Trigonometric Functions

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LEVEL
High school when students studying trigonometry functions.

OBJECTIVES
(i) To understand how to use of the amplitude and period of a function such as *f(x)=a*sin(bx+c)+d* to find the maximum or minimum of a function.
(ii) To understand the use of trigonometric identities.

Corresponding eActivity
T0101.g1e

OVERVIEW
Trigonometric identities occur frequently and students need to memorize them. In this note, we show some examples why trigonometric identities are important in some cases.

EXPLORATORY ACTIVITIES
[Note]
We shall use small letter *x* instead of capital X as shown on the calculator throughout the paper.

Remark: We assume readers already know the following facts:

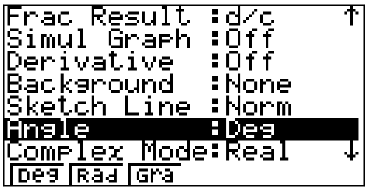
(i) The minimum period for y = sin(ax) is $\frac{2\pi}{a}$, where a is a positive integer.

(ii) The graph of *y*= $A\sin(a(x-b)+c$ has amplitude *A*, and its graph is being shifted horizontally to the right *b* unit(s) and up *c* unit(s) from that of *y =sin(x)* respectively. But *y*= $A\sin(a(x-b)+c$ will still have minimum period $\frac{2\pi}{a}$.

(iii) If a trigonometric function *f* has a period *L*, horizontal or veridical shifting from *f* will have the same period *L*.

Activity 1: Find the minimum period for the function of *f(x)* =sin 2x + cos 4x.
Solution:
We note that this problem can be solved by manipulating trigonometric identities, which we show here: $\sin 2x + \cos 4x = \sin 2x + (1 - 2\sin^2 2x) = -2(\sin 2x - \frac{1}{4})^2 + \frac{9}{8}$, we conclude that the minimum period for this function is $\frac{2\pi}{2} = \pi$. Next we show how technology can be handy to explore this type of problem if one forgot a trig identify.

A. Tabular Approach (open **act1-table 1)** We may explore the answer by tabulating the function as follows. We use the degree mode instead of radian mode for the angles. [Press (SHIFT MENU and change the angle from radian to degree], we show the screen shot below:



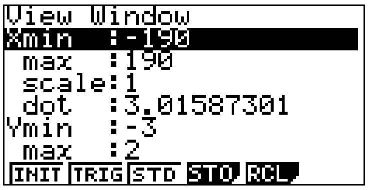
We set the angle value between X=0 and X=190 with step size=10. Explore the table using ▲▼◀▶ and we shall see a table of values as shown below:

Table 1

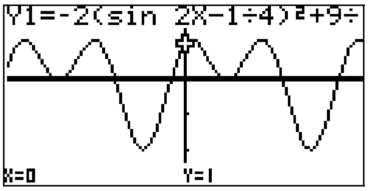
X	**F(X)**
0	**1**
10	**1.10806**
20	0.81644
30	0.36603
40	0.04512
50	0.04512
60	0.36603
70	0.81644
80	1.10806
90	1
100	0.42402
110	-0.46914
120	-1.136603
130	-1.92450
140	-1.92450
150	-1.36603
160	-0.46914
170	0.42402
180	**1**
190	**1.10806**

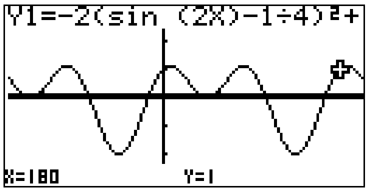
We notice that the values of the function repeat every 180 degrees.

B. Graphical Approach (open **act1-graph 1)** Again we use the degree mode ([SHIFT] [MENU]). By pressing [F1] to trace the graph. With the following V-Window:



We obtain the following screen dumps to reconfirm the period of this function to be 180 degrees or π .



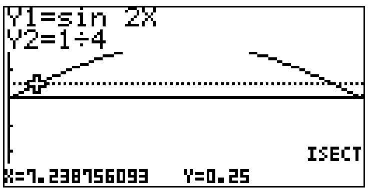


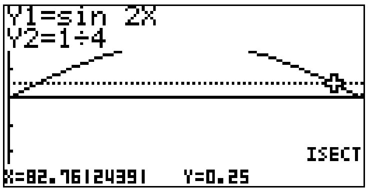
Graphs of y = $\sin 2x + \cos 4x$

Activity 2: For the function above, f(x) =sin 2x + cos 4x, (a) find an x in which *f(x)* achieves the maximum when x is in [0, π/2], (b) find its maximum value.
Solution:

(a) Since *f(x)* = $\sin 2x + \cos 4x = -2(\sin 2x - \frac{1}{4})^2 + \frac{9}{8}$ as discussed in **Activity 1**, and we observe from the Table 1 above that there are two maximum values near x=10 and x=80. In addition, since f(x) = $-2(\sin 2x - \frac{1}{4})^2 + \frac{9}{8}$, it is a quadratic equation in sin 2x. In other words, f(x) has a maximum when sin 2x = 1/4.

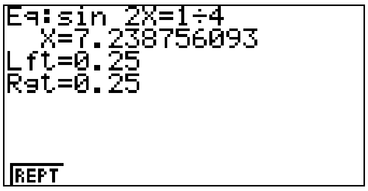
<u>Graphical Approach</u>: Open **act2. graph1**: We observe from the following screen shots that we have two intersections for sin 2x =1/4 (After plotting the graphs, press [F5] and ISCT (for intersections)).



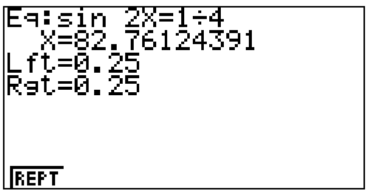


Analytical approach: To solve sin(2x)= $\frac{1}{4}$, we have x= $\frac{\sin^{-1}\left(\frac{1}{4}\right)}{2}$. However, this does not say immediately that x takes two different values. This demonstrates why graphics calculator is beneficial to learners in this case. Next we use fx-9860 to solve sin2x= $\frac{1}{4}$ or x= $\frac{\sin^{-1}\left(\frac{1}{4}\right)}{2}$ below.

Algebraic approach: Open **act2. solver:** We again use the '**degree mode**'. Type sin(2x)= $\frac{1}{4}$ and choose initial value X=0. After solving the equation, the calculator displays the solution, in this case *We get X to be about **7.2388** degrees.*, and also the difference between the left and right sides of the equation when *x* has the value shown as the solution. If this value is zero, or very close to zero, this will confirm that a good approximation to the solution has been found. We show its screen shot below:



If we use the guessing value for X to be 90, we get **the second desired answer, which is about X=82.76 degrees**, which we show below:



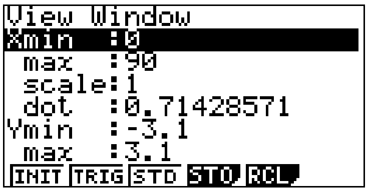
(b) The maximum value for $\sin 2x + \cos 4x = -2(\sin 2x - \frac{1}{4})^2 + \frac{9}{8}$ is $\frac{9}{8}$.

Remark: Alternatively, if student understands the concept of derivative, we can use the derivative function of *f* to analyze the local max for *f* as follows:

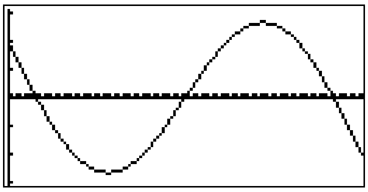
Step 1. We find the derivative for $\sin 2x + \cos 4x$ by hand, which gives

$$f'(x) = 2\cos 2x - 4\sin 4x$$

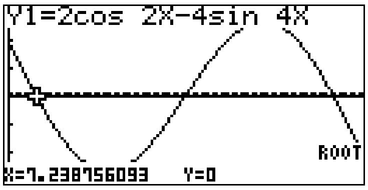
Step 2. We plot *y=f′(x)* below. Open **act2. graph 2:** Use the 'degree mode' and use the following V-Window:

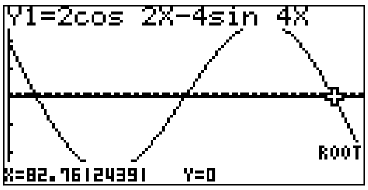


The graph is shown below:



Step 3. By tracing the graph F1, we note that there are two places where the derivative changes from positive to negative, one is between X=0 and X=10 and the other is around X=80 and X=90. These are the places where *f* achieves its maximum. We use F5[G-Solv] command to find the roots of the function. We confirm the desirable answers are about **7.2388** and **82.76**, which we show below:





EXERCISES

Exercise 1. If *y* = sin 2x + cos 2x. (a) Find the minimum period for the function. (b) Find the maximum value of y. (c) Use calculator and methods described in this note to verify your answers for (a) and (b).

Exercise 2. If $y = \sin(x - \frac{\pi}{6})\cos x$. (a) Find the minimum period for *y*. (b) Find the minimum value for *y*. (c) Use calculator and methods described in this note to verify your answers for (a) and (b).

SOLUTIONS

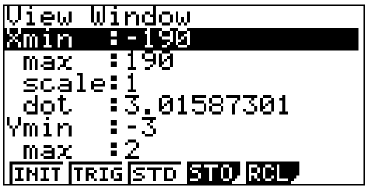
Exercise 1

(a) and (b): Since $\sin 2x + \cos 2x = \sqrt{2}\sin(2x + \frac{\pi}{4})$, the minimum period is $\frac{2\pi}{2} = \pi$. And the maximum value is $\sqrt{2}$.

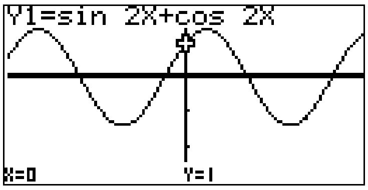
(c)

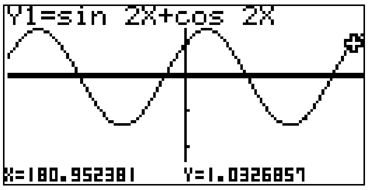
A. Tabular Approach (Open **ex1-table)** We explore the answer by tabulating the function as follows. We use the degree mode instead of radian mode for the angles and set the range for X from 0 to 200 with step=10. We confirm the period of this function to be 180 degrees or π.

B. Graphical Approach (open **ex1-graph)** Again we use the degree mode. ([SHIFT] [MENU]) By pressing [SHIFT] [F1] to trace the graph. With the following V-Window:



We obtain the following graphs to reconfirm the period of this function to be 180 degrees or π.



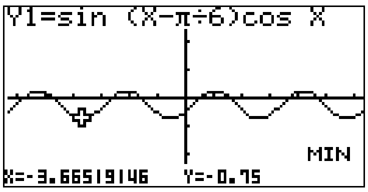


Exercise2.
(a) and (b): By using the identity

$$\sin x \cdot \cos y = \frac{1}{2}\left[(\sin(x+y) + \sin(x-y)\right].$$

We write $y = \sin(x - \frac{\pi}{6})\cos x = \frac{1}{2}\left[\sin(2x - \frac{\pi}{6}) - \sin\frac{\pi}{6}\right] = \frac{1}{2}\sin(2x - \frac{\pi}{6}) - \frac{1}{4}$. Therefore, the minimum period is π and the minimum value for *y* is $-\frac{1}{2} - \frac{1}{4} = -\frac{3}{4}$.

(c) Open **ex2. graph1**: (We use radian mode here) By plotting the graphs of Y1= $\sin(x - \frac{\pi}{6})\cos x$ and Y2= $\frac{1}{2}\sin(2x - \frac{\pi}{6}) - \frac{1}{4}$ together, we see that these two are identical. Next by selecting either one of them and press [F5] and MIN after plotting, we find the minimum value is indeed -3/4 or -0.75 as shown in the screen shot below:



Open **ex2. graph2**: By plotting Y1=sin 2x and Y2= $\sin(2x-\frac{\pi}{6})$ together, we conjecture that these two have the same period, which is π. Indeed, this is the case because

$$\sin(2x-\frac{\pi}{6})=\sin(2(x-\frac{\pi}{12})),$$

and y=sin(2x) and $y=\sin(2(x-\frac{\pi}{12})$ is only a horizontal shifting away from each other (which will not change their respective periods). We open **ex2. graph3**, if the period of y=sin(2x) is π, then same as $y=\sin(2(x-\frac{\pi}{12})$.

By plotting Y2= $\sin(2x-\frac{\pi}{6})$ and Y3= $\frac{1}{2}\sin(2x-\frac{\pi}{6})$ together, we see that Y2 and Y3 have the same period, this is because the graph of $y=\frac{1}{2}\sin(2x-\frac{\pi}{6})$ changes only the amplitude from 1 to 1/2. Consequently, the minimum for $y=\frac{1}{2}\sin(2x-\frac{\pi}{6})$ is -1/2 and thus the minimum for $y=\frac{1}{2}\sin(2x-\frac{\pi}{6})-\frac{1}{4}=-\frac{1}{2}-\frac{1}{4}=-\frac{3}{4}$.

Remark: The Exercise 2 above gives us a reason to learn trigonometric identity such as

$$\sin x\cdot\cos y=\frac{1}{2}\left[(\sin(x+y)+\sin(x-y)\right].$$

One will agree it is easier to find the period and its extremum for $y=\frac{1}{2}\sin(2x-\frac{\pi}{6})-\frac{1}{4}$ than those of $y=\sin(x-\frac{\pi}{6})\cos x$.

