Matrices

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LEVEL
High School and University

OBJECTIVES
In this paper, we use matrices in row echelon form to solve systems of linear equations with the aid of the calculator.

Corresponding eActivity
Matrix.g1e

OVERVIEW
A matrix is said to be in **row echelon form** if the following conditions are satisfied:

a. the first nonzero entry in each row is a 1.
b. the index *j* of the column in which the first nonzero entry of a row occurs is less than the column index of the first nonzero entry of the next row.
c. Any rows consisting entirely of 0s occur at the bottom of the matrix.

In this paper, we show how systems of equations are solved by reducing the matrix corresponding to the system of equations to row echelon form, using elementary row operations aided by the calculator. There are other ways to solve systems of equations using the calculator. One way is by setting the calculator to EQUATION mode or by solving the inverse of the coefficient matrix in the RUN editor. This paper will deal with systems which cannot be solved using these methods.

ACTIVITIES
Example 1. Solve the system

$$x + y + z = 3$$
$$2x + y + 4z = 8$$

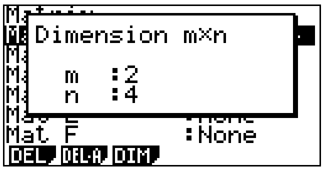
Matrices

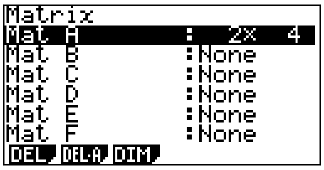
Solution:

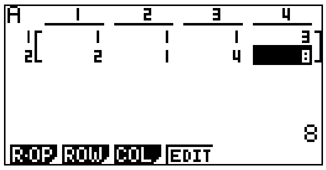
We begin by writing the system in matrix form as follows:

$$\begin{pmatrix} 1 & 1 & 1 & 3 \\ 2 & 1 & 4 & 8 \end{pmatrix}$$

We access the RUN editor and solve this matrix. We first set the dimension of the matrix and store the matrix entries as follows:





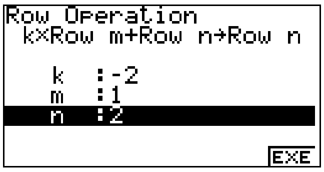


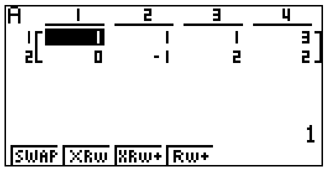
We reduce the matrix of the system to row echelon form. We illustrate how the calculator is used to perform elementary operations.

1. We first multiply row 1 with -2 and add to row 2, entering the results to row 2.

$$\begin{pmatrix} 1 & 1 & 1 & 3 \\ 2 & 1 & 4 & 8 \end{pmatrix} \xrightarrow{-2r_1+r_2} \begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & -1 & 2 & 2 \end{pmatrix}$$

In the calculator we enter the information in xRw+ to get the resulting matrix:

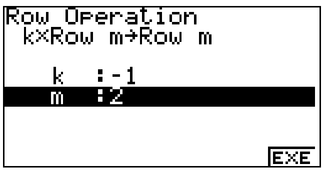


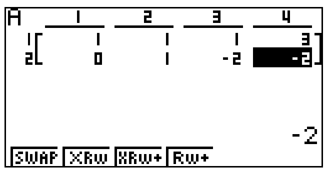


2. The next step would be to multiply -1 to row 2 and replace row 2 with the results.

$$\begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & -1 & 2 & 2 \end{pmatrix} \xrightarrow{-r_2} \begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & -2 & -2 \end{pmatrix}$$

This time we enter the information in xRw to get the resulting matrix:

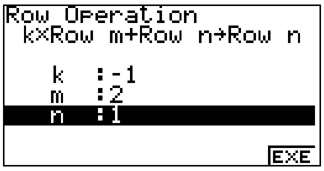


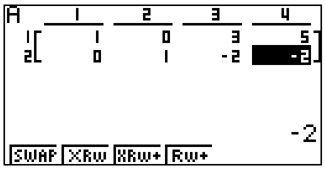


3. Lastly, we multiply -1 to row 2 and add to row 1 and enter the results in row 1.

$$\begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & -2 & -2 \end{pmatrix} \xrightarrow{-r_2+r_1} \begin{pmatrix} 1 & 0 & 3 & 5 \\ 0 & 1 & -2 & -2 \end{pmatrix}$$

We enter the results in xRw+ to get the final matrix:





The resulting matrix $\begin{pmatrix} 1 & 0 & 3 & 5 \\ 0 & 1 & -2 & -2 \end{pmatrix}$ corresponds to the system

$$x + 3z = 5$$
$$y - 2z = -2$$

or equivalently, $x = -3z + 5$ and $y = 2z - 2$.

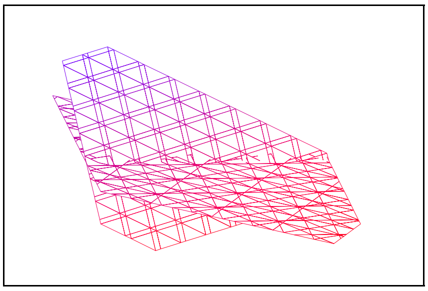
The system consists of all ordered triples $(-3z+5, 2z-2, z)$ where z can be any real number. The system has an **infinite number of solutions**. The graphs of the linear equations

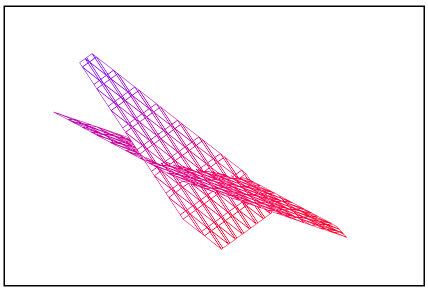
$$x + y + z = 3 \text{ and } 2x + y + 4z = 8$$

are planes in three-dimensional space. For a system of two equations and three variables, a solution exists if the two planes intersect in a line. In this example, the set of solutions consists of points on the line *l* where *l* is obtained by translating the line of scalar multiples of the vector (-3,2,1) by the vector (5,-2,0). Note that

$$(-3z+5, 2z-2, z) = (5,-2,0) + z(-3,2,1)$$

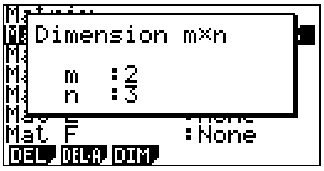
The intersection of the two planes are shown below:

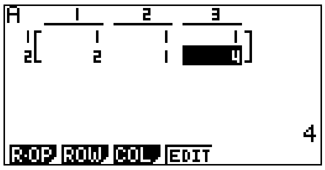


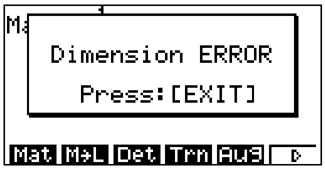


Remarks: The above system consists of two equations in 3 unknowns. Note that the calculator can solve for 3 unknown variables in its equation editor only when given 3 equations. Thus, we cannot solve the system using this approach.

Moreover the coefficient matrix of the system is given by $\begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 4 \end{pmatrix}$, which is not a square matrix, so that no inverse exists. A dimension error will occur when we attempt to obtain the inverse of such a matrix in the RUN editor:







Hence, the system is solved by rewriting the augmented matrix $\begin{pmatrix} 1 & 1 & 1 & 3 \\ 2 & 1 & 4 & 8 \end{pmatrix}$ of the system of equations in its row echelon form.

Example 2. Solve the system

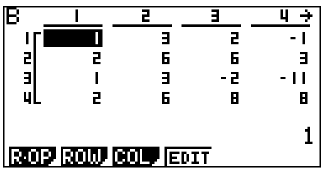
$$\begin{gathered} x + 3y + 2z - w = 6 \\ 2x + 6y + 6z + 3w = 16 \\ x + 3y - 2z - 11w = -2 \\ 2x + 6y + 8z + 8w = 20 \end{gathered}$$

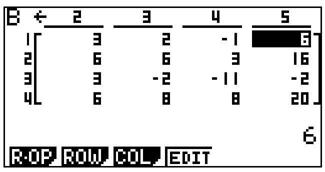
Solution:

We write the system in matrix form.

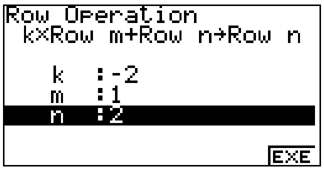
$$\begin{pmatrix} 1 & 3 & 2 & -1 & 6 \\ 2 & 6 & 6 & 3 & 16 \\ 1 & 3 & -2 & -11 & -2 \\ 2 & 6 & 8 & 8 & 20 \end{pmatrix}$$

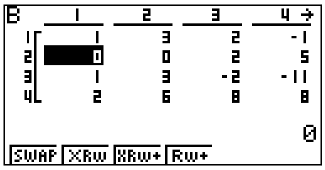
This is entered in the <u>RUN editor</u> as follows:

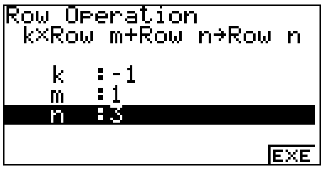


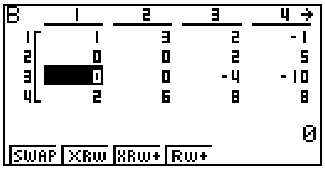


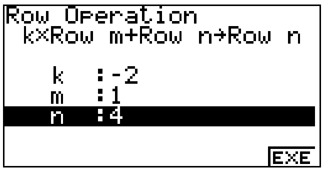
We use elementary row operations on the system, until we obtain a matrix in row echelon form. Here are the screen dumps showing the row operations performed in a particular step and the resulting matrix:

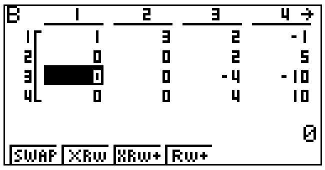


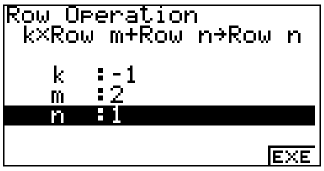


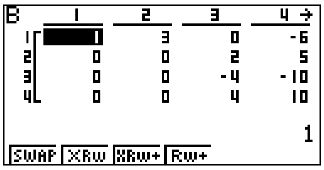


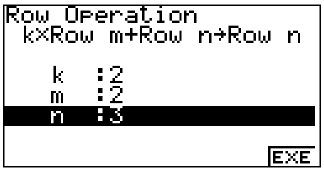


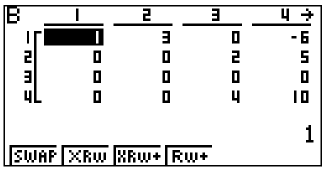


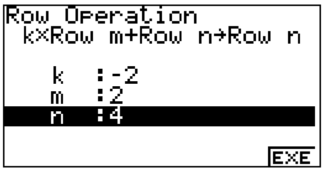


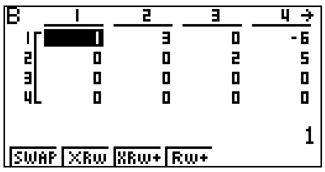


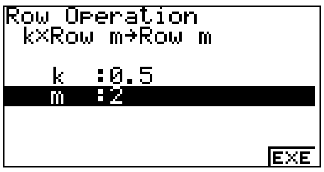


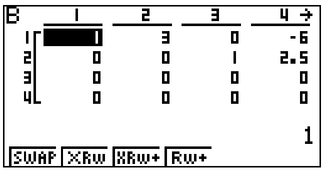


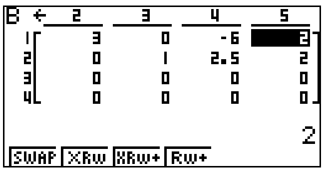












The last matrix is

$$\begin{pmatrix} 1 & 3 & 0 & -6 & 2 \\ 0 & 0 & 1 & \frac{5}{2} & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

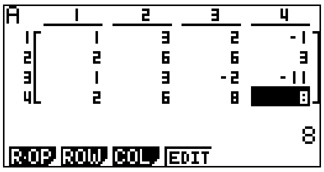
Translating back into equations, we get

$$x+3y-6w=2$$
$$z+5/2w=2$$

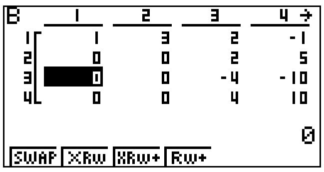
or equivalently, $x = 2-3y+6w$
$z = 2-5/2w$

The solutions to the system consists of all ordered 4-tuples of the form $(2-3y+6w,y,2-5/2w,w) = (2,0,2,0)+ y(-3,1,0,0) + w(6,0,-5/2,1)$ where y and w can be any real numbers. These 4-tuples are called a two-parameter family of solutions because the two variables y and w can be chosen independently.

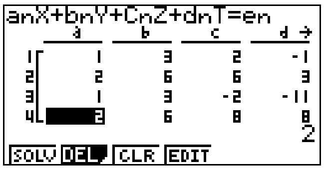
Remarks: In this example, the matrix of coefficients $\begin{pmatrix} 1 & 3 & 2 & -1 \\ 2 & 6 & 6 & 3 \\ 1 & 3 & -2 & -11 \\ 2 & 6 & 8 & 8 \end{pmatrix}$ does not have an inverse. Observe that a mathematical(MA ERROR) error occurs when we get its inverse:



Matrices that do not have inverses are singular, which indicate that at least one of the rows is linearly dependent on the other rows. This is evident in the 3rd row operation, note that the third and fourth rows are negatives of each other:



For the same reason, the calculator will give a mathematical error(MA ERROR) if you try to solve the equation in EQUA mode or SIML mode when in eactivity mode:



Exercise. A casting company produces three different bronze sculptures. The casting department has available a maximum of 140 labor hours per week, and the finishing department has a maximum of 180 labor hours available per week. Sculpture 1 requires 30 hours for casting and 10 hours for finishing. Sculpture 2 requires 10 hours for casting and 10 hours for finishing. Sculpture 3 requires 10 hours for casting and 30 hours for finishing. If the plant is to operate at maximum capacity, how many of each sculpture should be produced each week?

Solution:
Let x = number of sculpture 1 produced each week
y = number of sculpture 2 produced each week
z = number of sculpture 3 produced each week

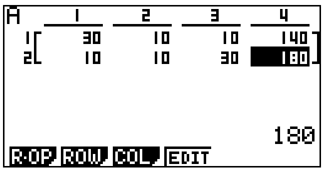
We have the following system of equations:

$$30x + 10y + 10z = 140$$
$$10x + 10y + 30z = 180$$

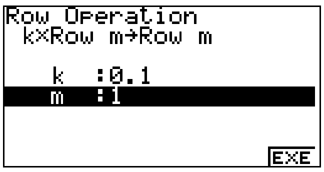
The system can be written in matrix form as follows:

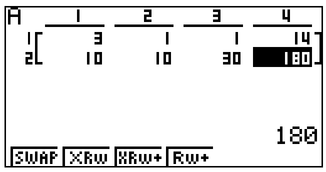
$$\begin{pmatrix} 30 & 10 & 10 & 140 \\ 10 & 10 & 30 & 180 \end{pmatrix}$$

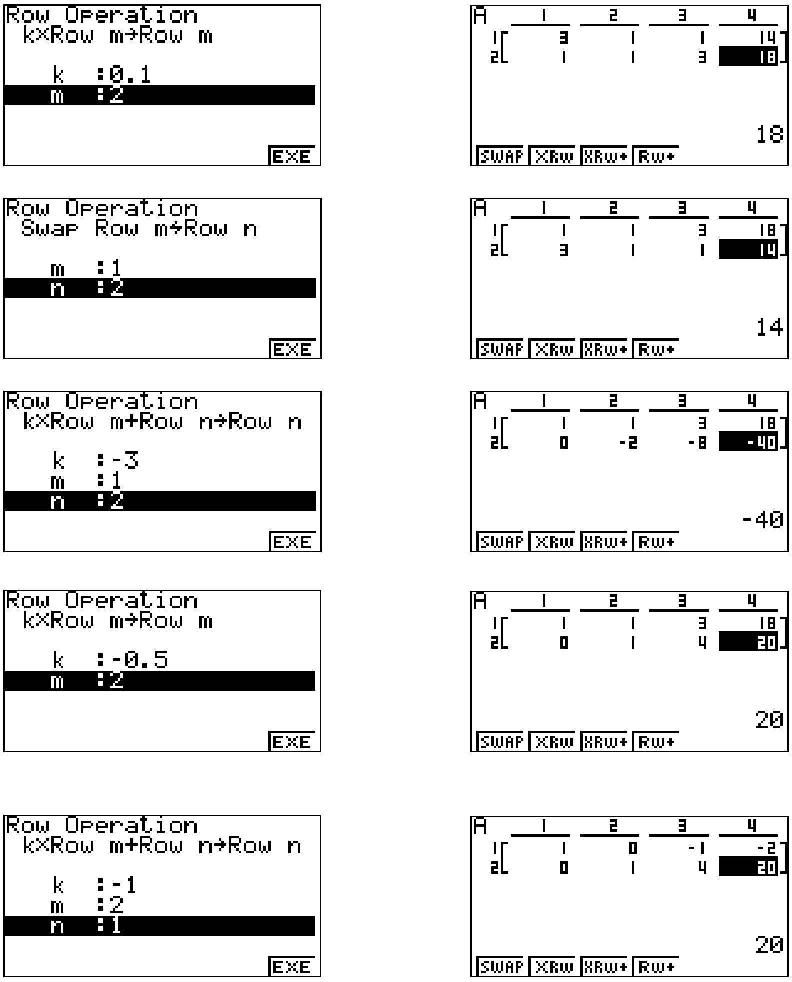
We enter the matrix in the RUN editor:



The following screen dumps illustrate the row operations performed on the matrix:







Translating back into equations, we get

$$x - z = -2$$
$$y + 4z = 20$$

or equivalently,

$$x = z - 2$$
$$y = 20 - 4z$$

The solutions to the system consists of all ordered triples of the form (z-2,-4z + 20, z). Since the variables x, y and z represent numbers of sculptures, they must be non-negative. Thus $z-2 \geq 0$ or $z \geq 2$. Also $20-4z \geq 0$ implies $z \leq 5$. Thus, the only possible values of z are 2, 3, 4 and 5.

We have the following table that shows the various possibilities:

SCULPTURE 1	SCULPTURE 2	SCULPTURE 3
0	12	2
1	8	3
2	4	4
3	0	5

REFERENCE

[1] Barnett, et al. *Applied Mathematics*, 7th Edition, Prentice Hall, 2000,ISBN: 0-13-083120-4.

