N^{th} Roots of Complex Number

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LEVEL
High schools and first year of university after students have studied complex number.

OBJECTIVES
To discuss finding the n^{th} root of complex number.

CORRESPONDING eActivity
COMPLEX.g1e

OVERVIEW
Finding the n^{th} roots of complex number is a challenging task even for an advanced student. In this activity we discuss finding the n^{th} roots of complex number where the solution process is made simpler with the aid of graphics calculator.

EXPLORATORY ACTIVITIES
[Note]
We shall use small letter *x*, *y, z* instead of capital X, Y, Z as shown on the calculator throughout the paper.

Activity 1: Find the third roots of i.

Solution:
The traditional approach is to express i in polar form, apply the De Moivre's Theorem and find the general form and finally all the roots. For this discussion, our approach still needs application of De Moivre's Theorem, but the large part of solution process will be supported with the graphics calculator.
Suppose $z^3 = 0 + i$, then modulus of z^3 is $\sqrt{0+1^2} = 1$ and the argument $\theta = \tan^{-1}\frac{1}{0}$ is undefined, which implies that $\theta = \frac{\pi}{2}$.

Hence, the polar form of z^3 is $1(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}) = \cos(\frac{\pi}{2} + 2N\pi) + i\sin(\frac{\pi}{2} + 2N\pi)$, and the general form of the roots is

$$z = \left(\cos(\frac{\pi}{2} + 2N\pi) + i\sin(\frac{\pi}{2} + 2N\pi)\right)^{\frac{1}{3}}$$

Moreover, we can write $\left(\cos(\frac{\pi}{2} + 2N\pi) + i\sin(\frac{\pi}{2} + 2N\pi)\right)^{\frac{1}{3}}$ as $\left[e^{(\frac{\pi}{2}+2N\pi)}\right]^{\frac{1}{3}}$, so

$$z = \left[e^{(\frac{\pi}{2}+2N\pi)}\right]^{\frac{1}{3}} = e^{\frac{1}{3}(\frac{\pi}{2}+2N\pi)}$$

And finally we can re-express the above as

$$z = \cos\frac{1}{3}\left(\frac{\pi}{2} + 2N\pi\right) + i\sin\frac{1}{3}\left(\frac{\pi}{2} + 2N\pi\right) \qquad \text{where } N=0,\ 1,\ 2.$$

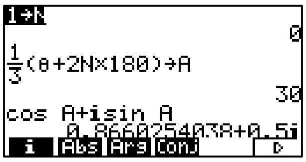
<u>Graphics Calculator Support</u>

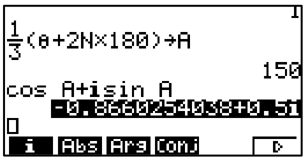
We now place the calculation of the three roots at the calculator through variables substitution. Open the <u>Run</u> strip "**Act1A**" and store the modulus and argument of the complex number to r and θ respectively. Here we set the calculator to display angles in unit of degree, so we replace π with $180°$.

Allow values to be stored into the variable N and allow the value calculated from the term $\frac{1}{3}\left(\frac{\pi}{2} + 2N\pi\right)$ to be stored into a variable. Recall that $\frac{\pi}{2}$ is the argument θ, and replace π with $180°$.

Now calculate the first root when N=1.

After that scroll up to assign 1 to *N* to find the second root, then repeat this by assigning 2 to *N* to find the third root.





Hence the three roots of i are $\pm 0.8660 + 0.5i$ and $-i$.

<u>Verify the Solutions</u>
While still inside the "**Act1A**" strip, scroll to the last line and use Answer Memory to verify the solutions. For this activity, we want to show that Ans^3 is indeed i. Verify for a value of *N* first then scroll up to verify the solutions obtained from other values of *N*.

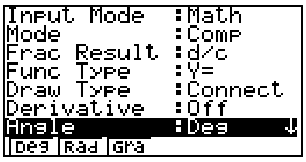
- When *N*=2, i.e. when root is i

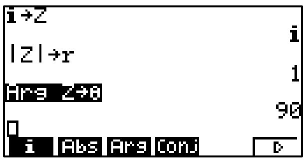
- When *N*=1 and 0, i.e. when root is $-0.8660 + 0.5i$ and $0.8660 + 0.5i$ respectively.

Therefore we have shown the third roots of i are $\pm 0.8660 + 0.5i$ and $-i$. □

The calculations arrangement in Activity 1 can be improved to be more general and thus able to calculate for n^{th} roots of other complex numbers. Open the <u>Run</u> strip "**Act1B**" and enter the following sequence of operations to redo Activity 1.

- Set the angle to display in unit of degree and calculate the modulus and argument of i in the calculator, assigning them to the variables *r* and θ. Also, set up for values to be stored into *N* and the general term.





- Set up operation to find the roots and verifying them immediately.

This systematic calculations sequence requires only a few changes if we want to solve for the n^{th} roots of a different complex number.

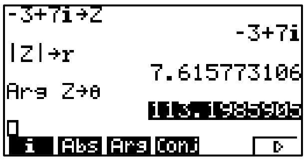
Activity 2: Find the seventh roots of $-3+7i$ and verify the answer.

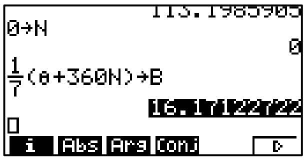
<u>Solution</u>:
With the improved calculation sequence we do not need to find the modulus or argument of the complex number first. However, we do need the general form of the roots.

Suppose $z^7 = -3+7i$ where the modulus of $-3+7i$ is r and the argument is θ. So the polar form of z^7 is $r(\cos(\theta+2N\pi)+i\sin(\theta+2N\pi))$, and the general form of the roots is

$$\begin{aligned} z &= r^{\frac{1}{7}}(\cos(\theta+2N\pi)+i\sin(\theta+2N\pi))^{\frac{1}{7}} \\ &= r^{\frac{1}{7}}\left(\cos\frac{1}{7}(\theta+2N\pi)+i\sin\frac{1}{7}(\theta+2N\pi)\right) \quad \text{where } N=0, 1, 2\ldots 6 \end{aligned}$$

Open the <u>Run</u> strip "**Act2A**", calculate and store the modulus and argument of $-3+7i$ to r and θ respectively. You can find r and θ at the [x^2] and [^] keys. Be sure to set up the calculator to display angles in unit of degree.





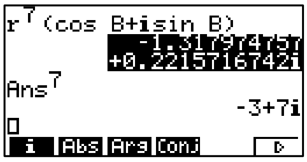
Access [Complex] with [OPTN] [F3] and tap [F2] and [F3] to use the absolute and argument features. Set up the variable N, then calculate and store the value from the term $\frac{1}{7}(\theta+360N)$ to a variable.

After we have entered the calculations to find the roots and verifying them, find the roots by assigning 0, 1, 2…6 to N. The first root when N=1 is

The next three roots when *N*=1, 2, 3.

N=1

N=2

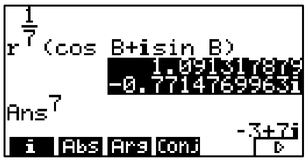


N=3

Finally the roots when *N*=4, 5, 6.

N=4

N=5

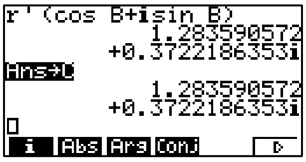


N=6

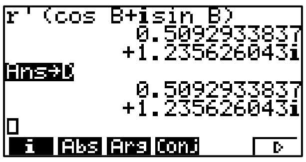
Thus we have found all seven roots and verified the answers. □

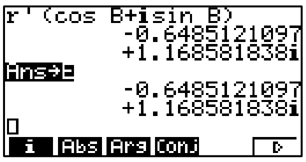
We can plot the seventh roots of $-3+7i$ on the same Argand plane to give us visualization of the solutions. The graphics calculator can do the plotting but the process is slightly indirect.

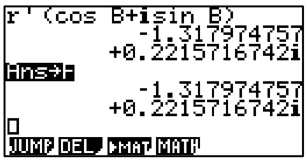
Open the strip "**Act2B**" and redo the calculations sequence of Activity 2. However, instead of checking the answer, enter an operation to assign the answer to a variable. In the display below we are storing the root when *N*=0 to the variable C.

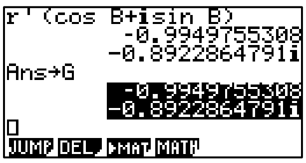


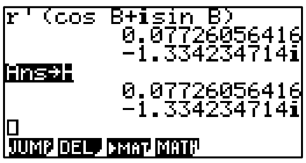
Continue by inputting the next values of *N*=1, 2…6 and store the answers to the variables D, E, F…I. Be sure to first change the variable, scroll up to change input to *N*, only then tap EXE while cursor is on the *N* line.

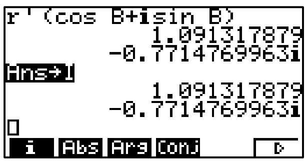








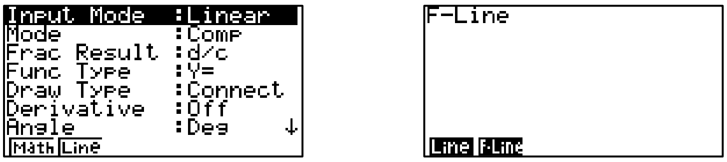




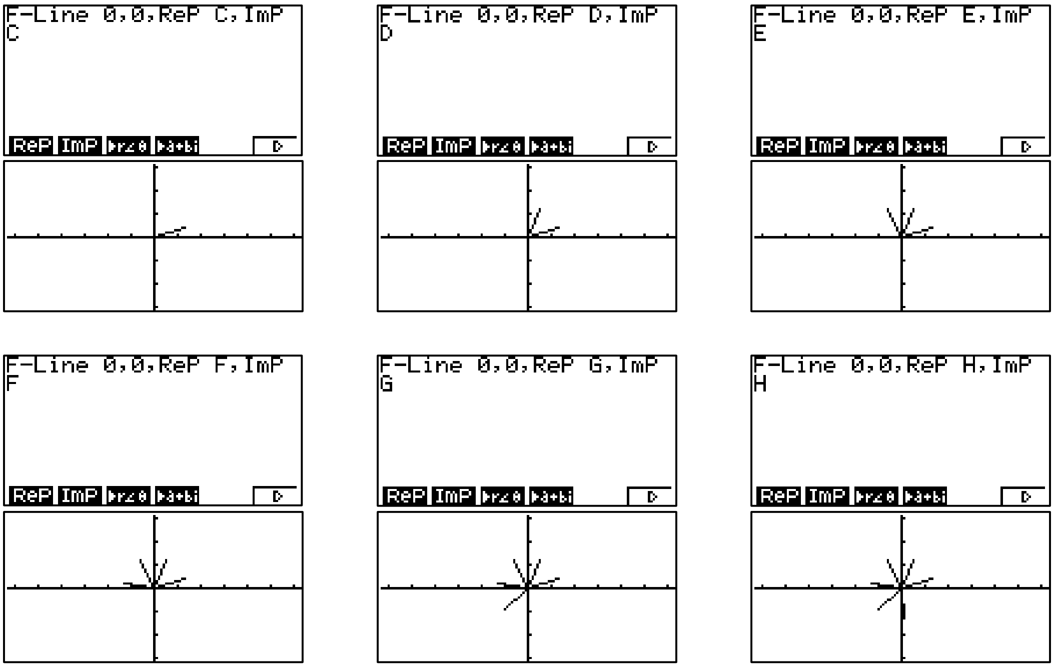
The variables and the corresponding roots stored can be summarized as follow:

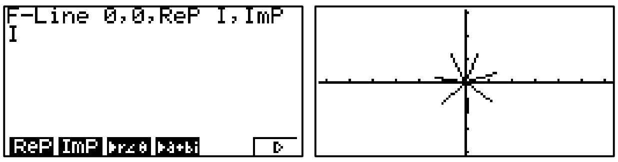
Root when *N*=	0	1	2	3	4	5	6
Variable stored	C	D	E	F	G	H	I

Having done that, change the input mode to [Linear]. Then EXIT and tap SHIFT F4 F6 F2 to access the [F-Line] function. Also, set the Viewing Window to [INI].



Tap OPTN F3 F6 to access [ReP] and [ImP] to plot the roots on the Argand plane with the real and imaginary parts of the roots, beginning with the root stored in C, followed by the remaining roots stored in D, E, F…I.





Finally all roots of $-3+7i$ are plotted on the Argand diagram for visualisation.

EXERCISES

Exercise 1

Find the fourth roots of $2 - i\sqrt{5}$. Verify your answer and plot the complex numbers and roots on the same Argand plane.

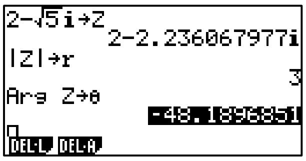
SOLUTIONS to EXERCISES

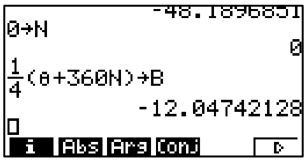
Exercise 1

Suppose $z^4 = 2 - i\sqrt{5}$ where the modulus of $2 - i\sqrt{5}$ is r and the argument is θ. The polar form of z^4 is $r(\cos(\theta + 2N\pi) + i\sin(\theta + 2N\pi))$, and the general form of the roots is

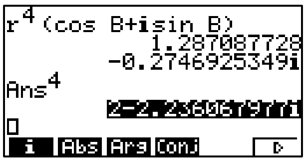
$$\begin{aligned} z &= r^{\frac{1}{4}}(\cos(\theta + 2N\pi) + i\sin(\theta + 2N\pi))^{\frac{1}{4}} \\ &= r^{\frac{1}{4}}\left(\cos\frac{1}{4}(\theta + 2N\pi) + i\sin\frac{1}{4}(\theta + 2N\pi)\right) \quad \text{where } N=0, 1, 2, 3 \end{aligned}$$

Open the Run strip "**Ex-1A**", calculate and store the modulus and argument of $2 - i\sqrt{5}$ to r and θ respectively. Set up the calculator to display angles in unit of degree. Also set up the variable N, then calculate and store the value from the term $\frac{1}{4}(\theta + 360N)$ to B.



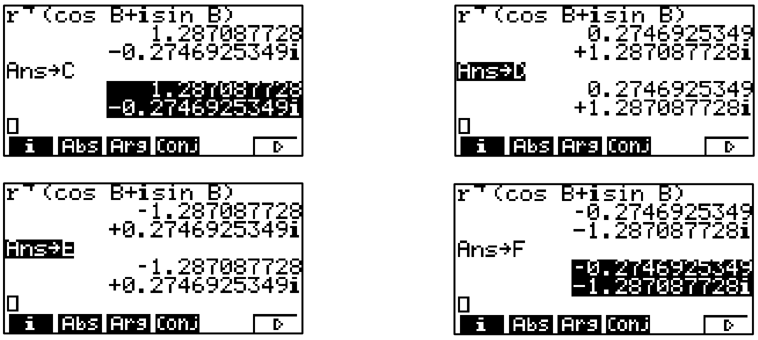


The root when N=0 and the verification of the answer.

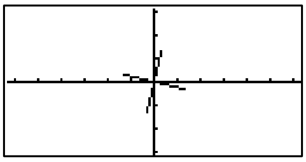


The roots when N=1, 2, 3 and the verification of the answers.

After finding all roots and verifying them, scroll to the last line and change the operation to store the answers produced when N=0, 1, 2, 3, to variables C, D, E and F respectively. These variables will be used to plot the solutions on the same Argand plane.



Having done that, change the input mode to [Linear] and set the Viewing Window to [INI]. Now plot the roots on the Argand plane with [F-Line], beginning with the root stored in C, then followed by plotting the remaining roots stored in D, E and finally F.



The final Argand diagram would look like the diagram above. □

REFERENCE

[1] Ong B.S. and Abdul Aziz Jemain, *Matematik STPM Jilid 1-Tulen*, Penerbit Fajar Bakti, 2001. ISBN: 967 65 6445 1 .

