



Solving System of Linear Equations

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LEVEL

High schools after students have studied system of linear equations and matrices.

OBJECTIVES

To discuss solving system of linear equations using matrix algebra with a graphics calculator.

CORRESPONDING eActivity

SIMUL.g1e

OVERVIEW

In this exploration we discuss solving system of linear equations using the inverse matrix method and the Gauss-Jordan method. We also explore on using the EQUA function in the graphics calculator like a "black box" to verify the solution obtained. The graphics calculator is used to facilitate the exploration and row operations.

EXPLORATORY ACTIVITIES

[Note]

We shall use small letter x , y , z instead of capital X , Y , Z as shown on the calculator throughout the paper.

Activity 1: Suppose we are given three linear equations as follow:

$$3x - 4y + 2z = 12, \quad x + 2y + 4.6z = 1, \quad -2x + 3.5y + z = 5.$$

Solve this linear system using inverse matrix method and verify the solution obtained.

Solution:

It is always useful to arrange the equations accordingly as follow.

Solving System of Linear Equations

$$\begin{aligned} 3x - 4y + 2z &= 12 \\ x + 2y + 4.6 &= 1 \\ -2x + 3.5y + z &= 5 \end{aligned}$$

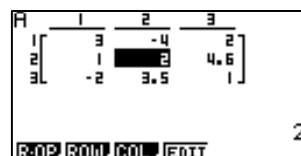
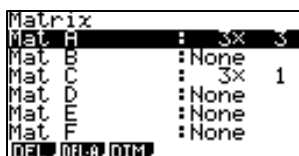
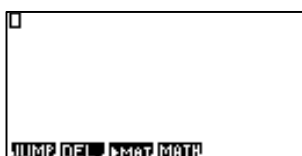
Now we can easily identify the coefficient matrix as $A = \begin{bmatrix} 3 & -4 & 2 \\ 1 & 2 & 4.6 \\ -2 & 3.5 & 1 \end{bmatrix}$ and the constants

matrix is $C = \begin{bmatrix} 12 \\ 1 \\ 5 \end{bmatrix}$.

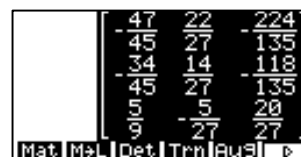
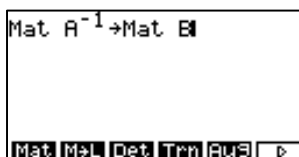
Now open the eActivity SIMUL.g1e.

A. Inverse Matrix Method

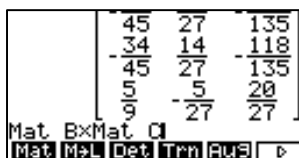
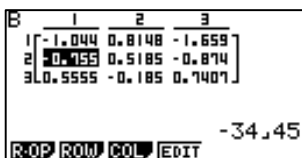
(a) Open the Run strip "**Act1A**". I have defined Mat A as coefficient matrix A and Mat C as the constant matrix C. You should explore these two matrices by tapping on **F3**.



(b) Tap **EXIT** to return to Run window, then tap **OPTN** **F2** to access the matrix functions. Determine the inverse matrix by calculating Mat A^{-1} as shown in middle display below. The calculation is made more efficient by assigning the output to the matrix Mat B.



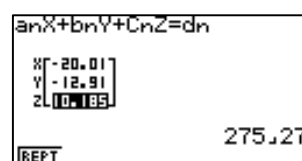
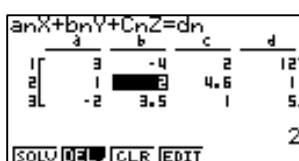
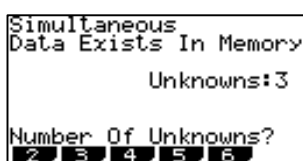
(c) Explore the inverse matrix Mat B obtained above, using **F-D** to toggle between decimal and fraction form. Then, find solution to the linear system by calculating $\text{Mat B} \times \text{Mat C}$.



Solving System of Linear Equations

The 3×1 matrix produced, $\begin{bmatrix} -\frac{2702}{135} \\ -\frac{1744}{135} \\ \frac{275}{27} \end{bmatrix}$ is the unique solution set to the simultaneous linear equations.

(d) We should always verify solution obtained whenever possible. Return to the eActivity and open SIMUL strip "Act1B" and tap on $\boxed{F2}$ (3) to view the coefficients and constant which are already keyed in. Press \boxed{EXE} to solve the system of linear equations.



Note that answer produced from this 'black box' is equivalent to the solution set.

Discussion: Inverse Matrix Method

A linear system can be represented in matrix algebra as $AX = C$ where A , C are the coefficient matrix and constant matrix respectively, and X is the variable matrix. In this

case we have $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. If the inverse of A , denote as A^{-1} exists, then the solution to the

simultaneous equations can be produced by multiplying both sides with A^{-1} as follow:

$$AX=C \rightarrow A^{-1}AX=A^{-1}C \rightarrow (A^{-1}A)X=A^{-1}C \rightarrow X=A^{-1}C$$

The inverse matrix method is much more efficient than substitution method when working with a linear system of three or more variables. You can find the inverse matrix by hand and verify your answer using the calculator.

Verifying the solution set using the SIMUL strip "Act1B" confirmed the solution set. Note that the Simultaneous "black box" can only solve linear system of up to 6 variables. \square

EXERCISES A

Exercise 1

Suppose we have the following linear system:

$$\begin{aligned} 3x - 2y + 4z &= 6 \\ -1.5x + y - 2z &= 11 \\ 9x - 6y + 12z &= -3.5 \end{aligned}$$

State the coefficient matrix and constant matrix. Find the inverse of the coefficient matrix, if exist, then use the inverse matrix method to solve this system of linear equations.

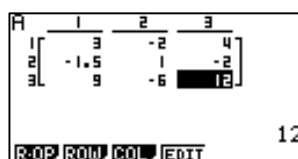
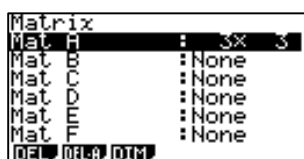
Solving System of Linear Equations

SOLUTIONS to EXERCISES A

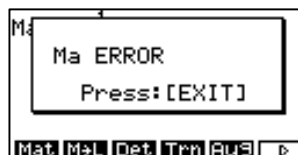
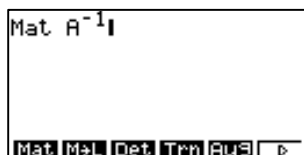
Exercise 1

Obviously the coefficient matrix is $\begin{bmatrix} 3 & -2 & 4 \\ -1.5 & 1 & -2 \\ 9 & -6 & 12 \end{bmatrix}$ and constant matrix is $\begin{bmatrix} 6 \\ 11 \\ -3.5 \end{bmatrix}$.

We now open the RUN strip "Ex1" and define Mat A as coefficient matrix.



Having done that, we could find the inverse of coefficient matrix by calculating Mat A⁻¹.



The reason the error message appears is because the coefficient matrix $\begin{bmatrix} 3 & -2 & 4 \\ -1.5 & 1 & -2 \\ 9 & -6 & 12 \end{bmatrix}$

is not invertible. This means the system of linear equations given has either infinitely many solutions (dependent) or has no solution (inconsistent) at all. Alternatively, we can also check the invertibility of the coefficient matrix by finding its determinant.



If the determinant is 0 then the coefficient matrix is not invertible and this follows that the linear system has no unique solution. □

Note in the exercise above the inverse matrix method does not tell us whether the linear system is dependent or inconsistent because the determinant of its coefficient matrix is always 0 whenever a linear system is inconsistent or dependent. In the next activity we explore the Gauss-Jordan method which is efficient and also able to tell us whether a system is dependent or inconsistent. This method requires us to perform row operations on the augmented matrix of a linear system.

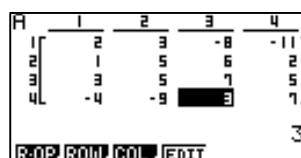
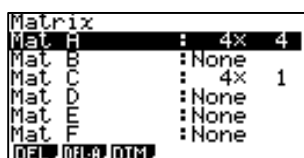
Solving System of Linear Equations

Activity 2: A linear system of 4 variables is showed in matrix form below. Form an augmented matrix from this system and use row operations to find the reduced row echelon form.

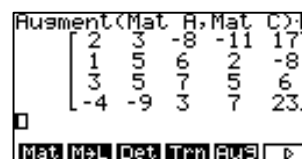
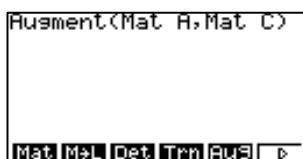
$$\begin{bmatrix} 2 & 3 & -8 & -11 \\ 1 & 5 & 6 & 2 \\ 3 & 5 & 7 & 5 \\ -4 & -9 & 3 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 17 \\ -8 \\ 6 \\ 23 \end{bmatrix}$$

Solution:

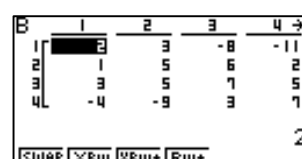
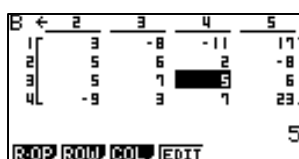
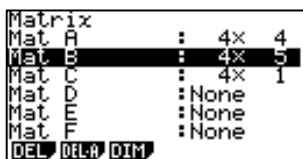
(a) We can easily see the coefficient and constant matrices. Open the RUN strip "Act2A". I have defined Mat A and Mat C as the coefficient matrix and constant matrix respectively. You can explore the two matrices using $\boxed{F3}$.



(b) Return to Run window and access the matrix functions through \boxed{OPTN} $\boxed{F2}$. Form the augmented matrix using $\boxed{F5}$ and assign the augmented matrix to Mat B.

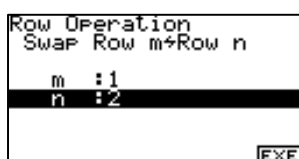
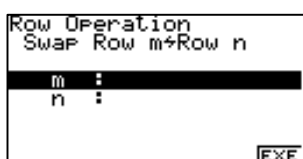


(c) Now return to the Run window to explore Mat B. We can access the row operation functions by tapping $\boxed{F1}$ and perform the appropriate sequence of row operations. The goal is to get the simplest form of augmented matrix.



(d) The augmented matrix has four rows labelled from 1 to 4. Here is a possible sequence of the row operations. The operating keystrokes for the first two operations are given.

- Swapping the position of row 1 and row 2: $\boxed{F1}$ $\boxed{1}$ \boxed{EXE} $\boxed{2}$ \boxed{EXE} \boxed{EXE}



Solving System of Linear Equations

- Adding to Row 2 a -2 multiple of Row 1: **F3** **(←)** **2** **EXE** **1** **EXE** **2** **EXE** **EXE**

```

Row Operation
k×Row m+Row n→Row n
k :
m :
n :
EXE
    
```

```

Row Operation
k×Row m+Row n→Row n
k :-2
m :1
n :2
EXE
    
```

```

B 1 2 3 4 →
1 | 1 5 6 2
2 | 0 -7 -20 -15
3 | 3 5 7 5
4 | -4 -9 3 7
1
SWAP XRow RRow+ Rw+
    
```

- Similarly, add to Row 3 the $(-3) \times \text{Row 1}$ and then add to Row 4 the $4 \times \text{Row 1}$. Both operations will simplify the first column of the matrix.

```

Row Operation
k×Row m+Row n→Row n
k :-3
m :1
n :3
EXE
    
```

```

B 1 2 3 4 →
1 | 1 5 6 2
2 | 0 -7 -20 -15
3 | 0 -10 -11 -1
4 | -4 -9 3 7
1
SWAP XRow RRow+ Rw+
    
```

```

Row Operation
k×Row m+Row n→Row n
k :4
m :1
n :4
EXE
    
```

```

B 1 2 3 4 →
1 | 1 5 6 2
2 | 0 -7 -20 -15
3 | 0 -10 -11 -1
4 | 0 11 27 15
0
SWAP XRow RRow+ Rw+
    
```

Systematically we move on to simplify the second column. This is very similar to the process of simplifying the first column.

- Multiply Row 2 with $-1/7$.

```

Row Operation
k×Row m→Row m
k :-0.1428571
m :2
EXE
    
```

```

B 1 2 3 4 →
1 | 1 5 6 2
2 | 0 1 2.8571 2.1428
3 | 0 -10 -11 -1
4 | 0 11 27 15
20.7
SWAP XRow RRow+ Rw+
    
```

- Add $10 \times \text{Row 2}$ to Row 3, then add $(-11) \times \text{Row 2}$ to Row 4 and finally add $(-5) \times \text{Row 2}$ to Row 1.

```

Row Operation
k×Row m+Row n→Row n
k :10
m :2
n :3
EXE
    
```

```

B 1 2 3 4 →
1 | 1 5 6 2
2 | 0 1 2.8571 2.1428
3 | 0 0 17.571 20.428
4 | 0 11 27 15
123.7
SWAP XRow RRow+ Rw+
    
```

```

Row Operation
k×Row m+Row n→Row n
k :-11
m :2
n :4
EXE
    
```

```

B 1 2 3 4 →
1 | 1 5 6 2
2 | 0 1 2.8571 2.1428
3 | 0 0 17.571 20.428
4 | 0 0 -4.428 -8.571
-31.7
SWAP XRow RRow+ Rw+
    
```

```

Row Operation
k×Row m+Row n→Row n
k :-5
m :2
n :1
EXE
    
```

```

B 1 2 3 4 →
1 | 1 0 -8.285 -8.714
2 | 0 1 2.8571 2.1428
3 | 0 0 17.571 20.428
4 | 0 0 -4.428 -8.571
1
SWAP XRow RRow+ Rw+
    
```

Solving System of Linear Equations

Using such systematic column reduction approach on the third and fourth column we will eventually have the following reduced row echelon form.

	← 2	3	4	5
1	0	0	0	17.831
2	1	0	0	15.19
3	0	1	0	12.114
4	0	0	1	-11.25
-6399.421				

SWAP XRow RRow+ RRow-

The last column of this matrix gives us the unique solution as

$$\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} \frac{7507}{421} \\ -\frac{6399}{421} \\ \frac{5100}{421} \\ -\frac{4740}{421} \end{bmatrix}.$$

(e) To verify the solution obtained, open SIMUL strip "**Act2B**" and tap on **F4**(4) to view the coefficients and constant keyed in. Press **EXE** to solve the system of linear equations.

Simultaneous
Data Exists In Memory
Unknowns:4
Number Of Unknowns?
2 3 4 5 6

anX+bnY+CnZ+dnT=en
← b c d e
1 3 -8 -11 17
2 5 8 2 -8
3 5 7 5 6
4 -9 3 7 23
6
SOLV DEL CLR EDIT

anX+bnY+CnZ+dnT=en
X 17.831
Y 15.19
Z 12.114
T -11.25
-6399.421
REPT

We should see that the answer produced is same as the solution set.

Discussion: Gauss-Jordan Method

Gauss-Jordan is essentially an elimination method. The row operations discussed can also be performed by hand where the calculator can help check each row operation. Note that we must arrange the equations according to the position of their respective variables before we put them in augmented matrix form.

When the reduced row echelon form (RREF) is attained the last column entries formed the solution set. RREF is always attained when the linear system is consistent (unique solution set.) When the system is inconsistent (has no solution) we usually get a row which shows a false statement when it is in RREF (see Exercise 1 of EXERCISE B). □

Activity 3: The following system of equations models the desired traffic flow of a certain busy 4-road section of the city where each of w , x , y and z represents the number of cars per hour passing four particular points at this section. Find the values of w , x , y and z that would realized the desired traffic flow.

Solving System of Linear Equations

$$\begin{aligned} 3x + 3z &= 2550 \\ w + 2x + y + 2z &= 2700 \\ 2w + 2x &= 1800 \\ 3x + y + 4z &= 3500 \end{aligned}$$

Solution:

(a) Open the RUN strip "Act3A" where Mat A and Mat C as defined as the coefficient matrix and constant matrix respectively.

Matrix	
Mat A	: 4x 4
Mat B	: None
Mat C	: 4x 1
Mat D	: None
Mat E	: None
Mat F	: None
DEL DELA DIM	

A	1	2	3	4
1	2	3	-8	-11
2	1	5	6	2
3	3	5	7	5
4	-4	-9	3	7
				3
R-OP ROW COL EDIT				

(b) Access the matrix functions through **OPTN** **F2** and form the augmented matrix using **F5** and assign the augmented matrix to Mat B. Then explore Mat B and perform the appropriate sequence of row operations to get RREF.

Augment(Mat A, Mat C)				
0	3	0	3	2550
1	2	1	2	2700
2	2	0	0	1800
0	3	1	4	3500
Mat M+L Det Trn RUS				

B	1	2	3	4	5
1	3	0	3	2550	
2	2	1	2	2700	
3	2	0	0	1800	
4	3	1	4	3500	
					2
R-OP ROW COL EDIT					

(c) A possible sequence of row operations is as follow.

- Swapping the positions of row 1 and row 2:

Row Operation	
Swap Row m→Row n	
m	: 1
n	: 2
EXE	

B	1	2	3	4	→
1	2	1	2	2700	
2	3	0	3	2550	
3	2	0	0	1800	
4	3	1	4	3500	
					1
SWAP XRW RRW+ RW+					

- Adding to Row 3 a -2 multiple of Row 1:

Row Operation	
k×Row m→Row n	
k	: -2
m	: 1
n	: 3
EXE	

B	1	2	3	4	→
1	2	1	2	2700	
2	3	0	3	2550	
3	0	-2	-2	-4	
4	3	1	4	3500	
					0
SWAP XRW RRW+ RW+					

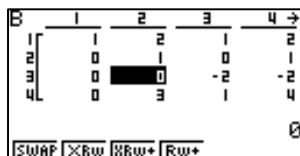
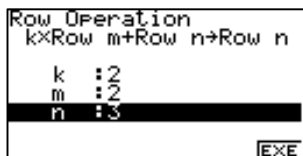
- Systematically we move on to simplify the second column. Multiply Row 2 with 1/3.

Row Operation	
k×Row m→Row m	
k	: 0.33333333
m	: 2
EXE	

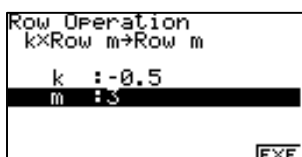
B	1	2	3	4	→
1	2	1	2	2700	
2	1	0	1	850	
3	0	-2	-2	-4	
4	3	1	4	3500	
					-2
SWAP XRW RRW+ RW+					

Solving System of Linear Equations

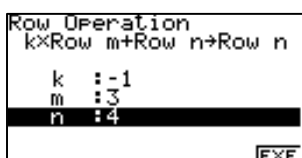
- Add $2 \times \text{Row 2}$ to Row 3, add $(-3) \times \text{Row 2}$ to Row 4 and finally add $(-2) \times \text{Row 2}$ to Row 1.



- So column 2 is simplified. Now multiply Row 3 with $-1/2$.



Notice at this stage that Row 3 and Row 4 have same entries. IF we now add $(-1) \times \text{Row 3}$ to Row 4, we get the following matrix.



From the matrix we get $w + y = 1000$, $x + z = 800$, and $y + z = 950$ or can be further expressed as $x = 800 - z$, $y = 950 - z$, and $w = 1000 - y = 50 + z$. Obviously the system does not have a unique solution and the problem is solved by any ordered 4-tuple of the form $(50 + z, 850 - z, 950 - z, z)$ where z is an integer ≥ 0 .

Discussion: Dependent System

For a dependent system its RREF will have 0 in all its entries in the last row, as the linear system of this activity demonstrates.

For this activity we can assert that the system has many solutions (but not infinite) with the condition that z is an integer ≥ 0 since we do not have negative cars passing a certain point. Also, z has an upper bound because if say $z=850$, then we shall have x negative which is not possible based on the same rationale.

EXERCISES B

Exercise 1.

Redo the problem in Exercises A using the Gauss-Jordan method.

Exercise 2.

Redo Activity 2 using the inverse matrix method.

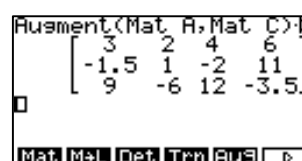
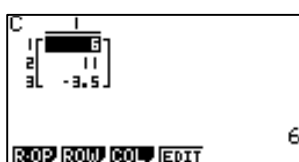
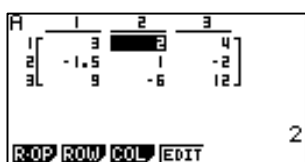
SOLUTIONS to EXERCISES B

Solving System of Linear Equations

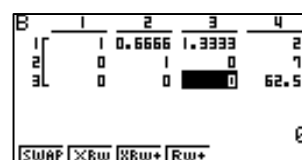
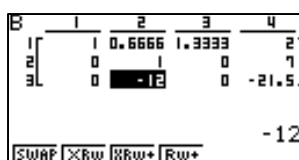
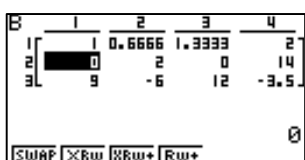
Exercise 1

Recall that the coefficient matrix is $\begin{bmatrix} 3 & -2 & 4 \\ -1.5 & 1 & -2 \\ 9 & -6 & 12 \end{bmatrix}$ and constant matrix is $\begin{bmatrix} 6 \\ 11 \\ -3.5 \end{bmatrix}$.

Open the RUN strip "**Ex2**" and define Mat A as coefficient matrix and Mat C as constant matrix, then form an augmented matrix using them, assigning the outcome to Mat B.



Return to Run window and open Mat B. Perform the row operations systematically. You can use the approach in Activity 2, simplifying column by column.

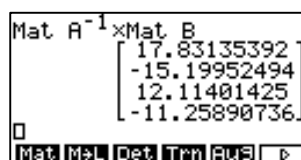
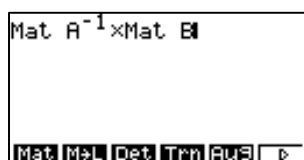


Even before we have reached the reduced row echelon form we can see that the third row displays a false statement of $0=62.5$. Therefore we do not need to continue doing row operations and conclude that the linear system has no solution (or inconsistent.) □

Exercise 2.

We have the coefficient matrix as $\begin{bmatrix} 2 & 3 & -8 & -11 \\ 1 & 5 & 6 & 2 \\ 3 & 5 & 7 & 5 \\ -4 & -9 & 3 & 7 \end{bmatrix}$ and the constant matrix as $\begin{bmatrix} 17 \\ -8 \\ 6 \\ 23 \end{bmatrix}$

Open RUN strip "**Ex3**" and define these two matrices in the calculator. In the strip I have define Mat A as the coefficient matrix and Mat B as the constant matrix. Having done that, calculate the product of inverse of Mat A and the constant matrix.



The solution set produced should be the same with the one in Activity 2. □

Solving System of Linear Equations

REFERENCE

[1] Howard Anton, *Elementary Linear Algebra 8thE*, John Wiley & Sons, 2000. ISBN: 047-1385-79-4 .