6 Equations of Lines and Circles

CASIO
Essential Materials

Introduction

These teaching materials were created with the hope of conveying to many teachers and students the appeal of scientific calculators.

(1) Change awareness (emphasizing the thinking process) and boost efficiency in learning mathematics

- By reducing the time spent on manual calculations, we can have learning with a focus on the thinking process that is more efficient.
- This reduces the aversion to mathematics caused by complicated calculations, and allows students to experience the joy of thinking, which is the essence of mathematics.

(2) Diversification of learning materials and problem-solving methods

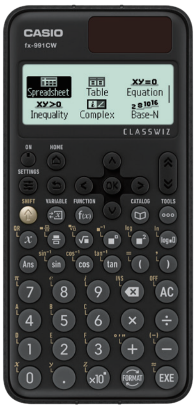
- Making it possible to do difficult calculations manually allows for diversity in learning materials and problem-solving methods.

(3) Promoting understanding of mathematical concepts

- By using the various functions of the scientific calculator in creative ways, students are able to deepen their understanding of mathematical concepts through calculations and discussions from different perspectives than before.
- This allows for exploratory learning through easy trial and error of questions.
- Listing and graphing of numerical values by means of tables allows students to discover laws and to understand visually.

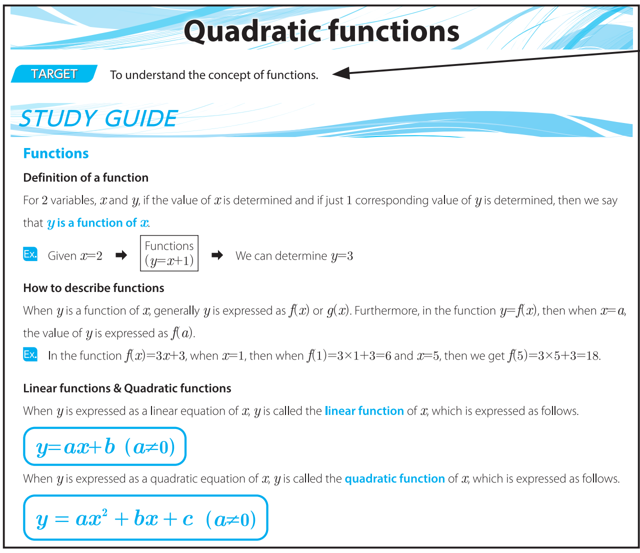
Features of this book

- As well as providing first-time scientific calculator users with opportunities to learn basic scientific calculator functions from the ground up, the book also has material to show people who already use scientific calculators the appeal of scientific calculators described above.
- You can also learn about functions and techniques that are not available on conventional Casio models or other brands of scientific calculators.
- This book covers many units of high school mathematics, allowing students to learn how to use the scientific calculator as they study each topic.
- This book can be used in a variety of situations, from classroom activities to independent study and homework by students.



Better Mathematics Learning with Scientific Calculator

Structure



Quadratic functions

TARGET To understand the concept of functions.

STUDY GUIDE

Functions

Definition of a function

For 2 variables, x and y, if the value of x is determined and if just 1 corresponding value of y is determined, then we say that **y is a function of x**.

Ex. Given $x=2$ ➡ Functions ($y=x+1$) ➡ We can determine $y=3$

How to describe functions

When y is a function of x, generally y is expressed as $f(x)$ or $g(x)$. Furthermore, in the function $y=f(x)$, then when $x=a$, the value of y is expressed as $f(a)$.

Ex. In the function $f(x)=3x+3$, when $x=1$, then when $f(1)=3\times1+3=6$ and $x=5$, then we get $f(5)=3\times5+3=18$.

Linear functions & Quadratic functions

When y is expressed as a linear equation of x, y is called the **linear function** of x, which is expressed as follows.

$$y=ax+b \quad (a\neq0)$$

When y is expressed as a quadratic equation of x, y is called the **quadratic function** of x, which is expressed as follows.

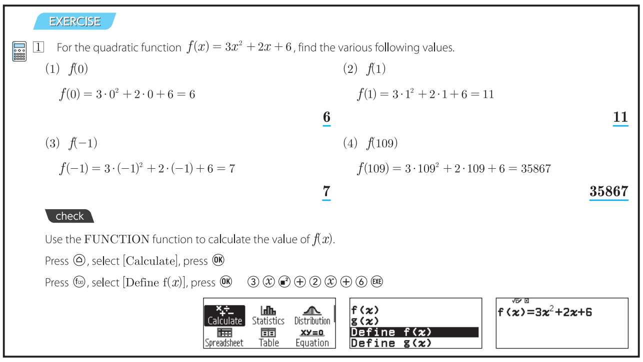
$$y=ax^2+bx+c \quad (a\neq0)$$

TARGET

Students can identify the objective to learn in each section.

STUDY GUIDE

Mathematical theorems and concepts are explained in detail. A scientific calculator is used to check and derive formulas according to the topic.



EXERCISE

1 For the quadratic function $f(x)=3x^2+2x+6$, find the various following values.

(1) $f(0)$

$f(0)=3\cdot0^2+2\cdot0+6=6$ **6**

(2) $f(1)$

$f(1)=3\cdot1^2+2\cdot1+6=11$ **11**

(3) $f(-1)$

$f(-1)=3\cdot(-1)^2+2\cdot(-1)+6=7$ **7**

(4) $f(109)$

$f(109)=3\cdot109^2+2\cdot109+6=35867$ **35867**

check

Use the FUNCTION function to calculate the value of $f(x)$.

Press ⓢ, select [Calculate], press OK

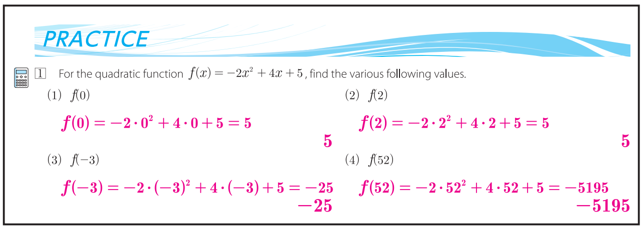
Press f(x), select [Define f(x)], press OK ③ⓧ ⊕ ② ⓧ ⊕ ⑥ EXE

EXERCISE

Students learn basic examples based on the explanation in Study Guide.

check

Explains how to use the scientific calculator to solve problems and check answers.



PRACTICE

1 For the quadratic function $f(x)=-2x^2+4x+5$, find the various following values.

(1) $f(0)$

$f(0)=-2\cdot0^2+4\cdot0+5=5$ **5**

(2) $f(2)$

$f(2)=-2\cdot2^2+4\cdot2+5=5$ **5**

(3) $f(-3)$

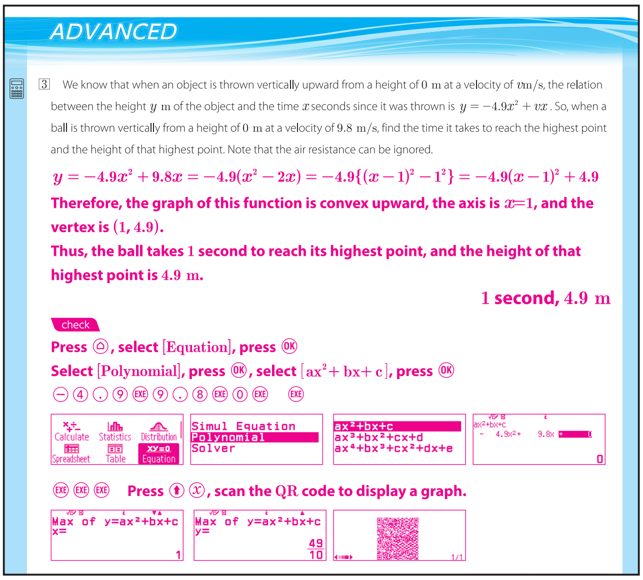
$f(-3)=-2\cdot(-3)^2+4\cdot(-3)+5=-25$ **−25**

(4) $f(52)$

$f(52)=-2\cdot52^2+4\cdot52+5=-5195$ **−5195**

PRACTICE

Students can do practice problems similar to those in EXERCISE. They can also practice using the scientific calculator as they learned to in Check.



ADVANCED

3 We know that when an object is thrown vertically upward from a height of 0 m at a velocity of v m/s, the relation between the height y m of the object and the time x seconds since it was thrown is $y=-4.9x^2+vx$. So, when a ball is thrown vertically from a height of 0 m at a velocity of 9.8 m/s, find the time it takes to reach the highest point and the height of that highest point. Note that the air resistance can be ignored.

$y=-4.9x^2+9.8x=-4.9(x^2-2x)=-4.9\{(x-1)^2-1^2\}=-4.9(x-1)^2+4.9$

Therefore, the graph of this function is convex upward, the axis is $x=1$, and the vertex is (1, 4.9).

Thus, the ball takes 1 second to reach its highest point, and the height of that highest point is 4.9 m.

1 second, 4.9 m

check

Press ⓢ, select [Equation], press OK

Select [Polynomial], press OK, select [ax²+bx+c], press OK

⊖ ④ ⊙ ⑨ EXE ⑨ ⊙ ⑧ EXE ⓪ EXE EXE

EXE EXE EXE **Press ⇧ ⓧ, scan the QR code to display a graph.**

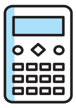
ADVANCED

Practical problems have been included in several topics. Solutions using a scientific calculator are also presented as necessary.

Other marks

Ex.	Simple examples on how to apply equations and theorems
explanation	Formulas and their supplementary explanations
proof	Proofs and checks of mathematical formulas
EXTRA Info.	Knowledge and information on formulas and other supplementary information in other units
OTHER METHODS	Alternative solutions and different verification methods for previously presented problems

Calculator mark



Where to use the scientific calculator

Colors of fonts in the teaching materials

- In STUDY GUIDE, important mathematical terms and formulas are printed in blue.
- In PRACTICE and ADVANCED the answers are printed in red.
  (Separate data is also available without the red parts, so it can be used for exercises.)

Applicable models

The applicable model is fx-991CW.
(Instructions on how to do input are for the fx-991CW, but in many cases similar calculations can be done on other models.)

Related Links

- Information and educational materials relevant to scientific calculators can be viewed on the following site.
  https://edu.casio.com
- The following video can be viewed to learn about the multiple functions of scientific calculators.
  https://www.youtube.com/playlist?list=PLRgxo9AwbIZLurUCZnrbr4cLfZdqY6aZA

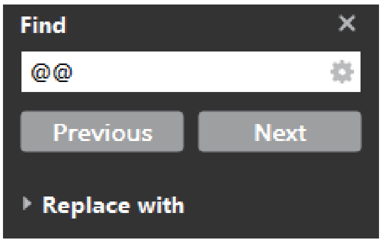
How to use PDF data

About types of data

- Data for all unit editions and data for each unit are available.
- For the above data, the PRACTICE and ADVANCED data without the answers in red is also available.

How to find where the scientific calculator is used

(1) Open a search window in the PDF Viewer.
(2) Type in "@@" as a search term.
(3) You can sequentially check where the calculator marks appear in the data.



How to search for a unit and section

(1) Search for units of data in all unit editions
- The data in all unit editions has a unit table of contents.
- Selecting a unit in the table of contents lets you jump to the first page of that unit.
- There is a bookmark on the first page of each unit, so you can jump from there also.

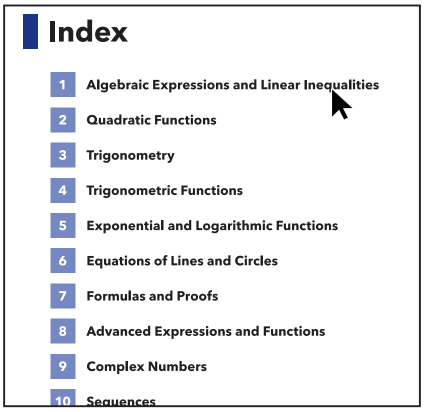
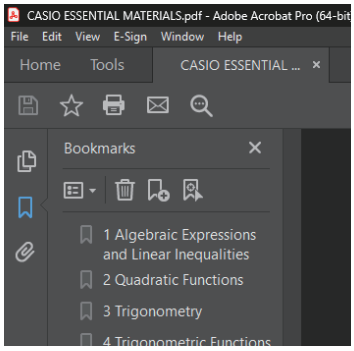


Table of contents of unit



Bookmark of unit

(2) Search for sections
- There are tables of contents for sections on the first page of units.
- Selecting a section in the table of contents takes you to the first page of that section.

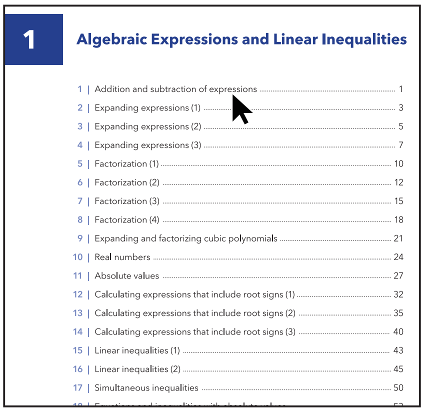


Table of contents of section

Internal division and external division

TARGET To understand how to find the coordinates of internal dividing points and external dividing points of line segments.

STUDY GUIDE

Distance between 2 points on a number line

On a number line, we say that when a real number a corresponds to a point P, that a is called the **coordinate** of that point P, and we write it as $\mathrm{P}(a)$. We can express the distance between 2 points on a number line as follows.

(1) Distance from the origin O to point $\mathrm{P}(a)$ is $\mathrm{OP}=|a|$

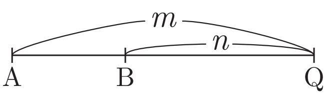
(2) Distance between 2 points $\mathrm{A}(a)$ and $\mathrm{B}(b)$ is $\mathrm{AB}=|b-a|$

Internal/external dividing points of line segments on number lines

Let m and n be positive numbers.

When point P is on line segment AB, such that $\mathrm{AP}:\mathrm{PB}=m:n$ holds, then we say that point P **internally divides** line segment AB into $m:n$, and that point P is the **internal dividing point**. When $m=n$, then point P is the **midpoint** of line segment AB.

Furthermore, when point Q is an extension of the line segment AB, such that $\mathrm{AQ}:\mathrm{QB}=m:n$ holds, then we say that point Q **externally divides** line segment AB into $m:n$, and that point Q is the **external dividing point**. In the case of external division, $m\neq n$.



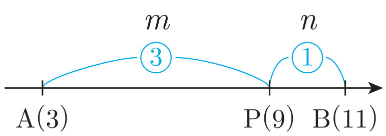
Ex. On a number line, for 2 points $\mathrm{A}(3)$ and $\mathrm{B}(11)$ connected by the line segment AB, we can find the coordinates of the following points.

(1) Point P internally dividing line segment AB by $3:1$

The length of line segment AB is $\mathrm{AB}=|11-3|=8$

From $\mathrm{AP}:\mathrm{PB}=3:1$, the length of line segment AP is

$$\mathrm{AP}=\mathrm{AB}\times\frac{3}{3+1}=8\times\frac{3}{4}=6$$

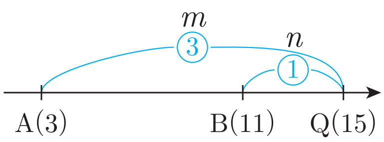


Therefore, since point P was moved 6 in the positive direction from point A on line segment AB, we get $\mathrm{P}(9)$.

(2) Point Q externally dividing line segment AB by $3:1$

From $\mathrm{AQ}:\mathrm{QB}=3:1$, the length of line segment AQ is

$$\mathrm{AQ}=\mathrm{AB}\times\frac{3}{3-1}=8\times\frac{3}{2}=12$$

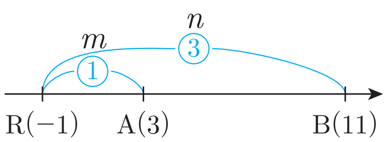


Therefore, since point Q was moved 12 in the positive direction from point A on line segment AB, we get $\mathrm{Q}(15)$.

(3) Point R externally dividing line segment AB by $1:3$

From $\mathrm{AR}:\mathrm{RB}=1:3$, the length of line segment AR is

$$\mathrm{AR}=\mathrm{AB}\times\frac{1}{3-1}=8\times\frac{1}{2}=4$$



However, since $\mathrm{AR}<\mathrm{RB}$, the point R is in the negative direction from point A.

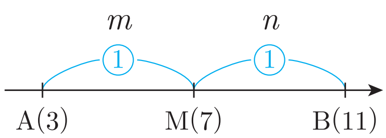
Therefore, since point R was moved 4 in the negative direction from point A on line segment AB, we get $\mathrm{R}(-1)$.

(4) Midpoint M on line segment AB

The midpoint of line segment AB is a point that internally divides line segment AB by $1:1$.

From $\mathrm{AM}:\mathrm{MB}=1:1$, the length of line segment AM is

$$\mathrm{AM}=\mathrm{AB}\times\frac{1}{1+1}=8\times\frac{1}{2}=4$$



Therefore, since point M was moved 4 in the positive direction from point A on line segment AB, we get $\mathrm{M}(7)$.

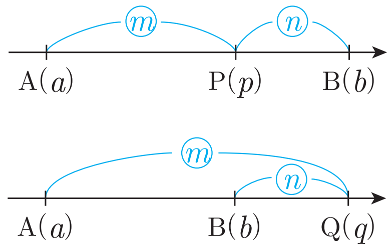
On a number line, when 2 points $\mathrm{A}(a)$ and $\mathrm{B}(b)$ are joined by line segment AB, then $m:n$ is internally divided by point P and externally divided by point Q. In this case, the coordinates of P and Q are expressed as follows.

Coordinates of internal dividing point P

$$\frac{na+mb}{m+n}$$

Coordinates of external dividing point Q

$$\frac{-na+mb}{m-n}$$ **(however, $m \neq n$)**



The formula for finding the external dividing point is the formula for finding the internal dividing point, but with the n replaced by $-n$.

explanation

Coordinates of internal dividing point P

$\mathrm{AP}:\mathrm{PB}=m:n \rightarrow n\mathrm{AP}=m\mathrm{PB} \rightarrow n(p-a)=m(b-p) \rightarrow np-na=mb-mp \rightarrow mp+np=na+mb$

$\rightarrow (m+n)p=na+mb \rightarrow p=\dfrac{na+mb}{m+n}$

Coordinates of external dividing point Q (when $m>n$)

$\mathrm{AQ}:\mathrm{QB}=m:n \rightarrow n\mathrm{AQ}=m\mathrm{QB} \rightarrow n(q-a)=m(q-b) \rightarrow nq-na=mq-mb \rightarrow mq-nq=-na+mb$

$\rightarrow (m-n)q=-na+mb \rightarrow q=\dfrac{-na+mb}{m-n}$

We can get similar results when $m<n$.

EXERCISE

◆ Find the coordinates of the following points on a number line, such that 2 points $\mathrm{A}(3)$ and $\mathrm{B}(11)$ are connected by the line segment AB.

(1) Point P internally dividing line segment AB by 3:1

$$\frac{1\times 3+3\times 11}{3+1}=9$$

9

(2) Point Q externally dividing line segment AB by 3:1

$$\frac{-1\times 3+3\times 11}{3-1}=15$$

15

(3) Point R externally dividing line segment AB by 1:3

$$\frac{-3\times 3+1\times 11}{1-3}=-1$$

−1

(4) Midpoint M on line segment AB

$$\frac{1\times 3+1\times 11}{1+1}=7$$

7

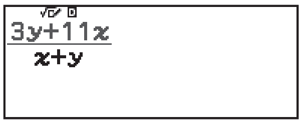
check

On the scientific calculator, use the VARIABLE function to calculate the coordinates of the internal dividing point and external dividing point.

Press [HOME], select [Calculate], press [OK]

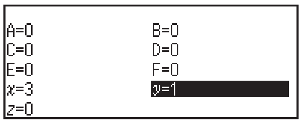
Input the formula for the internal dividing point (external dividing point) of line segment AB.

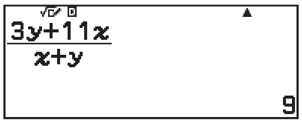
[FRACTION] [3] [SHIFT] [.] [+] [1] [1] [SHIFT] [0] [∨] [SHIFT] [0] [+] [SHIFT] [.]



(1) In the VARIABLE screen, from the formula for the internal dividing point, input [x=3, y=1], and then calculate.

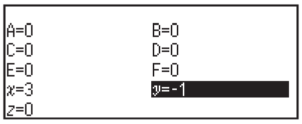
[VARIABLE] [∨] [∨] [∨] [3] [EXE] [>] [1] [EXE] [BACK] [EXE]

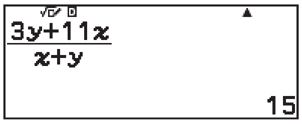




(2) In the same way, from the formula for the external dividing point, input [x=3, y=−1], and then calculate.

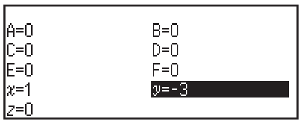
[VARIABLE] [∨] [∨] [∨] [3] [>] [−] [1] [EXE] [BACK] [EXE]

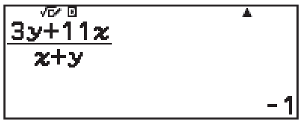




(3) In the same way, from the formula for the external dividing point, input [x=1, y=−3], and then calculate.

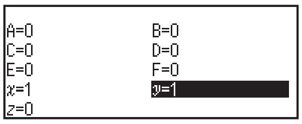
[VARIABLE] [∨] [∨] [∨] [1] [EXE] [>] [−] [3] [EXE] [BACK] [EXE]

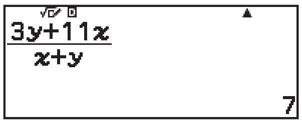




(4) In the same way, from the formula for the internal dividing point, input [x=1, y=1], and then calculate.

[VARIABLE] [∨] [∨] [∨] [1] [>] [1] [EXE] [BACK] [EXE]





OTHER METHODS

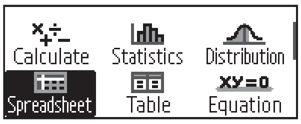
(1) - (3)

Press [HOME], select [Spreadsheet], press [OK], then clear the previous data by pressing [POWER]

Input the data in the table on the right in order from cell A1.

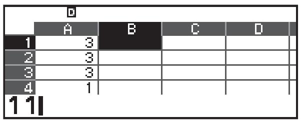
After inputting [A1:3, A2:3, A3:3, and A4:1] respectively, press [EXE], move to [B1].

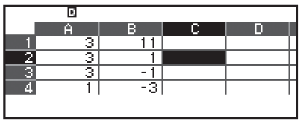
After inputting [B1:11, B2:1, B3:−1, and B4:−3] respectively, press [EXE], move to [C2].



Input value (__), output value (﹏)

	A	B	C
1	3(A)	11(B)	
2	3	1	(1)9
3	3	−1	(2)15
4	1	−3	(3)−1

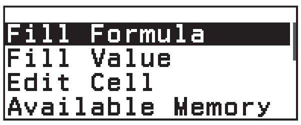


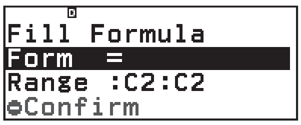


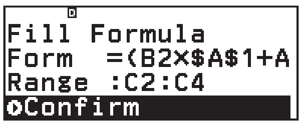
Press [TOOLS], select [Fill Formula], press [OK], after inputting [Form=$\frac{(B2\times\$A\$1+A2\times\$B\$1)}{(A2+B2)}$], press [EXE] (*)

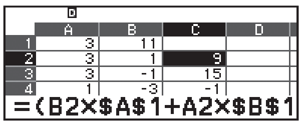
After inputting [Range:C2:C4], press [EXE], select [Confirm], press [OK]

The coordinates of the internal dividing point and external dividing point are calculated in column C.









Also, move to [A1] for use in PRACTICE.

* How to input [$] (refer to absolute values) → Press [CATALOG], select [Spreadsheet], press [OK], select [$], press [OK]
([CATALOG] [OK] [∨] [∨] [OK])

PRACTICE

◆ Find the coordinates of the following points on a number line, such that 2 points $\mathrm{A}(2)$ and $\mathrm{B}(6)$ are connected by the line segment AB.

(1) Point P internally dividing line segment AB by $4:3$

$$\frac{3\times2+4\times6}{4+3}=\frac{30}{7}$$

$\dfrac{30}{7}$

(2) Point Q externally dividing line segment AB by $5:1$

$$\frac{-1\times2+5\times6}{5-1}=7$$

7

(3) Point R externally dividing line segment AB by $3:5$

$$\frac{-5\times2+3\times6}{3-5}=-4$$

-4

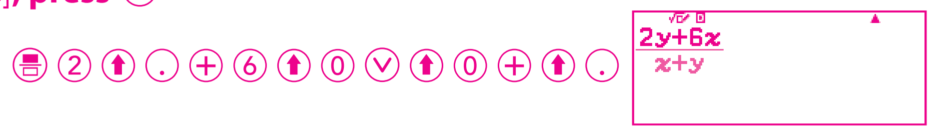
(4) Midpoint M on line segment AB

$$\frac{1\times2+1\times6}{1+1}=4$$

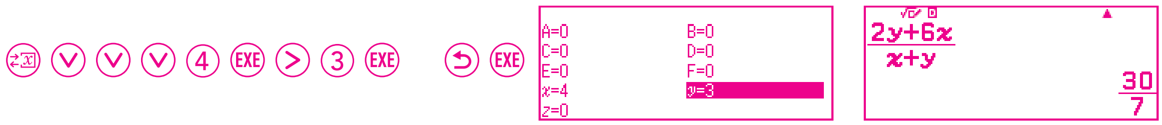
4

check

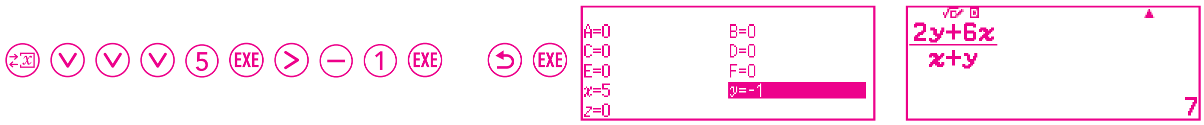
Press ⌂, select [Calculate], press OK



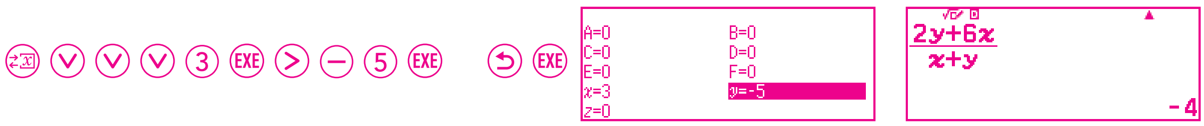
(1)



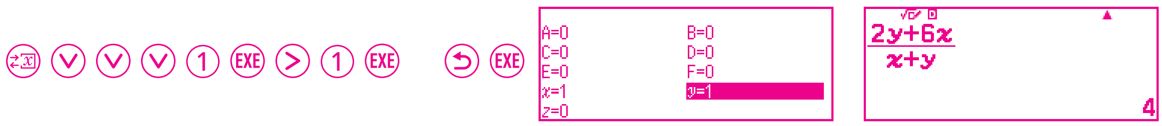
(2)



(3)



(4)

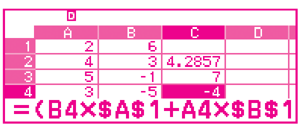


OTHER METHODS

Press ⌂, select [Spreadsheet], press OK

Input the data in the table on the right in order from cell A1.

The calculation formula is the same as in EXERCISE, so no input is required.



Input value (__), output value (__)

	A	B	C
1	2(A)	6(B)	
2	4	3	(1)4.2857
3	5	−1	(2)7
4	3	−5	(3)−4

Points on a plane (1)

TARGET To understand about the distance between points, internal dividing points, and external dividing points on coordinate planes.

STUDY GUIDE

Distance between 2 points on a coordinate plane

Distance between 2 points

We can find the distance between 2 points on a coordinate plane by using the following formulas.

(1) Distance between point $A(x_1, y_1)$ and point $B(x_2, y_2)$

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

(2) Distance between the origin O and point $A(x_1, y_1)$ $OA = \sqrt{{x_1}^2 + {y_1}^2}$

explanation

On a coordinate plane, there are point $A(x_1, y_1)$ and point $B(x_2, y_2)$, such that $x_1 \neq x_2$ and $y_1 \neq y_2$.

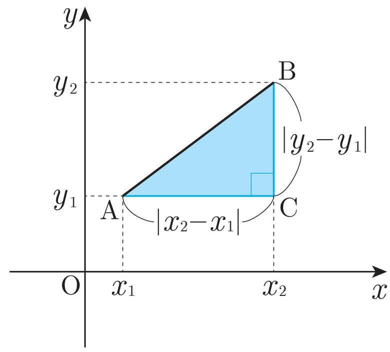
To get point $C(x_2, y_1)$, since $\triangle ABC$ is a right triangle,

then by using the Pythagorean theorem, we can derive

$AB^2 = AC^2 + BC^2 \rightarrow AB = \sqrt{AC^2 + BC^2}$.

Now, from $AC^2 = |x_2 - x_1|^2 = (x_2 - x_1)^2, BC^2 = |y_2 - y_1|^2 = (y_2 - y_1)^2$,

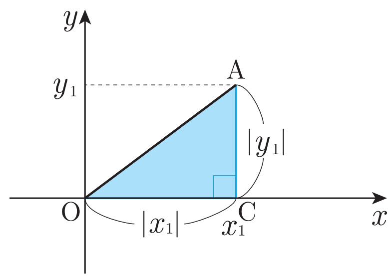
we get $AB = \sqrt{AC^2 + BC^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.



In this formula,

since $(x_2 - x_1)^2 = \{-(x_1 - x_2)\}^2 = (-1)^2(x_1 - x_2)^2 = (x_1 - x_2)^2$, the value of the coordinates of A and B can be subtracted either from one or the other.

Specifically, the distance between the origin O and point $A(x_1, y_1)$ is derived as $x_2=0$, $y_2=0$.



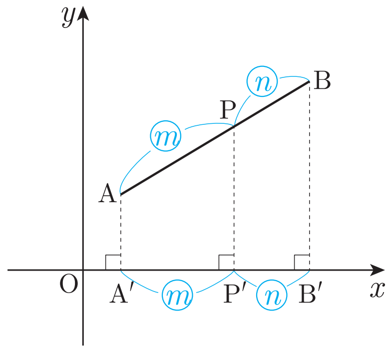
Internal/external dividing points of line segments on a plane

When 2 points $A(x_1, y_1)$ and $B(x_2, y_2)$ are joined by line segment AB, then $m:n$ is internally divided by point P and $m:n$ is externally divided by point Q. We can find the internal dividing point and external dividing point by using the following formulas.

(1) Coordinates of internal dividing point P $P\left(\dfrac{nx_1 + mx_2}{m+n}, \dfrac{ny_1 + my_2}{m+n}\right)$

(2) Coordinates of external dividing point Q $Q\left(\dfrac{-nx_1 + mx_2}{m-n}, \dfrac{-ny_1 + my_2}{m-n}\right)$

explanation



We can draw a perpendicular line from each of the points A, B, and P to the x axis, and designate the intersection with the x axis as A′, B′, and P′.

From the property of the ratio of parallel lines to line segments, we get

A′P′:B′P′=AP:BP=m:n

Therefore, point P′ internally divides the line segment A′B′ by m:n.

The x coordinates of points A′, B′, and P′ are respectively x_1, x_2, and x.

From the formula to find the internal dividing point of a line segment on a number line, we get $x=\dfrac{nx_1+mx_2}{m+n}$

By considering the y coordinates in the same way, we get $y=\dfrac{ny_1+my_2}{m+n}$

Therefore, the coordinates of the internal dividing point P are $\mathrm{P}\left(\dfrac{nx_1+mx_2}{m+n}, \dfrac{ny_1+my_2}{m+n}\right)$

We can derive the external dividing point in the same way.

Specifically, the coordinates of the midpoint, where m=1 and n=1, are derived as $\left(\dfrac{x_1+x_2}{2}, \dfrac{y_1+y_2}{2}\right)$.

EXERCISE

◆ Given 2 points A(1, 5) and B(4, 9) on a coordinate plane, solve the following problems.

(1) Find the distance between the origin O and point A.

$\mathrm{OA}=\sqrt{1^2+5^2}=\sqrt{26}$

$\sqrt{26}$

(2) Find the distance between point A and point B.

$\mathrm{AB}=\sqrt{(4-1)^2+(9-5)^2}=\sqrt{9+16}=\sqrt{25}=5$

5

(3) Find the coordinates of point P internally dividing line segment AB by 3:2.

$\left(\dfrac{2\times1+3\times4}{3+2}, \dfrac{2\times5+3\times9}{3+2}\right)=\left(\dfrac{14}{5}, \dfrac{37}{5}\right)$

$\left(\dfrac{14}{5}, \dfrac{37}{5}\right)$

(4) Find the coordinates of point Q externally dividing line segment AB by 5:3.

$\left(\dfrac{-3\times1+5\times4}{5-3}, \dfrac{-3\times5+5\times9}{5-3}\right)=\left(\dfrac{17}{2}, 15\right)$

$\left(\dfrac{17}{2}, 15\right)$

(5) Find the coordinates of the midpoint M on line segment AB.

$\left(\dfrac{1+4}{2}, \dfrac{5+9}{2}\right)=\left(\dfrac{5}{2}, 7\right)$

$\left(\dfrac{5}{2}, 7\right)$

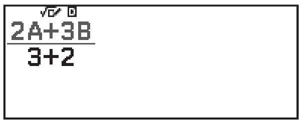
check

On the scientific calculator, use the VARIABLE function to calculate the x coordinates and y coordinates of the various points.

(3) Press [HOME], select [Calculate], press [OK]

Input the coordinates of the point that internally divides the line segment by 3:2.

[FRACTION] [2] [SHIFT] [4] [+] [3] [SHIFT] [5] [▽] [3] [+] [2] [EXE]



x coordinate [VARIABLE] [1] [EXE] [▷] [4] [EXE] [BACK] [EXE]

y coordinate [VARIABLE] [5] [EXE] [▷] [9] [EXE] [BACK] [EXE]

(4) In the same way, input the formula for externally dividing by 5:3 and simply substitute the coordinates of each point.

OTHER METHODS

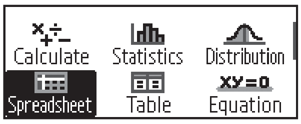
Press [HOME], select [Spreadsheet], press [OK], then clear the previous data by pressing [AC]

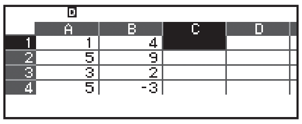
Input the data in the table on the right in order from cell A1.

After inputting [A1:1, A2:5, A3:3, A4:5, and A5:1] respectively, press [EXE], move to [B1].

After inputting [B1:4, B2:9, B3:2, B4:−3, and B5:1] respectively, press [EXE], move to [C1].

Input value (__), output value (__)				
	A	B	C	D
1	1(A_x)	4(B_x)	(1)OA	
2	5(A_y)	9(B_y)	(2)AB	
3	3	2	(3)x	(3)y
4	5	−3	(4)x	(4)y
5	1	1	(5)x	(5)y



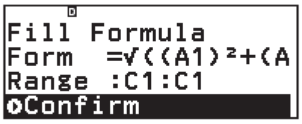


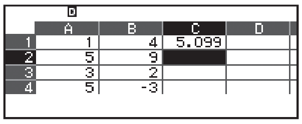
(1) Press [TOOLS], select [Fill Formula], press [OK], after inputting [Form= $\sqrt{((A1)^2+(A2)^2)}$], press [EXE]

After inputting [Range:C1:C1], press [EXE],

select [Confirm], press [OK]

When the sheet is displayed, move to [C2].



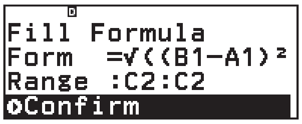


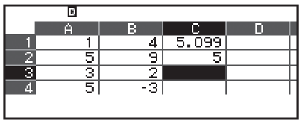
(2) Press [TOOLS], select [Fill Formula], press [OK], after inputting [Form= $\sqrt{((B1-A1)^2+(B2-A2)^2)}$], press [EXE]

After inputting [Range:C2:C2], press [EXE],

select [Confirm], press [OK]

When the sheet is displayed, move to [C3].





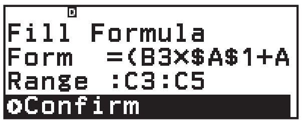
(3) - (5)

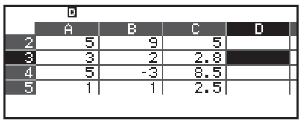
x coordinates Press [TOOLS], select [Fill Formula], press [OK], after inputting [Form= $\frac{(B3\times\$A\$1+A3\times\$B\$1)}{(A3+B3)}$], press [EXE]

After inputting [Range:C3:C5], press [EXE],

select [Confirm], press [OK]

When the sheet is displayed, move to [D3].



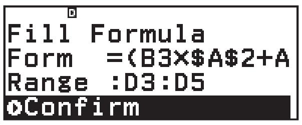


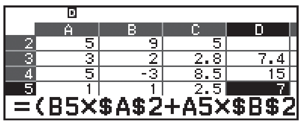
y coordinates Press [TOOLS], select [Fill Formula], press [OK], after inputting [Form= $\frac{(B3\times\$A\$2+A3\times\$B\$2)}{(A3+B3)}$], press [EXE]

After inputting [Range:D3:D5], press [EXE],

select [Confirm], press [OK]

When the sheet is displayed, move to [A1] for use in PRACTICE.





PRACTICE

◆ Given 2 points $A(2, 7)$ and $B(6, 13)$ on a coordinate plane, solve the following problems.

(1) Find the distance between the origin O and point A.

$$OA = \sqrt{2^2 + 7^2} = \sqrt{53}$$

$\sqrt{53}$

(2) Find the distance between point A and point B.

$$AB = \sqrt{(6-2)^2 + (13-7)^2} = \sqrt{16+36}$$
$$= \sqrt{52} = 2\sqrt{13}$$

$2\sqrt{13}$

(3) Find the coordinates of point P internally dividing line segment AB by 4:1.

$$\left(\frac{1\times2+4\times6}{4+1}, \frac{1\times7+4\times13}{4+1}\right)$$
$$=\left(\frac{26}{5}, \frac{59}{5}\right)$$

$\left(\frac{26}{5}, \frac{59}{5}\right)$

(4) Find the coordinates of point Q externally dividing line segment AB by 3:5.

$$\left(\frac{-5\times2+3\times6}{3-5}, \frac{-5\times7+3\times13}{3-5}\right)$$
$$=(-4, -2)$$

$(-4, -2)$

check

(3) Press ⌂, select [Calculate], press OK

[fraction] ⬆ 4 + 4 ⬆ 5 ∨ 4 + 1 EXE

x coordinate

[STO] 2 EXE > 6 EXE ↺ EXE

y coordinate

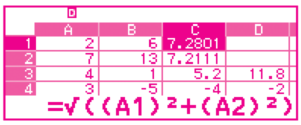
[STO] 7 EXE > 1 3 EXE ↺ EXE

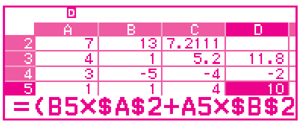
(4) In the same way, input the formula for externally dividing by 3:5 and simply substitute the coordinates of each point.

OTHER METHODS

Press ⌂, select [Spreadsheet], press OK
Input the data in the table on the right in order from cell A1.
The calculation formula is the same as in EXERCISE, so no input is required.
Note that, the coordinates of the midpoint of line segment AB are calculated in cells C5 and D5.

Input value (__), output value (~~)				
	A	B	C	D
1	$2(A_x)$	$6(B_x)$	(1)OA	
2	$7(A_y)$	$13(B_y)$	(2)AB	
3	4	1	(3)x	(3)y
4	3	−5	(4)x	(4)y





Points on a plane (2)

TARGET To understand the properties of shapes on a coordinate plane.

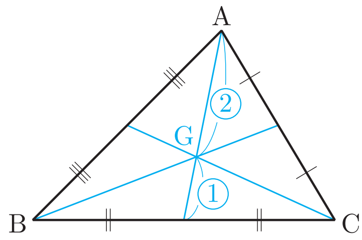
STUDY GUIDE

Shapes on a coordinate plane

Center of gravity of a triangle

The line segment connecting the vertex of a triangle and the midpoint of the opposite side is called the **median line** of the triangle. The 3 median lines of a triangle meet at 1 point, which is called the **center of gravity** of the triangle.

The center of gravity of the triangle is the point at which each line is **internally divided by 2:1**.



For $\triangle ABC$, having apexes at 3 points $A(x_1, y_1)$, $B(x_2, y_2)$, and $C(x_3, y_3)$, we can find the coordinates of the center of gravity G as follows.

Coordinates of center of gravity G $G\left(\dfrac{x_1+x_2+x_3}{3}, \dfrac{y_1+y_2+y_3}{3}\right)$

explanation

The coordinates of the midpoint M on side BC are $M\left(\dfrac{x_2+x_3}{2}, \dfrac{y_2+y_3}{2}\right)$.

The center of gravity G internally divides line segment AM by 2:1.

Therefore, the coordinates of the center of gravity G are

$$G\left(\frac{1\cdot x_1+2\cdot\dfrac{x_2+x_3}{2}}{2+1}, \frac{1\cdot y_1+2\cdot\dfrac{y_2+y_3}{2}}{2+1}\right)=\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$$

Points symmetric with respect to points (symmetric points)

When 2 points $P(p_1, p_2)$ and $Q(q_1, q_2)$ are symmetric with respect to a point $A(a, b)$ on a coordinate plane, the coordinates of midpoint A of line segment PQ can be found as follows.

$$a=\frac{p_1+q_1}{2}, b=\frac{p_2+q_2}{2}$$

EXERCISE

◆ Given 3 points $A(4, 8)$, $B(2, 1)$, and $C(9, 3)$ on a coordinate plane, such that G is the center of gravity of $\triangle ABC$, and that point D is symmetric to point B about point A, solve the following problems.

(1) Find the coordinates of point G.

$$\left(\frac{4+2+9}{3}, \frac{8+1+3}{3}\right) = (5, 4)$$

(5, 4)

(2) Find the coordinates of point D.

Let the coordinates of point D be (x, y), then the midpoint of line segment BD is A,

so from $4 = \frac{2+x}{2}, 8 = \frac{1+y}{2}$, we get $x=6$ and $y=15$

Therefore, D$(6, 15)$

(6, 15)

check

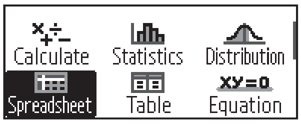
Press ⌂, select [Spreadsheet], press OK, then clear the previous data by pressing ⏻

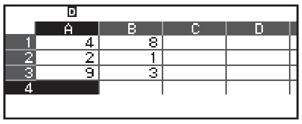
Input the data in the table on the right in order from cell A1.

After inputting [A1:4, A2:2, and A3:9] respectively, press EXE, move to [B1].

After inputting [B1:8, B2:1, and B3:3] respectively, press EXE, move to [A4].

Input value (__), output value (﹏)		
	A	B
1	$4(A_x)$	$8(A_y)$
2	$2(B_x)$	$1(B_y)$
3	$9(C_x)$	$3(C_y)$
4	$(1)x$	$(1)y$
5	$(2)x$	$(2)y$



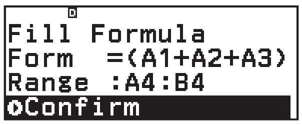


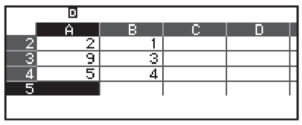
(1) Press ⋯, select [Fill Formula], press OK, after inputting [Form=$\frac{(A1+A2+A3)}{3}$], press EXE

After inputting [Range:A4:B4], press EXE,

select [Confirm], press OK

When the sheet is displayed, move to [A5].



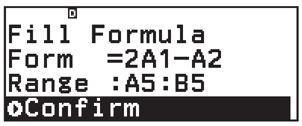


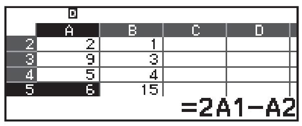
(2) Press ⋯, select [Fill Formula], press OK, after inputting [Form=2A1−A2], press EXE

After inputting [Range:A5:B5], press EXE,

select [Confirm], press OK

When the sheet is displayed, move to [A1] for use in PRACTICE.





PRACTICE

◆ Given 3 points $A(-3, 2)$, $B(5, -1)$, and $C(2, 9)$ on a coordinate plane, such that G is the center of gravity of $\triangle ABC$, and that point D is symmetric to point C about point B, solve the following problems.

(1) Find the coordinates of point G.

$$\left(\frac{-3+5+2}{3}, \frac{2-1+9}{3}\right) = \left(\frac{4}{3}, \frac{10}{3}\right)$$

$$\left(\frac{4}{3}, \frac{10}{3}\right)$$

(2) Find the coordinates of point D.

Let the coordinates of point D be (x, y), then the midpoint of line segment CD is B,

so from $5 = \frac{2+x}{2}, -1 = \frac{9+y}{2}$, we get $x=8$ and $y=-11$

Therefore, D $(8, -11)$

$$(8, -11)$$

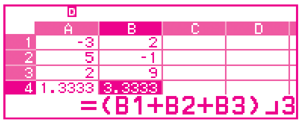
check

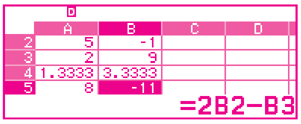
Press ⌂, select [Spreadsheet], press OK

Input the data in the table on the right in order from cell A1.

(1) The calculation formula is the same as in EXERCISE, so no input is required.

(2) Re-input [Form=2A2−A3], [Range:A5:B5] into cell A5.





Input value (__), output value (﹏)

	A	B
1	$-3(A_x)$	$2(A_y)$
2	$5(B_x)$	$-1(B_y)$
3	$2(C_x)$	$9(C_y)$
4	$(1)x$	$(1)y$
5	$(2)x$	$(2)y$

Equation of a line

TARGET To understand the equations of a line on a coordinate plane.

STUDY GUIDE

Equation of a line

General equation of a line

A line on a coordinate plane is expressed as a linear equation

$ax + by + c = 0$ ($a \neq 0$ or $b \neq 0$) of x and y.

If $b \neq 0$, it represents a line with a slope of $-\frac{a}{b}$ and an intercept of $-\frac{c}{b}$.

If $b = 0$, it represents a line that is perpendicular to the x axis.

> **EXTRA Info.**
> The graph of a linear function $y = mx + n$ ($m \neq 0$) is a line with a slope of m and an intercept of n.

Determining the equation of a line

A line on a coordinate plane can be identified by its slope and a point it passes through, which can be found as follows.

(1) A line passing through point $A(x_1, y_1)$ with slope m

$$y - y_1 = m(x - x_1)$$

When $x = x_1$, the line is perpendicular to the x axis

(2) A line passing through 2 different points $A(x_1, y_1)$ and $B(x_2, y_2)$ is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

When $x_1 = x_2$, the line is perpendicular to the x axis $x = x_1$

explanation

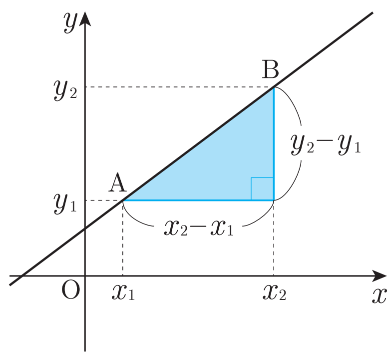
(1) For a line with slope m, the equation is $y = mx + n$.

It passes through point $A(x_1, y_1)$, so $y_1 = mx_1 + n$, $n = y_1 - mx_1$

Therefore, from $y = mx + y_1 - mx_1$, we get $y - y_1 = m(x - x_1)$

(2) The slope m of a line that passes through 2 different points $A(x_1, y_1)$ and $B(x_2, y_2)$ is $m = \frac{y_2 - y_1}{x_2 - x_1}$

Therefore, we can use (1) for $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$



EXERCISE

1 Find the equations for the following lines given the lines pass through point A $(2, 5)$.

(1) Line with slope of 3

$y-5=3(x-2)$, $y=3x-1$

$y=3x-1$

(2) Line perpendicular to x axis

It passes through point $A(2, 5)$, so $x=2$

$x=2$

2 Find the equation for a line that passes through the following 2 points A and B.

(1) $A(2, 1)$, $B(6, 9)$

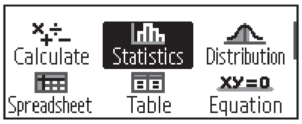
$$y-1=\frac{9-1}{6-2}(x-2),\ y=2x-3$$

$y=2x-3$

check

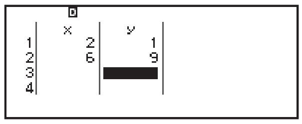
On the scientific calculator, use the Statistics function to show a line passing through 2 points.

Press ⌂, select [Statistics], press OK, select [2-Variable], press OK

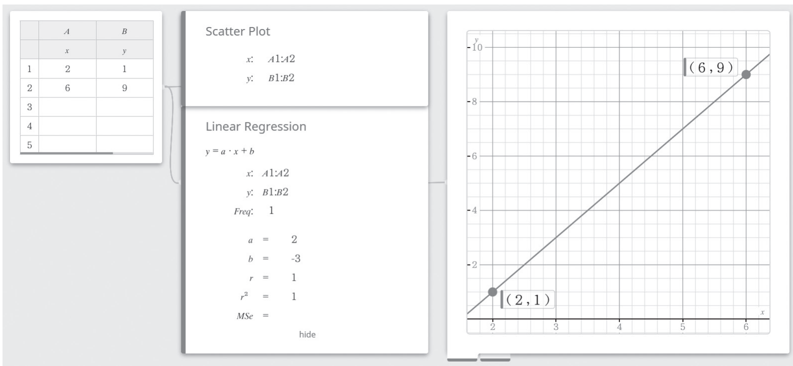


Input the x coordinates of 2 points A and B in the x column of the table, and input the y coordinates of 2 points A and B in the y column, respectively.

② EXE ⑥ EXE ∨ > ① EXE ⑨ EXE



Press ⇧ ⓧ, scan the QR code to display a graph.



$r=1$ of the linear regression shows the mathematical expression is correct.

(2) $A(4, -5)$, $B(4, 7)$

$x_1=x_2=4$, so $x=4$

$x=4$

PRACTICE

1 Find the equations for the following lines given the lines pass through point $A(-3, -2)$.

(1) Line with slope of -4

$y-(-2)=-4\{x-(-3)\}$

$y=-4x-14$

$y=-4x-14$

(2) Line perpendicular to x axis

It passes through point A$(-3, -2)$, so

$x=-3$

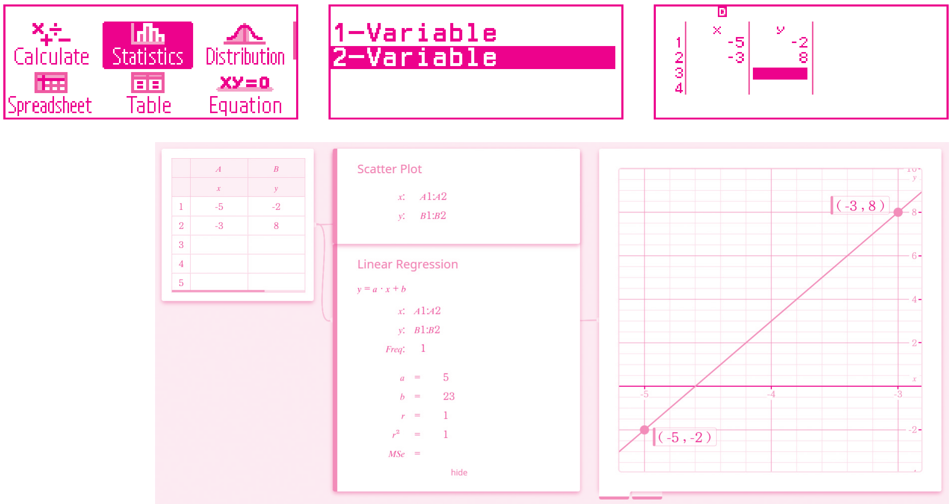
$x=-3$

2 Find the equation for a line that passes through the following 2 points A and B.

(1) $A(-5, -2), B(-3, 8)$

$$y-(-2)=\frac{8-(-2)}{-3-(-5)}\{x-(-5)\},\ y=5x+23$$

$y=5x+23$



(2) $A(-2, 1), B(-2, -6)$

$x_1=x_2=-2$, **so** $x=-2$

$x=-2$

The method of transposing all letter expressions of a line to the left side and expressing them as $ax+by+c=0$ will also be used in the future.

(Example)

$y=-3x+7 \leftrightarrow 3x+y-7=0$

$$y=\frac{1}{2}x-5 \leftrightarrow x-2y-10=0$$

Relation of 2 lines (1)

TARGET To understand the parallel and perpendicular relations of 2 lines.

STUDY GUIDE

Parallel and perpendicular relations of 2 lines

The conditions in which 2 lines $y=m_1x+n_1$ and $y=m_2x+n_2$ are parallel or perpendicular are as follows.

(i) 2 lines are parallel $\Leftrightarrow$ $m_1=m_2$

(ii) 2 lines are perpendicular $\Leftrightarrow$ $m_1m_2=-1$

Ex. 2 lines $y=2x-5$ and $y=2x+8$ both have a slope of 2, so they are parallel.

2 lines $y=3x+2$ and $y=-\frac{1}{3}x+7$ are perpendicular because the product of the slopes is $3\cdot\left(-\frac{1}{3}\right)=-1$.

Furthermore, when $n_1=n_2$ in (i), the 2 lines are the same, so they are also considered parallel.

EXERCISE

1 Find the equations for the following lines given the lines pass through point A $(1, 2)$.

(1) Line parallel to the line $y=4x-3$

The slope of the parallel lines is equal to the slope of the line $y=4x-3$, so it is 4.

It passes through the point $\mathrm{A}(1, 2)$, so we can find the equations $y-2=4(x-1)$, $y=4x-2$

$y=4x-2$

(2) Line perpendicular to the line $y=4x-3$

The slope of the perpendicular line is m, so $4m=-1, m=-\frac{1}{4}$

It passes through the point $\mathrm{A}(1, 2)$, so we can find the equations

$$y-2=-\frac{1}{4}(x-1), y=-\frac{1}{4}x+\frac{9}{4}$$

$y=-\frac{1}{4}x+\frac{9}{4}$

check

On the scientific calculator, use the Table function to check the positional relation of the lines.

Press [HOME], select [Table], press [OK], then clear the previous data by pressing [AC].

Press [CATALOG], select [Define $f(x)/g(x)$], press [OK], select [Define $f(x)$], press [OK].

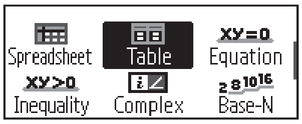
After inputting $f(x)=4x-2$, press [EXE].

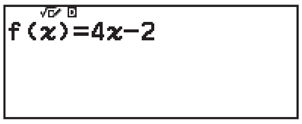
In the same way, input $g(x)=-\frac{1}{4}x+\frac{9}{4}$.

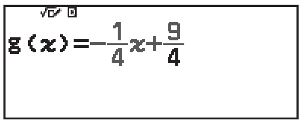
Press [CATALOG], select [Table Range], press [OK].

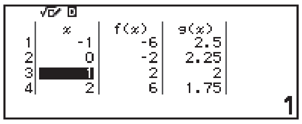
After inputting [Start:-1, End:2, Step:1], select [Execute], press [EXE].

Press [SHIFT] [x], scan the QR code to display a graph.







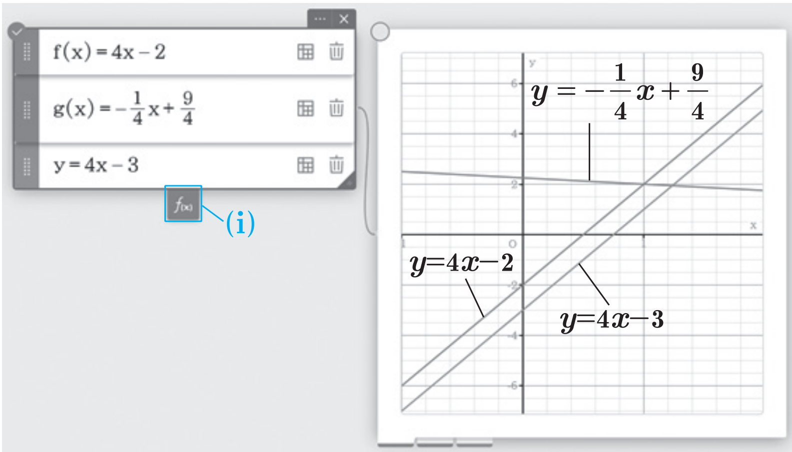


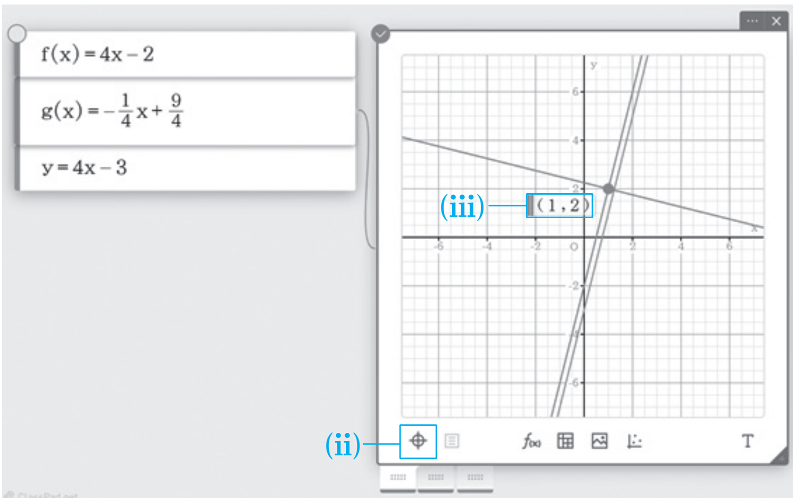
Also, you can use the following procedures (i) to (iii) to change the settings of the displayed graph.

(i) Tap the function sheet, then tap the function mark (Add Graph Function).
In the added function sheet, input $y=4x-3$.

(ii) Tap the graph sheet, tap the zoom mark (Zoom), tap [Default Zoom].

(iii) Tap the line in the graph sheet, and then tap the intersection point of the 2 lines to display the coordinates of the intersection point.





In this manual, we will continue to do graph operations on ClassPad.net for visualizing and studying graphs.

PRACTICE

1 Find the equations for the following lines given the lines pass through point $\mathrm{A}\,(3, 6)$.

(1) Line $y = -\frac{1}{3}x + 1$ and a parallel line

The slope of the parallel lines is equal to the slope of the line $y = -\frac{1}{3}x + 1$, so it is $-\frac{1}{3}$.

It passes through the point $\mathrm{A}(3, 6)$, so we can find the equations

$y - 6 = -\frac{1}{3}(x - 3), y = -\frac{1}{3}x + 7$

$$y = -\frac{1}{3}x + 7$$

(2) Line $y = -\frac{1}{3}x + 1$ and a perpendicular line

The slope of the perpendicular line is m, so $-\frac{1}{3}m = -1, m = 3$

It passes through the point $\mathrm{A}(3, 6)$, so we can find the equations $y-6=3(x-3)$, $y=3x-3$

$$y=3x-3$$

check

Press ⌂, select [Table], press OK, then clear the previous data by pressing ⏻

Press ⋯, select [Define f(x)/g(x)], press OK, select [Define f(x)], press OK

After inputting $f(x)=-\frac{1}{3}x + 7$, press EXE

Input $g(x)=3x-3$ in the same way.

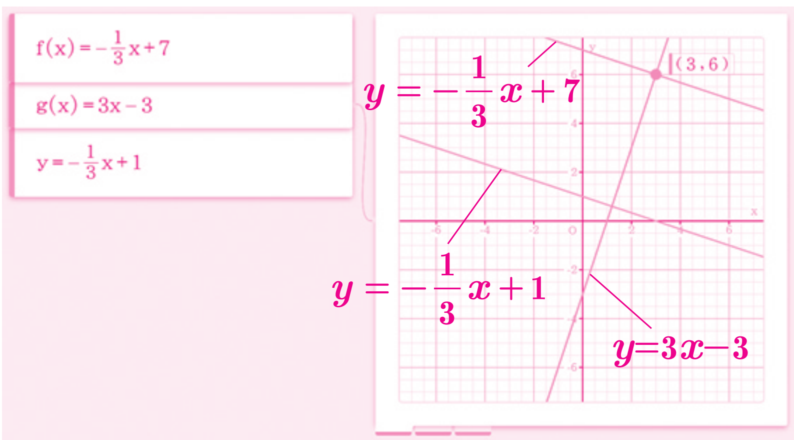
Press ⋯, select [Table Range], press OK

After inputting [Start:−1, End:6, Step:1], select [Execute], press EXE

Press ⇧ x, scan the QR code to display a graph.

Also, tap the function sheet, and input $y = -\frac{1}{3}x + 1$.

In the same way, tap the graph sheet, then, after setting zoom, display the coordinates of the intersection of the 2 lines.



Parallel and perpendicular relations of 2 lines (generally)

When $a_1b_1a_2b_2 \neq 0$, the conditions in which 2 lines $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$ are parallel or perpendicular are as follows.

(I) 2 lines are parallel $\Leftrightarrow$ $a_1b_2-a_2b_1=0$ $(a_1:b_1=a_2:b_2)$

(II) 2 lines are perpendicular $\Leftrightarrow$ $a_1a_2+b_1b_2=0$

explanation

By solving $a_1x+b_1y+c_1=0$ for y, we get $y=-\dfrac{a_1}{b_1}x-\dfrac{c_1}{b_1}$

By solving $a_2x+b_2y+c_2=0$ for y, we get $y=-\dfrac{a_2}{b_2}x-\dfrac{c_2}{b_2}$

If 2 lines are parallel, their slopes are equal, so from $-\dfrac{a_1}{b_1}=-\dfrac{a_2}{b_2}$, we get $a_1b_2-a_2b_1=0$

If 2 lines are perpendicular, the product of their slopes is -1, so from $\left(-\dfrac{a_1}{b_1}\right)\cdot\left(-\dfrac{a_2}{b_2}\right)=-1$, we get $a_1a_2+b_1b_2=0$

EXERCISE

2 Find the equations for the following lines given the lines pass through point $\mathrm{A}\ (1, 2)$.

(1) Line $4x-y-3=0$ and a parallel line

Since the coefficients of parallel lines are directly proportional, we find that the equation of the line is $4x-y+c=0$.

It passes through point $\mathrm{A}(1, 2)$, so $4\cdot 1-2+c=0$, $c=-2$

Therefore, $4x-y-2=0$

$4x-y-2=0$

(2) Line perpendicular to the line $y=4x-3$

Considering the coefficients of x and y of perpendicular lines, we find that the equation of the line is $x+4y+c=0$.

It passes through point $\mathrm{A}(1, 2)$, so $1+4\cdot 2+c=0$, $c=-9$

Therefore, $x+4y-9=0$

$x+4y-9=0$

PRACTICE

2 Find the equations for the following lines given the lines pass through point $\mathrm{A}\,(3, 5)$.

(1) Line $3x+2y+1=0$ and a parallel line

Since the coefficients of parallel lines are directly proportional, we find that the equation of the line is $3x+2y+c=0$.
It passes through point $\mathrm{A}(3, 5)$, so $3\cdot3+2\cdot5+c=0$, $c=-19$
Therefore, $3x+2y-19=0$

$3x+2y-19=0$

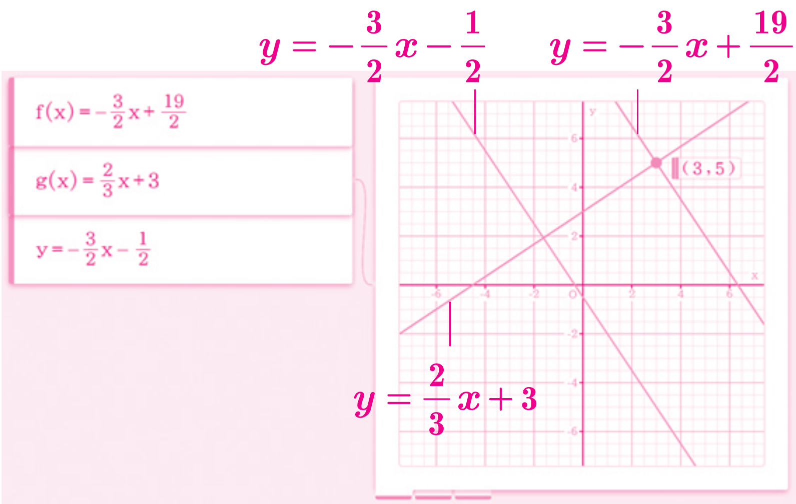
(2) Line $y=-\frac{3}{2}x-\frac{1}{2}$ and a perpendicular line

Considering the coefficients of x and y of perpendicular lines, we find that the equation of the line is $2x-3y+c=0$.
It passes through point $\mathrm{A}(3, 5)$, so $2\cdot3-3\cdot5+c=0$, $c=9$
Therefore, $2x-3y+9=0$

$2x-3y+9=0$

check

Press ⌂, select [Table], press (OK), then clear the previous data by pressing ⏻
Press (⋯), select [Define f(x)/g(x)], press (OK), select [Define f(x)], press (OK)
After inputting $\mathrm{f}(x)=-\frac{3}{2}x+\frac{19}{2}$, press (EXE)
In the same way, input $\mathrm{g}(x)=\frac{2}{3}x+3$.
Press (⋯), select [Table Range], press (OK)
After inputting [Start:−5, End:7, Step:1], select [Execute], press (EXE)
Press (⇧) (x), scan the QR code to display a graph.
Also, tap the function sheet, and input $y=-\frac{3}{2}x-\frac{1}{2}$.
In the same way, tap the graph sheet, then, after setting zoom, display the coordinates of the intersection of the 2 lines.



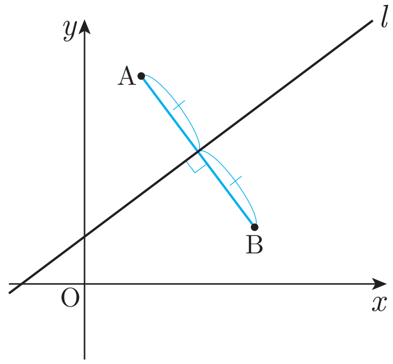
Relation of 2 lines (2)

TARGET To understand about points symmetric about a line.

STUDY GUIDE

Points symmetric about a line

The 2 points A and B are symmetric about the line l when the following (i) and (ii) hold.



(i) Line AB is perpendicular to line l.
(ii) Midpoint of line segment AB is on line l.

EXERCISE

◆ Given a line l is $y=3x$. Find the coordinates of point B symmetric to point $\mathrm{A}(-1, 2)$ with respect to line l.

Let (p, q) be the coordinates of point B.

The slope of the line l is 3, so the slope of line AB is $\dfrac{q-2}{p-(-1)}=\dfrac{q-2}{p+1}$.

$\mathrm{AB}\perp l$, so $3\cdot\dfrac{q-2}{p+1}=-1, p+3q=5 \quad \cdots\text{(i)}$

The coordinates of the midpoint of line AB are $\left(\dfrac{p-1}{2}, \dfrac{q+2}{2}\right)$.

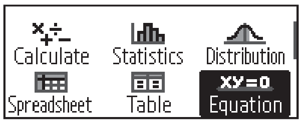
Since this is on the line l, we can substitute in $y=3x$ for $\dfrac{q+2}{2}=3\cdot\dfrac{p-1}{2}, 3p-q=5 \quad \cdots\text{(ii)}$

By solving (i) and (ii) simultaneously, we get $p=2$ and $q=1$, so the coordinates of point B are $(2, 1)$.

(2, 1)

check

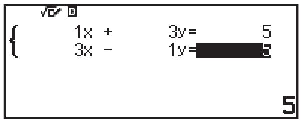
On the scientific calculator, use the Equation function to solve the simultaneous equations, and check the positional relation on the graph.



Press ⌂, select [Equation], press OK, select [Simul Equation], press OK, select [2 Unknowns], press OK

Input $\begin{cases} x+3y=5 \\ 3x-y=5 \end{cases}$.

① EXE ③ EXE ⑤ EXE ③ EXE ⊖ ① EXE ⑤ EXE



Display the solutions of the simultaneous equations and check them.

EXE EXE

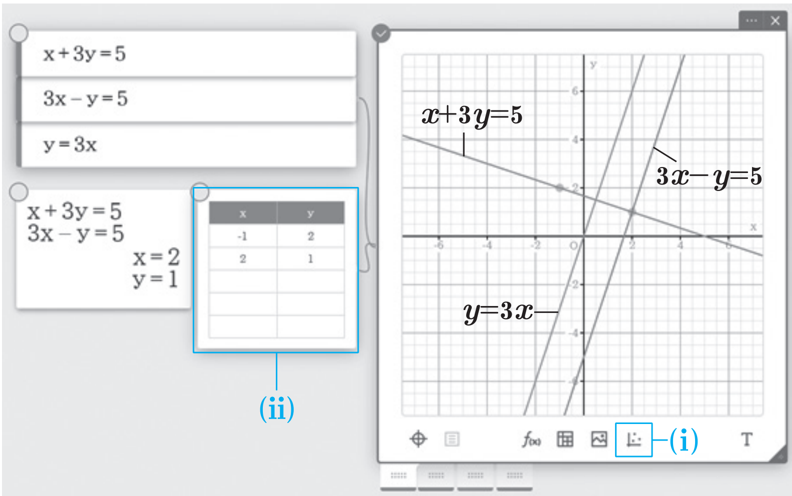
Press ⇧ ⓧ, scan the QR code to display a graph.

Also, you can use the following procedures (i) and (ii) to change the settings of the displayed graph.

(i) Tap the graph sheet, then tap the coordinates mark (Plot Points).

(ii) On the added coordinate sheet, input $(-1, 2)$ and $(2, 1)$.

In the same way, tap the function sheet, and after inputting $y=3x$, set the zoom.



PRACTICE

◆ Given a line l is $y=2x-5$. Find the coordinates of point B symmetric to point $\mathrm{A}(4, -3)$ with respect to the line l.

Let (p, q) be the coordinates of point B.

The slope of the line l is 2, so the slope of line AB is $\dfrac{q-(-3)}{p-4}=\dfrac{q+3}{p-4}$.

$\mathrm{AB}\perp l$, **so** $2\cdot\dfrac{q+3}{p-4}=-1, p+2q=-2 \quad \cdots(\mathrm{i})$

The coordinates of the midpoint of line AB are $\left(\dfrac{p+4}{2}, \dfrac{q-3}{2}\right)$.

Since this is on the line l, we can substitute in $y=2x-5$ for

$\dfrac{q-3}{2}=2\cdot\dfrac{p+4}{2}-5, 2p-q=-1 \quad \cdots(\mathrm{ii})$

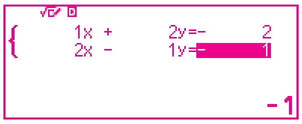
By solving (i) and (ii) simultaneously, we get $p=-\dfrac{4}{5}, q=-\dfrac{3}{5}$, **so the coordinates of point B are** $\left(-\dfrac{4}{5}, -\dfrac{3}{5}\right)$.

$$\left(-\frac{4}{5}, -\frac{3}{5}\right)$$

check

Press ⌂, **select** [Equation], **press** OK, **select** [Simul Equation], **press** OK, **select** [2 Unknowns], **press** OK

① EXE ② EXE ⊖ ② EXE ② EXE ⊖ ① EXE ⊖ ① EXE

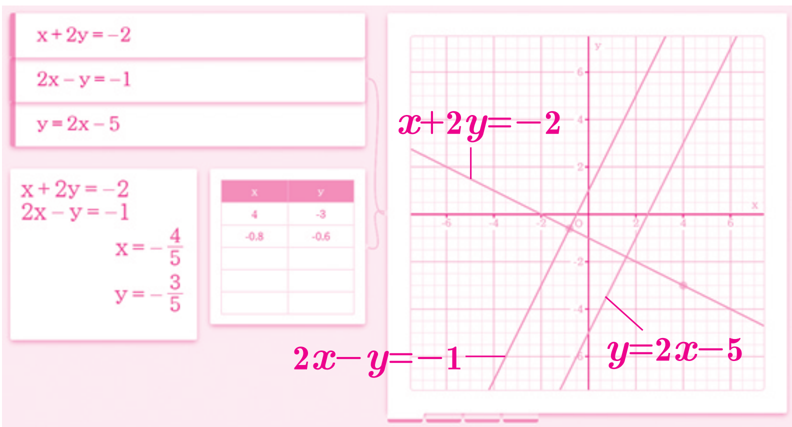


EXE EXE

Press ⇧ ⓧ, **scan the QR code to display a graph.**

Also, tap the graph sheet and enter $(4, -3)$ and $(-0.8, -0.6)$ in the coordinate sheet.

In the same way, tap the function sheet, and after inputting $y=2x-5$, set the zoom.



Relation of 2 lines (3)

TARGET To understand the equations of 2 lines that intersect.

STUDY GUIDE

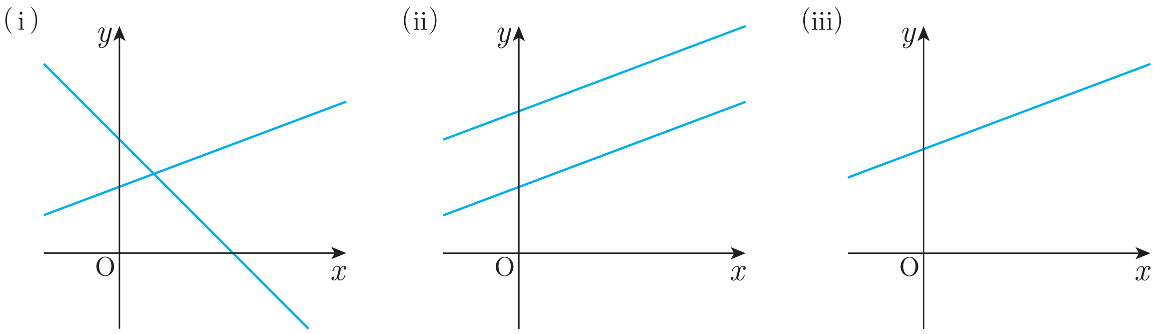
Relation of 2 lines and solving simultaneous equations

The positional relation of 2 lines can be separated into the following 3 cases according to the solution of the simultaneous equations of the 2 lines.

(i) 2 lines intersect at 1 point. ⇔ Simultaneous equations have only 1 solution.

(ii) 2 lines are parallel. ⇔ Simultaneous equations have no solution.

(iii) 2 lines coincide. ⇔ Simultaneous equations have infinite solutions.



2 straight lines $ax+by+c=0$ and $a'x+b'y+c'=0$ that intersect at 1 point are expressed by the following equation.

$$ax+by+c+k(a'x+b'y+c')=0 \quad (k \text{ is a constant})$$

EXERCISE

1 Select 1 correct answer A, B, or C for the simultaneous equations (1) to (3).

A…Have only 1 solution. B…Have no solution. C…Have infinite solutions.

(1) $\begin{cases} 2x+y=5 \\ 3x-y=7 \end{cases}$

By solving the equation $2x+y=5$ for y, we get $y=-2x+5$

By solving the equation $3x-y=7$ for y, we get $y=3x-7$

The straight lines given by the 2 equations have different slopes, so they are not parallel.

Therefore, these simultaneous equations have only 1 solution.

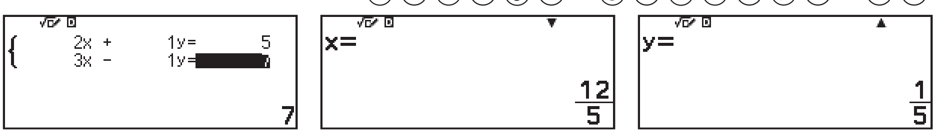
A

check

On the scientific calculator, use the Equation function to solve whether the simultaneous equations have solutions.

Press ⌂, select [Equation], press OK, select [Simul Equation], press OK, select [2 Unknowns], press OK

② EXE ① EXE ⑤ EXE ③ EXE (−) ① EXE ⑦ EXE EXE EXE



(2) $\begin{cases} 2x-3y=1 \\ 4x-6y=3 \end{cases}$

By solving the equation $2x-3y=1$ for y, we get $y=\frac{2}{3}x-\frac{1}{3}$

By solving the equation $4x-6y=3$ for y, we get $y=\frac{2}{3}x-\frac{1}{2}$

The straight lines given by the 2 equations have the same slope, so they are parallel.

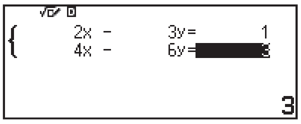
Therefore, these simultaneous equations have no solution.

B

check

Press (HOME), select [Equation], press (OK), select [Simul Equation], press (OK), select [2 Unknowns], press (OK)

(2) (EXE) (−) (3) (EXE) (1) (EXE)   (4) (EXE) (−) (6) (EXE) (3) (EXE)   (EXE)



You can confirm that these simultaneous equations have no solution (No Solution).

(3) $\begin{cases} 5x+2y=3 \\ \frac{5}{6}x+\frac{1}{3}y=\frac{1}{2} \end{cases}$

By solving the equation $5x+2y=3$ for y, we get $y=-\frac{5}{2}x+\frac{3}{2}$

By solving the equation $\frac{5}{6}x+\frac{1}{3}y=\frac{1}{2}$ for y, we get $y=-\frac{5}{2}x+\frac{3}{2}$

The 2 equations are the same, so the 2 lines coincide.

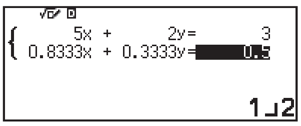
Therefore, these simultaneous equations have infinite solutions.

C

check

Press (HOME), select [Equation], press (OK), select [Simul Equation], press (OK), select [2 Unknowns], press (OK)

(5) (EXE) (2) (EXE) (3) (EXE)   (5) (FRACTION) (6) (EXE) (1) (FRACTION) (3) (EXE) (1) (FRACTION) (2) (EXE)   (EXE)



You can confirm that these simultaneous equations have infinite solutions (Infinite Solution).

2 Given 2 lines $2x-y-1=0$ and $x-y+2=0$ that intersect, find the equations that satisfy the following conditions.

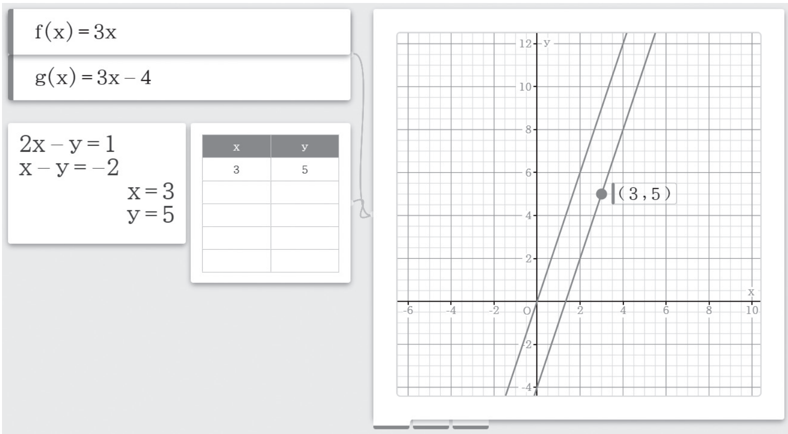
(1) Parallel to line $y=3x$.

By solving $\begin{cases} 2x-y-1=0 \\ x-y+2=0 \end{cases}$, we get $x=3$ and $y=5$, so the coordinates of the intersection of the 2 lines are $(3, 5)$.

Since the slope of the line $y=3x$ is 3, the slope of a parallel line is also 3.

Therefore, the equation of the line we find passes through point $(3, 5)$ with a slope of 3, such that $y-5=3(x-3)$, so $y=3x-4$

$y=3x-4$



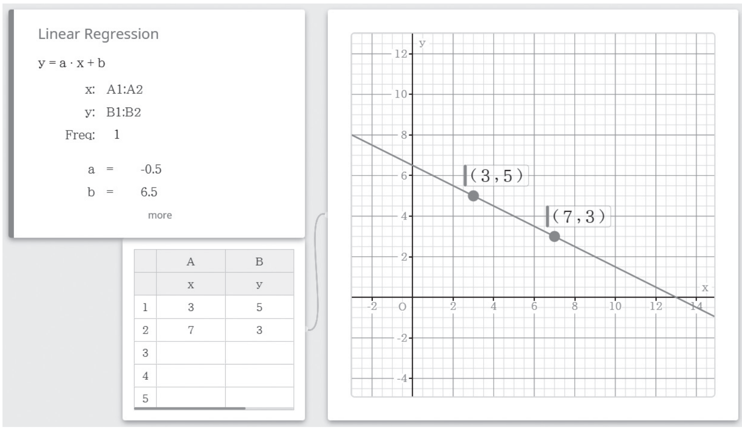
(2) Passes through point $(7, 3)$.

The equation of a line passing through the intersection of 2 lines $2x-y-1=0$ and $x-y+2=0$ can be expressed as $2x-y-1+k(x-y+2)=0$ (k is a constant).

It passes through point $(7, 3)$, so $2\cdot7-3-1+k(7-3+2)=0$, $k=-\frac{5}{3}$

Therefore, the equation of the line we find is $2x-y-1-\frac{5}{3}(x-y+2)=0$, so $x+2y-13=0$

$x+2y-13=0$



3 Given the simultaneous equations $\begin{cases} 4x+5y-1=0 \\ ax+5y+c=0 \end{cases}$ have no solution, find the conditions that a and c satisfy.

Simultaneous equations have no solution when the 2 lines represented by the equations are parallel but do not coincide.

Therefore, $a=4$ and $c\neq-1$

$a=4$ and $c\neq-1$

PRACTICE

1 Select 1 correct answer A, B, or C for the simultaneous equations (1) to (3).

A…Have only 1 solution. B…Have no solution. C…Have infinite solutions.

(1) $\begin{cases} 5x+6y=2 \\ 2x-3y=4 \end{cases}$

By solving the equation $5x+6y=2$ for y, we get $y=-\frac{5}{6}x+\frac{1}{3}$

By solving the equation $2x-3y=4$ for y, we get $y=\frac{2}{3}x-\frac{4}{3}$

The straight lines given by the 2 equations have different slopes, so they are not parallel.

Therefore, these simultaneous equations have only 1 solution.

A

check

Press ⌂, select [Equation], press OK, select [Simul Equation], press OK, select [2 Unknowns], press OK

5 EXE 6 EXE 2 EXE 2 EXE − 3 EXE 4 EXE EXE EXE

(2) $\begin{cases} 3x+4y=1 \\ 6x+8y=3 \end{cases}$

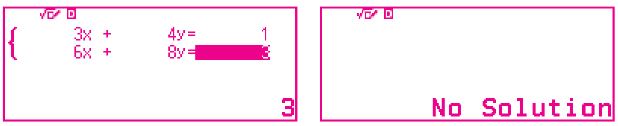
By solving the equation $3x+4y=1$ for y, we get $y=-\frac{3}{4}x+\frac{1}{4}$

By solving the equation $6x+8y=3$ for y, we get $y=-\frac{3}{4}x+\frac{3}{8}$

The straight lines given by the 2 equations have the same slope, so they are parallel.

Therefore, these simultaneous equations have no solution.

B



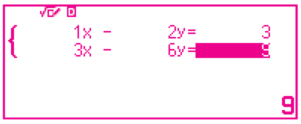
(3) $\begin{cases} x-2y=3 \\ 3x-6y=9 \end{cases}$

Divide both sides of the equation $3x-6y=9$ by 3 to get $x-2y=3$

The 2 equations are the same, so the 2 lines coincide.

Therefore, these simultaneous equations have infinite solutions.

C



2 Given 2 lines $3x+y-2=0$ and $x+y-4=0$ that intersect, find the equations that satisfy the following conditions.

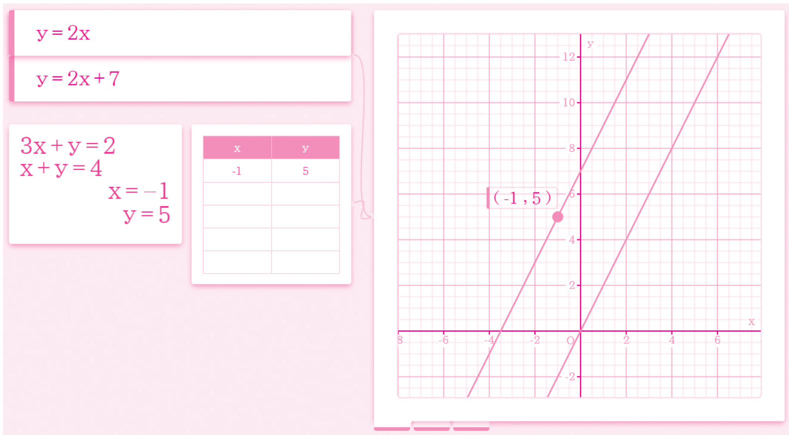
(1) Parallel to line $y=2x$.

By solving $\begin{cases} 3x+y-2=0 \\ x+y-4=0 \end{cases}$, we get $x=-1$ and $y=5$, so the coordinates of the intersection of the 2 lines are $(-1, 5)$.

Since the slope of the line $y=2x$ is 2, the slope of a parallel line is also 2.

Therefore, the equation of a line we find passes through point $(-1, 5)$ with a slope of 2, such that $y-5=2\{x-(-1)\}$, so $y=2x+7$

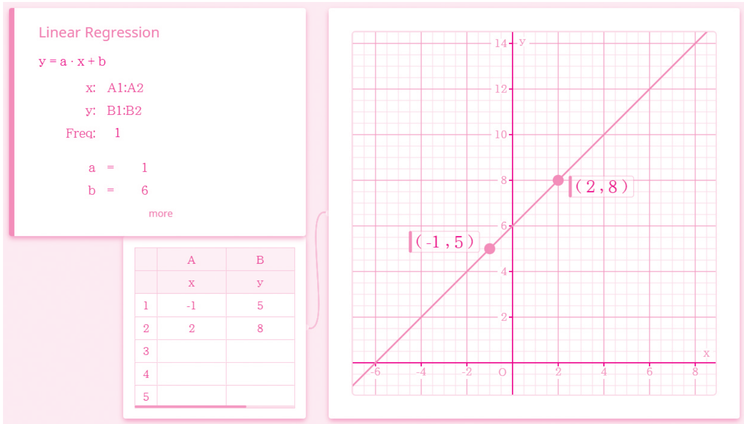
$y=2x+7$



(2) Passes through point $(2, 8)$.

The equation of a line passing through the intersection of 2 lines $3x+y-2=0$ and $x+y-4=0$ can be expressed as
$3x+y-2+k(x+y-4)=0$ (k is a constant).
It passes through point $(2, 8)$, so $3\cdot2+8-2+k(2+8-4)=0$, $k=-2$
Therefore, the equation of the line we find is $3x+y-2-2(x+y-4)=0$, such that $x-y+6=0$

$x-y+6=0$



3 Given the simultaneous equations $\begin{cases} 3x+2y-1=0 \\ ax+4y+3=0 \end{cases}$ have only 1 solution, find the conditions that a satisfies.

Simultaneous equations have only 1 solution when the 2 lines represented by the equations are not parallel.
Therefore, from $3\cdot4-a\cdot2\neq0$, we get $a\neq6$

$a\neq6$

Distance between points and lines

TARGET To understand about distances between points and lines.

STUDY GUIDE

Distance between points and lines

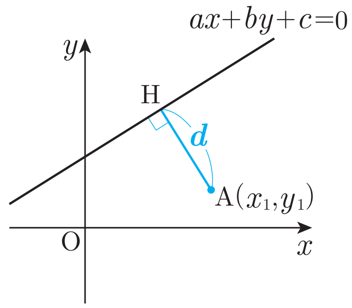
When we let H be the intersection of a perpendicular line drawn from point A to the line l, then the length d of AH is called the **distance** between point A and the line l, which we can find as follows.

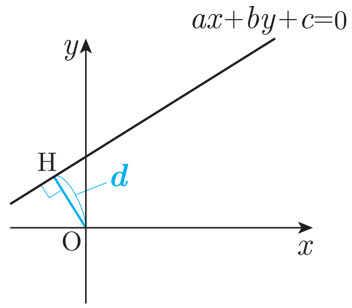
(1) d is the distance between $A(x_1, y_1)$ and line $ax+by+c=0$

$$d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

(2) d is the distance between the origin O and line $ax+by+c=0$

$$d = \frac{|c|}{\sqrt{a^2 + b^2}}$$





explanation

Let the points be $A(x_1, y_1)$ and $H(x_2, y_2)$, such that $d = AH = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ …(i)

Since the line AH is perpendicular to the line $ax+by+c=0$, we get $bx-ay+c'=0$ …(ii).

It passes through point $A(x_1, y_1)$, so we get $bx_1-ay_1+c'=0$, $c'=-bx_1+ay_1$

By substituting this into (ii), we get $bx-ay-bx_1+ay_1=0$, so $b(x-x_1)-a(y-y_1)=0$

The line AH passes through $H(x_2, y_2)$, so we get $b(x_2-x_1)-a(y_2-y_1)=0$ …(iii)

Since point $H(x_2, y_2)$ is a point on the line $ax+by+c=0$, we get $ax_2+by_2+c=0$

By transforming this, we get $a(x_2-x_1)+b(y_2-y_1)+ax_1+by_1+c=0$ …(iv)

If we solve (iii) and (iv) for x_2-x_1 and y_2-y_1, then we get

$$x_2 - x_1 = -\frac{a(ax_1 + by_1 + c)}{a^2 + b^2}, y_2 - y_1 = -\frac{b(ax_1 + by_1 + c)}{a^2 + b^2}$$

These are substituted into (i), for

$$d = \sqrt{\left\{-\frac{a(ax_1 + by_1 + c)}{a^2 + b^2}\right\}^2 + \left\{-\frac{b(ax_1 + by_1 + c)}{a^2 + b^2}\right\}^2} = \sqrt{\frac{(a^2 + b^2)(ax_1 + by_1 + c)^2}{(a^2 + b^2)^2}} = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

If the point A is the origin, then $x_1=0$ and $y_1=0$, so $d = \frac{|c|}{\sqrt{a^2 + b^2}}$

EXERCISE

◆ Find the distances between the following points and lines.

(1) Point $(2, 3)$ and line $5x-4y+8=0$

$$d=\frac{|5\cdot2-4\cdot3+8|}{\sqrt{5^2+(-4)^2}}=\frac{6}{\sqrt{41}}=\frac{6\sqrt{41}}{41}$$

$\dfrac{6\sqrt{41}}{41}$

(2) The origin and line $2x+3y-9=0$

$$d=\frac{|-9|}{\sqrt{2^2+3^2}}=\frac{9}{\sqrt{13}}=\frac{9\sqrt{13}}{13}$$

$\dfrac{9\sqrt{13}}{13}$

(3) Point $(1, -5)$ and line $y=2x+3$

By transforming $y=2x+3$, we get $2x-y+3=0$

$$d=\frac{|2\cdot1-1\cdot(-5)+3|}{\sqrt{2^2+(-1)^2}}=\frac{10}{\sqrt{5}}=2\sqrt{5}$$

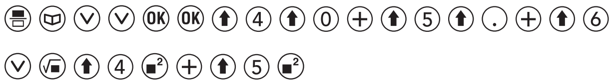
$2\sqrt{5}$

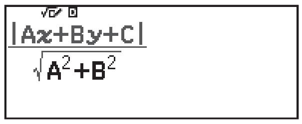
check

On the scientific calculator, use the VARIABLE function to calculate the distance between a point and a line.

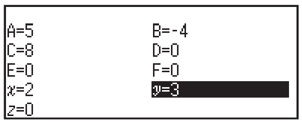
Press ⓐ, select [Calculate], press OK

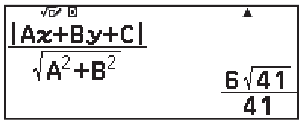
Input the formula for the distance between a point and a line.



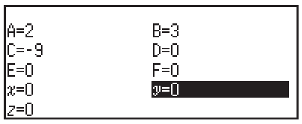


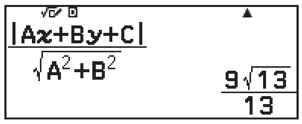
(1) In the VARIABLE screen, input [A=5, B=−4, C=8, x=2, and y=3], and then calculate.



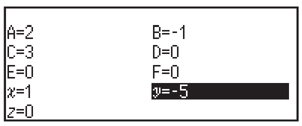


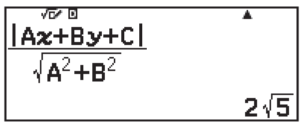
(2) In the same way, input [A=2, B=3, C=−9, x=0, and y=0], and then calculate.





(3) In the same way, input [A=2, B=−1, C=3, x=1, and y=−5], and then calculate.





PRACTICE

◆ Find the distances between the following points and lines.

(1) Point $(-1, -3)$ and line $6x-3y+2=0$

$$d=\frac{|6\cdot(-1)-3\cdot(-3)+2|}{\sqrt{6^2+(-3)^2}}=\frac{5}{\sqrt{45}}=\frac{\sqrt{5}}{3}$$

$\dfrac{\sqrt{5}}{3}$

(2) The origin and line $3x+4y-10=0$

$$d=\frac{|-10|}{\sqrt{3^2+4^2}}=\frac{10}{\sqrt{25}}=2$$

2

(3) Point $(3, 4)$ and line $y=-5x+7$

By transforming $y=-5x+7$, we get $5x+y-7=0$

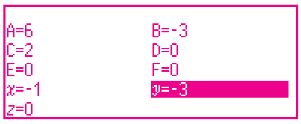
$$d=\frac{|5\cdot3+1\cdot4-7|}{\sqrt{5^2+1^2}}=\frac{12}{\sqrt{26}}=\frac{6\sqrt{26}}{13}$$

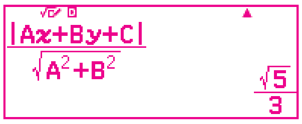
$\dfrac{6\sqrt{26}}{13}$

check

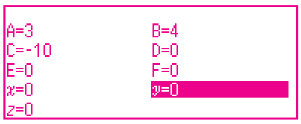
The calculation formula is the same as in EXERCISE, so no input is required.

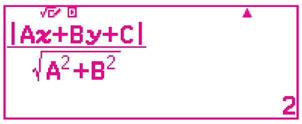
(1) ⇄x 6 EXE > (−) 3 EXE > 2 EXE ∨ ∨ (−) 1 EXE > (−) 3 EXE ↶ EXE



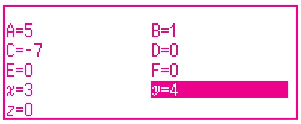


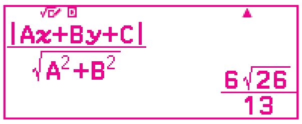
(2) ⇄x 3 EXE > 4 EXE > (−) 1 0 EXE ∨ ∨ 0 EXE > 0 EXE ↶ EXE





(3) ⇄x 5 EXE > 1 EXE > (−) 7 EXE ∨ ∨ 3 EXE > 4 EXE ↶ EXE





Equations for circles (1)

TARGET To understand the equations of circles.

STUDY GUIDE

Equations for circles

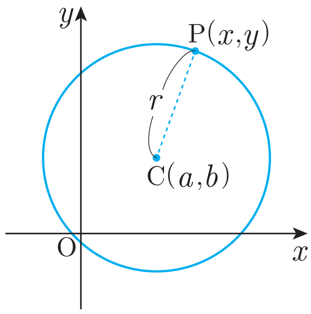
We can use the following equations to represent circles on the coordinate plane.

(1) Equation of a circle having a radius $r(r>0)$ centered on a point $C(a, b)$

$$(x-a)^2+(y-b)^2=r^2$$

(2) Equation of a circle having a radius r centered on the origin O

$$x^2+y^2=r^2$$



Furthermore, graphically, the circle (1) is the circle (2) parallel translated by a on the x axis and by b on the y axis.

explanation

Let (x, y) be the coordinates of an arbitrary point P on the circumference.

Express $CP=r$ in coordinates as $\sqrt{(x-a)^2+(y-b)^2}=r$

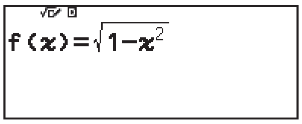
Square both sides to get $(x-a)^2+(y-b)^2=r^2$

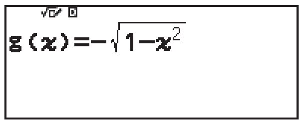
If the point C is the origin, then $a=0$ and $b=0$, so $x^2+y^2=r^2$

proof

Circles can be plotted in the scientific calculator by solving the equation for y.

$x^2+y^2=1, y^2=1-x^2$, so $y=\sqrt{1-x^2}$ or $y=-\sqrt{1-x^2}$



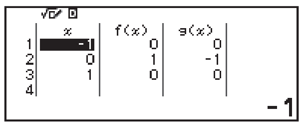


Press (⌂), select [Table], press (OK), then clear the previous data by pressing (⏻)

Press (⋯), select [Define f(x)/g(x)], press (OK), select [Define f(x)], press (OK)

After inputting f$(x)=\sqrt{1-x^2}$, press (EXE)

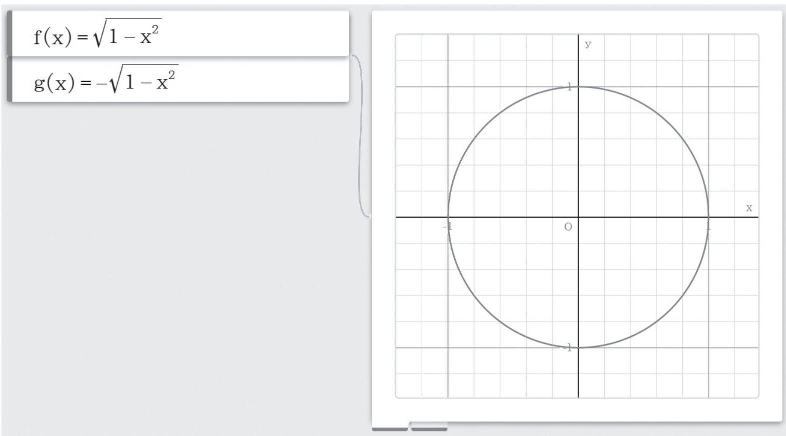
In the same way, input g$(x)=-\sqrt{1-x^2}$.



Press (⋯), select [Table Range], press (OK)

After inputting [Start:−1, End:1, Step:1], select [Execute], press (EXE)

Press (⇧) (x), scan the QR code to display a graph.



EXERCISE

1 Find the equations of the following circles.

(1) Center is at point $(1, 2)$ and radius is 3

$(x-1)^2+(y-2)^2=3^2, (x-1)^2+(y-2)^2=9$

$$(x-1)^2+(y-2)^2=9$$

(2) Center is the origin and radius is $\sqrt{5}$

$x^2+y^2=(\sqrt{5})^2, x^2+y^2=5$

$$x^2+y^2=5$$

2 Solve the following problems with regards to the equation $(x+3)^2+(y-2)^2=6$.

(1) What shape does the equation represent?

It can be transformed to $\{x-(-3)\}^2+(y-2)^2=(\sqrt{6})^2$, so it is a circle with its center at point $(-3, 2)$ and a radius of $\sqrt{6}$.

Circle with its center at point $(-3, 2)$ and a radius of $\sqrt{6}$

(2) How is the shape in (1) translated from the shape represented by the equation $x^2+y^2=6$?

Shape is parallel translated on the x axis by -3 and on the y axis by 2

PRACTICE

1 Find the equations of the following circles.

(1) Center is at point $(4, -5)$ and radius is $\sqrt{2}$

$(x-4)^2+\{y-(-5)\}^2=(\sqrt{2})^2$

$(x-4)^2+(y+5)^2=2$

$$(x-4)^2+(y+5)^2=2$$

(2) Center is the origin and radius is 4

$x^2+y^2=4^2$

$x^2+y^2=16$

$$x^2+y^2=16$$

2 Solve the following problems with regards to the equation $(x+2)^2+(y+5)^2=8$.

(1) What shape does the equation represent?

It can be transformed to $\{x-(-2)\}^2+\{y-(-5)\}^2=(2\sqrt{2})^2$, so it is a circle with a center at a point $(-2, -5)$ and a radius of $2\sqrt{2}$.

Circle with its center at point $(-2, -5)$ and a radius of $2\sqrt{2}$

(2) How is the shape in (1) translated from the shape represented by the equation $x^2+y^2=8$?

Shape is parallel translated on the x axis by -2 and on the y axis by -5

Equations for circles (2)

TARGET To understand the general equations of circles.

STUDY GUIDE

General equation of a circle

Transform the equation of a circle $(x-a)^2+(y-b)^2=r^2$ (standard equation of a circle).

$x^2-2ax+a^2+y^2-2by+b^2-r^2=0, x^2+y^2-2ax-2by+a^2+b^2-r^2=0$

Then, given $-2a=l$, $-2b=m$, and $a^2+b^2-r^2=n$, the general equation of a circle can also generally be expressed as follows.

$$x^2+y^2+lx+my+n=0 \quad (\text{However, } l^2+m^2-4n>0)$$

This form of expression is called the **general equation of a circle**.

Also, since $r^2>0$, by using $-2a=l$ and $-2b=m$ in $a^2+b^2-r^2=n$, we can solve for r^2, such that from

$r^2=a^2+b^2-n=\left(-\frac{l}{2}\right)^2+\left(-\frac{m}{2}\right)^2-n=\frac{l^2+m^2-4n}{4}>0$, we get $l^2+m^2-4n>0$.

When the equation of a circle is given by $x^2+y^2+lx+my+n=0$, we can find the center and radius of the circle by **completing the square** to transform it to $(x-a)^2+(y-b)^2=r^2$.

EXTRA Info.

The following transformation is called completing the square.

$$x^2+px=\left(x+\frac{p}{2}\right)^2-\left(\frac{p}{2}\right)^2$$

EXERCISE

1 What shape does the equation $x^2+y^2-4x+10y-7=0$ represent?

Transforming this equation gives us

$(x-2)^2-4+(y+5)^2-25-7=0, (x-2)^2+(y+5)^2=6^2$

Therefore, we get a circle with its center at point $(2, -5)$ and a radius of 6.

Circle with its center at point $(2, -5)$ and a radius of 6

2 Find the equation of a circle that passes through 3 points $\mathrm{A}(-4, 6)$, $\mathrm{B}(-5, 5)$, and $\mathrm{C}(4, 2)$.

Let the equation of the circle be $x^2+y^2+lx+my+n=0$.

It passes through $\mathrm{A}(-4, 6)$, so $(-4)^2+6^2-4l+6m+n=0, -4l+6m+n=-52 \quad \ldots(\mathrm{i})$

It passes through $\mathrm{B}(-5, 5)$, so $(-5)^2+5^2-5l+5m+n=0, -5l+5m+n=-50 \quad \ldots(\mathrm{ii})$

It passes through $\mathrm{C}(4, 2)$, so $4^2+2^2+4l+2m+n=0, 4l+2m+n=-20 \quad \ldots(\mathrm{iii})$

By solving (i), (ii), and (iii), we get $l=2$, $m=-4$, $n=-20$

Therefore, we find the equation for the circle is $x^2+y^2+2x-4y-20=0$

The circle we find here is called the **circumscribed circle** of $\triangle\mathrm{ABC}$, and the center of this circle is called the **circumcenter**.

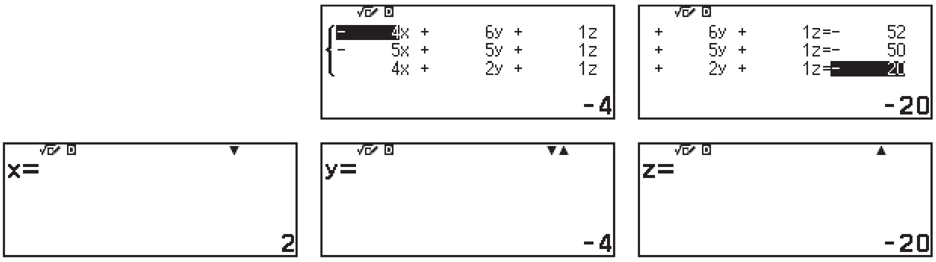
$$\boldsymbol{x^2+y^2+2x-4y-20=0}$$

check

Press ⌂, select $[\text{Equation}]$, press (OK), select $[\text{Simul Equation}]$, press (OK), select $[3\ \text{Unknowns}]$, press (OK)

(−) (4) (EXE) (6) (EXE) (1) (EXE) (−) (5) (2) (EXE)   (−) (5) (EXE) (5) (EXE) (1) (EXE) (−) (5) (0) (EXE)   (4) (EXE) (2) (EXE) (1) (EXE) (−) (2) (0) (EXE)

(EXE) (EXE) (EXE)



PRACTICE

1 What shape does the equation $x^2+y^2+6x-8y+21=0$ represent?

Transforming this equation gives us

$(x+3)^2-9+(y-4)^2-16+21=0, (x+3)^2+(y-4)^2=2^2$

Therefore, we get a circle with its center at point $(-3, 4)$ and a radius of 2.

Circle with its center at point $(-3, 4)$ and a radius of 2

2 Find the equation of a circle that passes through 3 points $\mathrm{A}(2, 4)$, $\mathrm{B}(5, 3)$, and $\mathrm{C}(6, 2)$.

Let the equation of the circle be $x^2+y^2+lx+my+n=0$.

It passes through $\mathrm{A}(2, 4)$, so $2^2+4^2+2l+4m+n=0, 2l+4m+n=-20$ ...(i)

It passes through $\mathrm{B}(5, 3)$, so $5^2+3^2+5l+3m+n=0, 5l+3m+n=-34$ ...(ii)

It passes through $\mathrm{C}(6, 2)$, so $6^2+2^2+6l+2m+n=0, 6l+2m+n=-40$ ...(iii)

By solving (i), (ii), and (iii), we get $l=-4$, $m=2$, $n=-20$

Therefore, we find the equation for the circle is $x^2+y^2-4x+2y-20=0$

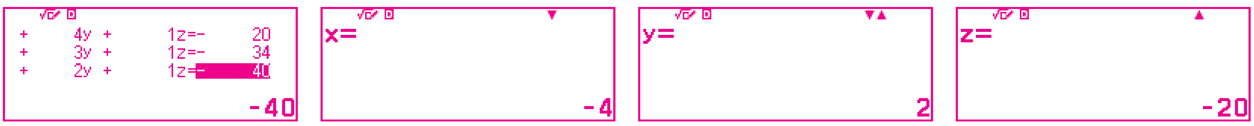
$$x^2+y^2-4x+2y-20=0$$

check

Press ⌂, select $[\text{Equation}]$, press (OK), select $[\text{Simul Equation}]$, press (OK), select $[3\ \text{Unknowns}]$, press (OK)

(2) (EXE) (4) (EXE) (1) (EXE) (−) (2) (0) (EXE)   (5) (EXE) (3) (EXE) (1) (EXE) (−) (3) (4) (EXE)

(6) (EXE) (2) (EXE) (1) (EXE) (−) (4) (0) (EXE)   (EXE) (EXE) (EXE)



Circles and lines (1)

TARGET To understand about positional relations of circles and lines.

STUDY GUIDE

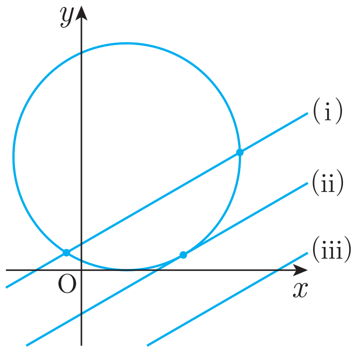
Positional relation of circles and lines

Common points on circles and lines

When a circle and a line have common points, the coordinates of those points are the real roots of the simultaneous equations of the circle and line.

From the number of solutions to the quadratic equation found by eliminating y in the simultaneous equations of the circle $(x-a)^2+(y-b)^2=r^2$ and the line $y=mx+n$, we know the following.

(i) 2 different real roots $\Rightarrow$ Intersect at 2 different points.
(ii) Multiple roots $\Rightarrow$ Tangent at 1 point.
(iii) Has no real roots. $\Rightarrow$ No common points.



Positional relation of circles and lines

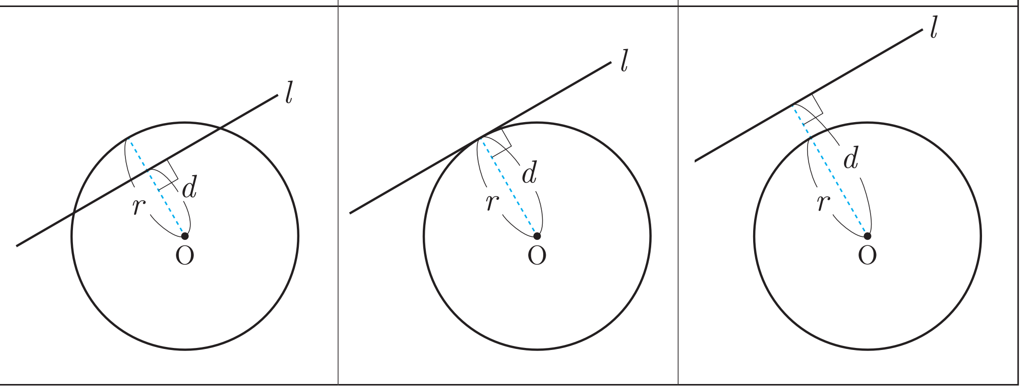
The following methods can be used to determine the positional relations of circles and lines.

(i) Determine the number of solutions to the quadratic equation for finding the coordinates of the common points.

(ii) Use the difference in the radius r of the circle and the distance d from the center of the circle to the line.

The following summarizes the positional relation of circles and lines.

Positional relation of circles and lines	Intersect at 2 different points	Tangent	No common points
Number of common points	2	1	0
Real roots of quadratic equation	2 different real roots	Multiple roots	No real roots
Sign of discriminant D	$D>0$	$D=0$	$D<0$
Difference in d and r	$d<r$	$d=r$	$d>r$



EXERCISE

1 Find the coordinates of the common points of the line and the circle below.

(1) $x^2+y^2=1, y=x+1$

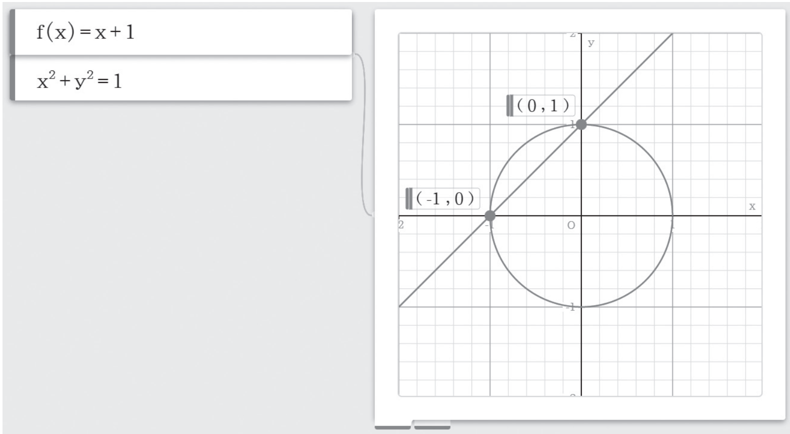
$$\begin{cases} x^2+y^2=1 \cdots \text{(i)} \\ y=x+1 \cdots \text{(ii)} \end{cases}$$

Substitute (ii) into (i), for $x^2+(x+1)^2=1, 2x^2+2x=0, 2x(x+1)=0, x=0, -1$

By substituting this into (ii), when $x=0$ then $y=1$, and when $x=-1$ then $y=0$

Therefore, the coordinates of the common points are $(0, 1)$ and $(-1, 0)$

$(0, 1), (-1, 0)$



(2) $x^2+y^2=8, x+y-4=0$

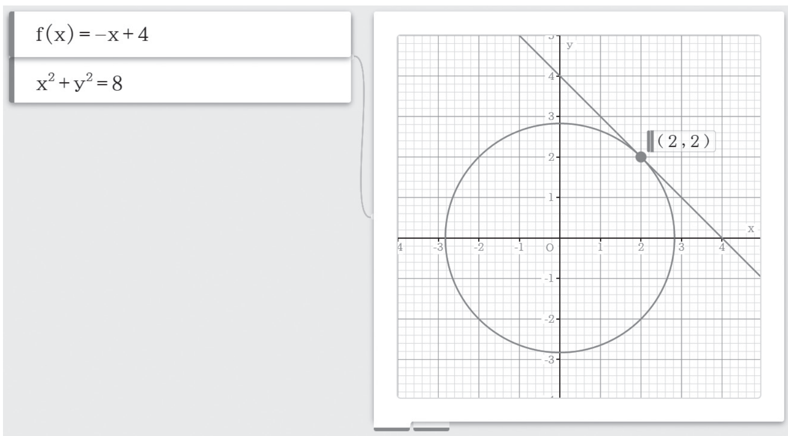
$$\begin{cases} x^2+y^2=8 \cdots \text{(i)} \\ x+y-4=0 \cdots \text{(ii)} \end{cases}$$

By transforming (ii) and substituting it into (i), we get $x^2+(-x+4)^2=8, 2x^2-8x+8=0, (x-2)^2=0, x=2$

By substituting this into (ii), when $x=2$ then $y=2$

Therefore, the coordinates of the common point are $(2, 2)$

$(2, 2)$



2 Find the number of common points of the line and the circle below.

(1) $x^2+y^2=3, y=-3x+1$

$$\begin{cases} x^2+y^2=3 \cdots \text{(i)} \\ y=-3x+1 \cdots \text{(ii)} \end{cases}$$

Substitute (ii) into (i) for $x^2+(-3x+1)^2=3, 5x^2-3x-1=0$

Let this discriminant be D, such that $D=(-3)^2-4\cdot 5\cdot(-1)=29>0$

Therefore, the circle and line intersect at 2 different points.

2

EXTRA Info.

The number of real roots of the quadratic equation $ax^2+bx+c=0$ can be determined by using the discriminant $D=b^2-4ac$.

$D>0$ → Has **2** different real roots. (Intersects at **2** different points.)

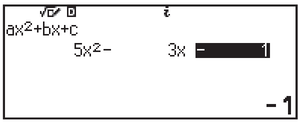
$D=0$ → Has multiple roots. (Is tangent.)

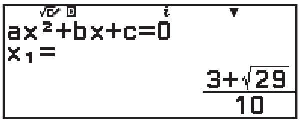
$D<0$ → No real roots. (No common points.)

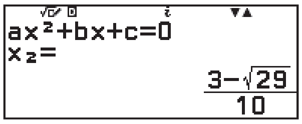
check

Press ⌂, select [Equation], press OK, select [Polynomial], press OK, select [ax²+bx+c], press OK

⑤ EXE ⊖ ③ EXE ⊖ ① EXE EXE EXE







OTHER METHODS

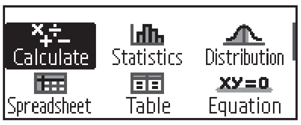
By transforming (ii), we get $3x+y-1=0$ …(iii)

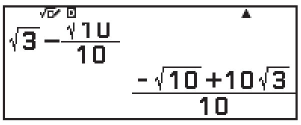
The circle in (i) has a radius of $\sqrt{3}$, and the coordinates of the center are $(0, 0)$.

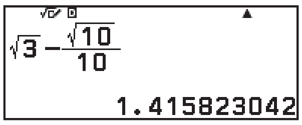
The distance d between the point $(0, 0)$ and the line in (iii) is $d=\dfrac{|-1|}{\sqrt{3^2+1^2}}=\dfrac{\sqrt{10}}{10}$

$$r-d=\sqrt{3}-\frac{\sqrt{10}}{10}=\frac{10\sqrt{3}-\sqrt{10}}{10}=\frac{\sqrt{300}-\sqrt{10}}{10}>0$$

Therefore, since $d<r$, the circle and line intersect at 2 different points.







(2) $x^2+y^2=10, x-3y+10=0$

$$\begin{cases} x^2+y^2=10 \cdots \text{(i)} \\ x-3y+10=0 \cdots \text{(ii)} \end{cases}$$

The circle in (i) has a radius of $\sqrt{10}$, and the coordinates of the center are $(0, 0)$.

The distance d between the point $(0, 0)$ and the line in (ii) is $d=\dfrac{|10|}{\sqrt{1^2+(-3)^2}}=\sqrt{10}$

$r-d=\sqrt{10}-\sqrt{10}=0$

Therefore, since $d=r$, the circle and line are tangent.

1

(3) $x^2+y^2=20, y=-x+8$

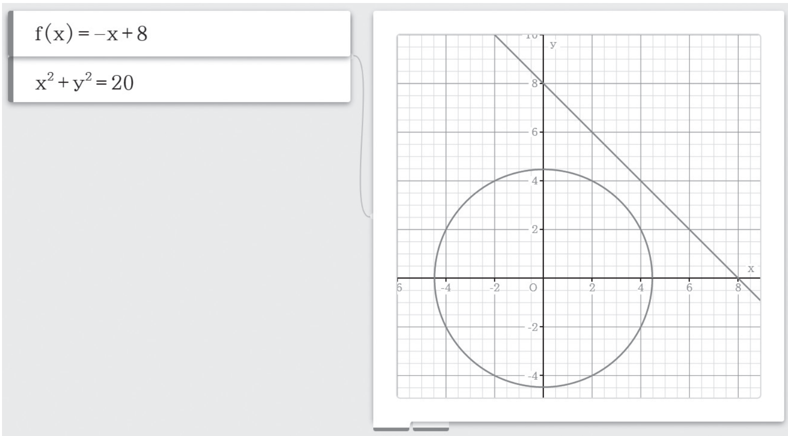
$$\begin{cases} x^2+y^2=20 \cdots \text{(i)} \\ y=-x+8 \cdots \text{(ii)} \end{cases}$$

Substitute (ii) into (i) for $x^2+(-x+8)^2=20, x^2-8x+22=0$

Let this discriminant be D, such that $D=(-8)^2-4\cdot 1\cdot 22=-24<0$

Therefore, the circle and line have no common points.

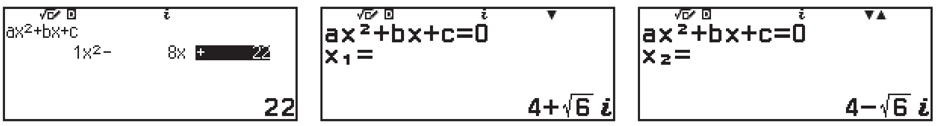
0



check

Press ⌂, select [Equation], press OK, select [Polynomial], press OK, select [ax²+bx+c], press OK

1 EXE (−) 8 EXE 2 2 EXE EXE EXE



The solution is an imaginary number because it contains an imaginary unit i. That is to say, the simultaneous equations have no real root.

3 Solve the following problems.

(1) Given a circle $x^2 + y^2 = 4$ and a line $y = -x + m$ are tangent, find the value of the constant m.

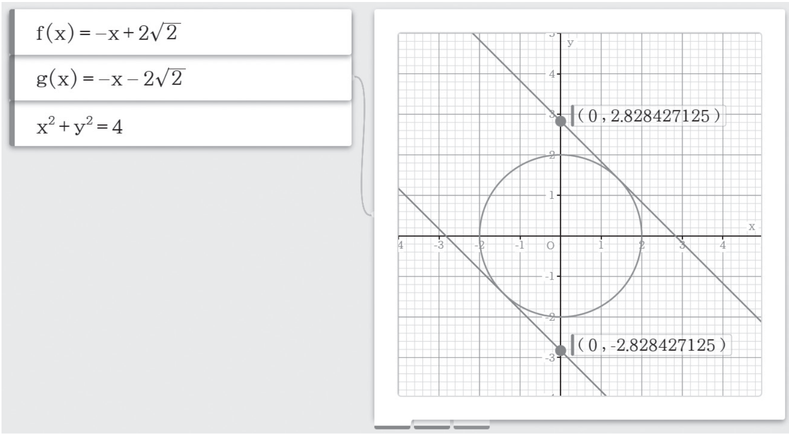
$$\begin{cases} x^2 + y^2 = 4 \cdots \text{(i)} \\ y = -x + m \cdots \text{(ii)} \end{cases}$$

Substitute (ii) into (i) for $x^2 + (-x + m)^2 = 4, 2x^2 - 2mx + m^2 - 4 = 0$

Let this discriminant be D, such that $\dfrac{D}{4} = (-m)^2 - 2 \cdot (m^2 - 4) = -m^2 + 8$

Since the circle and line are tangent at $D = 0$, we get $-m^2 + 8 = 0, m = \pm 2\sqrt{2}$

$$\boldsymbol{m = \pm 2\sqrt{2}}$$



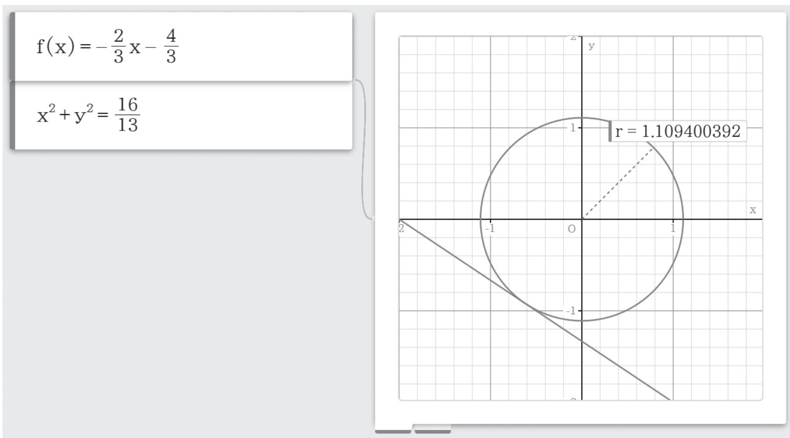
(2) Given a circle $x^2 + y^2 = r^2$ and line $2x + 3y + 4 = 0$ are tangent, find the value of the radius r.

The coordinates of the center of the circle are $(0, 0)$.

The distance d between the point $(0, 0)$ and the line is $d = \dfrac{|4|}{\sqrt{2^2 + 3^2}} = \dfrac{4}{\sqrt{13}} = \dfrac{4\sqrt{13}}{13}$

Since the circle and line are tangent at $d = r$, we get $r = \dfrac{4\sqrt{13}}{13}$

$$\boldsymbol{r = \frac{4\sqrt{13}}{13}}$$



PRACTICE

1 Find the coordinates of the common points of the line and the circle below.

(1) $x^2+y^2=5, y=x-1$

$$\begin{cases} x^2+y^2=5 \cdots \text{(i)} \\ y=x-1 \cdots \text{(ii)} \end{cases}$$

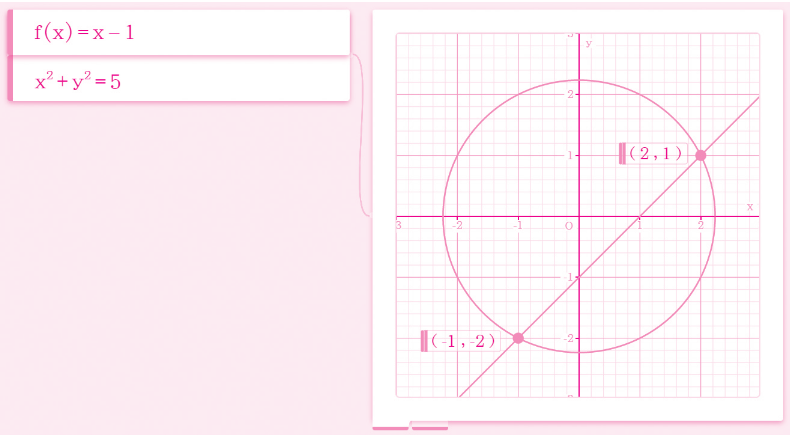
Substitute (ii) into (i), for

$$x^2+(x-1)^2=5, x^2-x-2=0, (x+1)(x-2)=0, x=-1, 2$$

By substituting this into (ii), when $x=-1$ then $y=-2$, and when $x=2$ then $y=1$

Therefore, the coordinates of the common points are $(-1, -2), (2, 1)$

$(-1, -2), (2, 1)$



(2) $x^2+y^2=50, x+y+10=0$

$$\begin{cases} x^2+y^2=50 \cdots \text{(i)} \\ x+y+10=0 \cdots \text{(ii)} \end{cases}$$

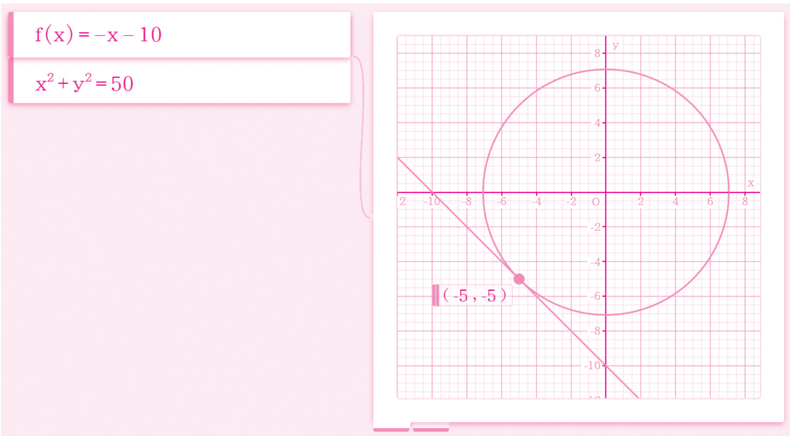
By transforming (ii) and substituting it into (i), we get

$$x^2+(-x-10)^2=50, x^2+10x+25=0, (x+5)^2=0, x=-5$$

By substituting this into (ii), when $x=-5$ then $y=-5$

Therefore, the coordinates of the common point are $(-5, -5)$

$(-5, -5)$



[2] Find the number of common points of the line and the circle below.

(1) $x^2+y^2=6, y=2x+3$

$$\begin{cases} x^2+y^2=6 \cdots \text{(i)} \\ y=2x+3 \cdots \text{(ii)} \end{cases}$$

Substitute (ii) into (i), for $x^2+(2x+3)^2=6, 5x^2+12x+3=0$

Let this discriminant be D, such that $D=12^2-4\cdot5\cdot3=84>0$

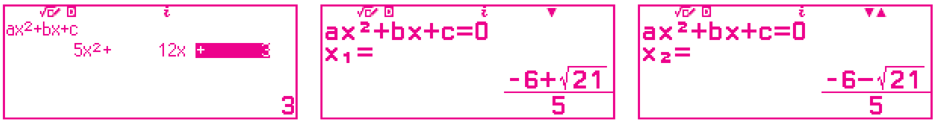
Therefore, the circle and line intersect at 2 different points.

2

check

Press ⌂, select [Equation], press OK, select [Polynomial], press OK, select [ax²+bx+c], press OK

5 EXE 1 2 EXE 3 EXE   EXE EXE



(2) $x^2+y^2=1, 3x+4y+10=0$

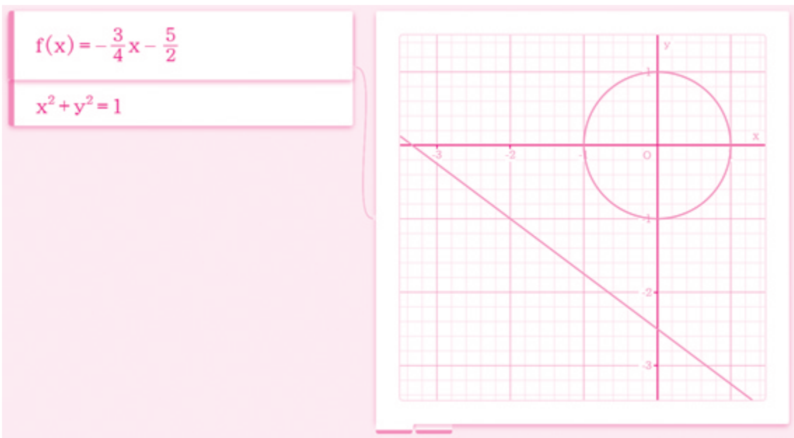
The circle has a radius of 1, and the coordinates of the center are $(0, 0)$.

The distance d between the point $(0, 0)$ and the line is $d=\dfrac{|10|}{\sqrt{3^2+4^2}}=\dfrac{10}{\sqrt{25}}=2$

$$r-d=1-2=-1<0$$

Therefore, since $d>r$, the circle and line have no common points.

0



(3) $x^2+y^2=10, y=3x-10$

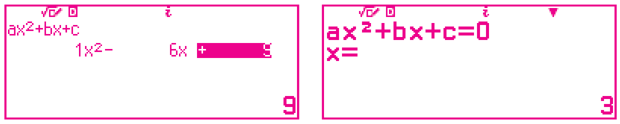
$$\begin{cases} x^2+y^2=10\cdots(\text{i}) \\ y=3x-10\cdots(\text{ii}) \end{cases}$$

Substitute (ii) into (i), for $x^2+(3x-10)^2=10, x^2-6x+9=0$

Let this discriminant be D, such that $D=(-6)^2-4\cdot1\cdot9=0$

Therefore, the circle and line are tangent.

1



3 Solve the following problems.

(1) Given a circle $x^2+y^2=5$ and a line $y=-2x+m$ are tangent, find the value of the constant m.

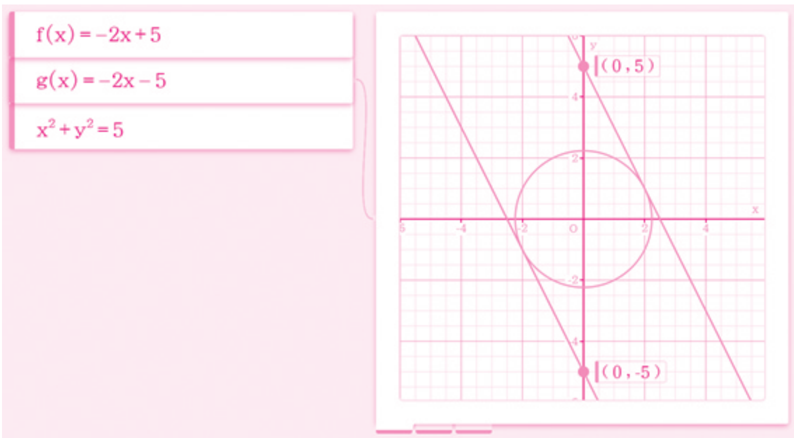
$$\begin{cases} x^2+y^2=5\cdots(\text{i}) \\ y=-2x+m\cdots(\text{ii}) \end{cases}$$

Substitute (ii) into (i) for $x^2+(-2x+m)^2=5, 5x^2-4mx+m^2-5=0$

Let this discriminant be D, such that $\frac{D}{4}=(-2m)^2-5\cdot(m^2-5)=-m^2+25$

Since the circle and line are tangent at $D=0$, we get $-m^2+25=0, m=\pm5$

$m=\pm5$



(2) Given a circle $x^2+y^2=r^2$ and line $3x-2y-26=0$ are tangent, find the value of the radius r.

The coordinates of the center of the circle are $(0, 0)$.

The distance d between the point $(0, 0)$ and the line is

$$d=\frac{|-26|}{\sqrt{3^2+(-2)^2}}=\frac{26}{\sqrt{13}}=2\sqrt{13}$$

Since the circle and line are tangent at $d=r$, we get $r=2\sqrt{13}$

$r=2\sqrt{13}$

Circles and lines (2)

TARGET To understand the equations of tangents on circles.

STUDY GUIDE

Equations for tangents to circles

We can use the following equations to express that a line is tangent to a circle.

(1) Equation for a tangent to a point $P(x_1, y_1)$ on a circle $x^2 + y^2 = r^2$

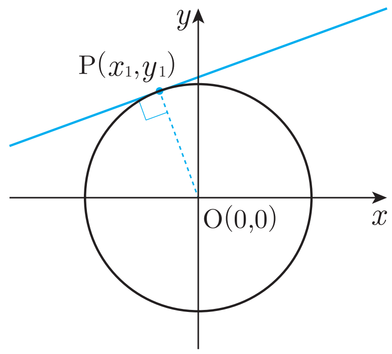
$$x_1x + y_1y = r^2$$

(2) Equation for a tangent to a point $P(x_1, y_1)$ on a circle

$$(x - a)^2 + (y - b)^2 = r^2$$

$$(x_1 - a)(x - a) + (y_1 - b)(y - b) = r^2$$

explanation



The tangent passes through the point $P(x_1, y_1)$.

Since the point $P(x_1, y_1)$ is on the circle $x^2 + y^2 = r^2$, we get $x_1^2 + y_1^2 = r^2$

Furthermore, if we let m be the slope of the tangent, and the tangent is perpendicular to the line OP, then we get

(slope of line OP)$\cdot m = -1$, so $\frac{y_1}{x_1} \cdot m = -1, m = -\frac{x_1}{y_1}$

The equations of the tangent are

$$y - y_1 = -\frac{x_1}{y_1}(x - x_1), y_1(y - y_1) = -x_1(x - x_1), y_1y + x_1x = x_1^2 + y_1^2 = r^2$$

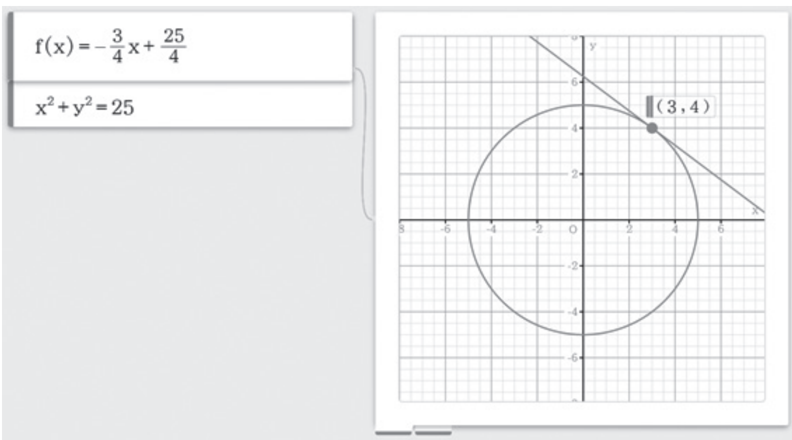
EXERCISE

◆ Find the equation of a tangent at point P on the circumference, as shown below.

(1) Circle $x^2 + y^2 = 25$ and point $P(3, 4)$

The equation of the tangent we find is $3x + 4y = 25$

$3x+4y=25$



(2) Circle $x^2 + y^2 = 4$ and point $\mathrm{P}(-2, 0)$

The equation of the tangent we find is $-2x+0\cdot y=4$, $x=-2$

$$x=-2$$

(3) Circle $(x-1)^2 + (y-2)^2 = 10$ and point $\mathrm{P}(2, 5)$

The equation of the tangent we find is $(2-1)(x-1)+(5-2)(y-2)=10$, $x+3y-17=0$

$$x+3y-17=0$$

PRACTICE

◆ Find the equation of a tangent at point P on the circumference, as shown below.

(1) Circle $x^2 + y^2 = 40$ and point $\mathrm{P}(2, 6)$

The equation of the tangent we find is $2x+6y=40$

$$2x+6y=40$$

(2) Circle $x^2 + y^2 = 9$ and point $\mathrm{P}(0, -3)$

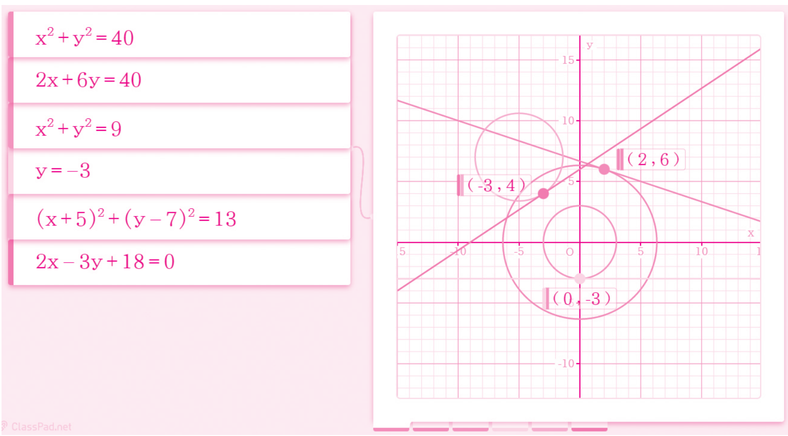
The equation of the tangent we find is $0\cdot x-3y=9$, $y=-3$

$$y=-3$$

(3) Circle $(x+5)^2 + (y-7)^2 = 13$ and point $\mathrm{P}(-3, 4)$

The equation of the tangent we find is $(-3+5)(x+5)+(4-7)(y-7)=13$, $2x-3y+18=0$

$$2x-3y+18=0$$



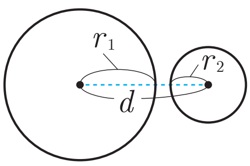
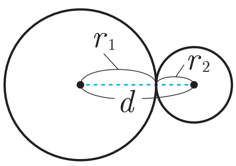
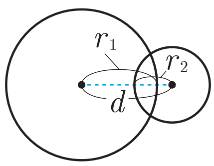
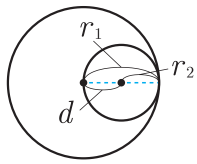
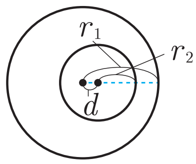
2 circles (1)

TARGET To understand about positional relations of **2** circles.

STUDY GUIDE

Positional relation of 2 circles

We can classify the radii r_1 and r_2 $(r_1>r_2)$ of 2 circles and the distance d between their 2 centers as follows.

(1) $d>r_1+r_2$	(2) $d=r_1+r_2$	(3) $r_1-r_2<d<r_1+r_2$	(4) $d=r_1-r_2$	(5) $d<r_1-r_2$
One is outside the other.	They are externally tangent.	They intersect at 2 different points.	They are internally tangent.	One is inside the other.
				

EXERCISE

1 Determine the positional relation of the 2 circles below.

(1) Circle $x^2+y^2=9$ …(i), and circle $(x+4)^2+(y-3)^2=4$ …(ii)

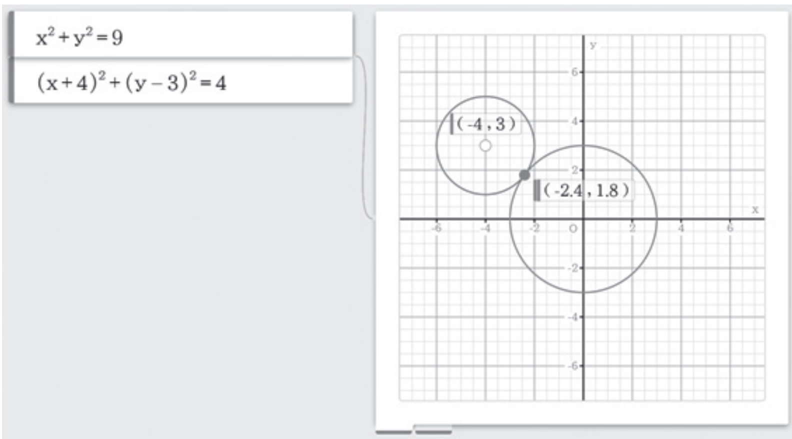
Circle (i) has its center at the origin and a radius of $r_1=3$, and circle (ii) has its center at the point $(-4, 3)$ and a radius of $r_2=2$

The distance between the centers of the 2 circles is $d=\sqrt{(-4)^2+3^2}=5$

The sum of the 2 radii of the circles is $r_1+r_2=3+2=5$

Therefore, since $d=r_1+r_2$, the 2 circles are externally tangent.

2 circles are externally tangent.



(2) Circle $x^2+y^2=40$ …(i), and circle $(x-3)^2+(y+1)^2=10$ …(ii)

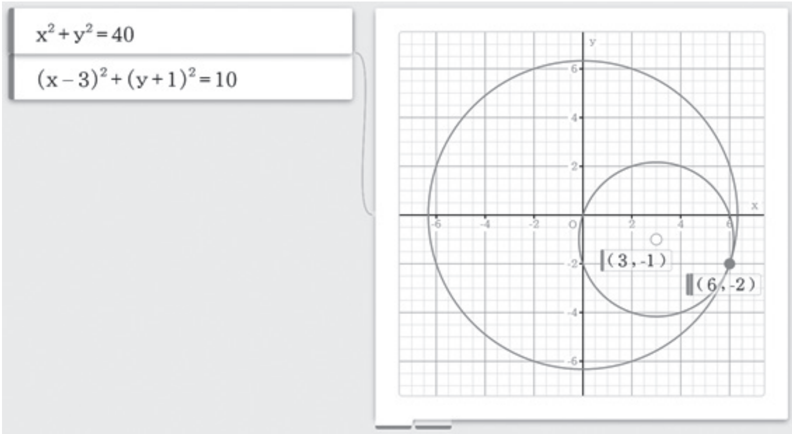
Circle (i) has its center at the origin and a radius of $r_1=\sqrt{40}=2\sqrt{10}$, and circle (ii) has its center at the point $(3, -1)$ and a radius of $r_2=\sqrt{10}$

The distance between the centers of the 2 circles is $d=\sqrt{3^2+(-1)^2}=\sqrt{10}$

The difference of the 2 radii of the circles is $r_1-r_2=2\sqrt{10}-\sqrt{10}=\sqrt{10}$

Therefore, since $d=r_1-r_2$, the 2 circles are internally tangent.

2 circles are internally tangent.



(3) Circle $x^2+y^2=4$ …(i), and circle $(x+1)^2+(y-3)^2=1$ …(ii)

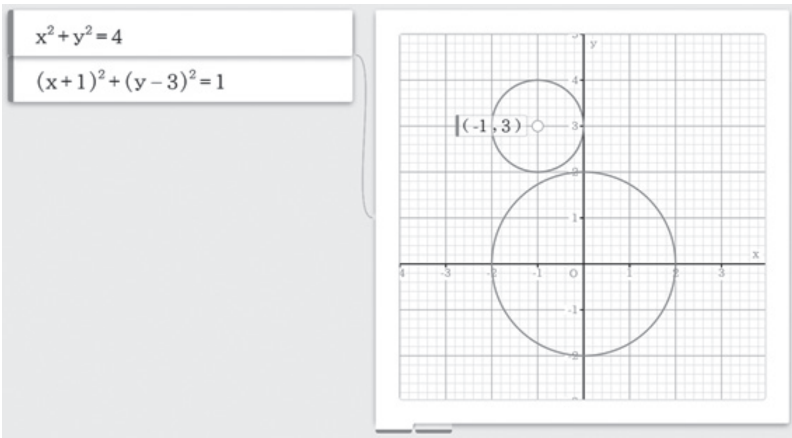
Circle (i) has its center at the origin and a radius of $r_1=2$, and circle (ii) has its center at the point $(-1, 3)$ and a radius of $r_2=1$

The distance between the centers of the 2 circles is $d=\sqrt{(-1)^2+3^2}=\sqrt{10}$

The sum of the 2 radii of the circles is $r_1+r_2=2+1=3$

Therefore, since $d>r_1+r_2$, circle (ii) is outside of circle (i).

Circle (ii) is outside of circle (i).



2 Find the equation of a circle that has its center at $(3, 6)$ and is externally tangent to the circle $x^2+y^2=5$.

Let r_2 be the radius of the circle to be found.

Circle $x^2+y^2=5$ has its center at the origin and a radius of $r_1=\sqrt{5}$

The distance between the centers of the 2 circles is $d=\sqrt{3^2+6^2}=\sqrt{45}=3\sqrt{5}$

Given the 2 circles are externally tangent, then $d=r_1+r_2$, so we get $3\sqrt{5}=\sqrt{5}+r_2, r_2=2\sqrt{5}$

Therefore, we find the equation for the circle is $(x-3)^2+(y-6)^2=(2\sqrt{5})^2=20$

$$\boldsymbol{(x-3)^2+(y-6)^2=20}$$

PRACTICE

1 Determine the positional relation of the 2 circles below.

(1) Circle $x^2+y^2=16$ …(i), and circle $(x-4)^2+(y+3)^2=1$ …(ii)

Circle (i) has its center at the origin and a radius of $r_1=4$, and circle (ii) has its center at the point $(4, -3)$ and a radius of $r_2=1$

The distance between the centers of the 2 circles is $d=\sqrt{4^2+(-3)^2}=5$

The sum of the 2 radii of the circles is $r_1+r_2=4+1=5$

Therefore, since $d=r_1+r_2$, the 2 circles are externally tangent.

2 circles are externally tangent.

(2) Circle $x^2+y^2=4$ …(i), and circle $(x+2)^2+(y-1)^2=25$ …(ii)

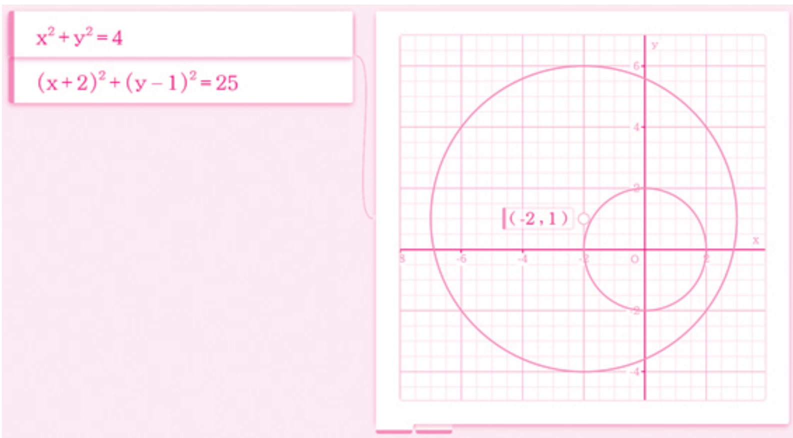
Circle (i) has its center at the origin and a radius of $r_1=2$, and circle (ii) has its center at the point $(-2, 1)$ and a radius of $r_2=5$

The distance between the centers of the 2 circles is $d=\sqrt{(-2)^2+1^2}=\sqrt{5}$

The difference of the 2 radii of the circles is $r_2-r_1=5-2=3$

Therefore, since $d<r_2-r_1$, circle (i) is inside of circle (ii).

Circle (i) is inside of circle (ii).



(3) Circle $x^2+y^2=9$ $\dots$(i), and circle $(x-1)^2+(y-2)^2=1$ $\dots$(ii)

Circle (i) has its center at the origin and a radius of $r_1=3$, and circle (ii) has its center at the point $(1, 2)$ and a radius of $r_2=1$

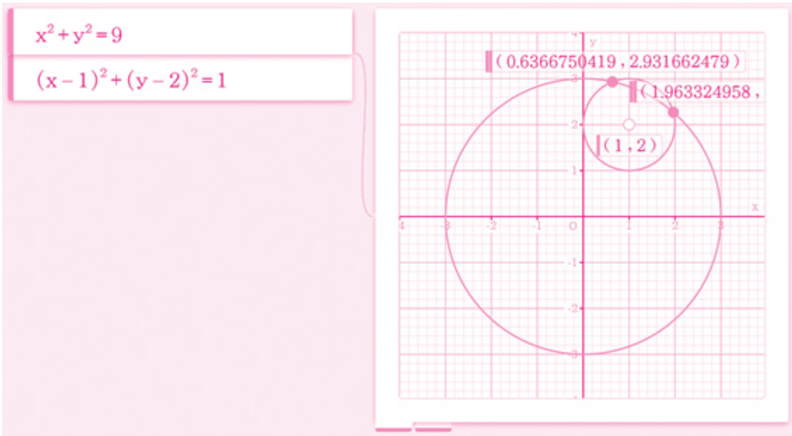
The distance between the centers of the 2 circles is $d=\sqrt{1^2+2^2}=\sqrt{5}$

The difference of the 2 radii of the circles is $r_1-r_2=3-1=2$

The sum of the 2 radii of the circles is $r_1+r_2=3+1=4$

Therefore, since $r_1-r_2<d<r_1+r_2$, they intersect at 2 different points.

2 circles intersect at 2 different points.



2 Find the equation of a circle that has its center at $(-3, -4)$ and is externally tangent to the circle $x^2+y^2=1$.

Let r_2 be the radius of the circle to be found.

Circle $x^2+y^2=1$ has its center at the origin and a radius of $r_1=1$

The distance between the centers of the 2 circles is $d=\sqrt{(-3)^2+(-4)^2}=5$

Given the 2 circles are externally tangent, then $d=r_1+r_2$, so we get $5=1+r_2$, so $r_2=4$

Therefore, we find the equation for the circle is $(x+3)^2+(y+4)^2=4^2=16$

$$(x+3)^2+(y+4)^2=16$$

2 circles (2)

TARGET To understand about shapes that pass through the intersection of 2 circles.

STUDY GUIDE

Circles or lines passing through the intersection of 2 circles

When 2 circles have common points, the **coordinates of the common points** are the **real roots of the simultaneous equations** of the 2 circles.

Given 2 circles $x^2+y^2+l_1x+m_1y+n_1$ ...(i) and $x^2+y^2+l_2x+m_2y+n_2$ ...(ii) that intersect at 2 points, the circle or line that passes through the intersection of the 2 circles (i) and (ii) can be expressed by the following equation.

$$(x^2+y^2+l_1x+m_1y+n_1)+k(x^2+y^2+l_2x+m_2y+n_2)=0$$

(k is a constant)

EXERCISE

◆ Solve the following problems with regards to the 2 circles $x^2+y^2=25$ ...(i) and $(x-4)^2+(y-3)^2=2$ ...(ii).

(1) Find the equations for the line that passes through the 2 common points of the 2 circles.

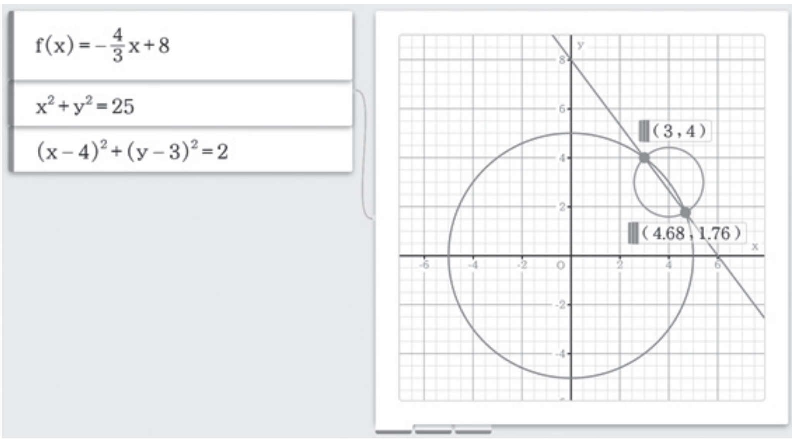
Given k is a constant, then $(x^2+y^2-25)+k\{(x-4)^2+(y-3)^2-2\}=0$ ...(iii) expresses the shape that passes through the common points of the 2 circles.

Given that (iii) is a line, and the coefficient of the terms x^2, y^2 is 0, then $k=-1$

Substitute this into (iii), and we can rearrange them.

$(x^2+y^2-25)-\{(x-4)^2+(y-3)^2-2\}=0, (x^2+y^2-25)-(x^2+y^2-8x-6y+23)=0, 4x+3y-24=0$

$4x+3y-24=0$



(2) Find the equations for the line that passes through the 2 common points of the 2 circles and point (3, 1).

Since (iii) passes through point (3, 1), then by substituting $x=3$ and $y=1$ into (iii), we get $-15+k\cdot3=0$, $k=5$

Substitute this into (iii), and we can rearrange them.

$(x^2+y^2-25)+5\{(x-4)^2+(y-3)^2-2\}=0, 6x^2+6y^2-40x-30y+90=0, x^2+y^2-\frac{20}{3}x-5y+15=0$

$$x^2+y^2-\frac{20}{3}x-5y+15=0$$

PRACTICE

◆ Solve the following problems with regards to the 2 circles $x^2+y^2=5 \quad \dots\text{(i)}$ and $(x+2)^2+(y+2)^2=9 \quad \dots\text{(ii)}$.

(1) Find the equations for the line that passes through the 2 common points of the 2 circles.

Given k is a constant, then $(x^2+y^2-5)+k\{(x+2)^2+(y+2)^2-9\}=0 \quad \dots\text{(iii)}$ expresses the shape that passes through the common points of the 2 circles.

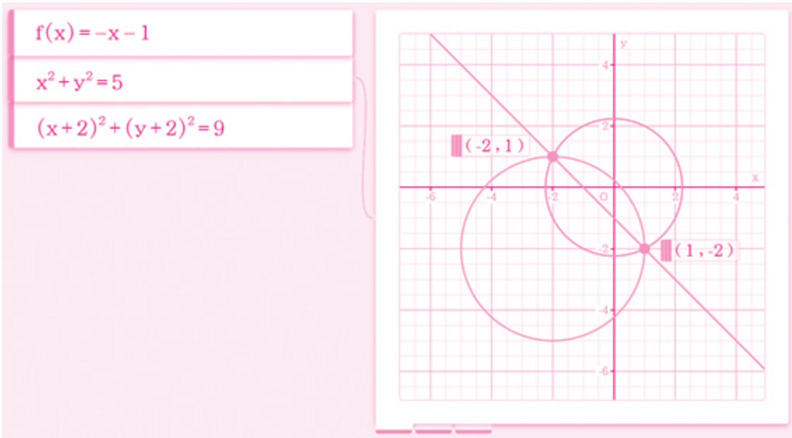
Given that (iii) is a line, and the coefficient of the terms x^2, y^2 is 0, then $k=-1$

Substitute this into (iii), and we can rearrange them.

$(x^2+y^2-5)-\{(x+2)^2+(y+2)^2-9\}=0$

$(x^2+y^2-5)-(x^2+y^2+4x+4y-1)=0, x+y+1=0$

$$x+y+1=0$$



(2) Find the equations for the line that passes through the 2 common points of the 2 circles and point $(-2, -2)$.

Since (iii) passes through point $(-2, -2)$, then by substituting $x=-2$ and $y=-2$ into (iii), we get $3+k\cdot(-9)=0$, $k=\dfrac{1}{3}$

Substitute this into (iii), and we can rearrange them.

$(x^2+y^2-5)+\dfrac{1}{3}\{(x+2)^2+(y+2)^2-9\}=0$

$4x^2+4y^2+4x+4y-16=0, x^2+y^2+x+y-4=0$

$$x^2+y^2+x+y-4=0$$

Loci and equations (1)

TARGET To understand loci and how to find them.

STUDY GUIDE

Loci

The set of all the points that satisfy certain conditions to describe a figure are called the **loci** of points that satisfy those conditions. We use the following procedures to find loci.

> **(1) Express the conditions for the point P in a formula for x and y such that the coordinates of the point P satisfying the conditions are (x, y)**
> **(2) Find the shape expressed by the formula in (1).**
> **(3) Confirm that the point P on the shape that is found satisfies the given conditions.**

EXERCISE

◆ Find the loci of point P that are the same distance from the 2 points $\mathrm{A}(-1, 2)$ and $\mathrm{B}(3, -4)$.

Let (x, y) be the coordinates of the point P.

Since point P is the same distance from the 2 points A and B, we know that $\mathrm{AP}=\mathrm{BP}$

Specifically, $\mathrm{AP}^2 = \mathrm{BP}^2 \quad \ldots(\mathrm{i})$

$\mathrm{AP}^2 = (x+1)^2 + (y-2)^2 = x^2 + y^2 + 2x - 4y + 5$

$\mathrm{BP}^2 = (x-3)^2 + (y+4)^2 = x^2 + y^2 - 6x + 8y + 25$

By substituting this into (i), we get

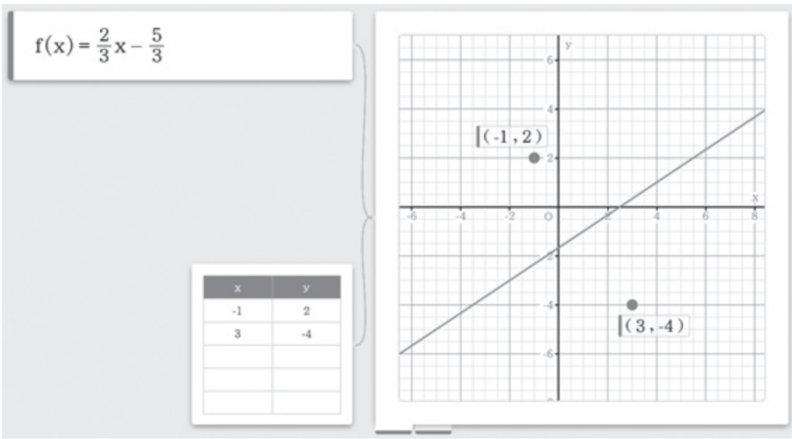
$x^2 + y^2 + 2x - 4y + 5 = x^2 + y^2 - 6x + 8y + 25, 8x - 12y - 20 = 0, 2x - 3y - 5 = 0$

Therefore, point P is on the line $2x-3y-5=0 \quad \ldots(\mathrm{ii})$.

Conversely, the point $\mathrm{P}(x, y)$ on line (ii) satisfies the conditions.

Therefore, the loci of point P is the line $2x-3y-5=0$.

Line $2x-3y-5=0$



PRACTICE

◆ Find the loci of point P that are the same distance from the 2 points $A(-2, -3)$ and $B(1, 4)$.

Let (x, y) be the coordinates of the point P.

Since point P is the same distance from the 2 points A and B, we know that $AP=BP$

Specifically, $AP^2 = BP^2$ ...(i)

$$AP^2 = (x+2)^2 + (y+3)^2 = x^2 + y^2 + 4x + 6y + 13$$

$$BP^2 = (x-1)^2 + (y-4)^2 = x^2 + y^2 - 2x - 8y + 17$$

By substituting this into (i), we get

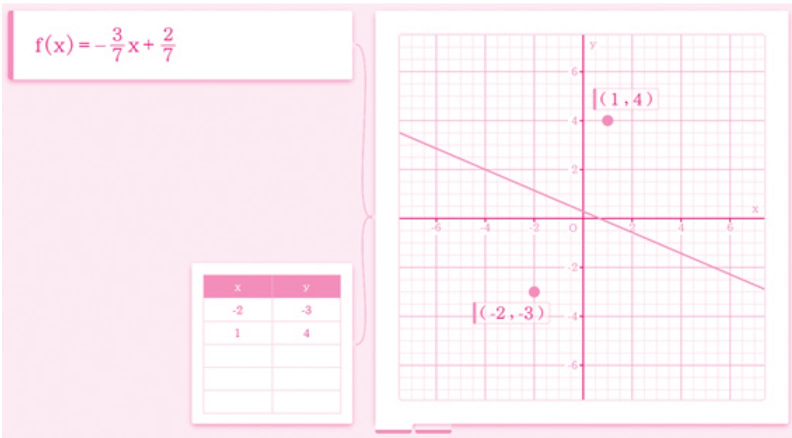
$$x^2 + y^2 + 4x + 6y + 13 = x^2 + y^2 - 2x - 8y + 17, 6x + 14y - 4 = 0, 3x + 7y - 2 = 0$$

Therefore, point P is on the line $3x+7y-2=0$ ...(ii).

Conversely, the point $P(x, y)$ on line (ii) satisfies the conditions.

Therefore, the loci of point P is the line $3x+7y-2=0$.

Line $3x+7y-2=0$



Loci and equations (2)

TARGET To understand the loci of points that correspond to moving points.

STUDY GUIDE

How to find the loci of a moving point

We use the following procedures to find loci of moving points.

(1) Use s and t to express the conditions for moving point $\mathrm{Q}(s, t)$.

(2) Find the loci of the point $\mathrm{P}(x, y)$ by expressing s and t in the formula of x and y in relation to P and Q.

(3) Eliminate s and t from the expressions in (1) and (2) to find the relational expression of x and y.

(4) Find the shape expressed by the equation in (3).

(5) Conversely, confirm that the point P on the shape satisfies the given conditions.

EXERCISE

1 Given that point Q moves on the circle $x^2+y^2=4$, find the loci of the midpoint P of the line segment OQ connecting the origin O and the point Q.

Let the coordinates of the point Q be (s, t), then since point Q is on $x^2+y^2=4$, we get $s^2+t^2=4 \quad \ldots(\mathrm{i})$

Let the coordinates of the point P be (x, y), then since point P is the midpoint of line segment OQ, we get

$x=\frac{s}{2}, y=\frac{t}{2}$; specifically $s=2x, t=2y \quad \ldots(\mathrm{ii})$

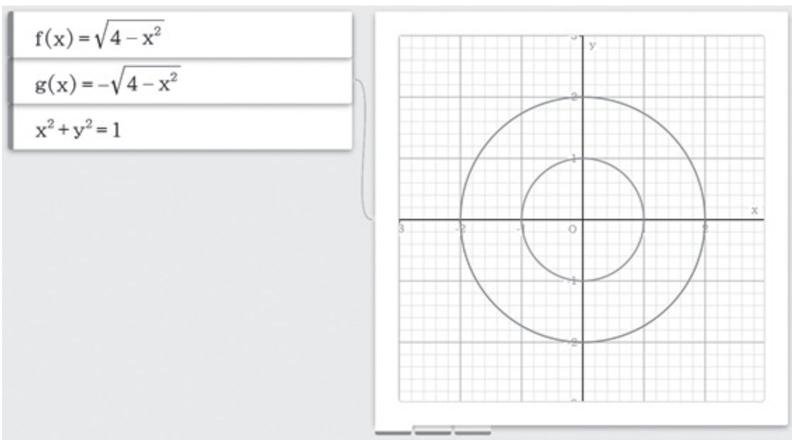
Substitute (ii) into (i), for $(2x)^2+(2y)^2=4, 4x^2+4y^2=4, x^2+y^2=1$

Therefore, point P is on the circle $x^2+y^2=1 \quad \ldots(\mathrm{iii})$.

Conversely, the point $\mathrm{P}(x, y)$ on the circle (iii) satisfies the conditions.

Therefore, the loci of point P is a circle with its center at the origin and a radius of 1.

Circle with its center at origin and a radius of 1



2 Given the value of the real number a changes, find the loci of the vertex of the parabola $y = x^2 - 2ax + 3a^2 + 4a - 1$.

To find the coordinates of the vertex of the parabola, complete the square to transform the equation.

$y = (x - a)^2 - a^2 + 3a^2 + 4a - 1 = (x - a)^2 + 2a^2 + 4a - 1$

Let the coordinates of the vertex of the parabola be $\mathrm{P}(x, y)$, such that $x = a$ ...(i), and $y = 2a^2 + 4a - 1$ ...(ii)

Substitute (i) into (ii), for $y = 2x^2 + 4x - 1$

Therefore, point P is on the parabola $y = 2x^2 + 4x - 1$ ...(iii).

Conversely, the point $\mathrm{P}(x, y)$ on the parabola (iii) satisfies the conditions.

Therefore, the loci of point P is the parabola $y = 2x^2 + 4x - 1$.

Parabola $y = 2x^2 + 4x - 1$

PRACTICE

1 Given that point Q moves on the circle $x^2 + y^2 = 16$, find the loci of the midpoint P of the line segment AQ connecting the point $\mathrm{A}(6, 0)$ and the point Q.

Let the coordinates of the point Q be (s, t), then since point Q is on $x^2 + y^2 = 16$, we get $s^2 + t^2 = 16$ ...(i)

Let the coordinates of the point P be (x, y), then since point P is the midpoint of line segment AQ, we get $x = \frac{s+6}{2}, y = \frac{t}{2}$;

specifically, $s = 2x - 6$ and $t = 2y$ ...(ii)

Substitute (ii) into (i), for

$(2x - 6)^2 + (2y)^2 = 16, 4(x - 3)^2 + 4y^2 = 16, (x - 3)^2 + y^2 = 4$

Therefore, point P is on the circle $(x - 3)^2 + y^2 = 4$ ...(iii).

Conversely, the point $\mathrm{P}(x, y)$ on the circle (iii) satisfies the conditions.

Therefore, the loci of point P is a circle with its center at $(3, 0)$ and a radius of 2.

Circle with its center at $(3, 0)$ and a radius of 2

2 Given the value of the real number a changes, find the loci of the vertex of the parabola $y = x^2 - 4ax + 8a^2 - 4a + 3$.

To find the coordinates of the vertex of the parabola, complete the square to transform the equation.

$y = (x - 2a)^2 - 4a^2 + 8a^2 - 4a + 3 = (x - 2a)^2 + 4a^2 - 4a + 3$

Let the coordinates of the vertex of the parabola be $\mathrm{P}(x, y)$, such that $x = 2a$ ...(i), and $y = 4a^2 - 4a + 3$ ...(ii)

Substitute (i) into (ii), for $y = x^2 - 2x + 3$

Therefore, point P is on the parabola $y = x^2 - 2x + 3$ ...(iii).

Conversely, the point $\mathrm{P}(x, y)$ on the parabola (iii) satisfies the conditions.

Therefore, the loci of point P is the parabola $y = x^2 - 2x + 3$.

Parabola $y = x^2 - 2x + 3$

Domains expressed by inequalities (1)

TARGET To understand about domains bounded by lines or circles.

STUDY GUIDE

Lines, circles, and domains

When a set of points satisfy certain conditions across a plane, we call that set a **domain**. Furthermore, the boundary between the domain and the other part is called the **boundary line**.

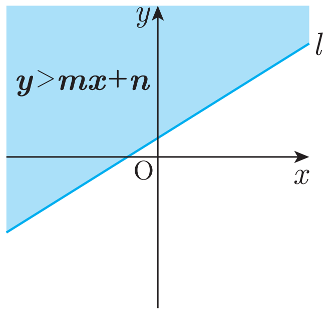
Lines and domains

If we let the line $y=mx+n$ be l, then for the domain expressed by the inequalities $y>mx+n$ and $y<mx+n$, we get the following.

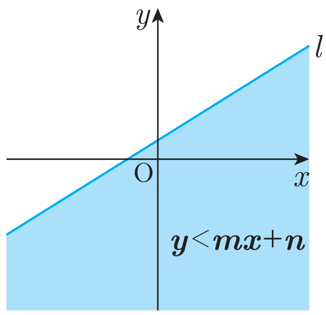
(i) Domain expressed by the inequality $y>mx+n$ → Above line l

(ii) Domain expressed by the inequality $y<mx+n$ → Below line l

(i)



(ii)



In both domains (i) and (ii), the boundary line is a straight line l, and when we include the equality sign as in $y \geq mx+n$, then we also include the points on the line l.

Circles and domains

If we let the circle $(x-a)^2+(y-b)^2=r^2$ be C, then for the domain expressed by the inequalities $(x-a)^2+(y-b)^2<r^2, (x-a)^2+(y-b)^2>r^2$, we get the following.

(i) Domain expressed by the inequality $(x-a)^2+(y-b)^2<r^2$ → Inside circle C

(ii) Domain expressed by the inequality $(x-a)^2+(y-b)^2>r^2$ → Outside circle C

In both domains (i) and (ii), the boundary line is a circle C, and when we include the equality sign as in $(x-a)^2+(y-b)^2 \geq r^2$, then we also include the points on the circle C.

EXERCISE

1 Draw the domains expressed by the following inequalities. Also, determine whether the domain includes the boundary line.

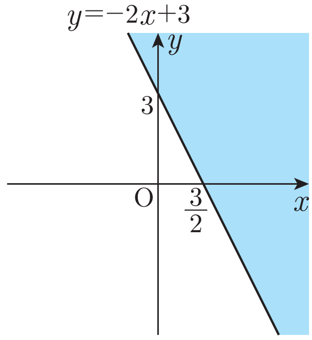
(1) Inequality $2x+y-3>0$

By transforming the inequality, we get $y>-2x+3$

Therefore, the domain expressed by this inequality is above the line $y=-2x+3$, which is the shaded area in the figure on the right.

However, the boundary line is not included.

Boundary line is not included.



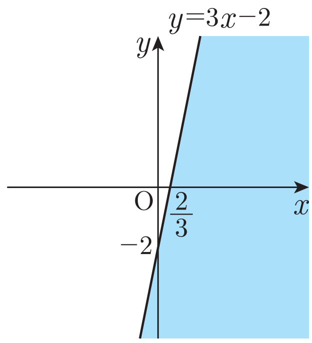
(2) Inequality $-3x+y+2\leq 0$

By transforming the inequality, we get $y\leq 3x-2$

Therefore, the domain expressed by this inequality is below the line $y=3x-2$, which is the shaded area in the figure on the right.

However, the boundary line is included.

Boundary line is included.



check

On the scientific calculator, use the Table function to check whether the points in the domain satisfy the inequality.

Press ⌂, select [Table], press OK, then clear the previous data by pressing ⏻

Press ⋯, select [Define f(x)/g(x)], press OK, select [Define f(x)], press OK

After inputting $f(x)=-2x+3$, press EXE

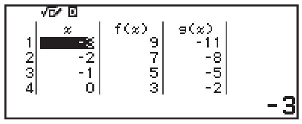
In the same way, input $g(x)=3x-2$.

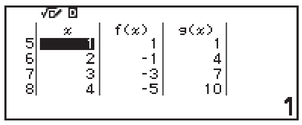
Press ⋯, select [Table Range], press OK

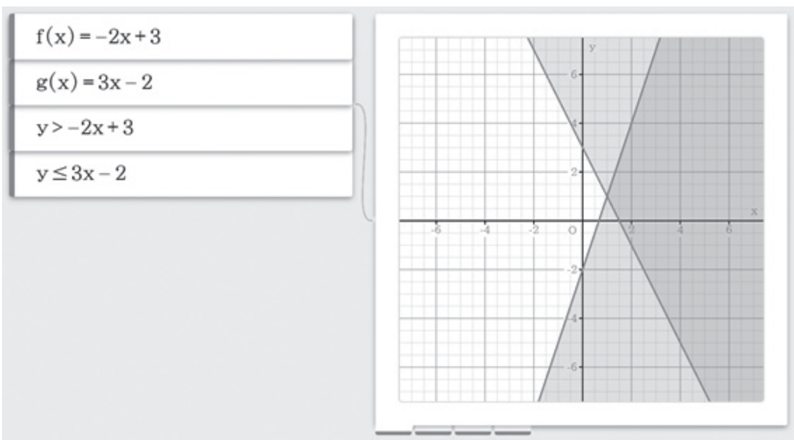
After inputting [Start:−3, End:4, Step:1], select [Execute], press EXE

Press ⇧ ⓧ, scan the QR code to display a graph.

Also, tap the function sheet, and input $y>-2x+3$ and $y\leq 3x-2$, respectively.







2 Solve the following problems with regards to the domain expressed by the inequality $x^2 + y^2 > 16$.

(1) Determine whether the 2 points $A(\sqrt{5}, \sqrt{10})$ and $B(3, 3)$ are points in the domain, respectively.

Point $A(\sqrt{5}, \sqrt{10})$, from $(\sqrt{5})^2 + (\sqrt{10})^2 = 5 + 10 = 15 < 16$, does not satisfy $x^2 + y^2 > 16$, so it is not a point in the domain.

Point $B(3, 3)$, from $3^2 + 3^2 = 9 + 9 = 18 > 16$, satisfies $x^2 + y^2 > 16$, so it is a point in the domain.

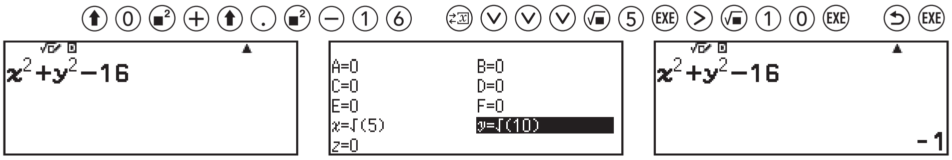
Point A is not in the domain, and point B is in the domain.

check

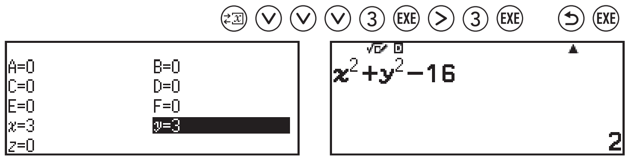
On the scientific calculator, use the VARIABLE function to check whether the 2 points A and B satisfy the inequality.

Press ⌂, select [Calculate], press OK

After inputting $x^2 + y^2 - 16$, input the coordinates for point A, and then calculate.



In the same way, input the coordinates for point B, and then calculate.



OTHER METHODS

On the scientific calculator, use the Table function to check a diagram of the circle.

Press ⌂, select [Table], press OK, then clear the previous data by pressing ⏻

Press ⋯, select [Define $f(x)/g(x)$], press OK, select [Define $f(x)$], press OK

After inputting $f(x) = \sqrt{16 - x^2}$, press EXE

In the same way, input $g(x) = -\sqrt{16 - x^2}$.

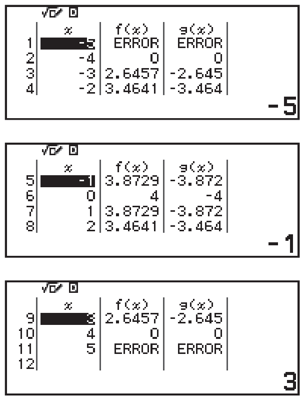
Press ⋯, select [Table Range], press OK

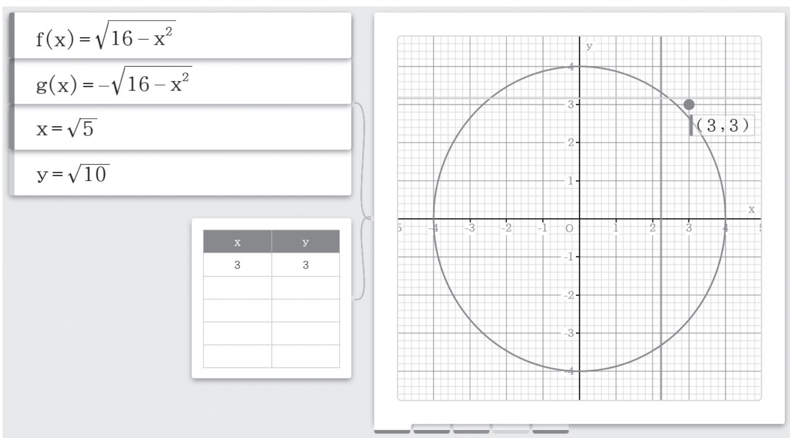
After inputting [Start:−5, End:5, Step:1], select [Execute], press EXE

Press ⇧ ⓧ, scan the QR code to display a graph.

Also, tap the function sheet, and input $x = \sqrt{5}$ and $y = \sqrt{10}$, respectively.

Finally, add a coordinates sheet, and input $(3, 3)$.

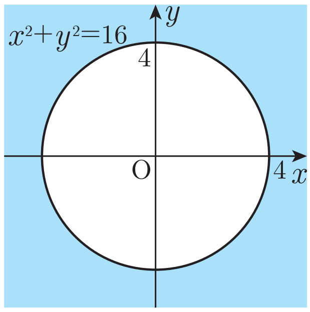




(2) Draw the domain. Also, determine whether the domain includes the boundary line.

The domain is the area outside the circle $x^2 + y^2 = 16$, which is the shaded area in the figure on the right.

However, the boundary line is not included.



Boundary line is not included.

[3] Solve the following problems with regards to the domain expressed by the inequality $x^2 + y^2 - 2x - 2y - 2 \leq 0$.

(1) Determine whether the 2 points $A(2, 2)$ and $B(3, 3)$ are points in the domain, respectively.

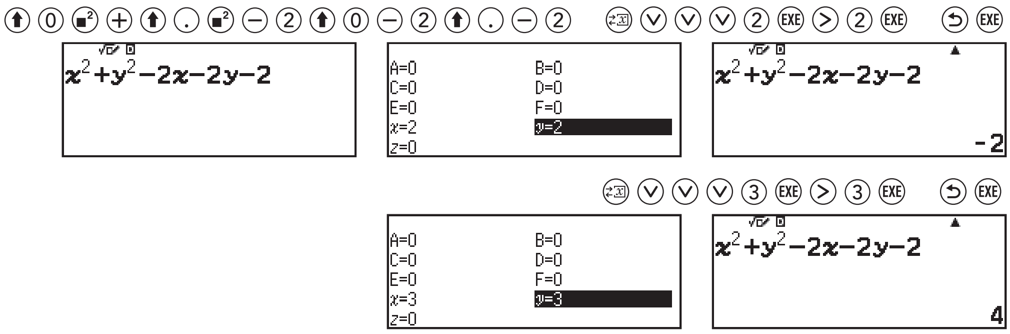
Point $A(2, 2)$, from $2^2 + 2^2 - 2 \cdot 2 - 2 \cdot 2 - 2 = -2 < 0$, satisfies $x^2 + y^2 - 2x - 2y - 2 \leq 0$, so it is a point in the domain.

Point $B(3, 3)$, from $3^2 + 3^2 - 2 \cdot 3 - 2 \cdot 3 - 2 = 4 > 0$, does not satisfy $x^2 + y^2 - 2x - 2y - 2 \leq 0$, so it is not a point in the domain.

Point A is in the domain, and point B is not in the domain.

check

Press ⓐ, select [Calculate], press OK



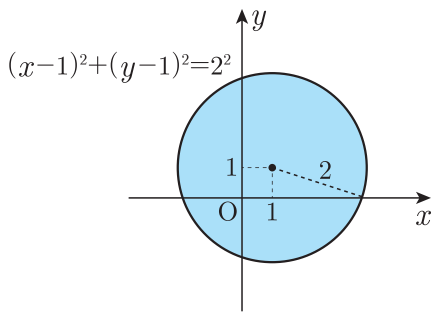
(2) Draw the domain. Also, determine whether the domain includes the boundary line.

Transform the left side of the inequality.

$$x^2 + y^2 - 2x - 2y - 2 = (x-1)^2 - 1 + (y-1)^2 - 1 - 2$$
$$= (x-1)^2 + (y-1)^2 - 2^2$$

Therefore, the domain is the area inside the circle $(x-1)^2 + (y-1)^2 = 2^2$, which is the shaded area in the figure on the right.

However, the boundary line is included.



Boundary line is included.

PRACTICE

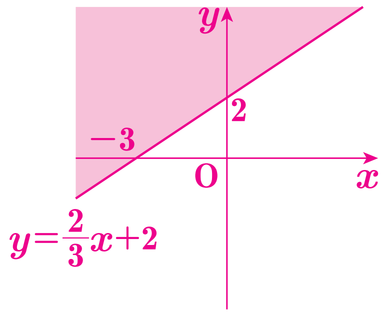
1 Draw the domains expressed by the following inequalities. Also, determine whether the domain includes the boundary line.

(1) Inequality $2x-3y+6\leq 0$

By transforming the equality, we get $y\geq \frac{2}{3}x+2$

Therefore, the domain expressed by this inequality is above the line $y=\frac{2}{3}x+2$, which is the shaded area in the figure on the right.

However, the boundary line is included.



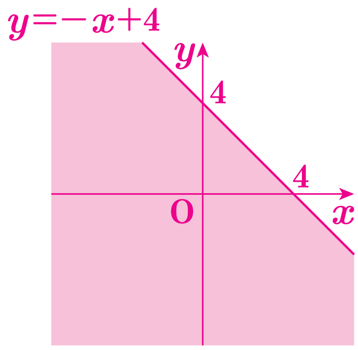
Boundary line is included.

(2) Inequality $x+y-4<0$

By transforming the inequality, we get $y<-x+4$

Therefore, the domain expressed by this inequality is below the line $y=-x+4$, which is the shaded area in the figure on the right.

However, the boundary line is not included.



Boundary line is not included.

check

Press (Home), select [Table], press (OK), then clear the previous data by pressing (Clear)

Press (Menu), select [Define f(x)/g(x)], press (OK), select [Define f(x)], press (OK)

After inputting $f(x)=\frac{2}{3}x+2$, press (EXE)

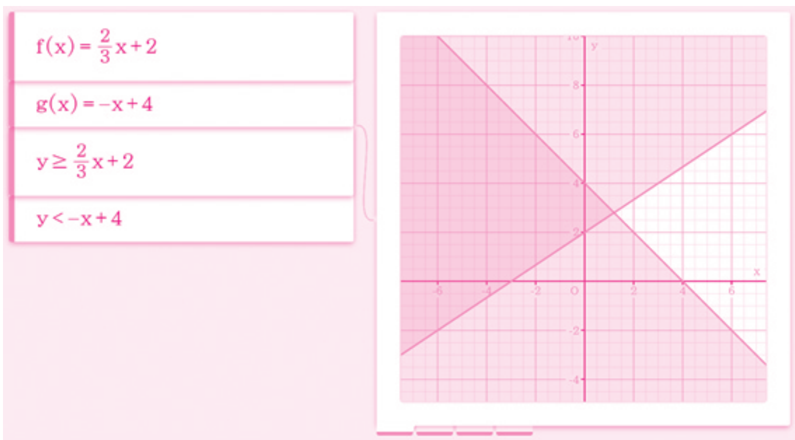
In the same way, input $g(x)=-x+4$.

Press (Menu), select [Table Range], press (OK)

After inputting [Start:−5, End:5, Step:1], select [Execute], press (EXE)

Press (Shift) (x), scan the QR code to display a graph.

Also, tap the function sheet, and input $y\geq \frac{2}{3}x+2$ and $y<-x+4$, respectively.



2 Solve the following problems with regards to the domain expressed by the inequality $x^2+y^2 \leq 36$.

(1) Determine whether the 2 points $A(3, 5)$ and $B(5, 4)$ are points in the domain, respectively.

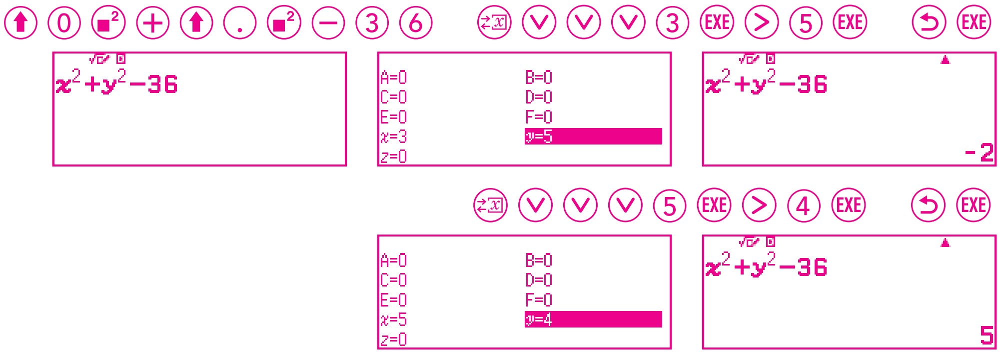
Point $A(3, 5)$, from $3^2+5^2=34<36$, satisfies $x^2+y^2 \leq 36$, so it is a point in the domain.

Point $B(5, 4)$, from $5^2+4^2=41>36$, does not satisfy $x^2+y^2 \leq 36$, so it is not a point in the domain.

Point A is in the domain, and point B is not in the domain.

check

Press ⌂, select [Calculate], press OK

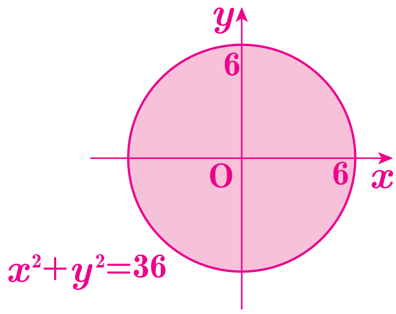


(2) Draw the domain. Also, determine whether the domain includes the boundary line.

The domain is the area inside the circle $x^2+y^2=36$, which is the shaded area in the figure on the right.

However, the boundary line is included.

Boundary line is included.

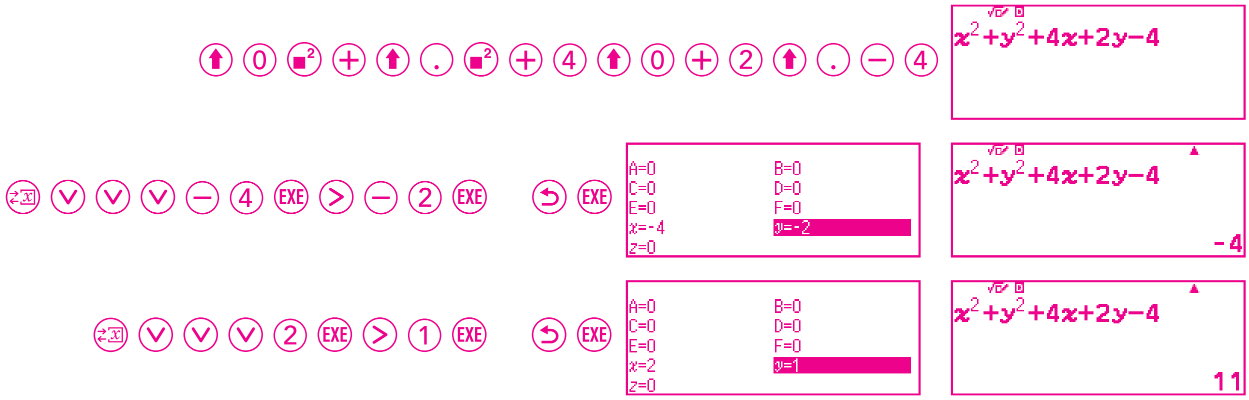


3 Solve the following problems with regards to the domain expressed by the inequality $x^2+y^2+4x+2y-4>0$.

(1) Determine whether the 2 points $\mathrm{A}(-4,-2)$ and $\mathrm{B}(2,1)$ are points in the domain, respectively.

Point $\mathrm{A}(-4,-2)$, from $(-4)^2+(-2)^2+4\cdot(-4)+2\cdot(-2)-4=-4<0$, does not satisfy $x^2+y^2+4x+2y-4>0$, so it is not a point in the domain.
Point $\mathrm{B}(2,1)$, from $2^2+1^2+4\cdot 2+2\cdot 1-4=11>0$, satisfies $x^2+y^2+4x+2y-4>0$, so it is a point in the domain.

Point A is not in the domain, and point B is in the domain.

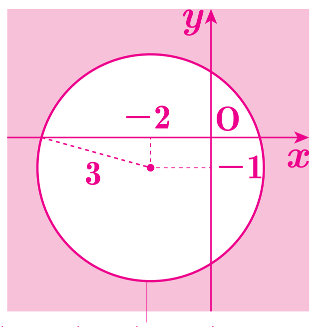


(2) Draw the domain. Also, determine whether the domain includes the boundary line.

Transform the left side of the inequality.

$$x^2+y^2+4x+2y-4$$
$$=(x+2)^2-4+(y+1)^2-1-4$$
$$=(x+2)^2+(y+1)^2-3^2$$

Therefore, the domain is the area outside the circle $(x+2)^2+(y+1)^2=3^2$, which is the shaded area in the figure on the right.
However, the boundary line is not included.



$(x+2)^2+(y+1)^2=3^2$

Boundary line is not included.

Domains expressed by inequalities (2)

TARGET To understand about domains expressed by simultaneous inequalities.

STUDY GUIDE

Domains expressed by simultaneous inequalities

A set of points that satisfy multiple inequalities simultaneously is called a domain expressed by simultaneous inequalities.

For the simultaneous inequalities $\begin{cases}\text{Inequality (i)}\\ \text{Inequality (ii)}\end{cases}$, if we let A be the domain expressed by the inequality (i) and B be the domain expressed by the inequality (ii), then the domain expressed by the simultaneous inequalities is as follows.

Common parts of A and B $A \cap B$

EXERCISE

◆ Draw the domains expressed by the following simultaneous inequalities. Also, determine whether the domain includes the boundary line.

(1) $\begin{cases} y > x-2 \\ y > -x+2 \end{cases}$

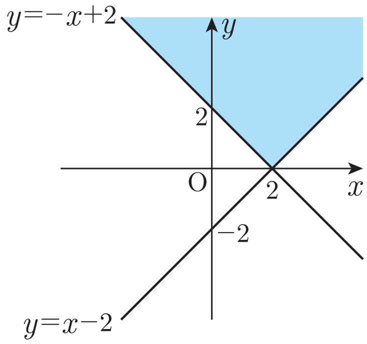
The domain expressed by the inequality $y>x-2$ is above the line $y=x-2$.

The domain expressed by the inequality $y>-x+2$ is above the line $y=-x+2$.

The common part of this is the shaded area in the figure on the right.

However, the boundary line is not included.

Boundary line is not included.



(2) $\begin{cases} x^2+y^2<9 \\ x-y+2>0 \end{cases}$

The domain expressed by the inequality $x^2+y^2<9$ is inside the circle $x^2+y^2=9$.

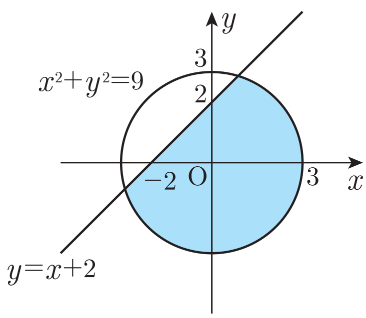
By transforming the inequality $x-y+2>0$, we get $y<x+2$

The domain expressed by this inequality is below the line $y=x+2$.

The common part of this is the shaded area in the figure on the right.

However, the boundary line is not included.

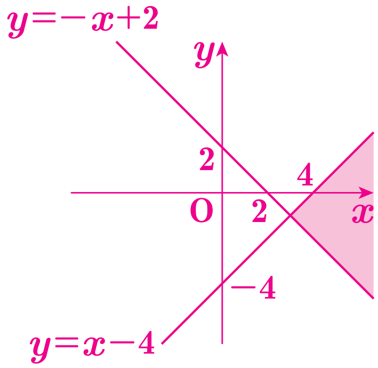
Boundary line is not included.



PRACTICE

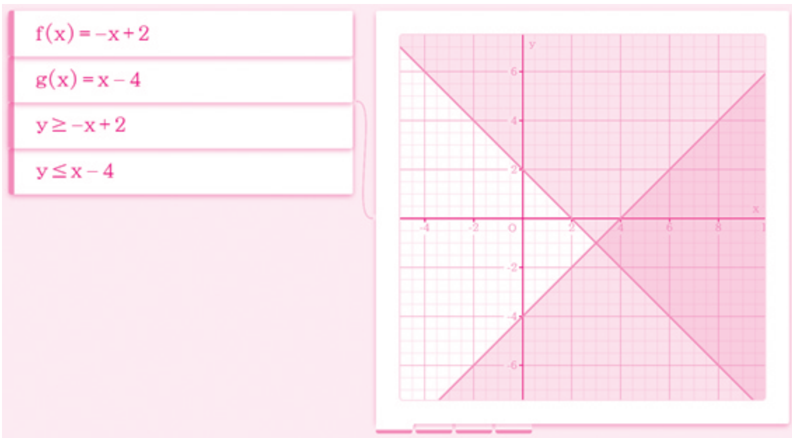
1 Draw the domains expressed by the following simultaneous inequalities. Also, determine whether the domain includes the boundary line.

(1) $\begin{cases} x+y-2\geq 0 \\ x-y-4\geq 0 \end{cases}$

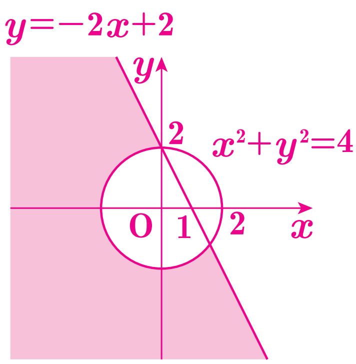


By transforming the inequality $x+y-2\geq 0$, we get
$y\geq -x+2$
The domain expressed by this inequality is above the line $y=-x+2$.
By transforming the inequality $x-y-4\geq 0$, we get
$y\leq x-4$
The domain expressed by this inequality is below the line $y=x-4$.
The common part of this is the shaded area in the figure on the right.
However, the boundary line is included.

Boundary line is included.



(2) $\begin{cases} x^2+y^2\geq 4 \\ 2x+y-2\leq 0 \end{cases}$



The domain expressed by the inequality $x^2+y^2\geq 4$ is outside the circle $x^2+y^2=4$.
By transforming the inequality $2x+y-2\leq 0$, we get
$y\leq -2x+2$
The domain expressed by this inequality is below the line $y=-2x+2$.
The common part of this is the shaded area in the figure on the right.
However, the boundary line is included.

Boundary line is included.

ADVANCED

2 Solve the following problems with regards to the simultaneous inequalities $\begin{cases} x-2y\leq 0 \\ x+2y-8\leq 0 \\ 3x-2y\geq 0 \end{cases}$.

(1) Draw the domains expressed by the simultaneous inequalities.

By transforming the inequality $x-2y\leq 0$, we get $y\geq \frac{1}{2}x$

The domain expressed by this inequality is above the line $y=\frac{1}{2}x$ ...(i).

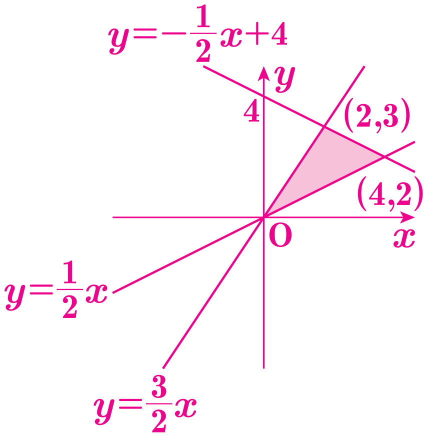
By transforming the inequality $x+2y-8\leq 0$, we get $y\leq -\frac{1}{2}x+4$

The domain expressed by this inequality is below the line $y=-\frac{1}{2}x+4$ ...(ii).

By transforming the inequality $3x-2y\geq 0$, we get $y\leq \frac{3}{2}x$

The domain expressed by this inequality is below the line $y=\frac{3}{2}x$ ...(iii).

The common part of this is the shaded area in the figure on the right.
However, the boundary line is included.
For the intersection points, we simultaneously solve (i) and (ii) to get (4, 2)
By simultaneously solving (ii) and (iii), we get (2, 3)
The intersection point of (i) and (iii) is the origin.



Boundary line is included.

(2) Find the area of the domain found in (1).

Since the intercept of the line (ii) is 4 and the intersection of the x axis is 8, we find the area is $\frac{1}{2}\cdot 8\cdot 4-\frac{1}{2}\cdot 4\cdot 2-\frac{1}{2}\cdot 8\cdot 2=4$

4

check

Press ⌂, select [Table], press OK, then clear the previous data by pressing ⏻

Press ⊙⊙⊙, select [Define f(x)/g(x)], press OK, select [Define f(x)], press OK

After inputting $f(x)=\frac{1}{2}x$, press EXE

In the same way, input $g(x)=-\frac{1}{2}x+4$.

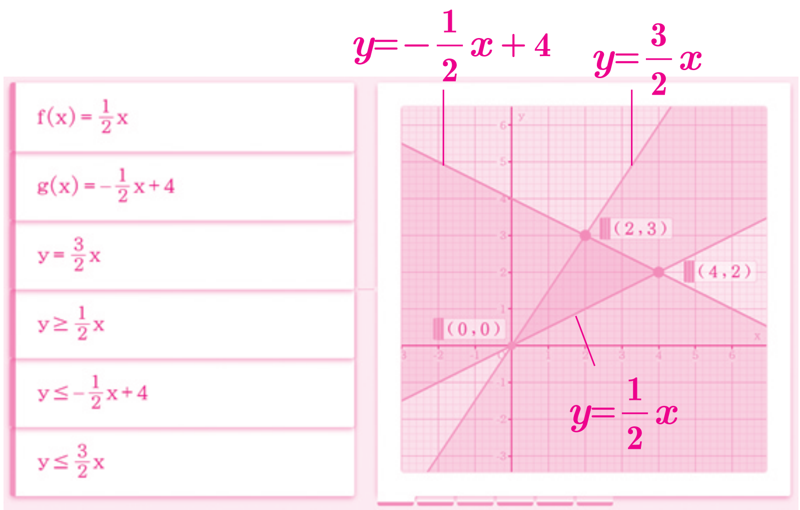
Press ⊙⊙⊙, select [Table Range], press OK

After inputting [Start:−5, End:5, Step:1], select [Execute], press EXE

Press ⇧ ⓧ, scan the QR code to display a graph.

Also, tap the function sheet, and input $y=\frac{3}{2}x$, $y\geq\frac{1}{2}x$, $y\leq-\frac{1}{2}x+4$, and $y\leq\frac{3}{2}x$, respectively.

In the same way, tap the graph sheet, then display the coordinates of the intersection of the lines.



Domains expressed by inequalities (3)

TARGET To understand about maximums/minimums in domains expressed by inequalities.

STUDY GUIDE

How to find maximum values and minimum values using domains

When a point (x, y) is in a domain that satisfies simultaneous inequalities, then we can find the maximum values and minimum values of the expression $P(x, y)$ given by x and y by using the domain as follows.

(1) Draw the domain D expressed by the simultaneous inequalities.

(2) Assume $P(x, y)=k$, then determine the range of possible values of k such that the graph of $P(x, y)=k$ and domain D have common points.

EXERCISE

◆ Given x and y satisfy the 4 inequalities $x\geq 0$, $y\geq 0$, $2x+y\leq 6$, and $x+2y\leq 6$, solve the following problems.

(1) Draw the domains that satisfy the 4 inequalities.

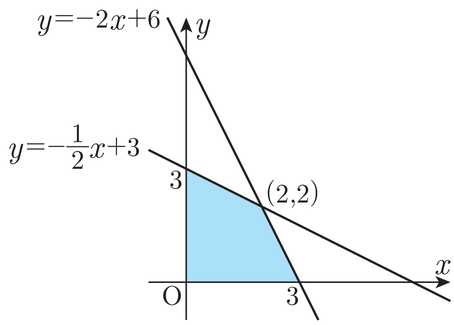
Let D be a domain that satisfies the 4 given inequalities.

By transforming the inequality $2x+y\leq 6$, we get $y\leq -2x+6$ …(i)

By transforming the inequality $x+2y\leq 6$, we get $y\leq -\frac{1}{2}x+3$ …(ii)

For the point of intersection of the boundary lines of (i) and (ii),

solve $\begin{cases} y=-2x+6 \\ y=-\frac{1}{2}x+3 \end{cases}$ to get $(x, y)=(2, 2)$



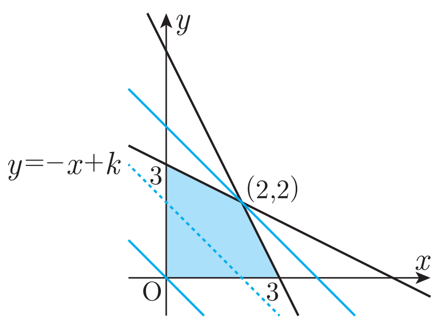
The domain D is the inside and perimeter of a quadrangle with 4 vertices at $(0, 0)$, $(0, 3)$, $(3, 0)$, and $(2, 2)$, which is the shaded area in the figure on the right.

(2) Find the maximum value and minimum value of $x+y$.

Assume $x+y=k$, then transform it such that $y=-x+k$ for a line with a slope of -1 and a y-intercept at k.

Find the maximum value and minimum value for k such that the line has common points with the domain D.

From the figure on the right, we can see that the value of k is maximum as it passes through point $(2, 2)$ and minimum as it passes through point $(0, 0)$.



Therefore, $x+y$ has a maximum value of $2+2=4$ when $x=2$ and $y=2$, and a minimum value of $0+0=0$ when $x=0$ and $y=0$.

When $x=2$ and $y=2$, the maximum value is 4, and when $x=0$ and $y=0$, the minimum value is 0

check

Press ⌂, select [Table], press OK, then clear the previous data by pressing ⏻

Press ⋯, select [Define f(x)/g(x)], press OK, select [Define f(x)], press OK

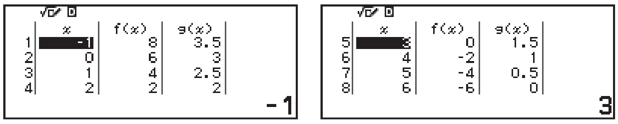
After inputting f(x)$=-2x+6$, press EXE

In the same way, input g(x)$=-\frac{1}{2}x+3$.

Press ⋯, select [Table Range], press OK

After inputting [Start:-1, End:6, Step:1], select [Execute], press EXE

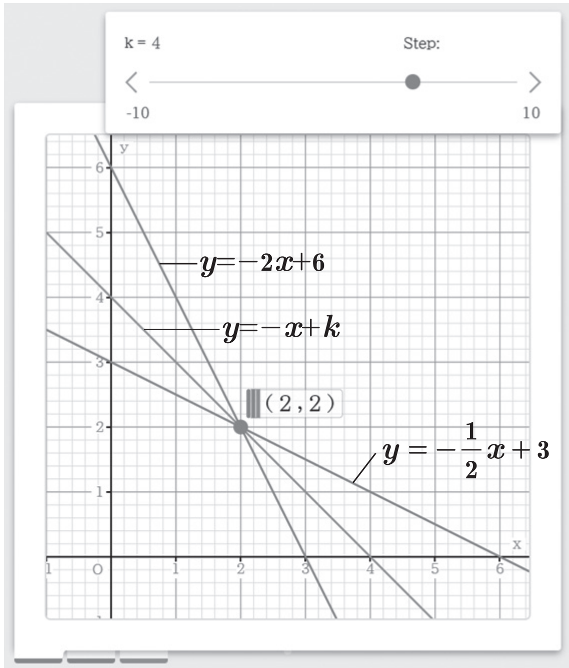
Press ⬆ x, scan the QR code to display a graph.



Also, tap the function sheet, and input $y=-x+k$.

In the same way, tap the graph sheet, then display the coordinates of the intersection of the lines.

By changing the y-intercept (variable k) of a line with a slope of -1 on the graph, we can understand how to approach the solution.



PRACTICE

1 Given that x and y satisfy the 4 inequalities $x\geq0$, $y\geq0$, $2x+y\leq7$, and $x+2y\leq8$, solve the following problems.

(1) Draw the domains that satisfy the 4 inequalities.

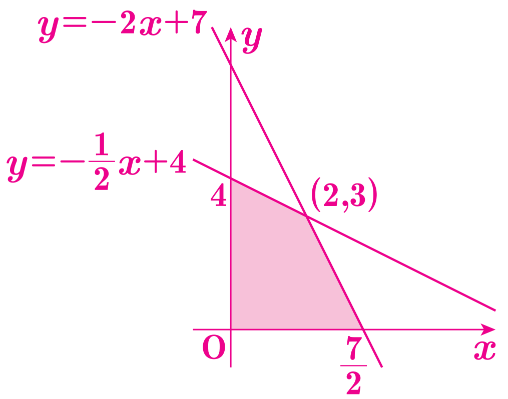
Let D be a domain that satisfies the 4 given inequalities.

By transforming the inequality $2x+y\leq7$,

we get $y\leq-2x+7$ ...(i)

By transforming the inequality $x+2y\leq8$,

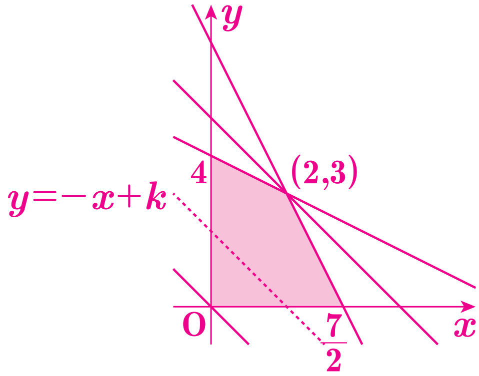
we get $y\leq-\frac{1}{2}x+4$ ...(ii)



For the point of intersection of the boundary lines of (i) and (ii), solve $\begin{cases} y=-2x+7 \\ y=-\frac{1}{2}x+4 \end{cases}$ **to get $(x, y)=(2, 3)$**

The domain D is the inside and perimeter of a quadrangle with 4 vertices at $(0, 0)$, $(0, 4)$, $\left(\frac{7}{2}, 0\right)$, and $(2, 3)$, which is the shaded area in the figure above.

(2) Find the maximum value and minimum value of $x+y$.

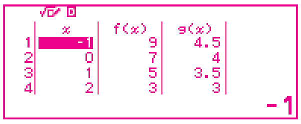


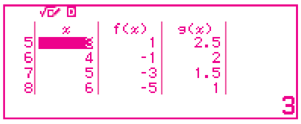
Assume $x+y=k$, then transform it such that $y=-x+k$ for a line with a slope of -1 and a y-intercept at k.
Find the maximum value and minimum value for k such that the line has common points with the domain D.
From the figure on the right, we can see that the value of k is maximum as it passes through point $(2, 3)$ and minimum as it passes through point $(0, 0)$.
Therefore, $x+y$ has a maximum value of $2+3=5$ when $x=2$ and $y=3$, and a minimum value of $0+0=0$ when $x=0$ and $y=0$.

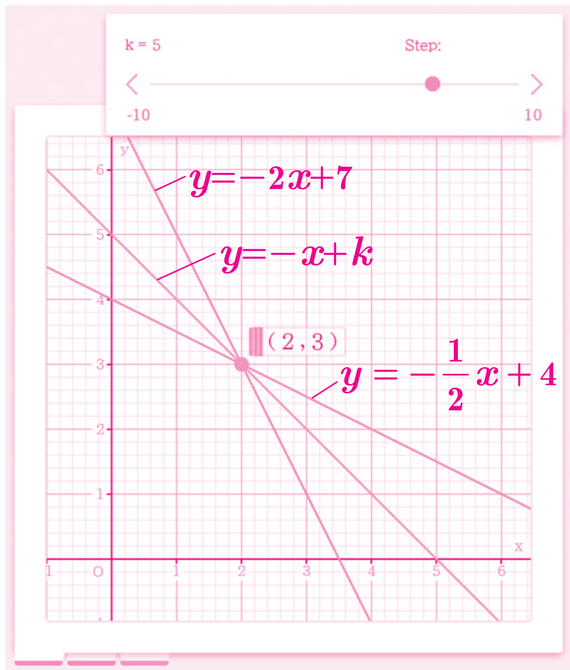
When $x=2$ and $y=3$, the maximum value is 5, and when $x=0$ and $y=0$, the minimum value is 0

check

Press (⌂), select [Table], press (OK), then clear the previous data by pressing (⏻)
Press (⋯), select [Define f(x)/g(x)], press (OK), select [Define f(x)], press (OK)
After inputting f(x)$=-2x+7$, press (EXE)
In the same way, input g(x)$=-\frac{1}{2}x+4$.
Press (⋯), select [Table Range], press (OK)
After inputting [Start:-1, End:6, Step:1], select [Execute], press (EXE)
Press (⇧) (x), scan the QR code to display a graph.







Also, tap the function sheet, and input $y=-x+k$.
In the same way, tap the graph sheet, then display the coordinates of the intersection of the lines.

ADVANCED

2 A manufacturer has 2 types of products, A and B. To produce 1t, A requires 60kWh of electricity and $2m^3$ of gas, and B requires 40kWh of electricity and $6m^3$ of gas. The profit per 1t is \$20,000 for A \$30,000 for B. Given that this manufacturer receives up to 2200kWh of electric power and $120m^3$ of gas each day, then how many t each of A and B should be produced per day to maximize total profits for products produced in 1 day. Also, find the total profit at that time.

	Electric power (kWh)	Gas (m^3)	Profit (\$10,000)
Product A	60	2	2
Product B	40	6	3
Maximum	2200	120	

Let production volume for 1 day be xt for A and yt for B.

Production volume is positive, so $x \geq 0$ and $y \geq 0$ …(i)

From the relation of electric power supply for 1 day, we get $60x+40y \leq 2200$ …(ii)

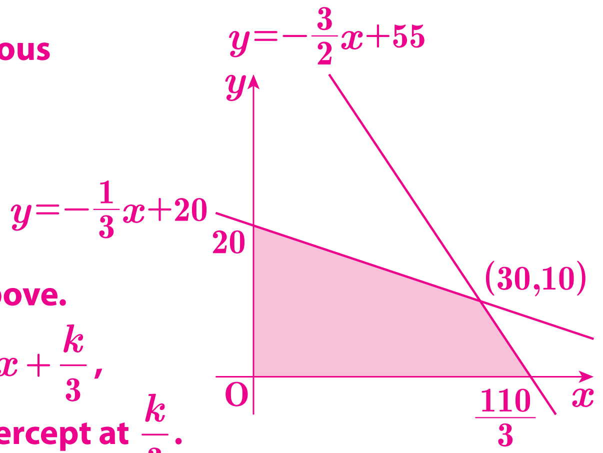
From the relation of gas supply for 1 day, we get $2x+6y \leq 120$ …(iii)

Therefore, find values for x and y that maximize total profits $k=2x+3y$ under the conditions stated in (i), (ii), and (iii).

By transforming (ii), we get $y \leq -\frac{3}{2}x+55$ …(iv)

By transforming (iii), we get $y \leq -\frac{1}{3}x+20$ …(v)

For the point the boundary lines of (iv) and (v) intersect, solve $\begin{cases} 3x+2y=110 \\ x+3y=60 \end{cases}$ to get $(x, y)=(30, 10)$



The domain expressed by the simultaneous inequalities is the inside and perimeter of a quadrangle with 4 vertices at $(0, 0)$, $(0, 20)$, $\left(\frac{110}{3}, 0\right)$, and $(30, 10)$, which is the shaded area in the figure above.

$k=2x+3y$ was transformed to $y=-\frac{2}{3}x+\frac{k}{3}$, which is a line with slope $-\frac{2}{3}$ and y-intercept at $\frac{k}{3}$.

Thus, the value of k reaches a maximum when it passes through the point (30, 10).

Therefore, $2x+3y$ reaches a maximum value when $x=30$ and $y=10$, which is $2 \cdot 30+3 \cdot 10=90$.

The maximum total profit reaches \$900,000 when 30t of A and 10t of B are produced.

Domains expressed by inequalities (4)

TARGET To understand about proofs by using domains expressed by inequalities.

STUDY GUIDE

Proofs using domains

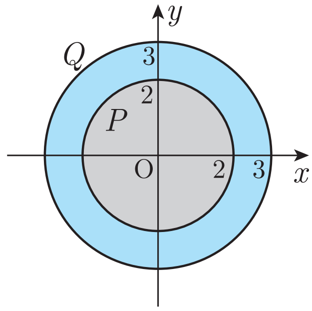
For 2 conditions p and q, assuming that all of P satisfy the condition p, and all of Q satisfy the condition q, then the following holds.

If p then q $\Leftrightarrow$ $P \subset Q$

Ex. If $x^2+y^2<4$ then $x^2+y^2<9$

If we can show that $P \subset Q$ holds for a set P that satisfies condition p: $x^2+y^2<4$ and a set Q satisfies condition q: $x^2+y^2<9$, then we can prove that if $x^2+y^2<4$ then $x^2+y^2<9$.

Because of this, we simply use a diagram to show that the domain of the set P is included in the domain of the set Q.



EXERCISE

◆ Prove that if $x^2+y^2 \leq 1$ then $x+y \leq 2$.

[Proof]

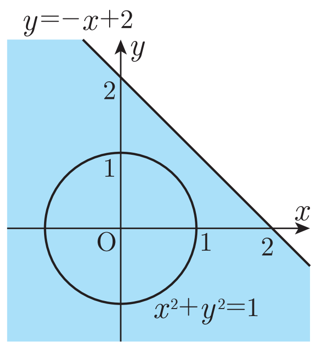
Assume a domain P expressed by the inequality $x^2+y^2 \leq 1$ and a domain Q expressed by the inequality $x+y \leq 2$.

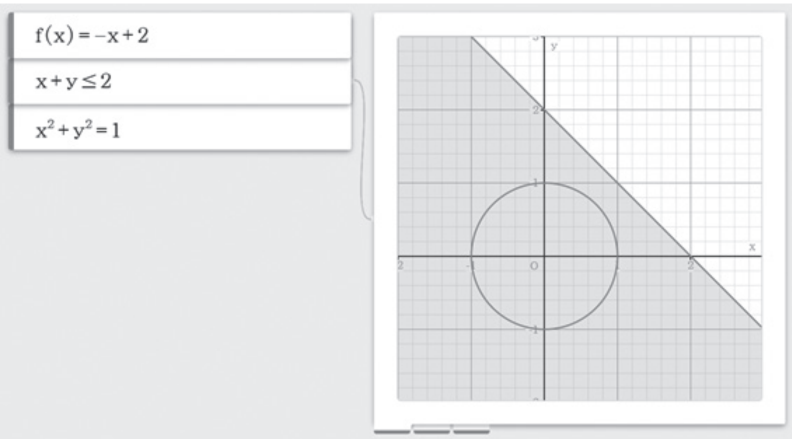
As shown in the figure on the right, P is the inside and perimeter of the circle $x^2+y^2=1$, and Q is on and below the line $y=-x+2$.

Also, $x^2+y^2=1$ and $y=-x+2$ have no common points.

Thus, $P \subset Q$ holds.

Therefore, if $x^2+y^2 \leq 1$ then $x+y \leq 2$ holds.





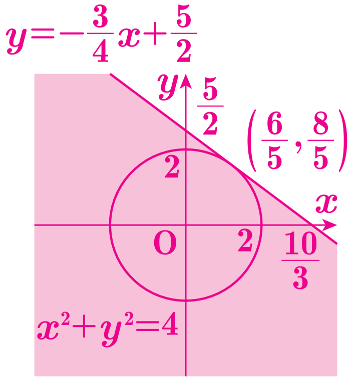
PRACTICE

Prove that if $x^2+y^2\leq 4$ then $3x+4y\leq 10$.

[Proof]

Assume a domain P expressed by the inequality $x^2+y^2\leq 4$ and a domain Q expressed by the inequality $3x+4y\leq 10$.

As shown in the figure on the right, P is the inside and perimeter of the circle $x^2+y^2=4$.

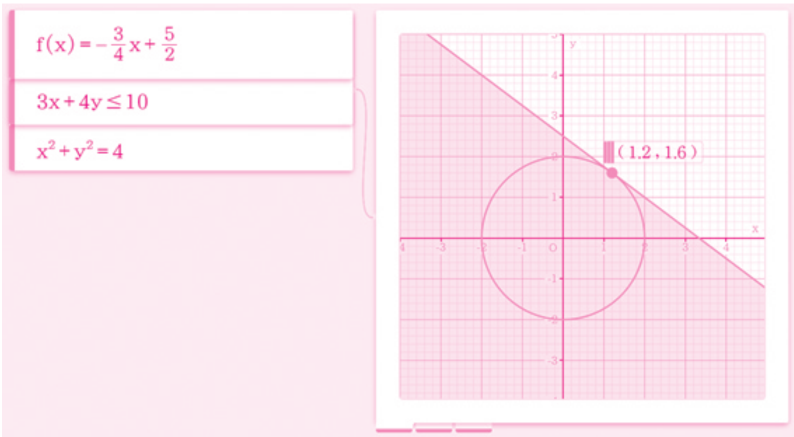


Q is on and below the line $y=-\frac{3}{4}x+\frac{5}{2}$.

Also, $x^2+y^2=4$ and $y=-\frac{3}{4}x+\frac{5}{2}$ are tangent at 1 point.

Thus, $P\subset Q$ holds.

Therefore, if $x^2+y^2\leq 4$ then $3x+4y\leq 10$ holds.



Domains expressed by inequalities (5)

TARGET To understand about domains bounded by parabolas.

STUDY GUIDE

Parabolas and domains

If we let the parabola $y=f(x)$ be C, then for the domain expressed by the inequalities $y>f(x)$ and $y<f(x)$, we get the following.

(i) Domain expressed by the inequality $y>f(x)$
$\rightarrow$ Above parabola C
(ii) Domain expressed by the inequality $y<f(x)$
$\rightarrow$ Below parabola C

In both domains (i) and (ii), the boundary line is a parabola C, and when we include the equality sign as in $y\geq f(x)$, then we also include the points on the parabola C.

proof

Use the scientific calculator to check the domain P of $y\geq x^2$ and the domain Q of $y\leq x^2$ (both of which include the boundary line).

Press ⌂, select [Table], press OK, then clear the previous data by pressing ⏻

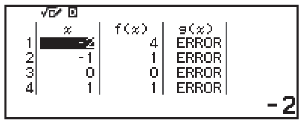
Press ⋯, select [Define f(x)/g(x)], press OK, select [Define f(x)], press OK

After inputting f(x)$=x^2$, press EXE

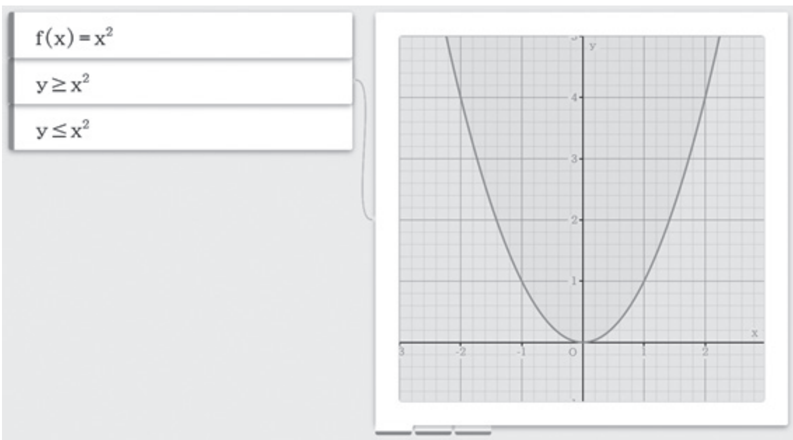
Press ⋯, select [Table Range], press OK

After inputting [Start:-2, End:1, Step:1], select [Execute], press EXE

Press ⇧ x, scan the QR code to display a graph.



Also, tap the function sheet, and input $y\geq x^2$ and $y\leq x^2$.



EXERCISE

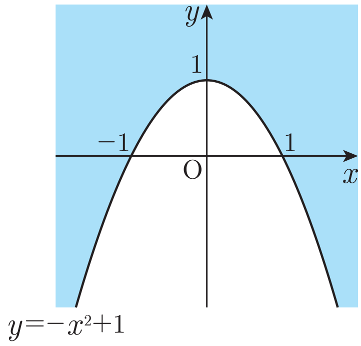
◆ Draw the domain expressed by the inequality $y > -x^2+1$. Also, determine whether the domain includes the boundary line.

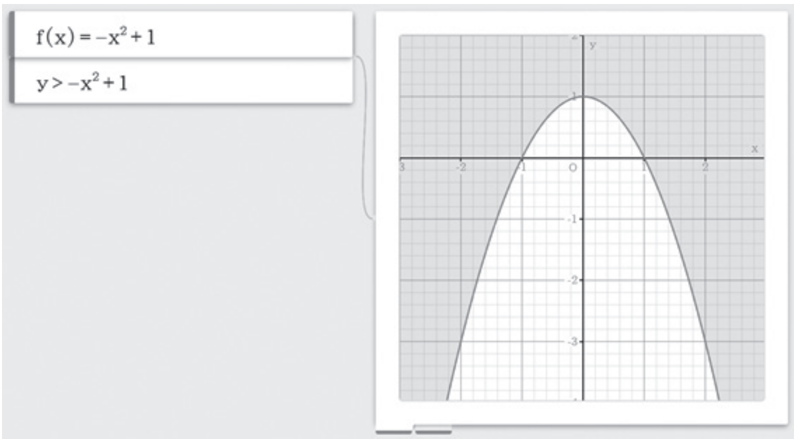
The boundary line $y=-x^2+1$ expresses a parabola that is upward convex with a vertex at $(0, 1)$ and an axis at $x=0$.

The domain expressed by this inequality is above the parabola $y=-x^2+1$, which is the shaded area in the figure on the right.

However, the boundary line is not included.

Boundary line is not included.





PRACTICE

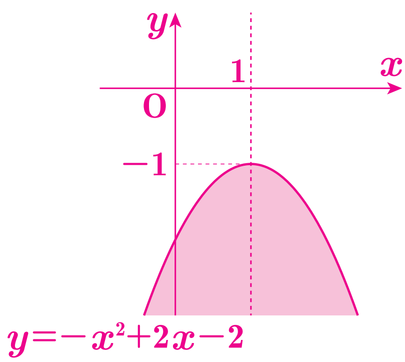
◆ Draw the domain expressed by the inequality $y \leq -x^2+2x-2$. Also, determine whether the domain includes the boundary line.

The boundary line $y=-x^2+2x-2=-(x-1)^2-1$ expresses a parabola that is upward convex with a vertex at $(1, -1)$ and an axis at $x=1$.

The domain expressed by this inequality is below the parabola $y=-(x-1)^2-1$, which is the shaded area in the figure on the right.

However, the boundary line is included.

Boundary line is included.



CASIO Essential Materials

Publisher: CASIO Institute for Educational Development

Date of Publication: 2023/12/22 (1st edition)

https://edu.casio.com