Sequences in Mathematics

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LEVEL
High school or university students with basic knowledge in Algebra.

OBJECTIVES
To explain how sequences of numbers can be generated and how they relate to real-life problems.

Corresponding eActivities
A04ARITH.g1e (for Activity1), A04GEOM.g1e (for Activity2), A04FIBO.g1e (for Activity 3)

OVERVIEW
Arithmetic and geometric progressions will be generated in spreadsheets. We show basic differences between them and how their parameters influence their behavior. The Fibonnaci numbers are shown as an example of a different type of numerical sequence.

[Note] The solutions to the exercises are included into eActivities.

EXPLORATORY ACTIVITIES
Here we describe three activities. For their mathematical background refer for example to [DW], page 459-471.

Activity 1 (A04ARITH.g1e):
Let us assume building a structure with many floors. The basic floor is built for a certain (basic) price. Building higher structures becomes more expensive as the material must be carried higher, the safety precautions must be stronger, and the risk is higher. For these reasons, insurance companies usually request higher payments to balance the risks.

The ARITHMETIC insurance company gives you its proposal: *Our basic insurance per floor is $15. For each floor above the ground you pay $2 more than for the previous floor*. How much are we to pay total insurance for a 10-level building (i.e. for a building with a ground floor and 9 floors above it)?

Solution:

(a) (Refer to Initial Values)[1] The column A contains the floor numbers (with the ground floor having its order number equal to 0).

SHEET	A	B	C	D
1	0	15		
2	1			
3	2			
4	3			
5	4			
6	5			
7	6			
8	7			
9	8			
10	9			

In the cell B1 we type the basic insurance cost i.e. 15. In every other cell the previous price is increased by 2. Into B2 we therefore type the formula =B1+2. Pressing [EXE] results in its calculation with the result 17.

(b) (Refer to Partial Payments) All elements of the sequence are generated using the same rule:

$$b_n = b_{n-1} + d$$

where b_i is the i-th element of the sequence and *d* is the difference between two consecutive elements.

As we know the difference, and the value in B2 has been calculated using this formula, the formula from B2 can now be copied into all discussed cells. For copying, press [F2] (EDIT) and again [F2] (COPY). The menu now contains only one option [F1] (PASTE). In a stepwise manner, move the cursor to the cell B2, B3, B4, …, B9 and always press [F1].

SHEET	A	B	C	D
1	0	15		
2	1	17		
3	2	19		
4	3	21		
5	4	23		
6	5	25		
7	6	27		
8	7	29		
9	8	31		
10	9	33		

=B9+2

Notice that copying is not literal. If it would be, the result of the calculation had to be identical (17 in all cells). The standard copying in spreadsheets acts in our desired way – it copies the relationship between the cells. What we have copied in reality is the relationship "Add two to the content of the above cell". One can easily see it. The active cell B10 contains the formula =B9+2.

[1] Some pictures of the calculator display have been artificially expanded by means of a graphic editor.

EXERCISES A

***Exercise 1*.**
We can also claim that the column A of the spreadsheet also contains an arithmetic progression. Why?

SOLUTIONS to EXERCISES A

***Exercise 1*.**
Its initial element is 1 and its difference is also 1.

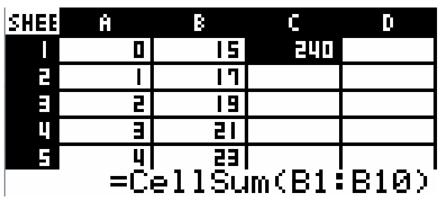
(c) (Refer to Total Arithm) We would like to know the total payment expected by the ARITHMETIC insurance company. To do it, we are supposed to make a long summation

$$S_n = b_1 + b_2 + b_3 + b_4 + b_5 + b_6 + b_7 + b_8 + b_9 + b_{10} = \sum_{i=1}^{10} b_i$$

Luckily, the spreadsheet contains the *Sum* function that substantially simplifies the process.
Let us calculate it in the cell C1. Move to the cursor to it. Type "=" to indicate the beginning of a formula. (Press [SHIFT] [•] to get the symbol "="). The menu gets the form:

Now press [F5] (CEL) to get the menu with six built-in spreadsheet functions:

Press [F5] (SUM). Type the range of its values – B1 to B10 – and press [EXE]. The result appears.



Notice that typing the function name and parameters "=CellSum(B1:B10)" gives an optically identical result, but ends up with an error message. The function name *must* be selected from the menu.

(d) (Refer to Another Total) Into the cell D1, type the formula
=(B1+B10)*5
To our surprise, the result is also 240.

EXERCISES A

***Exercise 2*.**
Why do the formulas =CellSum(B1:B10) and =(B1+B10)*5 give same results?

Exercise 3.
One can characterize the sequence -11, -7, -3, 1, 5, 9, 13 also as an arithmetic one. Explain why.

SOLUTIONS to EXERCISES A
Exercise 2.
The entire sum can be decomposed into 5 sections:
B1+B10
B2+B9
B3+B8
B4+B7
B5+B6
All of them produce the same result 48. There are five such pairs: 5 x 48 = 240.

Exercise 3.
Its initial element is -11. The difference between all elements is same (four).

(e) (Refer to Graphing it) Make a graph of the sequence -11, -7, -3, 1, 5, 9, 13. What type of line do you get?
Every graph must be based on two variables. The ordering number of the element is the independent variable. The value of the element is the dependent variable.

EXERCISES A
Exercise 4.
The sequence 37, 27, 17, 7, -3, -13, -23, -33 can also be called an arithmetic sequence. Explain why.

Exercise 5.
In your head, calculate the sum of the arithmetic sequence 37, 27, 17, 7, -3, -13, -23, -33. Search for the easiest possible way.

SOLUTIONS to EXERCISES A
Exercise 4.
The difference between all elements is same: -10.

Exercise 5.
It is an arithmetic sequence. Let us add its first and last elements: 37+(-33) = 4. The sequence has eight elements. They form 4 pairs. 4 x 4 = 16. The sum is 16.

Activity 2 (A04GEOM.g1e):

To have a choice of offers, we visited another insurance company named GEOMETRIC. Its manager gave as a different proposal: *Our basic insurance per floor is $1. For each floor above the ground you will pay 2-times more than for the previous floor*.

The offer starts at much lower level than the one from the ARITHMETIC. Will we pay more or less for the whole building?

Solution:

Notice that the production rule of the next element is now different. We start with an initial (constant) value

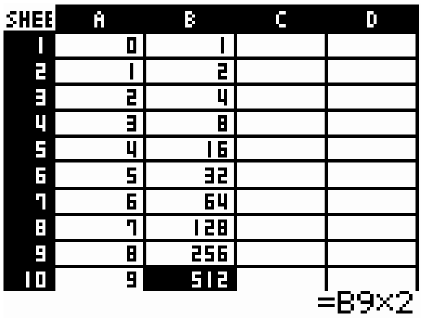
$$b_1 = a$$

but all other values b_2, b_3, b_4, …, b_n are generated as products:

$$b_n = b_{n-1} \times r$$

where r is a common ratio.

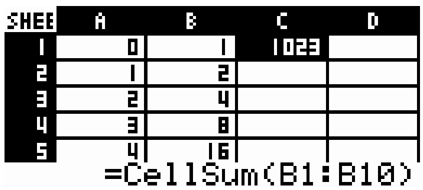
(a) (Refer to Initial Values)[2] The column A contains the floor numbers (with the ground floor having its order number equal to 0). The below picture shows payments for individual floors.



SHEE	A	B	C	D
1	0	1		
2	1	2		
3	2	4		
4	3	8		
5	4	16		
6	5	32		
7	6	64		
8	7	128		
9	8	256		
10	9	512		

=B9×2

(b) (Refer to Total Geom) To know the total payment expected by the GEOMETRIC insurance company, sum the values in the column B. Despite its very low starting value ($1 for the ground floor), the total is $1023, which is much higher than the proposal offered by ARITHMETIC insurance company.



SHEE	A	B	C	D
1	0	1	1023	
2	1	2		
3	2	4		
4	3	8		
5	4	16		

=CellSum(B1:B10)

Notice in the previous figure that payment for 8th floor is already greater than $240, the total requested by the ARITHMETIC.

(c) (Refer to Summation) The sum of the geometric sequence can also be expressed using a formula

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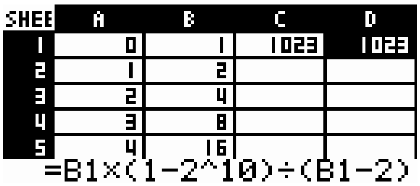
[2] Again, "big" pictures of the calculator display have been artificially expanded by means of a graphic editor.

$$S_n = \frac{a(1-r^n)}{1-r}$$

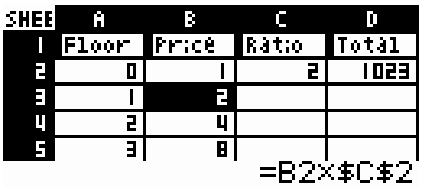
Here, S_n represents the sum of the first *n* elements; *a* is the value of the first element and *r* is the common ratio (the multiplying factor producing the next element from the previous one). Naturally, the formula produces the same result as the CellSum function above.
In the eActivity, you find both calculations: in C1, using the CellSum function; in D1, using the formula.



SHEE	A	B	C	D
1	0	1	1023	1023
2	1	2		
3	2	4		
4	3	8		
5	4	16		

=B1×(1-2^10)÷(B1-2)

(d) (Refer to Equal Offers) We started bargaining with a manager of the GEOMETRIC. Our condition is achieving a payment not higher than $240. Naturally, the manager wants to keep it as close to $240 as possible. What ratio should he use to get to it as close as possible?
The display looks as follows.

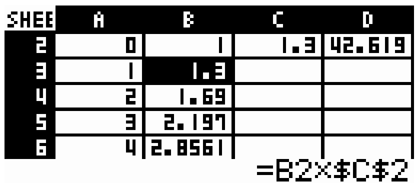


SHEE	A	B	C	D
1	Floor	Price	Ratio	Total
2	0	1	2	1023
3	1	2		
4	2	4		
5	3	8		

=B2×C2

Experiment with the content of C2 (the sequence ratio) in order to get in D2 the value as close as possible to 240.

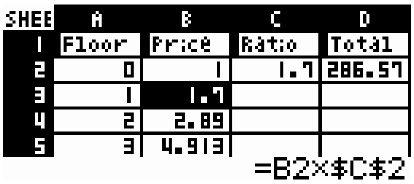
(e) (Refer to First Guess) Our first guess for the ration is 1.3. So we type the value into the C2 cell. All values in the column B and the sum in D2 are instantly recalculated. The new sum 42.619 is displayed.



SHEE	A	B	C	D
2	0	1	1.3	42.619
3	1	1.3		
4	2	1.69		
5	3	2.197		
6	4	2.8561		

=B2×C2

(f) (Refer to Second Guess) The value is much lower than presumed $240. Thus, the ratio must be increased. Let us set it up to 1.7.



Now the result is 283.57. It is much closer to $240 than the previous result. As it is still more than needed, the value in B2 must be decreased (say to 1.65). In a stepwise manner, we can progress to a value sufficiently close to $240.

EXERCISES B

***Exercise 1*.**
Graph the geometric sequence. What type of line do you get?

SOLUTIONS to EXERCISES B
***Exercise 1*.**
A strongly growing curve.

(f) (Refer to <u>Interest Growth</u>) You deposit $1000. You do not plan to make any other deposits or withdrawals during the next 10 years because your bank offers 5% interest rate for such clients. How much will you have on your account in the end of the period?

If the initial deposit is D_0 and the interest rate is r percent, after one year the new deposit D_1 is increased by the interest. So the new one-year period start with the deposit calculated using the formula

$$D_1 = D_0 + rD_0 = D_0(1+r)$$

The same process repeats every year. Thus,

$$D_2 = D_1 + rD_1 = D_1(1+r) = D_0(1+r)(1+r) = D_0(1+r)^2$$

It is easy to see that the deposit grows in the form of a geometric progression

$$\begin{aligned} D_1 &= D_0(1+r) \\ D_2 &= D_0(1+r)^2 \\ D_3 &= D_0(1+r)^3 \\ &\ldots \\ D_n &= D_0(1+r)^n \end{aligned}$$

The next element is always produced from the previous one by the factor (1+r). This can be observed in the column B of the spreadsheet.

SHEE	A	B	C	D
1	2005	1000	0.05	
2	2006	1050		
3	2007	1102.5		
4	2008	1157.6		
5	2009	1215.5		
6	2010	1276.2		
7	2011	1340		
8	2012	1407.1		
9	2013	1477.4		
10	2014	1551.3		
11	2015	1628.8		

(g) (Refer to Double Deposit) What interest rate would double your deposit within 10 years?

Similarly to the activities (e) – First Guess and (f) – Second Guess, modify the value of the C1 cell. Increase or decrease it to get the value 2000 into B11. (To speed up your operation, the cell D2 contains the formula =B11 – the content of B11 is automatically copied here. Thus, there is no need to move to the end of the column B.)

SHEE	A	B	C	D
1	Year	DepoSi	Rate	Output
2	2005	1000	0.05	1551.3
3	2006	1050		
4	2007	1102.5		
5	2008	1157.6		

=B11

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EXERCISES B

***Exercise 2*.**

Why is the sequence of natural numbers (1, 2, 3, 4, 5, …) not a geometric progression?

SOLUTIONS to EXERCISES B

***Exercise 2*.**

Calculate the ratio of its different elements:
2/1,
3/2,
4/3,
etc.
All ratios are different.

Activity 3 (A04FIBO.g1e):

The arithmetic and geometric progressions are the most popular series in mathematics. On the other hand, series can be generated using other rules, too. A popular series of numbers refers to a 12th century mathematician Fibonnaci. He proposed and solved the following problem:
A farmer buys a pair of rabbits. Rabbits mate every month and within the next month deliver a pair of young ones. It takes another month for the youngsters to mature and be

capable of mating and delivering babies. Let us assume that this repeats a year. How many pairs of rabbits will the farmer have?

Solution:

(a) (Refer to <u>Fibonnaci seq.</u>) For simplicity let us assume that the farmer bought his first pair in January. In February they mate. During this month, the female rabbit bears the baby pair, so the farmer still has just one. In March, the puppies are born and the old pair mates again – expecting another pair in April. Thus, in March the farmer has got two pairs. In April, the old pair delivers puppies raising the total of pairs to three.
During March, the young ones mature and are capable of mating in April. So, in May the oldest pair and their older of their descendants deliver babies. It makes 5 pairs. One can observe regularity in the sequence 1, 1, 2, 3, 5 – the next number is generated as the sum of its two predecessors. The formula looks as follows:

$$F_n = \begin{cases} 1 & \text{if} \quad n = 1 \\ 1 & \text{if} \quad n = 2 \\ F_{n-1} + F_{n-2} & \text{if} \quad n \geq 3 \end{cases}$$

Implementing the formula in a spreadsheet table produces the required result.

The values 1 are typed in the cells B1 and B2. B3 is the first cell containing a formula (the sum of two cells above it – see the below figure). Then, this formula can be copied to all other cells in the B column.

SHEE	A	B	C	D
1	Jan	1		
2	Feb	1		
3	March	2		
4	April	3		
5	May	5		
6	June	8		
7	July	13		
8	Aug	21		
9	Sept	34		
10	Oct	55		
11	Nov	89		
12	Dec	144		

=B1+B2

EXERCISE C

Exercise 1.
Graph the sequence the B column. Which type of progression does it remind more: arithmetic or geometric?

SOLUTION to EXERCISE C
Exercise 1.
The behavior of the graph resembles that of geometric progression.

REFERENCE

[DW] Andrew Demetropoulos and Kenneth C. Wolf, *Intermediate Algebra*, Macmillan Publishing Company, 1985. ISBN 0-02-328530-3

