



## Solving Quadratic Equations

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### **LEVEL**

High school or university students with basic knowledge in Algebra.

### **OBJECTIVES**

To demonstrate problems leading to quadratic equations and various methods of their solutions.

### **Corresponding eActivities**

A02BALL.g1e (for Activity 1), A02SHOES.g1e (for Activity 2),

### **OVERVIEW**

Real-life problems leading to quadratic equations are demonstrated and solved several methods. The equivalency of these solutions is shown.

### **EXPLORATORY ACTIVITIES**

Here we describe two activities. For their mathematical background refer for example to [DW], page 195-261.

#### **Activity 1 (A01BALL.g1e):**

A ball is kicked vertically with the initial speed 9 m/sec. How long will it fly?

#### **Solution:**

From the *Physics* course you likely know that the position of the ball depends on the initial velocity  $v_0$  and the gravitational constant ( $g = 9.81 \text{ m/s}^2$ ). The formula looks as follows:

$$y = -\frac{1}{2}gt^2 + v_0t$$

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We are interested in the situation with the ball on the ground i.e. when  $y = 0$ . The formula then looks as follows:

$$0 = -\frac{1}{2}gt^2 + v_0t$$

This can be simplified to the form

$$t \cdot \left(-\frac{1}{2}gt + v_0\right) = 0$$

The product equals zero when any of its factors equals to it. So, the equation has two roots:

$$t = 0$$

and

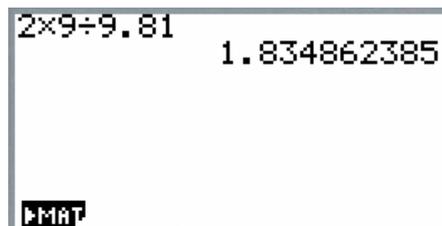
$$-\frac{1}{2}gt + v_0 = 0$$

$$-\frac{1}{2}gt = -2v_0$$

$$gt = 2v_0$$

$$t = \frac{2v_0}{g}$$

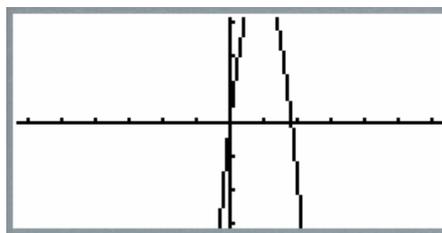
(a) (Refer to Duration) A simple calculation in the RUN-MAT mode produces the result. The ball will fly approximately 1.83 second.



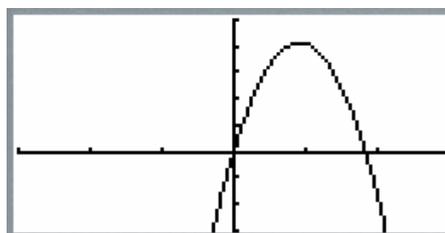
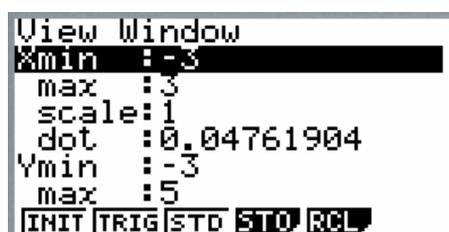
(b) (Refer to 2 zero points) How can there be two solutions?

A graph will help us to understand what happens. Select **MENU** and then GRAPH. Type the function  $-0.5 \cdot 9.81x^2 + 9x$ . Press **F6** to draw its graph.

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To improve the output, let us change the displayed area. Set up narrow the interval on the x axis and expand the interval on the y axis. The left window shows the new parameters, the right one the graph corresponding to them.



Now we see the meaning of 0 and 1.83 better. Zero represents the starting point. As we kick the ball from the ground, its initial y-coordinate is really 0. After 1.83 seconds it touches the ground again.

### EXERCISES A

#### Exercise 1.

The graph from 2 zero points may still cause confusions. In accordance to it, the ball is under ground before the kick and it again disappears below surface after 1.83 seconds. Modify the graph to represent a real kick more appropriately.

#### Exercise 2.

The gravitation on the Moon is six-times lower than that of on the Earth (i.e. its gravitational constant is also 6-times smaller). How long will it take for a stone to reach the ground if it has been thrown in the Moon with the speed 9m/s?

### SOLUTIONS to EXERCISES A

#### Exercise 1

Open V-Window. Assign Ymin to 0. The negative "height" will not be displayed.

#### Exercise 2.

Use the gravitational constant  $9.81/6$ . Otherwise repeat the above eActivity.

#### Activity 2 (A02SHOES.g1e):

A shoemaker has been asked to make 100 special-purpose shoes. During the first week he produces nine of them, on the next week eleven, on the third week thirteen. He sees that due to his growing experience and improved skills he will be capable of producing two shoes more during every next week than during the previous one. How long will it

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take for him to produce all pairs?

### Solution:

(a) (Refer to Estimation)<sup>1</sup> In a spreadsheet we will simulate the progress of shoemaker's production and calculate its total. The necessary data are placed into three columns. The titles in the first row indicate their purpose.

SHEET	A	B	C	D
1	Week	Shoes	Total	
2	1	9	9	
3	2			
4	3			
5	4			

=CellSum(\$B\$2:B2)

The column A contains the ordering number of the week.

The column B shows the number of shoes produced on the particular week. It starts with the constant 9. All other cells will contain values greater by two compared to their above neighbors.

In the C column, the total production (till the end of the particular week) is calculated using the CellSum function. Its argument is the interval from the lower bound B2 (fixed by \$B\$2) to its expanding upper bound. As the formula from C2 is copied to C3, C4, and so on, the upper bound is automatically changed to B3, B4, etc.

To start our estimation process, we fill in the empty cell in the third row. The formula =B2+2 is placed into the cell B3 and the formula = CellSum(\$B\$2:B2) is copied from C2 to C3. We repeat these two steps until the value in a cell of the column C exceeds 100.

SHEET	A	B	C	D
1	Week	Shoes	Total	
2	1	9	9	
3	2	11	20	
4	3	13	33	
5	4	15	48	
6	5	17	65	
7	6	19	84	
8	7	21	105	

As we see, seven weeks is enough for completing the work. Because the value in C8 is a bit bigger than the required number, we can conclude: "The shoemaker will need a bit less than seven weeks for producing 100 pairs of shoes."

## EXERCISES B

### Exercise 1.

Repeat the above estimation process by yourself.

## SOLUTIONS to EXERCISES B

<sup>1</sup> Some pictures of the calculator display have been artificially expanded by means of a graphic editor.

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### Exercise 1

Do not forget to increase the values in the column B by 2. Accumulate their sum in the column C. Repeat all until a value in a cell in the column C reaches 100 (or more).

(b) (Refer to Equation) The problem can be solved using another method. One can see that the total can be calculated using a "trick". Notice that the sum of the first and last element is equal to the sum of the second and the second to the last element and etc., for example:

$$\begin{aligned} B_2 + B_8 &= 9 + 21 = 30 \\ B_3 + B_7 &= 11 + 19 = 30, \text{ and etc.} \end{aligned}$$

It is easy to see that this property holds for all such pairs (with the elements closer and closer to the middle. The number of pairs is one half of their number (which also equals to one half of the number of days).

$$S_n = (b_1 + b_n) \frac{n}{2}$$

Here,  $S_n$  is the produced number of shoes,  $b_1$  is the first element of the sequence,  $b_n$  is its last element and  $n$  is the number of working days. Realize that the formula holds for any number of days and we are interested in the moment when  $S_n$  equals 100.

Despite the fact that we do not know the number of working days in advance, we can express the "last day production" as "Day 1 production increased (n-1) times". We also know that 2 is the difference between two consecutive elements. Thus,

$$b_n = b_1 + 2(n-1)$$

We substitute the last expression into the above formula and get

$$S_n = (b_1 + (b_1 + 2(n-1))) \frac{n}{2} = (2b_1 + 2n - 2) \frac{n}{2} = b_1 n + n^2 - 1 = n^2 + b_1 n - 1$$

As  $S_n = 100$  and  $b_1 = 9$ , the formula can be expressed either in the form

$$100 = n^2 + 9n - 1$$

or in its even more simplified version

$$n^2 + 9n - 101 = 0$$

This version is called a quadratic equation.

The calculator offers a simple method of solving it. As all quadratic equations belong to the category "polynomials", the polynomials are characterized by their by the highest exponent – in our case it is 2. So, our next selection is **F1**. The screen looks as follows:

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Similarities in the used notations are obvious. We are using  $n$  as the independent variable; the calculator prefers  $x$ . The parameters are then evident:  $a = 1$ ,  $b = 9$ ,  $c = -101$ . Type them in and press **[F1]** (SOLV). The result appears:



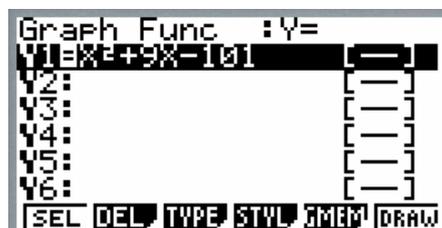
Notice that:

- The equation has two solutions: 6.5113 and -15.51. As any shoemaker cannot work a negative number of weeks, 6.5113 is our solution.
- By pressing the key **[1]** or **[2]**, you can select a solution and view its additional decimal places.

(c) (Refer to [Graphing](#)) What is the meaning of the two solutions?

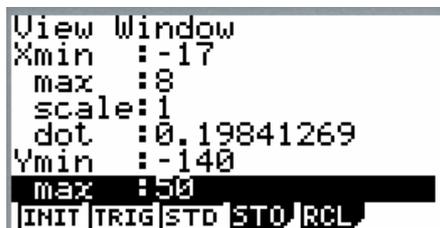
To understand it, let us consider the function  $f(x) = x^2 + 9x - 101$  on an appropriate interval (instead of discussing a set of numbers having their values equal to  $n^2 + 9n - 101$  for some specified  $n$  as we have been doing up to now). Evidently, we should be interested in an interval larger than  $(-15.51, 6.5113)$  to see “further to both directions” and get a better picture on its behavior.

Select **[MENU]** and then GRAPH. Type the function  $x^2 + 9x - 101$ .

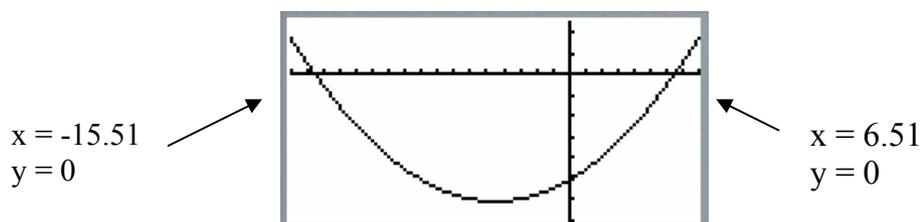


To view the proper picture, set up the intervals of the  $x$  and  $y$  values. The selection of  $x$  is simple – as we cover the requested interval, let us make it a bit larger e.g.  $(-17, 8)$ . The estimation of  $y$  values takes longer, so we recommend  $(-140, 50)$  for now. Press **[SHIFT]** and **[F3]** to do it:

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Press **EXIT** to return to the window with the function and press **EXIT** to draw the graph.



The graph shows a curve that intersects the x axis at two points. From our above calculation we can conclude that their x-coordinates are -15.51 and 6.51.

Remember that the right selection of the x and y maximum and minimum values is critical for viewing the graph properly.

## EXERCISES B

### Exercise 2.

The area of three squares is 194. How long are their sides if we know that they are consecutive integers? Solve the problem by estimation.

## SOLUTIONS to EXERCISES B

### Exercise 2

Use a spreadsheet. In the column A store integers 1, 2, 3, ... In the B column calculate their squares 1, 4, 9, .... Starting from the cell C3 calculate the sum of three neighboring squares. Repeat all until you find out the result.

SHEE	A	B	C	D
1	Side	Square	Sum	
2	1	1		
3	2	4		
4	3	9	14	
5	4	16	29	
6	5	25	50	
7	6	36	77	
8	7	49	110	
9	8	64	149	
10	9	81	194	

=B8+B9+B10

(d) (Refer to Opposite curves) The graphs of quadratic functions are oriented into two

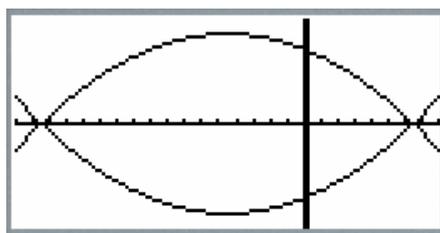
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different directions. The orientation depends on the sign of "a" – the coefficient of the quadratic element. Let us display  $x^2 + 9x - 101$  and  $-x^2 - 9x + 101$  to demonstrate the difference.

Type the both functions. Before drawing them, let us define settings that fit to both. By pressing **[SHIFT]** **[F3]** activate V-window. The interval for x use the same interval as above (i.e. -17 to 8). Expand the interval for y. It should cover -140 to 140.

```
View Window
Xmin :-17
max :8
scale:1
dot :0.19841269
Ymin :-140
max :140
[INIT|TRIG|STD|STO|RCL]
```

Use **[EXIT]** for returning to the GRAPH window. Pressing **[F6]** (DRAW) displays both of them.

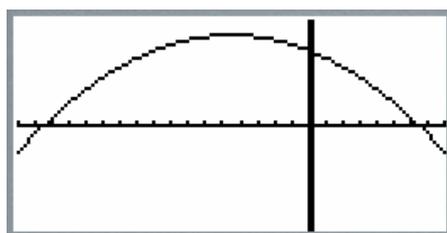
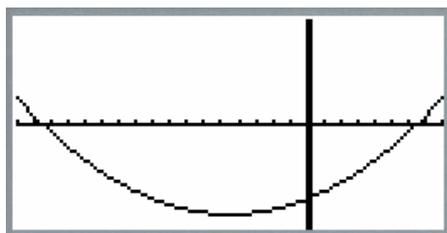


Evidently, they form a symmetric pair and have same roots.

(e) (Refer to [Separate curves](#)) The above combined graph does not allow us to decide which graph is which. It is easy to separate them. Even if both functions are typed, you can make a selection by pressing **[F1]** (SEL). In the next figure, the first function is selected, the second one is not.

```
Graph Func : Y=
Y1 X^2+9X-101  [—]
Y2 -X^2-9X+101 [—]
```

The choice is indicated by the highlighted "=" symbol". In the above picture it is  $y_1$ . The **[F1]** key functions as a toggle. The first click on the specified function selects it, the next one deselects it. Using this method, you can see each function separately and compare their behavior.



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### **EXERCISES B**

#### **Exercise 3.**

Study the below functions by graphing them:

$$y_1 = x^2$$

$$y_2 = -x^2$$

$$y_3 = x^2 + 2$$

$$y_4 = x^2 - 2$$

$$y_5 = -x^2 - 2$$

$$y_6 = x^2 + 2x - 2$$

$$y_7 = x^2 + 2x + 2$$

$$y_8 = x^2 - 2x - 2$$

$$y_9 = -x^2 - 2x - 2$$

$$y_{10} = -x^2 + 2x + 2$$

To view them the best, set up the view window arguments to  $x_{min} = -4$ ,  $x_{max} = 4$ ,  $y_{min} = -12$ ,  $y_{max} = 12$ . Select one or two of them and graph them.

Answer the below questions:

Which of them intersect the x axis?

Which of them form symmetric pairs?

What differences can you observe in their behavior?

#### **Exercise 4**

One by one, solve the above set of equations using the SOLV tool. Are there relationships between the SOLV results and your previous observations?

### **SOLUTIONS to EXERCISES B**

#### **Exercise 3**

If the function has its coefficients all positive or all negative, it does not intersect the x axis.

If the arguments of two functions have all their arguments "same" but with all signs opposite, they form a symmetric pair.

The constant moves the function "up" or "down". The coefficient of x moves it "to the right" and "to the left".

#### **Exercise 4**

Only the functions intersecting the x axis can be solved by SOLVER.

### **REFERENCE**

[DW] Andrew Demetropoulos and Kenneth C. Wolf, *Intermediate Algebra*, Macmillan Publishing Company, 1985. ISBN 0-02-328530-3