Parametric Equations of Conic Sections

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LEVEL
High schools and college after students have
- understood the concept of derivative and integral calculus;
- have a good grasp of conic section in parametric form.

OBJECTIVES
To solve problems on conic section represented parametrically with a graphics calculator.

CORRESPONDING eActivity
PARAMTRC.g1e

OVERVIEW
In this activity we will study two problems on parametric equations for conic sections. In particular the discussion is on using graphics calculator technology to find equations of tangent, normal and finding the arc length from different approaches.

EXPLORATORY ACTIVITIES
[Note]
We shall use small letter *x*, *y, z* instead of capital X, Y, Z as shown on the calculator throughout the paper.

Here let's begin with this exploration activity about graphing an ellipse.

Exploration: Suppose a parametric representation of an ellipse is as follow:

$$x = 5\cos t\ ,\ y = 2\sin t\ .$$

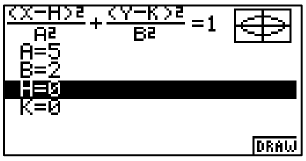
There are a few ways we can graph the ellipse with a graphics calculator.

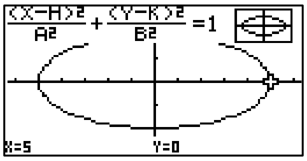
- **Cartesian Form**

One sure way is to express the parametric form to its Cartesian equivalent but the challenge is always about finding the right Cartesian equation. For these parametric equations this is not too difficult.
If we now square both sides of these equations then we shall have $x^2 = 5^2 \cos^2 t$, $y^2 = 2^2 \sin^2 t$ or $\frac{x^2}{5^2} = \cos^2 t$, $\frac{y^2}{2^2} = \sin^2 t$. Adding them up gives us the equation of $\frac{x^2}{5^2} + \frac{y^2}{2^2} = \cos^2 t + \sin^2 t$, or simply

$$\frac{x^2}{5^2} + \frac{y^2}{2^2} = 1.$$

Open the eActivity PARAMTRC.g1e then the Conic Editor strip "**Exp1**". Select to graph the ellipse of the form "$\frac{X^2}{A^2} + \frac{Y^2}{B^2} = 1$" by entering the appropriate values. In [Conic] mode we can choose to find properties of this ellipse such as its focus and eccentricity.

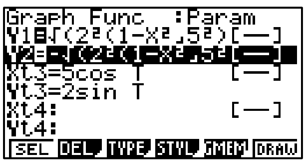


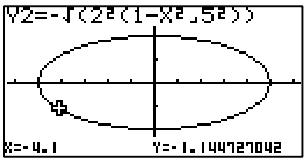


Alternatively, we could graph the ellipse in the Graph mode using a different Cartesian equation. If we solve the equation $\frac{x^2}{5^2} + \frac{y^2}{2^2} = 1$ for y then we would have the following:

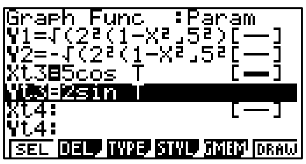
$$\frac{x^2}{5^2} + \frac{y^2}{2^2} = 1 \Rightarrow \frac{y^2}{2^2} = 1 - \frac{x^2}{5^2} \Rightarrow y = \pm\sqrt{2^2\left(1 - \frac{x^2}{5^2}\right)}.$$

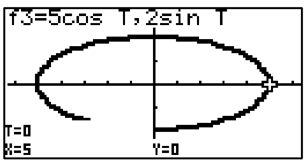
Graphing these 2 functions on the same set of axes gives us the graph of $\frac{x^2}{5^2} + \frac{y^2}{2^2} = 1$.
Open the Graph Editor strip "**Exp2**" to graph the two functions.





- **Parametric Form**
Finally we could graph the ellipse in its original parametric form. Using the strip of "**Exp2**", enter the parametric equations of $x = 5\cos t$, $y = 2\sin t$, then deselect the Cartesian functions above to graph the ellipse.



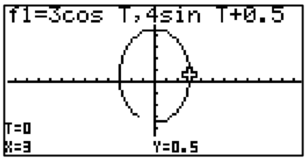


Any of these 3 approaches can be used to graph other conic sections, depending on the needs of the problems solving.

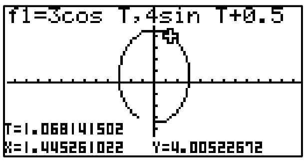
Activity 1: The parametric representation of an ellipse is given as $x = 3\cos t$, $y = 4\sin t + 0.5$. Find the equations of tangent to the ellipse at $t = \frac{\pi}{3}$.

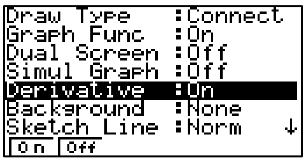
<u>**Solution**</u>:
A traditional approach is to employ differential calculus to evaluate slope of the ellipse at $t = \frac{\pi}{3}$, then find the corresponding equation of tangent. With a graphics technology we can choose to explore the problem graphically first then use a table of values. For this activity we explore the graph of ellipse in its parametric form.

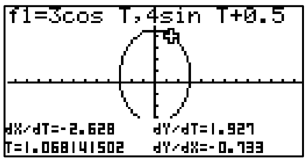
(a) Open the <u>Graph Editor</u> strip "**Act1A**", tap [F3] [F3] to set up [Type] to parametric for graphing $x = 3\cos t$, $y = 4\sin t + 0.5$ and use appropriate View Window setting to view the graph.



(b) Trace the graph and try locate where $t = \frac{\pi}{3}$ occurs, understanding that $\frac{\pi}{3} \approx 1.0472$. Having done that, tap [SHIFT] [MENU] to enter [Setup] to set [Derivative] on, and re-trace the graph again.

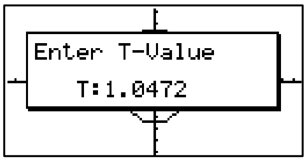


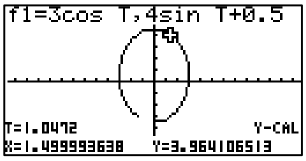




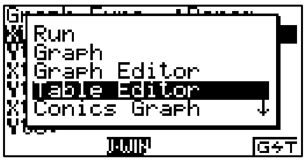
From the graphical exploration of the parametric equations of the said ellipse, we learned that the point when $t = \frac{\pi}{3}$ should occur in first quadrant, and clearly the slope (refer to $\frac{dy}{dx}$ in right screen above) is negative. We should also expect the equation of tangent has positive *y*-intercept.

(c) While tracing of the curve may not able to show the actual point where $t = \frac{\pi}{3}$, we could tap [F5] [F6] [F1] to engage [Y-CAL] to show the actual point. Using $\frac{\pi}{3} \approx 1.0472$, we should find that the corresponding point when $t = \frac{\pi}{3}$ is (1.5000, 3.9641), correct to 4 decimal places.

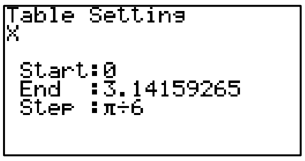


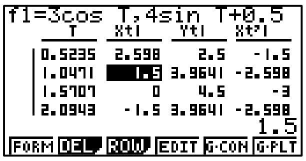


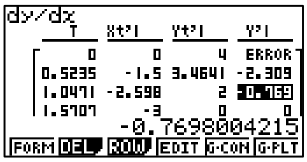
(d) Finally we need to determine the corresponding slope of the ellipse when $t = \frac{\pi}{3}$ which we could do so at the table of values. While still inside the "**Act1A**" strip, tap [SHIFT] [,] to access other functions where we shall choose to use the <u>Table Editor</u>.



Using [F5] to set up start value of 0 and end value of π, we also use a step size of $\pi/6$ to include the row of $t = \frac{\pi}{3}$. Make sure also that [Type] is set to parametric.



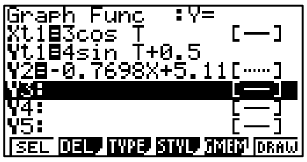


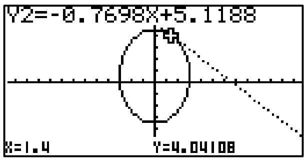


With the table made, use the arrow keys to explore the table in particular the row where $t = \frac{\pi}{3}$. We should again find the corresponding point as (1.5, 3.9641), with the slope as -0.7698, correct to 4 decimal places. So using the slope-point form, the equation of tangent to the ellipse $x = 3\cos t$, $y = 4\sin t + 0.5$ at $t = \frac{\pi}{3}$ is

$$y - 3.9641 = -0.7698(x - 1.5) \quad \Rightarrow y - 3.9641 = -0.7698x + 1.1547$$
$$\Rightarrow y = -0.7698x + 5.1188$$

We can graph this line equation against the parametric equations of the ellipse to visualise them (use <u>Graph Editor</u> strip "**Act1B**".)



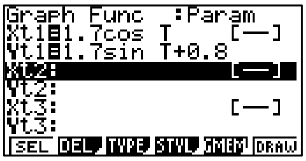


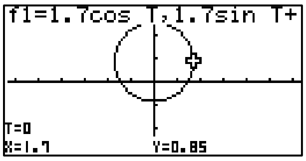
The graphs drawn support $y = -0.7698x + 5.1188$ as the tangent we sought. □

In this next problem we want to find the arc length of a section of the circle which is represented parametrically.

Activity 2: Graph the parametric equations of $x = 1.7\cos t$, $y = 1.7\sin t + 0.85$ for $\frac{4\pi}{5} \le t \le \frac{5\pi}{3}$. Then, find the arc length of the curve from $t = \frac{4\pi}{5}$ to $t = \frac{5\pi}{3}$.

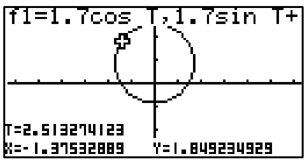
<u>Solution</u>
(a) We begin by exploring the graph of the parametric equations which obviously are the equations of a circle. Open the <u>Graph Editor</u> strip "**Act2A**", set [Type] to parametric for this matter and graph $x = 1.7\cos t$, $y = 1.7\sin t + 0.85$. Use appropriate View Window setting to view the graph.

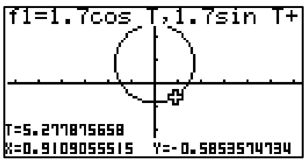




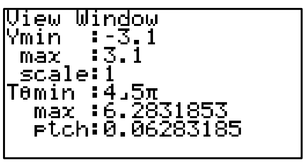
Clearly the graph drawn is a circle, and for the matter of discussion this circle has centre (0, 0.85) and radius of 1.7.

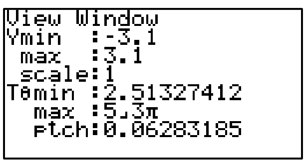
(b) Trace the graph especially along the section of $\frac{4\pi}{5} \le t \le \frac{5\pi}{3}$, understanding that $\frac{4\pi}{5} \approx 2.513$ and $\frac{5\pi}{3} \approx 5.236$.

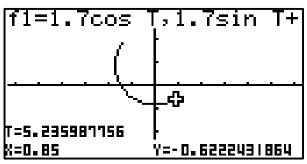




(c) With the circle still drawn, tap F3 to enter View Window, set [Tθmin] as $\frac{4\pi}{5}$ and [Tθmax] as $\frac{5\pi}{3}$, then graph the circle again.







The section of the circle drawn above is the graph of $x = 1.7\cos t$, $y = 1.7\sin t + 0.85$ for $\frac{4\pi}{5} \le t \le \frac{5\pi}{3}$.

To find the arc length of the curve for $\frac{4\pi}{5} \le t \le \frac{5\pi}{3}$ is to determine the distance traversed by section of the circle shown in the graphs above. To find the arc length we will use the integral formula of (see any calculus book)

$$S = \int_{t1}^{t2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt .$$

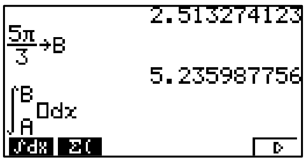
Here we know that $t_1 = \frac{4\pi}{5}$ and $t_2 = \frac{5\pi}{3}$; and the derivatives of $x = 1.7\cos t$ and $y = 1.7\sin t + 0.85$ with respect to t are

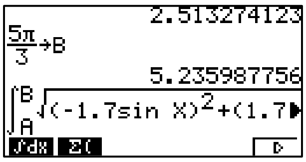
$$x = 1.7\cos t \qquad \Rightarrow \frac{dx}{dt} = -1.7\sin t$$

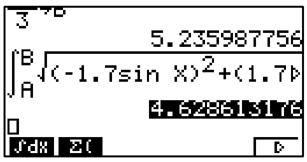
$$y = 1.7\sin t + 0.85 \quad \Rightarrow \frac{dy}{dt} = 1.7\cos t$$

(d) Open the <u>Run</u> strip "**Act2B**". Assign $\frac{4\pi}{5}$ and $\frac{5\pi}{3}$ to some variables; the reason for doing this is to have more control over the parameters and a tidier screen.

(e) Tap F4 F6 F1 to access the definite integral. As the graphics calculator only performed integration with respect to x, we use x to represent the parameter t in the following calculations. So now to find the arc length of the curve from $t = \frac{4\pi}{5}$ to $t = \frac{5\pi}{3}$ we enter the definite integral of $\int_A^B \sqrt{(-1.7\sin t)^2 + (1.7\cos t)^2}\, dt$.







And so the arc length of the section of circle from $t = \frac{4\pi}{5}$ to $t = \frac{5\pi}{3}$ is ≈ 4.6286. □

EXERCISES

Exercise 1

Redo Activity 1 but this time, find the equation of normal to $t = \frac{\pi}{3}$.

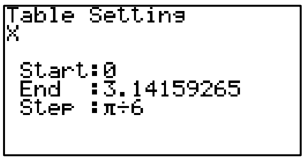
Exercise 2

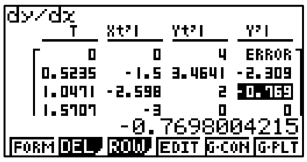
Sketch the curve represented parametrically as $x = 2\cosh t$, $y = 5\sinh t$, from $t = -\frac{\pi}{4}$ to $t = \frac{2\pi}{5}$. Then, find the arc length of the curve from $t = -\frac{\pi}{4}$ to $t = \frac{2\pi}{5}$.

SOLUTIONS to EXERCISES

Exercise 1

For this exercise we will not explore the ellipse graphically as we have done this at Activity 1. Open the Table Editor strip of "**Ex1A**", then make and explore the table of values of the ellipse. When $t = \frac{\pi}{3}$, we find the corresponding point as (1.5, 3.9641), with the slope -0.7698, correct up to 4 decimal places.

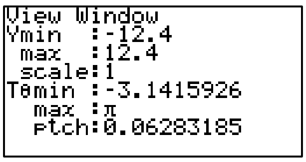


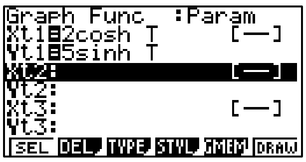


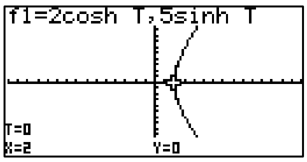
Based on the property that slope of normal is the negative reciprocal of slope of tangent, and using the slope-point form, the equation of normal to the ellipse $x = 3\cos t$, $y = 4\sin t + 0.5$ at $t = \frac{\pi}{3}$ is $y - 3.9641 = \frac{1}{0.7698}(x - 1.5)$. □

Exercise 2

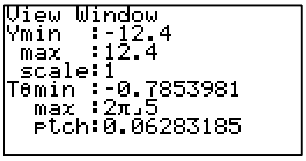
Open the Graph Editor strip "**Ex2A**", set [Type] to parametric and enter the parametric equations of $x = 2\cosh t$, $y = 5\sinh t$ (use OPTN F3 to access hyperbolic functions.) We could set [Tθmin] as $-\pi$ and [Tθmax] as π to view more of the graph.

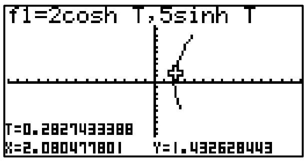




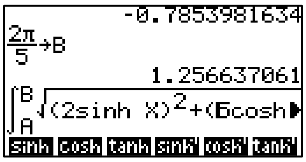


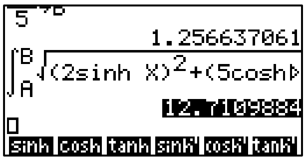
Apparently this is one branch of the hyperbola $x = 2\cosh t$, $y = 5\sinh t$. With the hyperbola still drawn, tap F3 to enter View Window, set [Tθmin] as $-\frac{\pi}{4}$ and [Tθmax] as $\frac{2\pi}{5}$ and graph again to view the section of the hyperbola for $t = -\frac{\pi}{4}$ and $t = \frac{2\pi}{5}$.





The derivatives of $x = 2\cosh t$ and $y = 5\sinh t$ are $\frac{dx}{dt} = 2\sinh t$ and $\frac{dy}{dt} = 5\cosh t$. Enter the definite integral of $\int_A^B \sqrt{(2\sinh t)^2 + (5\cosh t)^2}\,dt$ in the Run strip "**Ex2B**" to calculate the arc length. Here A and B are assigned the values of $-\frac{\pi}{4}$ and $\frac{2\pi}{5}$ respectively.





The arc length of the section of hyperbola from $t = -\frac{\pi}{4}$ to $t = \frac{2\pi}{5}$ is ≈ 12.71. □

REFERENCE

[1] Ong B.S. and Abdul Aziz Jemain, *Matematik STPM Jilid 1-Tulen*, Penerbit Fajar Bakti, 2001. ISBN: 967 65 6445 1.
[2] Elliot Mendelson, *Schaum's 3000 Solved Problems in Calculus*, McGraw-Hill, 1988. ISBN: 0-07-041480-7.

