{x^2-1}=\frac{1}{(x+1)(x-1)}=\frac{x}{x(x+1)(x-1)},\ \frac{1}{x^2+x}=\frac{1}{x(x+1)}=\frac{x-1}{x(x+1)(x-1)}$$

Addition and subtraction of fractional expressions

Addition $\dfrac{A}{C}+\dfrac{B}{C}=\dfrac{A+B}{C}$ **Subtraction** $\dfrac{A}{C}-\dfrac{B}{C}=\dfrac{A-B}{C}$

How to calculate

(1) **Factorize** the denominator and numerator.

(2) If the denominators are different, then **reduce to a common denominator**.

(3) After calculating, **reduce** if possible.

EXERCISE

◆ Calculate the following.

(1) $\dfrac{1}{x^2+2x}+\dfrac{1}{x^2+6x+8}$

$$=\frac{1}{x(x+2)}+\frac{1}{(x+2)(x+4)}$$
$$=\frac{x+4}{x(x+2)(x+4)}+\frac{x}{x(x+2)(x+4)}$$
$$=\frac{2x+4}{x(x+2)(x+4)}$$
$$=\frac{2\cancel{(x+2)}}{x\cancel{(x+2)}(x+4)}$$
$$=\frac{2}{x(x+4)}$$

$$\boldsymbol{\frac{2}{x(x+4)}}$$

(2) $\dfrac{x+4}{x+3}-\dfrac{x-3}{x-2}$

$$=\frac{(x+4)(x-2)}{(x+3)(x-2)}-\frac{(x+3)(x-3)}{(x+3)(x-2)}$$
$$=\frac{x^2+2x-8}{(x+3)(x-2)}-\frac{x^2-9}{(x+3)(x-2)}$$
$$=\frac{x^2+2x-8-x^2+9}{(x+3)(x-2)}$$
$$=\frac{2x+1}{(x+3)(x-2)}$$

$$\boldsymbol{\frac{2x+1}{(x+3)(x-2)}}$$

PRACTICE

◆ Calculate the following.

(1) $\dfrac{1}{x^2+9x+20}+\dfrac{1}{x^2+11x+30}$

$$=\frac{1}{(x+4)(x+5)}+\frac{1}{(x+5)(x+6)}=\frac{x+6}{(x+4)(x+5)(x+6)}+\frac{x+4}{(x+4)(x+5)(x+6)}$$

$$=\frac{2x+10}{(x+4)(x+5)(x+6)}=\frac{2\cancel{(x+5)}}{(x+4)\cancel{(x+5)}(x+6)}$$

$$=\frac{2}{(x+4)(x+6)}$$

$$\frac{2}{(x+4)(x+6)}$$

(2) $\dfrac{x+5}{x^2+4x-21}+\dfrac{x+8}{x^2+9x+14}$

$$=\frac{x+5}{(x-3)(x+7)}+\frac{x+8}{(x+2)(x+7)}=\frac{(x+5)(x+2)}{(x-3)(x+2)(x+7)}+\frac{(x-3)(x+8)}{(x-3)(x+2)(x+7)}$$

$$=\frac{x^2+7x+10}{(x-3)(x+2)(x+7)}+\frac{x^2+5x-24}{(x-3)(x+2)(x+7)}=\frac{2x^2+12x-14}{(x-3)(x+2)(x+7)}$$

$$=\frac{2(x-1)\cancel{(x+7)}}{(x-3)(x+2)\cancel{(x+7)}}=\frac{2(x-1)}{(x-3)(x+2)}$$

$$\frac{2(x-1)}{(x-3)(x+2)}$$

(3) $\dfrac{x-2}{2x^2-7x+3}-\dfrac{x+1}{2x^2+3x-2}$

$$=\frac{x-2}{(2x-1)(x-3)}-\frac{x+1}{(2x-1)(x+2)}=\frac{(x-2)(x+2)}{(2x-1)(x-3)(x+2)}-\frac{(x+1)(x-3)}{(2x-1)(x-3)(x+2)}$$

$$=\frac{x^2-4}{(2x-1)(x-3)(x+2)}-\frac{x^2-2x-3}{(2x-1)(x-3)(x+2)}=\frac{\cancel{2x-1}}{\cancel{(2x-1)}(x-3)(x+2)}$$

$$=\frac{1}{(x-3)(x+2)}$$

$$\frac{1}{(x-3)(x+2)}$$

Problems to find the values of sums and products

TARGET To understand how to find the value of equations from the value of the sums or products.

STUDY GUIDE

Values of expressions

x^2+y^2 is equal to y^2+x^2, which we get by changing the position of x and y. Expressions like this, which do not change when the variables are switched, are called **symmetric expressions**. We can use $x+y$ and xy to show symmetric expressions using x and y. The $x+y$ and xy here are called **elementary symmetric polynomials**.

$$\boxed{x^2+y^2=(x+y)^2-2xy, x^3+y^3=(x+y)^3-3xy(x+y)}$$

EXERCISE

◆ Find the value of the following equations.

(1) When $x+y=4$ and $xy=3$, find the value of x^2+y^2

$$\begin{aligned} x^2+y^2 &= (x+y)^2-2xy \\ &= 4^2-2\cdot 3 \\ &= 10 \end{aligned}$$

10

(2) When $x=\frac{1}{\sqrt{5}+2}, y=\frac{1}{\sqrt{5}-2}$, find the value of x^2+y^2

$$x+y=\frac{1}{\sqrt{5}+2}+\frac{1}{\sqrt{5}-2}=\frac{\sqrt{5}-2+\sqrt{5}+2}{(\sqrt{5}+2)(\sqrt{5}-2)}=\frac{2\sqrt{5}}{5-4}=2\sqrt{5}$$

$$xy=\frac{1}{\sqrt{5}+2}\cdot\frac{1}{\sqrt{5}-2}=\frac{1}{(\sqrt{5}+2)(\sqrt{5}-2)}=\frac{1}{5-4}=1$$

$$x^2+y^2=(x+y)^2-2xy=(2\sqrt{5})^2-2\cdot 1=18$$

18

PRACTICE

◆ Find the value of the following equations.

(1) When $x+y=2$ and $xy=-1$, find the value of x^3+y^3

$$\begin{aligned} x^3+y^3 &= (x+y)^3-3xy(x+y) \\ &= 2^3-3\cdot(-1)\cdot 2 \\ &= 14 \end{aligned}$$

14

(2) When $x=\frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}, y=\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}$, find the value of x^2+y^2

$$x+y=\frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}+\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}=\frac{(\sqrt{3}-\sqrt{2})^2+(\sqrt{3}+\sqrt{2})^2}{(\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{2})}=10$$

$$xy=\frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}\cdot\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}=1$$

$$x^2+y^2=(x+y)^2-2xy=10^2-2\cdot 1=98$$

98

Dividing integer polynomials

TARGET To understand how to divide one integer polynomial by another integer polynomial.

STUDY GUIDE

Dividing integer polynomials

Integer polynomials are added, subtracted, and multiplied in the same way as specified for regular numbers, so we can also consider dividing them in the same way.
When the integer polynomial A is divided by the nonzero integer polynomial B, we get the quotient Q and the remainder R, from which we can derive the following equation.

$$\boxed{A = BQ + R \quad (\text{exponent of } R) < (\text{exponent of } B)}$$

How to find the quotient and remainder by dividing integer polynomial A by integer polynomial B

(1) Arrange the letters that are the same in A and the same in B in **descending order of powers**.
(2) Continue dividing until the exponent of the **remainder R** is **less than** the **exponent** of the divisor expression B. In particular, when $R=0$, then A is divisible by B.
(Note) **Leave space** for the **missing exponent** when calculating on paper.

EXERCISE

◆ Find the quotient and remainder by dividing integer polynomial A by integer polynomial B.
$A = x^3 + 2x^2 - 5x + 2, B = x - 3$

$$
\begin{array}{r|rrrr}
 & x^2 & +5x & +10 & \\
\hline
x-3 & x^3 & +2x^2 & -5x & +2 \\
 & x^3 & -3x^2 & & \\
\hline
 & & 5x^2 & -5x & \\
 & & 5x^2 & -15x & \\
\hline
 & & & 10x & +2 \\
 & & & 10x & -30 \\
\hline
 & & & & 32
\end{array}
$$

Quotient is $x^2 + 5x + 10$ and remainder is 32

PRACTICE

◆ Find the quotient and remainder by dividing integer polynomial A by integer polynomial B.

(1) $A = 2x^3 + 5x^2 - 3x + 3, B = x - 2$

$$
\begin{array}{r|rrrr}
 & 2x^2 & +9x & +15 & \\
\hline
x-2 & 2x^3 & +5x^2 & -3x & +3 \\
 & 2x^3 & -4x^2 & & \\
\hline
 & & 9x^2 & -3x & \\
 & & 9x^2 & -18x & \\
\hline
 & & & 15x & +3 \\
 & & & 15x & -30 \\
\hline
 & & & & 33
\end{array}
$$

Quotient is $2x^2 + 9x + 15$ and remainder is 33

(2) $A = 3 - 2x + 4x^3, B = 2x^2 - x + 1$

$$
\begin{array}{r|rrrr}
 & & & 2x & +1 \\
\hline
2x^2-x+1 & 4x^3 & & -2x & +3 \\
 & 4x^3 & -2x^2 & +2x & \\
\hline
 & & 2x^2 & -4x & +3 \\
 & & 2x^2 & -x & +1 \\
\hline
 & & & -3x & +2
\end{array}
$$

Quotient is $2x+1$ and remainder is $-3x+2$

Dividing integer polynomials and the remainder theorem

TARGET To understand how to use the remainder theorem to find the remainder when dividing integer polynomials.

STUDY GUIDE

Remainder theorem

For an integer polynomial x, expressed as $P(x)$ or $Q(x)$ etc., we can substitute the x in $P(x)$ for a value k, so we express the obtained value as $P(k)$.

Given an integer polynomial $P(x)$ divided by the linear expression $x-k$ such that the remainder is a constant, when the quotient is $Q(x)$, then the remainder is R, expressed as $P(x)=(x-k)\,Q(x)+R$.

If we substitute $x=k$ in this equation, then we get $P(k)=(k-k)\,Q(k)+R=R$.

Therefore, the remainder $R=P(k)$, from which we derive the **remainder theorem** shown below.

(1) Dividing the integer polynomial $P(x)$ by $x-k$ gives the remainder $P(k)$.

(2) Dividing the integer polynomial $P(x)$ by $ax+b$ gives the remainder $P\left(-\dfrac{b}{a}\right)$.

EXERCISE

1 Find the remainder by dividing the following integer polynomials by the linear expression in [ ].

(1) $x^3-x^2+5 \quad [x-2]$

Given $P(x)=x^3-x^2+5$.

When divided by $x-2$, the remainder is $P(2)$, so

$P(2)=2^3-2^2+5=9$

9

(2) $3x^3-5x^2+4x+3 \quad [3x+1]$

Given $P(x)=3x^3-5x^2+4x+3$.

When divided by $3x+1$, the remainder is $P\left(-\dfrac{1}{3}\right)$, so

$$P\left(-\frac{1}{3}\right)=3\left(-\frac{1}{3}\right)^3-5\left(-\frac{1}{3}\right)^2+4\left(-\frac{1}{3}\right)+3=1$$

1

2 Find the value of the constant a by dividing x^3+2x^2-3x+a by $x+1$ leaving a remainder of 3.

Given $P(x)=x^3+2x^2-3x+a$.

Dividing $P(x)$ by $x+1$ leaves a remainder of 3, which gives us $P(-1)=3$.

Specifically, because $(-1)^3+2(-1)^2-3(-1)+a=3$, we get $a=-1$.

$a=-1$

PRACTICE

1 Find the remainder by dividing x^3-3x^2-2x+4 by $x+2$.

Given $P(x)=x^3-3x^2-2x+4$.

When divided by $x+2$, the remainder is $P(-2)$, so

we get $P(-2)=(-2)^3-3(-2)^2-2(-2)+4=-12$.

-12

2 Find the value of the constant a by dividing x^3+x^2-5x-a by $x-2$ leaving a remainder of 2.

Given $P(x)=x^3+x^2-5x-a$.

Dividing $P(x)$ by $x-2$ leaves a remainder of 2, which gives us $P(2)=2$.

Specifically, because $2^3+2^2-5\cdot 2-a=2$, we get $a=0$.

$a=0$

3 Dividing x^3-ax^2+bx+7 by $x-2$ leaves a remainder of 25, and dividing by $x+3$ leaves a remainder of 10. Find the value of constants a and b.

Given $P(x)=x^3-ax^2+bx+7$.

Dividing $P(x)$ by $x-2$ leaves a remainder of 25, so from $P(2)=25$, we get

$2^3-a\cdot 2^2+b\cdot 2+7=25, 2a-b=-5$ ...(i)

Dividing $P(x)$ by $x+3$ leaves a remainder of 10, so from $P(-3)=10$, we get

$(-3)^3-a(-3)^2+b(-3)+7=10, 3a+b=-10$ ...(ii)

Solving for (i) and (ii), gives us $a=-3$ and $b=-1$.

$a=-3, b=-1$

4 Find the value of the constant a by dividing $2x^3+a^2x^2+5x-a-1$ by $x-1$ leaving a remainder of 8.

Given $P(x)=2x^3+a^2x^2+5x-a-1$.

Dividing $P(x)$ by $x-1$ leaves a remainder of 8, so we get $P(1)=8$.

Specifically, we get $2\cdot 1^3+a^2\cdot 1^2+5\cdot 1-a-1=8, a^2-a+6=8$.

Solving this gives us $a=-1$ and 2.

$a=-1,2$

5 Dividing integer polynomial $P(x)$ by $x+1$ leaves a remainder of 5, and dividing by $x-3$ leaves a remainder of 9. Find the remainder by dividing integer polynomial $P(x)$ by $(x+1)(x-3)$.

Since the remainder of $P(x)$ divided by the quadratic equation $(x+1)(x-3)$ is either a linear expression or a constant, let the remainder be $ax+b$, (where a and b are constants). Given the quotient $Q(x)$, we can derive the following equality.

$P(x)=(x+1)(x-3)Q(x)+ax+b$ ...(i)

Here, dividing $P(x)$ by $x+1$ leaves a remainder of 5, so we get $P(-1)=5$.

In (i), substituting $x=-1$ gives $P(-1)=-a+b$, so $-a+b=5$ ...(ii)

Further, dividing $P(x)$ by $x-3$ leaves a remainder of 9, so we get $P(3)=9$.

In (i), substituting $x=3$ gives $P(3)=3a+b$, so we get $3a+b=9$ ...(iii)

Solving for (ii) and (iii), gives us $a=1$ and $b=6$, leaving a remainder of $x+6$.

$x+6$

Factor theorem and factorizing cubic polynomials

TARGET To understand how to factorize **cubic** polynomials using the factor theorem.

STUDY GUIDE

Factor theorem

If $x-k$ is a factor of integer polynomial $P(x)$, then $P(x)$ is divisible by $x-k$. That is to say, that the remainder $P(k)$ from dividing $P(x)$ by $x-k$ is 0.

Conversely, when the remainder $P(k)$ from dividing $P(x)$ by $x-k$ is 0, then $P(x)$ is divisible by $x-k$ and has $x-k$ as a factor.

(1) Integer polynomial $P(x)$ has $x-k$ as a factor $\Longleftrightarrow P(k)=0$

(2) Integer polynomial $P(x)$ has $ax+b$ as a factor $\Longleftrightarrow P\left(-\dfrac{b}{a}\right)=0$

How to find k such that $P(k)=0$ $\quad \pm\dfrac{\textbf{Divisor of constant term}}{\textbf{Divisor of the highest coefficient}}$

EXERCISE

◆ Factorize the following equations.

(1) x^3+2x+3

Let $P(x)=x^3+2x+3$.

Since $P(-1)=(-1)^3+2(-1)+3=0$, then $P(x)$ has $x+1$ as a factor.

From $(x^3+2x+3)\div(x+1)=x^2-x+3$, we get

$x^3+2x+3=(x+1)(x^2-x+3)$

$$
\begin{array}{r|rrrr}
 & x^2 & -x & +3 & \\
\hline
x+1 & x^3 & & +2x & +3 \\
 & x^3 & +x^2 & & \\
\hline
 & & -x^2 & +2x & \\
 & & -x^2 & -x & \\
\hline
 & & & 3x & +3 \\
 & & & 3x & +3 \\
\hline
 & & & & 0
\end{array}
$$

$$\boldsymbol{(x+1)(x^2-x+3)}$$

(2) $2x^3+x^2+3x-2$

Let $P(x)=2x^3+x^2+3x-2$.

Since $P\left(\dfrac{1}{2}\right)=2\left(\dfrac{1}{2}\right)^3+\left(\dfrac{1}{2}\right)^2+3\left(\dfrac{1}{2}\right)-2=0$, then $P(x)$ has $2x-1$ as a factor.

From $(2x^3+x^2+3x-2)\div(2x-1)=x^2+x+2$, we get

$2x^3+x^2+3x-2=(2x-1)(x^2+x+2)$

$$
\begin{array}{r|rrrr}
 & x^2 & +x & +2 & \\
\hline
2x-1 & 2x^3 & +x^2 & +3x & -2 \\
 & 2x^3 & -x^2 & & \\
\hline
 & & 2x^2 & +3x & \\
 & & 2x^2 & -x & \\
\hline
 & & & 4x & -2 \\
 & & & 4x & -2 \\
\hline
 & & & & 0
\end{array}
$$

$$\boldsymbol{(2x-1)(x^2+x+2)}$$

PRACTICE

1 Factorize the following equations.

(1) x^3-2x^2-6x+4

Let $P(x)=x^3-2x^2-6x+4$.

Since $P(-2)=(-2)^3-2(-2)^2-6(-2)+4=0$, **then** $P(x)$ **has** $x+2$ **as a factor.**

From $(x^3-2x^2-6x+4)\div(x+2)=x^2-4x+2$, **we get**

$(x^3-2x^2-6x+4)=(x+2)(x^2-4x+2)$

$$(x+2)(x^2-4x+2)$$

(2) $2x^3-5x^2-x+1$

Let $P(x)=2x^3-5x^2-x+1$.

Since $P\left(-\frac{1}{2}\right)=2\left(-\frac{1}{2}\right)^3-5\left(-\frac{1}{2}\right)^2-\left(-\frac{1}{2}\right)+1=0$, **then** $P(x)$ **has** $2x+1$ **as a factor.**

From $(2x^3-5x^2-x+1)\div(2x+1)=x^2-3x+1$, **we get**

$(2x^3-5x^2-x+1)=(2x+1)(x^2-3x+1)$

$$(2x+1)(x^2-3x+1)$$

(3) $2x^3+x^2-5x+2$

Let $P(x)=2x^3+x^2-5x+2$.

Since $P(1)=2\cdot1^3+1^2-5\cdot1+2=0$, **then** $P(x)$ **has** $x-1$ **as a factor.**

From $(2x^3+x^2-5x+2)\div(x-1)=2x^2+3x-2$, **we get**

$$\begin{aligned}2x^3+x^2-5x+2&=(x-1)(2x^2+3x-2)\\&=(x-1)(x+2)(2x-1)\end{aligned}$$

$$(x-1)(x+2)(2x-1)$$

2 Find the value of the constants a and b such that ax^3-3x^2+bx-3 is divisible by x^2-2x-3.

Let $P(x)=ax^3-3x^2+bx-3$.

Since $x^2-2x-3=(x+1)(x-3)$, **the conditions for** $P(x)$ **to be divisible by** x^2-2x-3 **are that** $P(x)$ **has** $x+1$ **and** $x-3$ **as factors. Therefore, from** $P(-1)=0$,

$P(-1)=a(-1)^3-3(-1)^2+b(-1)-3=0, -a-b-6=0, a+b=-6\ldots(\mathrm{i})$

Also, from $P(3)=0$,

$P(3)=a\cdot3^3-3\cdot3^2+b\cdot3-3=0, 27a+3b-30=0, 9a+b=10\ldots(\mathrm{ii})$

Therefore, solving for (i) **and** (ii), **gives us** $a=2$ **and** $b=-8$.

$$a=2,\ b=-8$$

EXTRA Info.

Use the scientific calculator to find the value of a function to use with the remainder theorem and factorization.

EXERCISE

◆ Solve each of the following problems.

(1) Find the remainder of x^3-4x^2+x+6 divided by $x+2$. Also, factorize x^3-4x^2+x+6.

Let $f(x)=x^3-4x^2+x+6$.

From the remainder theorem, the remainder of $f(x)$ divided by $x+2$ is $f(-2)$, so from $f(-2)=-20$, the remainder is -20

From the factor theorem, since $f(-1)=0$, $f(2)=0$, and $f(3)=0$, the 3rd order $f(x)$ has $x+1$, $x-2$, and $x-3$ as factors.

Therefore, $f(x)=(x+1)(x-2)(x-3)$

Remainder is -20, $(x+1)(x-2)(x-3)$

(2) Factorize $2x^3-9x^2+10x-3$.

Let $g(x)=2x^3-9x^2+10x-3$.

From the factor theorem, since $g\left(\frac{1}{2}\right)=0$, $g(1)=0$, and $g(3)=0$, the 3rd order $g(x)$ has $x-\frac{1}{2}$, $x-1$, and $x-3$ as factors.

$$g(x)=2x^3-9x^2+10x-3=2\left(x-\frac{1}{2}\right)(x-1)(x-3)=(2x-1)(x-1)(x-3)$$

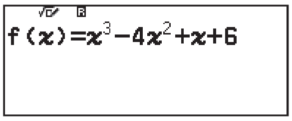
$(2x-1)(x-1)(x-3)$

check

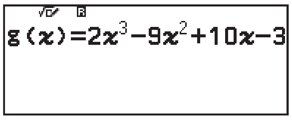
Press ⌂, select [Table], press OK, then clear the previous data by pressing ⏻

Press ⊙⊙⊙, select [Define f(x)/g(x)], press OK, select [Define f(x)], press OK

After inputting f$(x)=x^3-4x^2+x+6$, press EXE

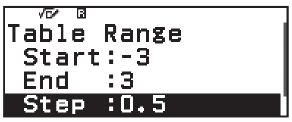


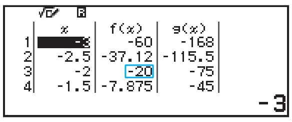
In the same way, input g$(x)=2x^3-9x^2+10x-3$.

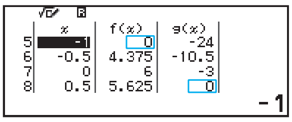


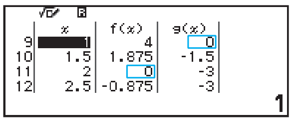
Press ⊙⊙⊙, select [Table Range], press OK

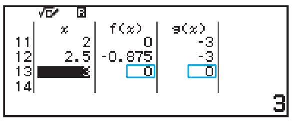
After inputting [Start:-3, End:3, and Step:0.5], select [Execute], press EXE



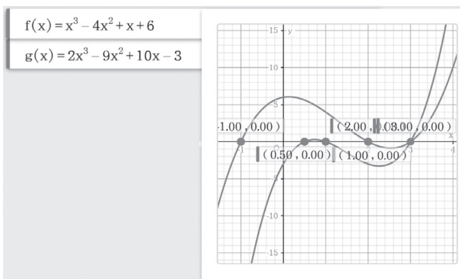








Press ⬆ ⓧ, scan the QR code to display a graph.



From the table and graph, we can confirm that the solution to the equation $f(x)=0$ is $x=-1$, 2, and 3, and the solution to the equation $g(x)=0$ is $x=\frac{1}{2}$, 1, and 3.

PRACTICE

◆ Solve each of the following problems.

(1) Find the remainder of x^3-2x^2-x+2 divided by $x-3$. Also, factorize x^3-2x^2-x+2.

Let $f(x)=x^3-2x^2-x+2$.
From the remainder theorem, the remainder of $f(x)$ divided by $x-3$ is $f(3)$, so from $f(3)=8$, the remainder is 8
From the factor theorem, since $f(-1)=0$, $f(1)=0$, and $f(2)=0$, the 3rd order $f(x)$ has $x+1$, $x-1$, and $x-2$ as factors.
Therefore, $f(x)=(x+1)(x-1)(x-2)$

Remainder is 8 and $(x+1)(x-1)(x-2)$

(2) Factorize $2x^3-7x^2+2x+3$.

Let $g(x)=2x^3-7x^2+2x+3$.
From the factor theorem, since $g\left(-\frac{1}{2}\right)=0$, $g(1)=0$, and $g(3)=0$, the 3rd order $g(x)$ has $x+\frac{1}{2}$, $x-1$, and $x-3$ as factors.
$$g(x)=2x^3-7x^2+2x+3=2\left(x+\frac{1}{2}\right)(x-1)(x-3)=(2x+1)(x-1)(x-3)$$

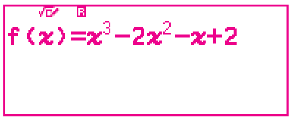
$(2x+1)(x-1)(x-3)$

check

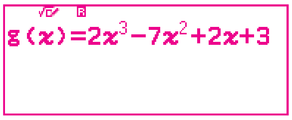
Press ⌂**, select** [Table]**, press** OK**, then clear the previous data by pressing** ⏻

Press ⋯**, select** [Define f(x)/g(x)]**, press** OK**, select** [Define f(x)]**, press** OK

After inputting f(x)$=x^3-2x^2-x+2$**, press** EXE



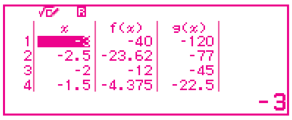
In the same way, input g(x)$=2x^3-7x^2+2x+3$.

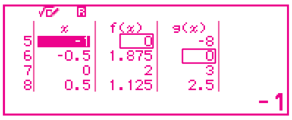


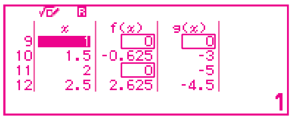
Press ⋯**, select** [Table Range]**, press** OK

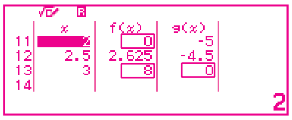
After inputting [Start:−3, End:3, and Step:0.5]**,**

select [Execute]**, press** EXE

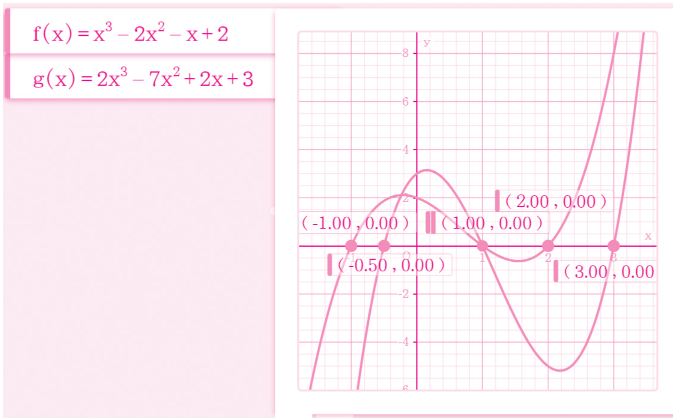








Press ⇧ x**, scan the QR code to display a graph.**



From the table and graph, we can confirm that the solution to the equation $f(x)=0$ **is** $x=-1$, 1, **and** 2**, and the solution to the equation** $g(x)=0$ **is** $x=-\dfrac{1}{2}$, 1, **and** 3.

Identities

TARGET To understand the meaning and nature of identities.

STUDY GUIDE

Identities

Equalities, which we call equations, are only true for a specific value of x. For example, the linear equation $x+2=5$ is only true if $x=3$. In contrast, the equality $(x+y)(x-y)=x^2-y^2$ is true for any value assigned to x and y. In this way, an equality that is true for any value assigned to the letters in it is called an **identity** for those letters.

(1) $ax^2+bx+c=a'x^2+b'x+c'$ **is an identity for x**
$\Longleftrightarrow$ $a=a', b=b', c=c'$

(2) $ax^2+bx+c=0$ **is an identity for x**
$\Longleftrightarrow$ $a=0, b=0, c=0$

EXERCISE

◆ Find the values of the constants a, b, and c such that the equality $3x^2+4x-2=a(x+1)^2+b(x+1)+c$ is an identity for x.

Expand the right side and arrange x in descending order of powers.

From $a(x^2+2x+1)+b(x+1)+c=ax^2+2ax+a+bx+b+c=ax^2+(2a+b)x+a+b+c$,

$3x^2+4x-2=ax^2+(2a+b)x+a+b+c$

Since this equality is an identity for x, compare the coefficients of terms of the same degree on both sides. This method is called the coefficient comparison method.

Solving the simultaneous equations $\begin{cases} 3=a \\ 4=2a+b \\ -2=a+b+c \end{cases}$ gives us $a=3, b=-2, c=-3$.

$$a=3, b=-2, c=-3$$

PRACTICE

◆ Find the values of the constants a, b, and c such that the equality $-x^2+4x+7=a(x-1)^2+b(x-1)+c$ is an identity for x.

Expand the right side and arrange x in descending order of powers.

From $a(x^2-2x+1)+b(x-1)+c=ax^2+(-2a+b)x+a-b+c$,

$-x^2+4x+7=ax^2+(-2a+b)x+a-b+c$

Since this equality is an identity for x, compare the coefficients of terms of the same degree on both sides.

Solving the simultaneous equations $\begin{cases} -1=a \\ 4=-2a+b \\ 7=a-b+c \end{cases}$ **gives us $a=-1, b=2, c=10$.**

$$a=-1, b=2, c=10$$

How to determine constants in identities

TARGET To understand how to substitute numerical values to determine the constants in an identity.

STUDY GUIDE

How to determine constants in identities

Identities are always true for any value assigned, so we can determine the constants by assigning values that are easy to calculate. This method is called the **value substitution method**. When doing this, only the necessary conditions are satisfied, so we need to reverse it to confirm the sufficient conditions also.

(1) Substitute some values for x on both sides of the equation.
(2) Solve with simultaneous equations, and determine the value of the constants.
(3) Confirm that the equality is an identity.

EXTRA Info.

Example in which substitution method does not hold

$a(x+1)=1$

If $x=0$, then $a=1$, but when $x=1$, we see that $a=1$ is not true.

$\Rightarrow$ Not an identity.

EXERCISE

◆ Find the values of the constants a, b, and c such that the equality $2x^2-14x+15=a(x-2)^2+b(x-2)+c$ is an identity for x.

To make the calculation easier, substitute a value for x such that $x-2$ is 0, 1, and -1, specifically, substitute $x=2$, 3, and 1 for x on both sides.

Substitute $x=2$, so that $2\cdot2^2-14\cdot2+15=a(2-2)^2+b(2-2)+c, -5=c \quad \ldots(\text{i})$

Substitute $x=3$, so that $2\cdot3^2-14\cdot3+15=a(3-2)^2+b(3-2)+c, -9=a+b+c \quad \ldots(\text{ii})$

Substitute $x=1$, so that $2\cdot1^2-14\cdot1+15=a(1-2)^2+b(1-2)+c, 3=a-b+c \quad \ldots(\text{iii})$

Use simultaneous equations to solve (i), (ii), and (iii) to get $a=2, b=-6, c=-5$.

When this value is substituted into the equation, the left and right sides become equal, so it is an identity.

$$a=2, b=-6, c=-5$$

PRACTICE

◆ Find the values of the constants a, b, and c such that the equality $x^2+2x+8=a(x+2)^2+b(x+2)+c$ is an identity for x.

To make the calculation easier, substitute a value for x such that $x+2$ is 0, 1, and -1, specifically, substitute $x=-2$, -1, and -3 for x on both sides.

Substitute $x=-2$, so that $(-2)^2+2(-2)+8=a(-2+2)^2+b(-2+2)+c, 8=c \ldots(\text{i})$

Substitute $x=-1$, so that

$(-1)^2+2(-1)+8=a(-1+2)^2+b(-1+2)+c, 7=a+b+c \ldots(\text{ii})$

Substitute $x=-3$, so that

$(-3)^2+2(-3)+8=a(-3+2)^2+b(-3+2)+c, 11=a-b+c \ldots(\text{iii})$

Use simultaneous equations to solve (i), (ii), and (iii), to get $a=1, b=-2, c=8$.

When this value is substituted into the equation, the left and right sides become equal, so it is an identity.

$$a=1, b=-2, c=8$$

Proving equalities

TARGET To understand how to prove equalities.

STUDY GUIDE

How to prove equalities

To prove that $A=B$, we have 3 methods, that we use separately according to the problem.

(1) Transform the more complex of A and B, then derive the other one.
(2) When both A and B are complex, transform both to derive the same expression C.
(3) Transform $A-B$ to show $A-B=0$.

Proving equalities that have conditional expressions

(1) Solve conditional expressions for 1 letter.

(2) Substitute the expression from (1) into the left and right sides to eliminate 1 letter.

(3) Transform the expression from (2) to show the expressions are the same.

Proving equalities when conditional expressions are fractional expressions (proportional expressions)

(1) Let fractional expressions and proportional expressions be k.

(2) Transform the expressions from (1) and substitute for the left and right sides to eliminate a letter.

(3) Transform the expression from (2) to show the expressions are the same.

EXERCISE

◆ Solve the following equalities.

(1) $x^3+y^3=(x+y)^3-3xy(x+y)$

[Proof] (Right side) $=x^3+3x^2y+3xy^2+y^3-(3x^2y+3xy^2)=x^3+y^3=$(Left side)

Therefore, (Left side)=(Right side)

(2) When $a+b+c=0$, we get $(a+b)^2+(b+c)^2+b^2=-2(ab+bc+ca)$

[Proof] From $a+b+c=0$, we get $c=-(a+b)$

(Left side) $=(a+b)^2+\{b-(a+b)\}^2+b^2=a^2+2ab+b^2+a^2+b^2=2a^2+2ab+2b^2$

(Right side) $=-2\{ab-b(a+b)-(a+b)a\}=-2(-b^2-a^2-ab)=2a^2+2ab+2b^2$

Therefore, (Left side)=(Right side)

(3) When $\dfrac{a}{b}=\dfrac{c}{d}$, we get $\dfrac{ad+bc}{2bd}=\dfrac{a+c}{b+d}$

[Proof] Let $\dfrac{a}{b}=\dfrac{c}{d}=k$, then $a=bk, c=dk$

(Left side) $=\dfrac{bdk+bdk}{2bd}=\dfrac{2bdk}{2bd}=k$

(Right side) $=\dfrac{bk+dk}{b+d}=\dfrac{(b+d)k}{b+d}=k$

Therefore, (Left side)=(Right side)

PRACTICE

◆ Solve the following equalities.

(1) $(3x+y)^2-(3x-y)^2=12xy$

[Proof]

(Left side) $= 9x^2+6xy+y^2-(9x^2-6xy+y^2)=12xy=$ **(Right side)**

Therefore, (Left side)=(Right side)

(2) $(2n+1)^2+(2n^2+2n)^2=(2n^2+2n+1)^2$

[Proof]

(Left side) $= 4n^2+4n+1+4n^4+8n^3+4n^2$

$= 4n^4+8n^3+8n^2+4n+1$

(Right side) $= 4n^4+4n^2+1+8n^3+4n+4n^2$

$= 4n^4+8n^3+8n^2+4n+1$

Therefore, (Left side)=(Right side)

(3) When $x+y+z=0$, we get $x^3+y^3+z^3=-3(x+y)(y+z)(z+x)$

[Proof]

From $x+y+z=0$**, we get** $z=-(x+y)$

(Left side) $= x^3+y^3+\{-(x+y)\}^3$

$= x^3+y^3-(x^3+3x^2y+3xy^2+y^3)$

$= x^3+y^3-x^3-3x^2y-3xy^2-y^3$

$= -3x^2y-3xy^2$

(Right side) $= -3(x+y)\{y-(x+y)\}\{-(x+y)+x\}$

$= -3(x+y)(-x)(-y)$

$= -3x^2y-3xy^2$

Therefore, (Left side)=(Right side)

(4) When $a:b=c:d$, we get $\frac{5a-3b}{5c-3d}=\frac{5a+3b}{5c+3d}$

[Proof]

From $a:b=c:d$**, then if we let** $\frac{a}{b}=\frac{c}{d}=k$**, we get** $a=bk, c=dk$

(Left side) $= \frac{5bk-3b}{5dk-3d}=\frac{(5k-3)b}{(5k-3)d}=\frac{b}{d}$

(Right side) $= \frac{5bk+3b}{5dk+3d}=\frac{(5k+3)b}{(5k+3)d}=\frac{b}{d}$

Therefore, (Left side)=(Right side)

Proving inequalities

TARGET To understand how to prove inequalities.

STUDY GUIDE

How to prove inequalities

To prove the inequality $A \geq B$, we subtract B from A and transform the expression to show it is greater than 0.

Proving inequalities that have root signs and absolute values

When 2 expressions A and B have radical signs or absolute values, use the following property to prove that $A \geq B$.

When $A \geq 0$ and $B \geq 0$, then $A \geq B \iff A^2 \geq B^2$

Relation between arithmetic mean and geometric mean

For 2 real numbers a and b, $\dfrac{a+b}{2}$ is called the **arithmetic mean** of a and b, and when $a>0$ and $b>0$, then $\sqrt{ab}$ is called the **geometric mean** of a and b. The arithmetic mean and geometric mean have the following relation.

When $a>0$ and $b>0$, then $\dfrac{a+b}{2} \geq \sqrt{ab}$ (forming an equality when $a=b$)

explanation

Given $a>0$ and $b>0$, we get $\dfrac{a+b}{2} - \sqrt{ab} = \dfrac{1}{2}(a - 2\sqrt{ab} + b) = \dfrac{1}{2}\{(\sqrt{a})^2 - 2\sqrt{a}\sqrt{b} + (\sqrt{b})^2\} = \dfrac{1}{2}(\sqrt{a} - \sqrt{b})^2 \geq 0$

Therefore, $\dfrac{a+b}{2} \geq \sqrt{ab}$

So, we have an equality $\sqrt{a} - \sqrt{b} = 0$, that is to say when we get $a=b$.

EXERCISE

◆ Solve the following inequalities.

(1) $a^2 + 4b^2 \geq 4ab$

[Proof] $a^2 + 4b^2 - 4ab = (a - 2b)^2 \geq 0$

Therefore, $a^2 + 4b^2 \geq 4ab$

So, we have an equality $a-2b=0$, that is to say when we get $a=2b$.

(2) When $a>0$ and $b>0$, $2\sqrt{a} + 3\sqrt{b} > \sqrt{4a+9b}$

[Proof] $(2\sqrt{a} + 3\sqrt{b})^2 - (\sqrt{4a+9b})^2 = 4a + 12\sqrt{ab} + 9b - (4a + 9b) = 12\sqrt{ab} > 0$

Therefore, $(2\sqrt{a} + 3\sqrt{b})^2 > (\sqrt{4a+9b})^2$

From $2\sqrt{a} + 3\sqrt{b} > 0, \sqrt{4a+9b} > 0$, we get $2\sqrt{a} + 3\sqrt{b} > \sqrt{4a+9b}$

(3) When $a>0$, then $\left(a+\frac{1}{a}\right)\left(a+\frac{9}{a}\right)\geq 16$

[Proof] (left side)$=a^2+a\cdot\frac{9}{a}+\frac{1}{a}\cdot a+\frac{9}{a^2}=a^2+\frac{9}{a^2}+10$

From $a>0$, we get $a^2>0, \frac{9}{a^2}>0$ so from the arithmetic mean and geometric mean relation, $a^2+\frac{9}{a^2}\geq 2\sqrt{a^2\cdot\frac{9}{a^2}}=2\cdot 3=6$

Therefore, $\left(a+\frac{1}{a}\right)\left(a+\frac{9}{a}\right)=a^2+\frac{9}{a^2}+10\geq 6+10=16$

So, we have an equality $a^2=\frac{9}{a^2}$, that is to say from $a>0$, when $a=\sqrt{3}$.

PRACTICE

◆ Solve the following inequalities.

(1) When $a>b>1$, then $a^2b-ab^2>a(a-1)-b(b-1)$

[Proof]

From $a>b>1$, then $a-b>0$ and $a-1>0$ and $b-1>0$, so

$a^2b-ab^2-\{a(a-1)-b(b-1)\}=a^2b-ab^2-(a^2-a-b^2+b)$

$=ab(a-b)-(a+b)(a-b)+(a-b)=(a-b)(ab-a-b+1)$

$=(a-b)(a-1)(b-1)>0$

Therefore, $a^2b-ab^2>a(a-1)-b(b-1)$

(2) $|a+b|\leq|a|+|b|$

[Proof]

$(|a|+|b|)^2-|a+b|^2=|a|^2+2|a||b|+|b|^2-(a^2+2ab+b^2)$

$=a^2+2|a||b|+b^2-(a^2+2ab+b^2)=2(|ab|-ab)\geq 0$

Therefore, $(|a|+|b|)^2\geq|a+b|^2$

From $|a+b|\geq 0$ and $|a|+|b|\geq 0$, then $|a+b|\leq|a|+|b|$

So, we have an equality $|ab|=ab$, that is to say when $ab\geq 0$.

(3) When $a>0$ and $b>0$, then $\left(1+\frac{b}{a}\right)\left(1+\frac{a}{b}\right)\geq 4$

[Proof]

(Left side)$=1+\frac{a}{b}+\frac{b}{a}+\frac{b}{a}\cdot\frac{a}{b}=\frac{a}{b}+\frac{b}{a}+2$

From $a>0$ and $b>0$, we get $\frac{a}{b}>0, \frac{b}{a}>0$ so from the arithmetic mean and geometric mean relation,

$\frac{a}{b}+\frac{b}{a}\geq 2\sqrt{\frac{a}{b}\cdot\frac{b}{a}}=2\cdot 1=2$

Therefore, $\left(1+\frac{b}{a}\right)\left(1+\frac{a}{b}\right)=\frac{a}{b}+\frac{b}{a}+2\geq 2+2=4$

So, we have an equality $\frac{a}{b}=\frac{b}{a}$, that is to say from $a>0$ and $b>0$, when $a=b$.

EXTRA Info.

Use the scientific calculator to confirm that equations and inequalities hold.

EXERCISE

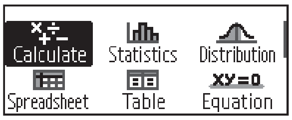
◆ Use the scientific calculator to confirm that the following equations and inequalities hold.

(1) $a^3+b^3=(a+b)^3-3ab(a+b)$

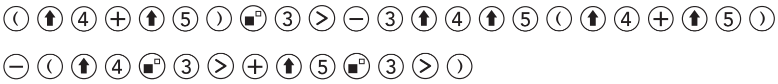
$a^3+b^3=(a+b)^3-3ab(a+b) \Leftrightarrow (a+b)^3-3ab(a+b)-(a^3+b^3)=0$

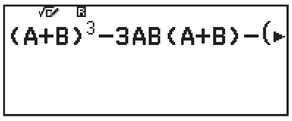
Therefore, we simply need to confirm that the right side − the left side =0 (or, the left side − the right side =0).

Press ⌂, select [Calculate], press OK



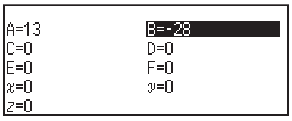
Input $(\mathrm{A}+\mathrm{B})^3-3\mathrm{AB}(\mathrm{A}+\mathrm{B})-(\mathrm{A}^3+\mathrm{B}^3)$.

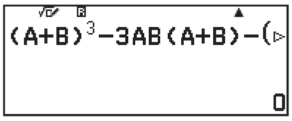




Use the VARIABLE function to assign any values to A and B.

(Example) To input A=13 and B=−28





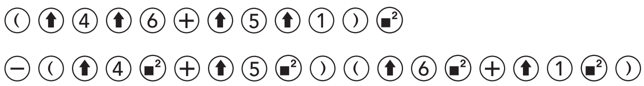
We can confirm $(\mathrm{A}+\mathrm{B})^3-3\mathrm{AB}(\mathrm{A}+\mathrm{B})-(\mathrm{A}^3+\mathrm{B}^3)=0$ using any values, so we can verify the validity of $a^3+b^3=(a+b)^3-3ab(a+b)$ holds. You may want to verify other examples.

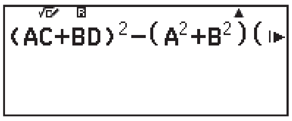
(2) $(ac+bd)^2 \le (a^2+b^2)(c^2+d^2)$

$(ac+bd)^2 \le (a^2+b^2)(c^2+d^2) \Leftrightarrow (ac+bd)^2-(a^2+b^2)(c^2+d^2) \le 0$

Therefore, we simply need to confirm that the left side − the right side ≤0 (or, the right side − the left side ≥ 0).

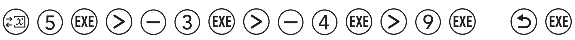
Input $(\mathrm{AC}+\mathrm{BD})^2-(\mathrm{A}^2+\mathrm{B}^2)(\mathrm{C}^2+\mathrm{D}^2)$.

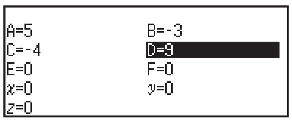


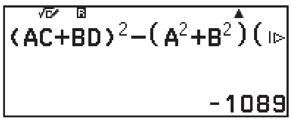


Use the VARIABLE function to assign any values to A, B, C, D.

(Example) To input A=5, B=−3, C=−4, and D=9







We can confirm $(\mathrm{AC}+\mathrm{BD})^2-(\mathrm{A}^2+\mathrm{B}^2)(\mathrm{C}^2+\mathrm{D}^2)<0$ using any values, so we can verify the validity of $(ac+bd)^2 \le (a^2+b^2)(c^2+d^2)$ holds. You may want to verify other examples.

PRACTICE

◆ Use the scientific calculator to confirm that the following equations and inequalities hold.

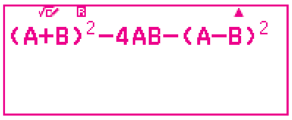
(1) $(a-b)^2=(a+b)^2-4ab$

Use the scientific calculator to confirm $(a-b)^2=(a+b)^2-4ab \Leftrightarrow$ $(a+b)^2-4ab-(a-b)^2=0$ **(or** $(a-b)^2-\{(a+b)^2-4ab\}=0$**).**

Press [HOME]**, select** [Calculate]**, press** [OK]

Input $(\mathrm{A}+\mathrm{B})^2-4\mathrm{AB}-(\mathrm{A}-\mathrm{B})^2$**.**

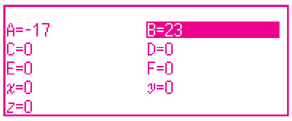
[(] [⇧] [4] [+] [⇧] [5] [)] [■²] [−] [4] [⇧] [4] [⇧] [5]
[−] [(] [⇧] [4] [−] [⇧] [5] [)] [■²]

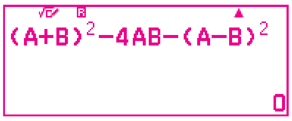


Use the $\mathrm{VARIABLE}$ **function to assign any values to** A **and** B**.**

(Example) To input $\mathrm{A}=-17$ **and** $\mathrm{B}=23$

[VARIABLE] [−] [1] [7] [EXE] [>] [2] [3] [EXE]   [↺] [EXE]





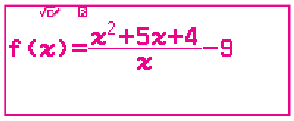
We can confirm $(\mathrm{A}+\mathrm{B})^2-4\mathrm{AB}-(\mathrm{A}-\mathrm{B})^2=0$ **using any values, so we can verify the validity of** $(a-b)^2=(a+b)^2-4ab$ **holds. You may want to verify other examples.**

(2) When $x>0$, then $\dfrac{x^2+5x+4}{x} \geq 9$

Use the scientific calculator to confirm $\dfrac{x^2+5x+4}{x} \geq 9 \Leftrightarrow \dfrac{x^2+5x+4}{x} - 9 \geq 0$ (or $9 - \dfrac{x^2+5x+4}{x} \leq 0$).

Press ⌂, select [Table], press OK, then clear the previous data by pressing ⏻

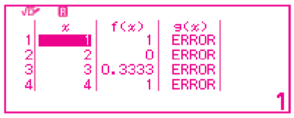
Press ⋯, select [Define f(x)/g(x)], press OK, select [Define f(x)], press OK



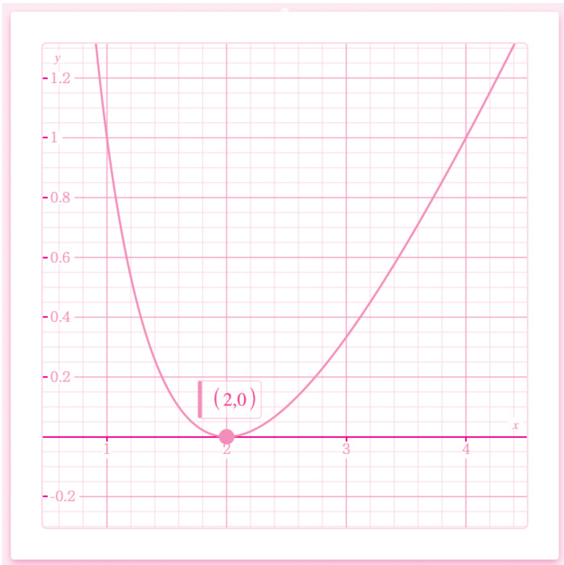
After inputting f(x)$=\dfrac{x^2+5x+4}{x} - 9$, press EXE

Press ⋯, select [Table Range], press OK
After inputting [Start:1, End:4, and Step:1], select [Execute], press EXE

Press ⇧ x, scan the QR code to display a graph.



From the graph, we can confirm the above function.
Furthermore, we can also see that the equality holds at $x=2$.



EXTRA Info.

Use the scientific calculator to infer a general rule from concrete examples.

EXERCISE

1 When $a > \sqrt{2}$, determine the magnitudes of the following 3 numbers.

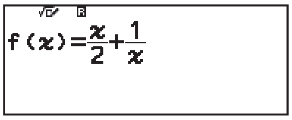
$$\sqrt{2},\ \frac{a}{2}+\frac{1}{a},\ \frac{a+2}{a+1}$$

Use the Table function in the scientific calculator to input specific values to infer the magnitude relationship.

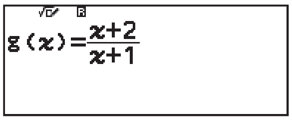
Press ⌂, select [Table], press OK, then clear the previous data by pressing ⏻

Press ⋯, select [Define $\mathrm{f}(x)/\mathrm{g}(x)$], press OK, select [Define $\mathrm{f}(x)$], press OK

After inputting $\mathrm{f}(x)=\dfrac{x}{2}+\dfrac{1}{x}$, press EXE

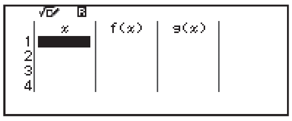


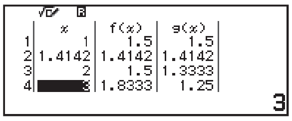
In the same way, input $\mathrm{g}(x)=\dfrac{x+2}{x+1}$.



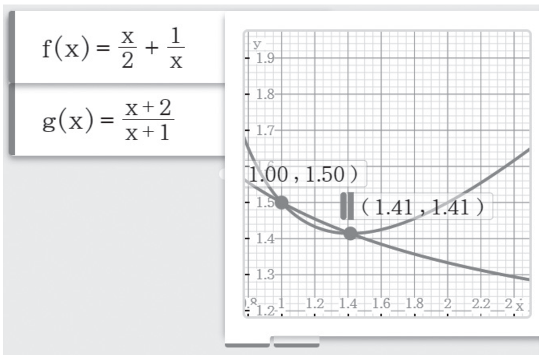
Move to row x line 1.

After inputting [x1:1, x2: $\sqrt{(2)}$, x3:2, and x4:3] respectively, press EXE





Press ⇧ ⓧ, scan the QR code to display a graph.



From the table and graph, when $a > \sqrt{2}$, we can infer that $\dfrac{a+2}{a+1} < \sqrt{2} < \dfrac{a}{2}+\dfrac{1}{a}$.

Furthermore, from here we can prove the above inference.

$$\sqrt{2}-\frac{a+2}{a+1}=\frac{\sqrt{2}(a+1)-(a+2)}{a+1}=\frac{(\sqrt{2}-1)a-(2-\sqrt{2})}{a+1}=\frac{(\sqrt{2}-1)a-\sqrt{2}(\sqrt{2}-1)}{a+1}=\frac{(\sqrt{2}-1)(a-\sqrt{2})}{a+1}>0$$

Therefore, $\sqrt{2} > \dfrac{a+2}{a+1}\ (a > \sqrt{2})$

Regarding $\dfrac{a}{2}$ and $\dfrac{1}{a}$, from the relation of the arithmetic mean to the geometric mean, we get $\dfrac{a}{2}+\dfrac{1}{a} \geq 2\sqrt{\dfrac{a}{2}\cdot\dfrac{1}{a}}=\sqrt{2}$.

We can derive the equal sign when $\dfrac{a}{2}=\dfrac{1}{a}$, that is to say $a=\sqrt{2}$, but we cannot derive it when $a > \sqrt{2}$.

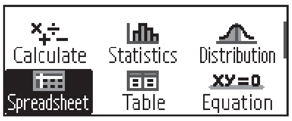
Therefore, $\dfrac{a}{2}+\dfrac{1}{a} > \sqrt{2}$

This gives us $\dfrac{a+2}{a+1} < \sqrt{2} < \dfrac{a}{2}+\dfrac{1}{a}$

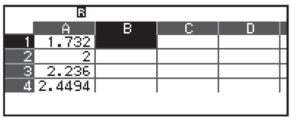
OTHER METHODS

There is also a method to infer magnitude relationships by using the $\mathrm{Spreadsheet}$ function in the scientific calculator.

Press ⌂, select $[\mathrm{Spreadsheet}]$, press OK, then clear the previous data by pressing ⏻



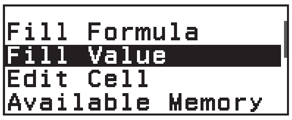
After inputting $[\mathrm{A1}: \sqrt{(3)}$, $\mathrm{A2}: \sqrt{(4)}$, $\mathrm{A3}: \sqrt{(5)}$, and $\mathrm{A4}: \sqrt{(6)}]$ respectively, press EXE, move to $[\mathrm{B1}]$.

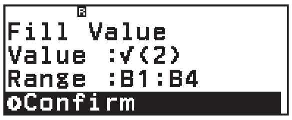


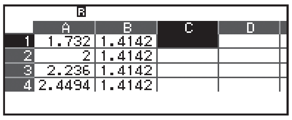
Press ⋯, select $[\mathrm{Fill\ Value}]$, press OK

After inputting $[\mathrm{Value}: \sqrt{(2)}]$, press EXE

After inputting $[\mathrm{Range:B1:B4}]$, press EXE, select $[\mathrm{Confirm}]$, press OK, move to $[\mathrm{C1}]$.



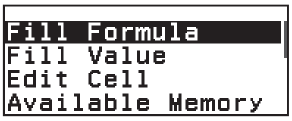


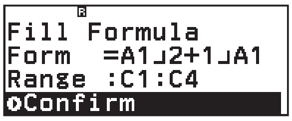


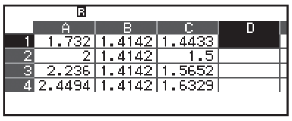
Press ⋯, select $[\mathrm{Fill\ Formula}]$, press OK

After inputting $[\mathrm{Form}{=}\dfrac{\mathrm{A1}}{2}+\dfrac{1}{\mathrm{A1}}]$, press EXE

After inputting $[\mathrm{Range:C1:C4}]$, press EXE, select $[\mathrm{Confirm}]$, press OK, move to $[\mathrm{D1}]$.



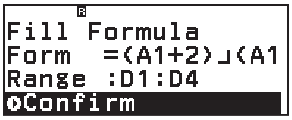


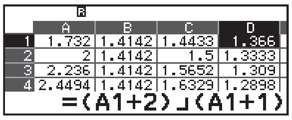


Press ⋯, select $[\mathrm{Fill\ Formula}]$, press OK

After inputting $[\mathrm{Form}{=}\dfrac{(\mathrm{A1}+2)}{(\mathrm{A1}+1)}]$, press EXE

After inputting $[\mathrm{Range:D1:D4}]$, press EXE, select $[\mathrm{Confirm}]$, press OK





From the calculated results, we can confirm that the value in column D < the value in column B < value in column C for all the values in column A $=\sqrt{3}$, $\sqrt{4}$, $\sqrt{5}$, and $\sqrt{6}$.

Therefore, when $a>\sqrt{2}$, we can infer $\dfrac{a+2}{a+1}<\sqrt{2}<\dfrac{a}{2}+\dfrac{1}{a}$.

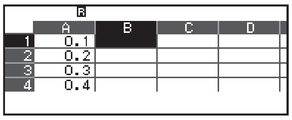
2 When $0<a<b$, $a+b=1$, determine the magnitudes of the following 3 numbers.

$\sqrt{a+b}$, $\sqrt{b}-\sqrt{a}$, $\sqrt{b-a}$

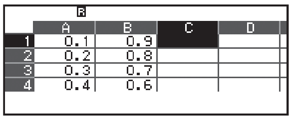
Use the Spreadsheet function in the scientific calculator to input specific values to infer the magnitude relationship.

Press ⌂, select [Spreadsheet], press OK, then clear the previous data by pressing ⏻

After inputting [A1:0.1, A2:0.2, A3:0.3, and A4:0.4] respectively, press EXE, move to [B1].



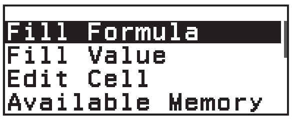
After inputting [B1:0.9, B2:0.8, B3:0.7, and B4:0.6] respectively, press EXE, move to [C1].

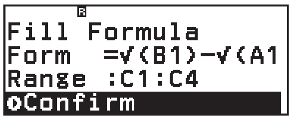


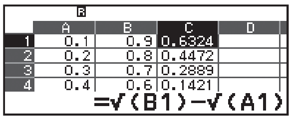
Press ⋯, select [Fill Formula], press OK

After inputting [Form= $\sqrt{(B1)}-\sqrt{(A1)}$], press EXE

After inputting [Range:C1:C4], press EXE, select [Confirm], press OK, move to [D1].





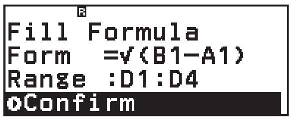


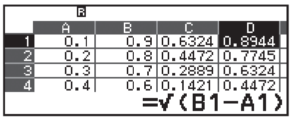
Press ⋯, select [Fill Formula], press OK

After inputting [Form= $\sqrt{(B1-A1)}$], press EXE

After inputting [Range:D1:D4], press EXE,

select [Confirm], press OK





From the calculated results, we can confirm that the value in column C < the value in column D<1 for all the values in column A=0.1, 0.2, 0.3, and 0.4.

Also, since $\sqrt{a+b}=\sqrt{1}=1$, we can infer that $\sqrt{b}-\sqrt{a}<\sqrt{b-a}<\sqrt{a+b}$.

We can prove that this inference holds when $0<a<b$, $a+b=1$.

From $0<\sqrt{b}-\sqrt{a}$, $0<\sqrt{b-a}$, and $0<\sqrt{a+b}$, compare the square of each expression.

$(\sqrt{b-a})^2-(\sqrt{b}-\sqrt{a})^2=b-a-(b-2\sqrt{b}\sqrt{a}+a)=2(\sqrt{a}\sqrt{b}-a)=2\sqrt{a}(\sqrt{b}-\sqrt{a})>0$

Therefore, from $(\sqrt{b-a})^2>(\sqrt{b}-\sqrt{a})^2$ we get $\sqrt{b-a}>\sqrt{b}-\sqrt{a}$

$(\sqrt{a+b})^2-(\sqrt{b-a})^2=a+b-(b-a)=2a>0$

Therefore, from $(\sqrt{a+b})^2>(\sqrt{b-a})^2$ we get $\sqrt{a+b}>\sqrt{b-a}$

This gives us $\sqrt{b}-\sqrt{a}<\sqrt{b-a}<\sqrt{a+b}$

PRACTICE

1 When $a>\sqrt{3}$, determine the magnitudes of the following 3 numbers.

$\sqrt{3},\ \dfrac{a}{4}+\dfrac{3}{a},\ \dfrac{a+3}{a+1}$

Press ⌂, select [Table], press OK, then clear the previous data by pressing ⏻

Press ⋯, select [Define $f(x)/g(x)$], press OK, after inputting [Define $f(x)$], press OK

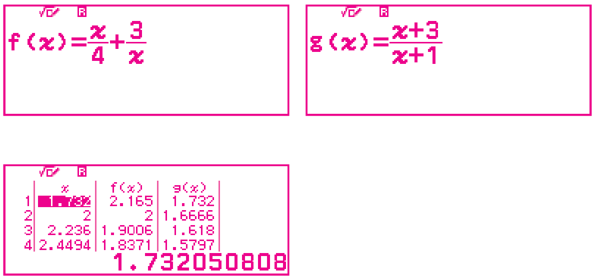
After inputting $f(x)=\dfrac{x}{4}+\dfrac{3}{x}$, select EXE

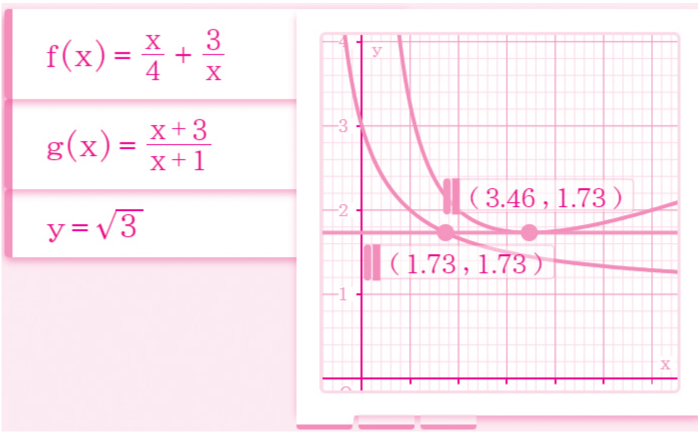
In the same way, input $g(x)=\dfrac{x+3}{x+1}$.

Move to row x line 1.

After inputting [$x1$: $\sqrt{(3)}$, $x2$: $\sqrt{(4)}$, $x3$: $\sqrt{(5)}$, and $x4$: $\sqrt{(6)}$] respectively, press EXE

Press ⇧ ⓧ, scan the QR code to display a graph.





From the table and graph, when $a>\sqrt{3}$, we can infer $\dfrac{a+3}{a+1}<\sqrt{3}\leq\dfrac{a}{4}+\dfrac{3}{a}$.

We can prove that this inference holds when $a>\sqrt{3}$.

$$\sqrt{3}-\frac{a+3}{a+1}=\frac{\sqrt{3}(a+1)-(a+3)}{a+1}=\frac{(\sqrt{3}-1)a-(3-\sqrt{3})}{a+1}$$

$$=\frac{(\sqrt{3}-1)a-\sqrt{3}(\sqrt{3}-1)}{a+1}=\frac{(\sqrt{3}-1)(a-\sqrt{3})}{a+1}>0$$

Therefore, $\sqrt{3}>\dfrac{a+3}{a+1}$

For $\dfrac{a}{4}$ and $\dfrac{3}{a}$, the relationship between the arithmetic mean and the geometric mean is $\dfrac{a}{4}+\dfrac{3}{a}\geq 2\sqrt{\dfrac{a}{4}\cdot\dfrac{3}{a}}=\sqrt{3}$

We can derive the equal sign when $\dfrac{a}{4}=\dfrac{3}{a}$, that is to say $a=2\sqrt{3}$.

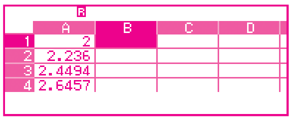
Therefore, $\dfrac{a}{4}+\dfrac{3}{a}\geq\sqrt{3}$

This gives us $\dfrac{a+3}{a+1}<\sqrt{3}\leq\dfrac{a}{4}+\dfrac{3}{a}$

OTHER METHODS

Press ⌂, select [Spreadsheet], press (OK), then clear the previous data by pressing ⏻

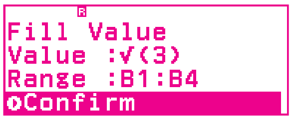
After inputting [A1: $\sqrt{(4)}$, A2: $\sqrt{(5)}$, A3: $\sqrt{(6)}$, and A4: $\sqrt{(7)}$] respectively, press (EXE), move to [B1].

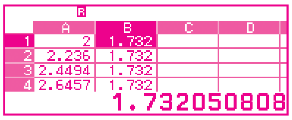


Press (⋯), select [Fill Value], press (OK)

After inputting [Value: $\sqrt{(3)}$], press (EXE)

After inputting [Range:B1:B4], press (EXE), select [Confirm], press (OK), move to [C1].

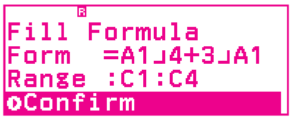


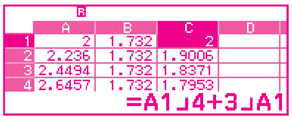


Press (⋯), select [Fill Formula], press (OK)

After inputting [Form= $\dfrac{\text{A1}}{4}+\dfrac{3}{\text{A1}}$], press (EXE)

After inputting [Range:C1:C4], press (EXE), select [Confirm], press (OK), move to [D1].

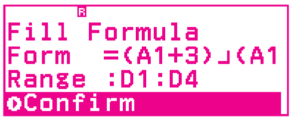


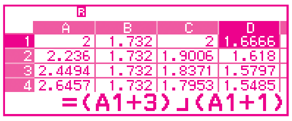


Press (⋯), select [Fill Formula], press (OK)

After inputting [Form= $\dfrac{(\text{A1}+3)}{(\text{A1}+1)}$], press (EXE)

After inputting [Range:D1:D4], press (EXE), select [Confirm], press (OK)





From the calculated results, we can confirm that the value in column D < the value in column B < value in column C for all the values in column A $=\sqrt{4}$, $\sqrt{5}$, $\sqrt{6}$, and $\sqrt{7}$.

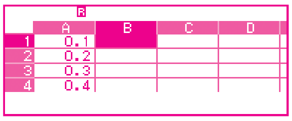
Therefore, when $a>\sqrt{3}$, we can infer $\dfrac{a+3}{a+1}<\sqrt{3}<\dfrac{a}{4}+\dfrac{3}{a}$.

2 When $0<a<b$, and $a+b=1$, determine the magnitudes of the following 3 numbers.

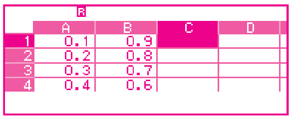
$\frac{1}{2}, a^2+b^2, 2ab$

Press , select [Spreadsheet], press OK, then clear the previous data by pressing

After inputting [A1:0.1, A2:0.2, A3:0.3, and A4:0.4] respectively, press EXE, move to [B1].



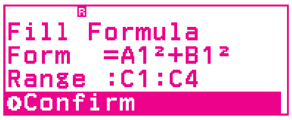
After inputting [B1:0.9, B2:0.8, B3:0.7, and B4:0.6] respectively, press EXE, move to [C1].

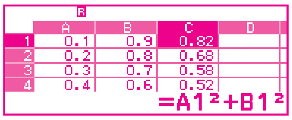


Press , select [Fill Formula], press OK

After inputting [Form= $A1^2+B1^2$], press EXE

After inputting [Range:C1:C4], press EXE, select [Confirm], press OK, move to [D1].

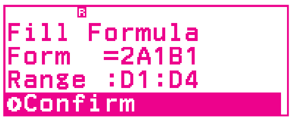


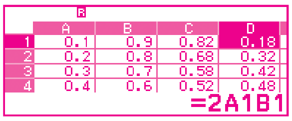


Press , select [Fill Formula], press OK

After inputting [Form=2A1B1], press EXE

After inputting [Range:D1:D4], press EXE, select [Confirm], press OK





From the calculated results, we can confirm that the value in column D $<0.5<$C for all the values in column $A=0.1, 0.2, 0.3$, and 0.4.

Therefore, we can infer $2ab<\frac{1}{2}<a^2+b^2$.

We can prove that this inference holds when $0<a<b$, $a+b=1$.

From $a+b=1$, we get $b=1-a$ then substitute this into $a<b$ for $a<1-a$, so we can get $a<\frac{1}{2}$

This means that

$$\frac{1}{2}-2ab=\frac{1}{2}-2a(1-a)=2a^2-2a+\frac{1}{2}=2\left(a^2-a+\frac{1}{4}\right)=2\left(a-\frac{1}{2}\right)^2>0$$

Therefore, $\frac{1}{2}>2ab$

Also, $a^2+b^2-\frac{1}{2}=a^2+(1-a)^2-\frac{1}{2}=2a^2-2a+\frac{1}{2}=2\left(a-\frac{1}{2}\right)^2>0$

Therefore, $a^2+b^2>\frac{1}{2}$

This gives us $2ab<\frac{1}{2}<.a^2+b^2$

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