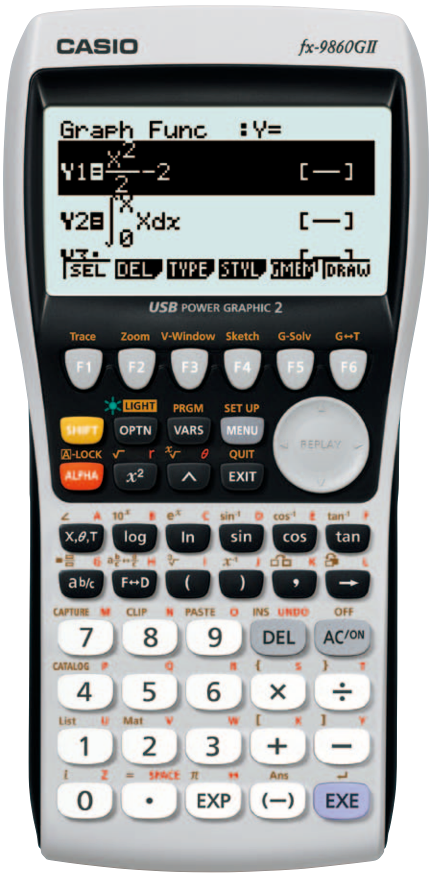
Learning Mathematics with Graphics Calculators



fx-9860GII

fx-CG20

Barry Kissane
&
Marian Kemp

Support Classroom with Technology

CASIO Worldwide Education Website

http://edu.casio.com

Reliable & Durable

PREFACE

Over 40 years, calculators have evolved from being computational devices for scientists and engineers to becoming important educational tools. What began as a tool to answer numerical questions has evolved to become an affordable, powerful and flexible environment for students and their teachers to explore mathematical ideas and relationships. CASIO's invention of a graphics screen on scientific calculators almost 30 years added significant educational power to these devices.

Following many years of advice from experienced teachers, the CASIO fx-9860GII series of calculators include substantial mathematical capabilities, innovative use of a graphics screen for mathematical purposes and extensive use of natural displays of mathematical notation. Continuing this evolutionary process, the CASIO fx-CG 20 includes a colour graphics screen and innovative applications to enhance its educational value.

This publication comprises a series of modules to help make best use of the opportunities for mathematics education afforded by these developments. The focus of the modules is on the use of the calculators in the development of students' understanding of mathematical concepts and relationships, as an integral part of the development of mathematical meaning for the students. The calculator is not just a device for computation, once the mathematics has been understood. It is now best thought of as a laboratory in which students can experiment with mathematical ideas and deepen their understanding of the mathematics involved. The similarities between the two calculator models allows users of either calculator to make use of the modules, except for three special modules which require the enhanced features of the CASIO fx-CG 20, the colour graphics calculator.

The mathematics involved in the modules spans a wide range from the early years of secondary school to the early undergraduate years, and we leave it to the reader to decide which modules suit their purposes. Although mathematics curricula vary across different countries, we are confident that the mathematical ideas included in the modules will be of interest to mathematics teachers and their students across international boundaries.

The modules are intended for use by both students and teachers. Each module contains a set of *Exercises*, focusing on calculator skills relevant to the mathematics associated with the module. In addition, a set of exploratory *Activities* is provided for each module, to illustrate some of the ways in which the calculator can be used to explore mathematical ideas through the use of the calculator; these are not intended to be exhaustive, and we expect that teachers will develop further activities of these kinds to suit their students and their particular curriculum. The *Notes for Teachers* in each module provide answers to exercises, as well as some advice about the classroom use of the activities (including answers where appropriate). Permission is given for the reproduction of any of the materials for educational purposes.

Some of these materials are based on earlier materials written by the authors and published by the Mathematical Association of Western Australia, to whom we express our gratitude for permitting the materials to be more widely published in the present form. We are also grateful to CASIO for supporting the development of these materials, and appreciate in particular the assistance of Mr Yoshino throughout the developmental process.

We hope that users of these materials enjoy working with the calculators as much as we have enjoyed developing the materials and we wish both teachers and their students a productive engagement with mathematics through the use of these fine calculators.

Barry Kissane and Marian Kemp

Murdoch University, Western Australia

Table of Contents

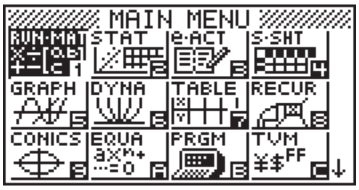
Module 1
Introduction to the calculators

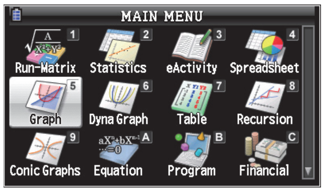
The CASIO fx-9860GII and CASIO fx-CG20 graphics calculators have many capabilities helpful for doing and learning mathematics. In this module, the general operations of the calculators are explained and illustrated to help you to become an efficient user of the calculator. Later modules assume that you have read this module.

Although the two calculators have a number of differences, they also share a number of similarities. In these modules, there may be slight differences between both the screens and commands used, but we are confident that you will be able to interpret these materials using one of these calculators. Most of the screens we use will show the CASIO fx-9860GII calculator, but you should find similar commands using the CASIO fx-CG 20.

Modes and [MENU]

Tap the [AC/ON] key to start. The calculator will show the most recent activity, but tap the [MENU] key to see the Main Menu. Graphics calculators allow many mathematical operations to be undertaken, and various modes of operation are shown here. Each of the icons refers to a particular mode.





As the two screens show, the two calculators represent similar information in slightly different ways. The names of the various modes are shown in full on the fx-CG 20 screen on the right, but are abbreviated in the fx-9860GII on the left.

To select a mode, use the keyboard to enter the number or letter key associated with it. Alternatively, move the cursor to highlight an icon for a mode and tap the [EXE] key. The screens above show the Run-Matrix mode highlighted on the left screen and the Graph mode highlighted on the right screen. These modes could be accessed directly by tapping [1] or [5] respectively. In brief, the various modes are for the following purposes:

Mode 1: Run-Matrix mode allows the calculator to be used like a scientific calculator, as described later in this and other modules. It is also the mode for conducting matrix operations, as you can define and use matrices and perform arithmetic with them.

Mode 2: *Statistics* mode is for various statistics, both univariate and bivariate, which are dealt with in the later Statistics modules. Both numerical and graphical analyses, as well as hypothesis testing, are made available by the calculators.

Mode 3: *eActivity* mode allows for educational activities to be constructed and used.

Mode 4: *Spreadsheet* mode provides spreadsheet capabilities.

Mode 5: *Graph* mode allows for functions of various kinds to be graphed and explored. As well as regular functions, polar functions and functions defined parametrically can be graphed.

Mode 6: *Dyna Graph* mode allows users to explore functions dynamically.

Mode 7: *Table* mode allows for functions of various kinds to be tabulated. Once functions are defined in either Graph or Table mode, they are accessible in both modes.

Mode 8: *Recursion* mode allows for sequences that have been defined recursively or explicitly to be tabulated and explored. Series can also be evaluated in this mode, which is a key aspect of Module 11.

Mode A: Equation mode is for solving equations of various kinds, including polynomial equations and systems of linear equations. We will use this mode extensively in Module 9.

Mode B: Programming mode is for writing and using calculator programs. These modules will not address this mode, but you will find detailed advice in the User Guide.

Mode C: Financial mode is various kinds of financial analysis. On the fx-9860GII calculator, the mode is described as TVM (Time Value of Money). We will not address in detail the use of this mode in these modules, but you will find detailed advice in the User Guide.

Mode H: Geometry mode allows for geometric objects to be defined and manipulated. We will rely on this mode in Module 5.

Mode I: Picture Plot mode is available only on the fx-CG 20 calculator. It allows for photographs to be analysed from a mathematical point of view, involving both mathematical modelling and data analysis. We will explore this mode in Module C3.

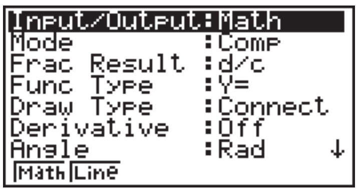
Mode K: Probability Simulation mode allows for experiments in probability to be conducted. We will explore this mode in Module 7.

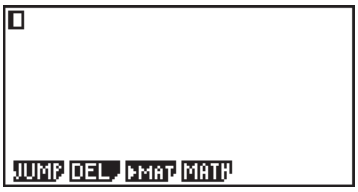
Notice that the calculator screen shows a maximum of twelve icons for modes Other modes, such as *Link*, *Memory* and *System* are used to define how the calculator works, communicates with other devices and stores information; these are explained in detail in the User Guide.

Further details on the use of many of these modes will be provided in later modules. For now, select Run-Matrix mode to see further how the calculator is operated.

Input mode

Tap MENU 1 to enter Run-Matrix mode, where the calculator works like a scientific calculator. We suggest that you set your calculator to use natural mathematical display, shown as Math on these two calculators. To do so, tap SHIFT and then MENU to access the SET UP menu, shown on the left below.





There are two choices for Input and output, described as *Math* and *Linear*. Tap the F1 key to select Math (written on the calculator screen directly above the F1 key). Notice that the choice of Math is shown at the top of the screen.

Tap EXIT to return to the home screen, shown at the right above. The screen is now ready for commands to be entered; don't be concerned if there are already some commands showing. Notice that there are four menu items showing at the bottom of the screen, which will remind you that Math mode has been chosen. [In the fx-CG 20, this choice is also printed at the top of the screen.]

Entering and editing commands

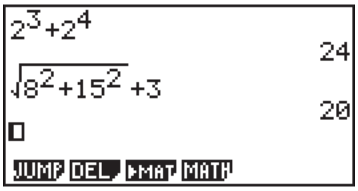
We will start with some computations. Enter a command followed by the [EXE] key to execute the command. There are some examples below, showing that you can enter many expressions, such as fractions and radicals, in the same ways in which you would write them by hand, because Math mode is chosen to represent expressions in ‘natural display’.

Notice that the calculator gives exact results when possible (such as for √12) and scientific notation when numbers are too large to fit the screen; the large number shown in the second screen represents $1.071476254 \times 10^{11}$.

Fractions can be entered into the calculator using the [a b/c] key. Tap [SHIFT] [a b/c] to enter a mixed number. Use the cursor keys [▲] and [▼] to move between numerator and denominator.

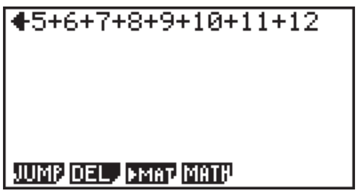
Exact or fractional answers can be changed to decimal answers (and vice versa) by tapping the [F↔D] key after the result is shown; if you tap the key several times, you can see how the result can be toggled back and forth between fractions and decimals. A result shown as an improper fraction can be changed to a proper fraction by using [SHIFT] [a b/c] (i.e. ($a\frac{b}{c} \leftrightarrow \frac{d}{c}$)).

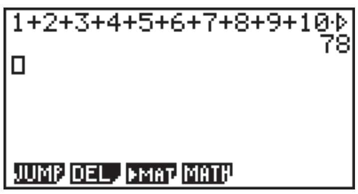
The power key is [∧] and there is a special key for squaring, [x^2]. With the calculator set in Math mode, complicated expressions can be written in the standard way. You will sometimes have to use the right cursor key [▶] to move out of an expression involving a fraction, power or radical. The examples below will help you to see how this works.



You should use the [(−)] key to enter negative numbers, as in the screen below. (The [−] key is for subtraction.) Look carefully at the screen below to see that a subtraction sign is slightly longer than a negative sign. Check for yourself to see that if you use the incorrect symbol, the results will often be incorrect.

If a command is too long to fit on the screen, it is still acceptable for entry, as the screens below show. The command entered is to add the first twelve counting numbers.





As you can see, the calculator automatically shows arrows on the display when a command is longer than the display. If you need to check or edit what has already been entered, you can move backwards and forwards with the two cursor keys ◀ and ▶ (These keys are on opposite sides of the large circular REPLAY key at the top right of the keyboard.). Note especially that if you tap ▶ when the cursor is at the right end of the display, it will jump to the left end; similarly, if you tap ◀ when the cursor is at the left end of the display, it will jump to the right end. Tap [EXE] at any stage to calculate the result.

It is not necessary to return the cursor to the end of the display before tapping [EXE]. Notice that only the first part of the command is shown, although the clear arrow indicates that there is more to be seen.

If you make an error when entering a command, you can erase it and start again using the [AC/ON] key or you can edit it using the [DEL] key. Position the cursor to the write of a character and tap [DEL] to delete a single character. You can then add another character by entering it from the keyboard.

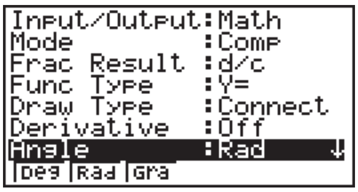
Characters can also be inserted using [SHIFT] [DEL], but it is generally not necessary to do this. Try this for yourself by entering the above command and then editing it to replace the 8 with a 3 before you tap [EXE].

Menus

Many calculator commands are not shown on the keyboard, since it would be much too cluttered and inefficient to do this. (There are many more commands than there are keys.) Most of the calculator commands are contained in *menus*, some of which you have already used.

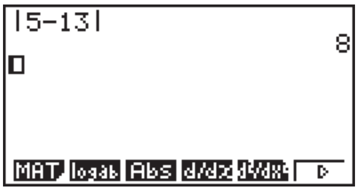
Up to six menu items are shown at once on a calculator screen, immediately above the function keys, labelled F1 to F6.

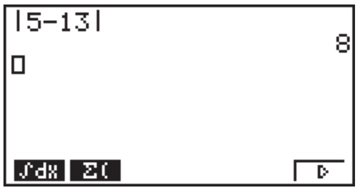
For example, in each of the modes of operation, the calculator can be set up to work in different ways. These are shown in the SET UP menu, which is accessed with [SHIFT] and [MENU]. You can use the ▲ and ▼ keys to see the various menus and their settings, and adjust them using the appropriate F keys to suit your needs. The example below shows that the calculator has been set to radians, using the F2 key.



Notice the arrow at the bottom of the screen, informing you that there are further menus below. Tap the [EXIT] key when finished.

In Run-Matrix mode, you may have noticed the four menu commands showing at the bottom of the screen. Look carefully at these to see that each command is in a shaded rectangle with the bottom right corner cut off. These show that the *name* of a menu is involved. Selecting the function key will reveal the contents of the menu; if you then select the [EXIT] key, you will return to the original menu name. The first screen below shows what happens if you tap F4 to access the MATH menu.





This menu contains another menu name (MAT, for dealing with matrices, which we will explore in Module 13) as well as some menu commands. The menu *commands* are shown as shaded rectangles without the corner cut off. If you tap the associated F key, the command will be pasted into the calculator screen. The middle screen above shows the absolute value function, obtained with F3; once the command is entered into the screen, it can be used, as shown.

A third kind of menu item involves a clear rectangle. Selecting these will have an *immediate* effect, such as those used in the SET UP menu earlier. In the example here, F6 will immediately ‘turn the corner’ to show further parts of the Math menu, shown in the third screen above.

These three kinds of menus (*names*, *commands* and *immediate*) are used throughout the calculator, and it is a good idea to experiment a bit with them now to become familiar with them. Remember to use the EXIT key to move back through a menu; notice that a menu is not removed from the screen with AC/ON or DEL keys.

An important set of menus is available through the OPTN key, with the first two pages of the menu shown below. (Use the F6 key to get the second page from the first page.)

We will explore many of these menus in later modules. For now, as an example, notice how the *Prob* menu (revealed with F3) provides some mathematical commands associated with probability:

The first command requires that 5 is entered followed by the factorial symbol (via F1) and EXE to obtain the result of $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$.

To get ${}_{20}C_2$, you need to enter 20, then the combinations command (via F3) then 2 and EXE to get the result of 190.

The RAND menu needs to be accessed with F4 to obtain a menu of commands. The first of these (with F1) gives the command *Ran#*, which generates a random number between 0 and 1; you will be most unlikely to get the same result as that shown above, of course.

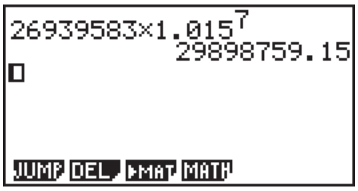
We will explore these commands in more detail in the Probability module.

There is a complete list of all the calculator commands in the Catalog, which is accessed with SHIFT 4. This is very helpful if you know the name of a command, but are not sure where it is located. Notice that you can access all commands starting with a certain letter by tapping the letter key. (Many of the commands will not make sense to you until you are familiar with them through use of later modules.)

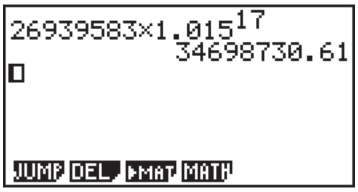
Recalling and editing commands

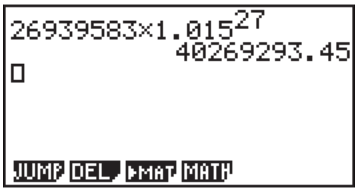
In Run-Matrix mode, you can access previous commands (provided you have not deleted them) using the ▲ key. If you move the cursor to a previous command, you can enter the command again by tapping [EXE], or can edit it using [DEL] and then tap [EXE]. This is a good way of performing several similar calculations in succession, without having to enter each complete command again.

For example, the following screen shows an estimate of almost 30 million for the 2020 population of Saudi Arabia, which had an estimated population of 26 939 583 in 2013, and a population growth rate of 1.5% per annum. Notice that a growth rate of 1.5% can be calculated by multiplying a number by 1.015.



The easiest way to obtain an estimate for later years, assuming the annual population growth rate stays the same, is to tap ▲ ▲ and then ◀ to edit the command to change the exponent of 10 to a different number each time. The screens below show the results for 2030 and 2040.





The growth rate of Saudi Arabia in 2013 will lead to a population of more than 40 million in 2040. The same calculator process could be used to predict the population if the growth rate was assumed to increase substantially from 1.5% to 2%, as shown below, where the number of years as well as the growth rate have both been edited.

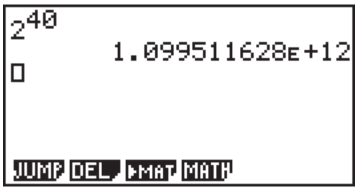
As you can see the population of Saudi Arabia is estimated to be almost 31 million in 2020, if the growth rate were to be increased to 2%, a figure around one million more than was predicted for a growth rate of 1.5%. Successive predictions of these kinds can be made efficiently in this way, without needing to enter long and complicated expressions more then once.

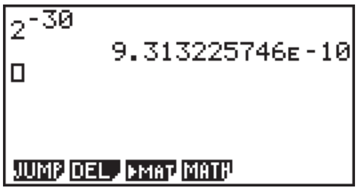
The list of commands will not be erased when you turn the calculator off, or change modes, but will be erased if you use the delete menu via F2 in the home screen, as shown below.

Notice that the delete menu allows you to delete a single line (with DEL.L via F1) or all lines (with DEL.A via F2).

Scientific notation

Scientific notation
When numbers become too large or too small to fit the screen, they will automatically be described in scientific notation, which involves a number between 1 and 10 and a power of 10. The precise way in which this happens depends on the decimal number format, which is described later in this module. To illustrate, the screen below shows two powers of 2 that require scientific notation to be expressed.



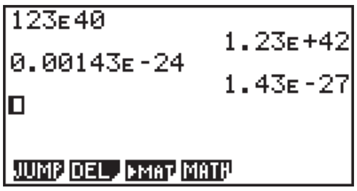


The precise value of the first result is 1 099 511 627 776, which does not fit on the screen, so it has been approximated using scientific notation. The number shown is a calculator representation of $1.099511628 \times 10^{12}$. Notice that the last digit has been rounded upwards. Similarly, the second result has been approximated from 0.000000000931322574615479… to fit the screen.

Numbers can be entered directly into the calculator using scientific notation. Start with the number between 1 and 10, tap the EXP key and then immediately enter the power of 10. For example, the average distance from the Earth to the Sun is 1.495978875×10^8 km, which can be entered in scientific notation as shown on the screen below.

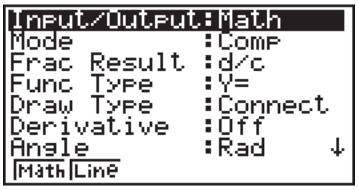
Notice that '$\times 10^8$' is shown on the screen as 'E8', although it is interpreted by the calculator as a power of ten. In the present mode used for display of results, notice also that the calculator does not regard this number as large enough to require scientific notation, and so it is represented as a number, indicating that the sun is on average about 149 597 887.5 km from the earth – an average distance of almost 150 million kilometres.

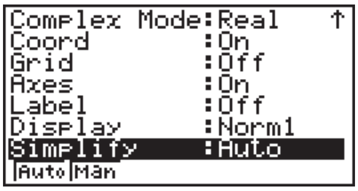
Scientific notation requires the first number to be between 1 and 10. So, if you use the EXP key to enter a number in scientific notation incorrectly (i.e. using a number that is not between 1 and 10), the calculator will represent it correctly in scientific notation, as the screen below shows.



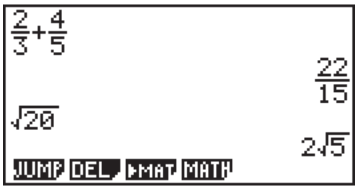
SET UP

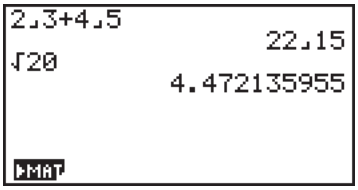
In any mode, the calculator can be set up in various ways by accessing SET UP (via SHIFT MENU). When you do this, you will notice that there are many items in the SET UP menu, and you can move from one to the other using the ▼ and ▲ cursor keys. The SET UP menu changes a little from one mode to another. The screens below show the items in Run-Matrix mode:





Input/Output
The calculator display can be set up to either natural display (*Math*) mode or single line (*Linear*) mode by tapping 1 or 2 respectively. Math mode allows for various mathematical expressions to be shown in the conventional way, and it is usually better to use this. In Linear mode, exact results will not usually be shown and the symbols will look a little different. It will also be harder to enter commands. For example, these two screens show the same information, the first in Math mode and the second in Line mode:





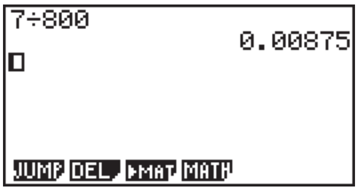
As well as looking different, it is slightly more difficult to enter the fractions in Linear mode, as the numbers need to be entered in precisely the same order as they are written. In addition, results are expressed exactly where possible in Math mode, but are shown as decimal approximations in Linear mode.

We suggest that you use natural display (Math) mode for almost all purposes.

Angles
The calculator can accept angles in degrees, radians or gradians, with the choice made in *Angle* in SET UP. We will discuss the differences between these and their respective uses in the Trigonometry module. Your choices can always be over-ridden in practice using the OPTN menu, as explained in Module 6. Most people leave their calculator in degrees if they are generally concerned with practical problems or radians if they are generally concerned with theoretical problems.

Display
There are a few choices for the way that numbers are displayed as decimals, accessed through *Display*. You can select *Fix* to specify the same number of decimal places for all results, *Sci* for scientific notation, *Eng* engineering notation, or *Norm* for all results in normal notation. It can sometimes be a useful idea to choose *Fix* or *Sci* (e.g., to ensure that all results are given in similar ways, especially if all results are money values), but we think it is generally best to choose *Normal* decimal formats, allowing the calculator to display as many decimal places as are appropriate.

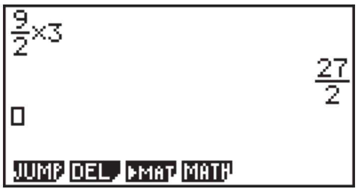
When *Normal* is chosen by tapping F3, there are two choices available, called *Norm1* and *Norm2*. (Tap F3 again to change the choice.) These are almost the same, except that using *Norm1* will result in scientific notation being used routinely for small numbers before *Norm2* will do so. For example, two screens below show the same calculation as a decimal after selecting *Norm1* and *Norm2* respectively, and using SHIFT = to force a decimal result.

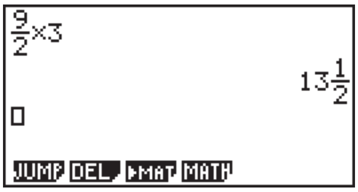


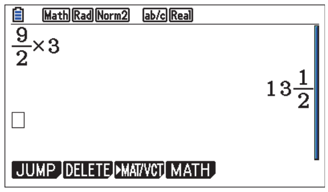
We suggest that it is generally better to choose *Norm2*, but you should decide this for yourself, as it is mostly a matter of personal preference and also depends on the kinds of calculations you generally wish to complete.

Fraction result format

The SET UP screen offers a choice of two ways of giving fraction results in the *Frac Result* menu: as proper fractions (represented as d/c) or as mixed fractions (represented as ab/c). In fact, results can easily be converted with ($a\frac{b}{c} \leftrightarrow \frac{d}{c}$) from one of these to the other (via [SHIFT] [F↔D]), so the decision is not very important.







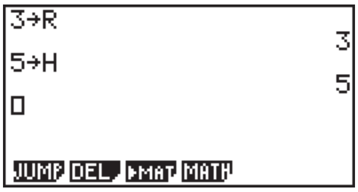
To illustrate the effects of the choices, the same fraction calculation has been completed in each of these two formats in the first two screens above.

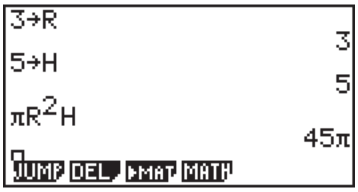
The third screen shows the same calculation on the fx-CG20. Notice that the top line of the screen displays some of the SET UP choices, while these can only be seen in the fx-9860GII model through accessing SET UP.

Memories

Calculator results can be stored in memories and retrieved later. This is convenient for recording values that you wish to use several times or for intermediate results. Variable memories, labelled A to Z, are available.

To store a result that is already showing on the calculator into a variable memory, first tap [→]. This applies to a number you have just entered or to the result of a calculation just completed. Then tap the pink [ALPHA] key followed by the memory key for the variable concerned, shown with pink letters above the keys on the keyboard. For example, the memory key for *H* is [F↔D] and that for *R* is [6]. The first screen below shows how to store a value of 3 into memory *R* and 5 into memory *H*.

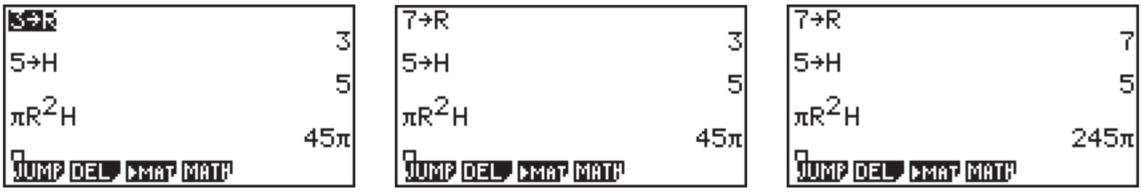




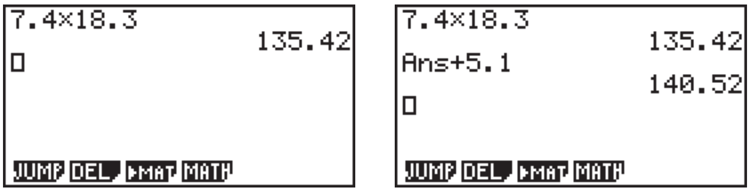
You can now regard *R* as a variable, with a present value of 3. To recall the present value of a variable, tap the [ALPHA] key, followed by the variable key. Variables are used on the calculator in the same way that they are in algebra. So, the second screen above shows the value of the expression $\pi R^2 H$, finding the volume of a cylinder with radius *R* and volume *H*.

Because the variable name *X* is used so often in mathematics, it can also be entered in the calculator and accessed without the [ALPHA] key, using the [X,θ,T] key. This is a more efficient process than using [ALPHA] [+], and we will use the [X,θ,T] key frequently in other modules.

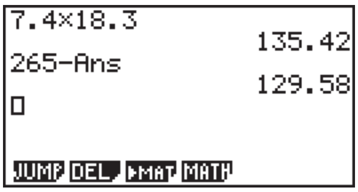
To change the value of a memory variable, you need to store a different number into the memory, as storing *replaces* any existing value. Turning off the calculator or changing modes will not delete the memory contents. The screen below shows that the value of *R* can be changed to 7 by using the calculator editing facilities explained earlier to edit the first store command. As soon as [EXE] is tapped (as shown in the third screen) *all* the command lines are executed, so that the result is changed also. Such techniques are very useful for repeated evaluations of a formula.



A very useful calculator memory is the Answer memory (*Ans*), which recalls the most recent calculator result. You might have seen this appearing when doing a succession of calculations. For example, the first screen below shows the calculator being used to find 7.4 x 18.3. When [+] [5] [•] [1] [EXE] is then pressed, the calculator assumes that the value of 5.1 is to be added to the previous result, which it refers to as *Ans*, since there is no number before the + sign. (*Ans* was not entered by the user.)



When a previous result is not to be used immediately, as it is in the above case, then the *Ans* memory can be recalled with [SHIFT] [(−)](*Ans*), as shown below to find 265 – (7.4 x 18.3) after first calculating the value in parentheses:



We will use the *Ans* memory extensively in Module 10, where it is especially useful.

Exercises

The main purpose of the exercises is to help you to develop your calculator skills.

1. Use the calculator to find 73 + 74 + 75 + 760 + 77 + 78. You should get a result of 1137. Then edit the previous command, changing the 760 to 76, and check that the resulting sum is now 453.

2. Express the square root of 32 as an exact number and as a decimal number.

3. Find cos 52^o.

4. The hypotenuse of a right triangle with shorter sides 7 and 11 can be found by calculating $\sqrt{7^2+11^2}$. Give this length as a decimal.

5. Use the calculator to evaluate $\frac{22}{3} \div \frac{14}{17}$. Then express the result as a decimal.

6. Find the seventh power of 17.

7. Use the MATH menu in the home screen to find $\log_3 81$.

8. When each person in a room of *n* people shakes hands with each other person in the room, there are $n\text{C}2$ handshakes. How many handshakes will there be if there are 38 students in a room and one teacher?

9. Evaluate 38!, which is the number of different orders in which the students in the previous question could line up outside their classroom. (Hint: tap OPTN and use the PROB menu).

10. Use the MATH menu or the NUM menu in OPTN to find the absolute value of 3.4 – 7.81, which is represented in standard mathematical notation as $|3.4-7.81|$.

11. Evaluate √2 + √3.

12. Evaluate $\left(\sqrt{4.1^2+5.3^2}\right)^5$.

13. As noted in the module, the population of Saudi Arabia was 26 939 583 in 2013, with a population growth rate of 1.5% per annum. If the population keeps growing at 1.5%, use the calculator to find out approximately when the population will reach 45 million. (Hint: To do this, enter a command and edit it successively until you get the desired result.)

14. Change the angle setting to radians and evaluate $\cos\frac{5\pi}{6}$ as both an exact number and as a decimal approximation.

15. Change the calculator Set Up to give all results with two decimal places. Check that this has been successful by finding the square root of 11.

16. Use the EXP key to enter and then find the square root of 1.4 x 10^{17}.

17. Give memory variables *A*, *B* and *C* the values of 7, 8 and 9 respectively. Then evaluate AB^2C.

18. Calculate the square of 34.5, but do not write down the result.
    Use the *Ans* memory to divide 8888 by your result.

Notes for teachers

This module is important for new users of the calculator, as it deals with many aspects of calculator use that are assumed (and so are not repeated) in other modules. The text of the module is intended to be read by students and will help them to start to see how the calculator can be used efficiently for various kinds of calculations.

Depending on the age and sophistication of the students, some parts of the introduction can safely be overlooked for later.

As a general principle, encourage students to look carefully at their calculator screens and to make sure that they understand what they are seeing. One valuable strategy is to ask students to predict what will happen before they tap the EXE key to complete a calculation. Making a prediction will help them to consider carefully what the calculator is being asked to do, and give them a stake (however small) in seeing that the answer produced is what they expected. Indeed, it can be a powerful and helpful lesson for students to make predictions that turn out to be incorrect, since this may encourage them to consider why their prediction did not eventuate.

It is generally a good idea for students to work with a partner, especially when they get stuck, so that they can discuss their ideas together and learn from each other. When students are exploring the calculator for the first time, make sure that they can easily return to the home screen in Run-Matrix mode using MENU 1 and that they can move through menus efficiently, especially with the EXIT key.

If an emulator and data projector are available to you, you may find it helpful to demonstrate some calculator operations to the whole class, or allow students to do this. This is also a good opportunity to emphasise the need to understand exactly what is showing on the screen and to predict what will be the effects of a particular operation.

The Exercises at the conclusion of this Module are best completed by students individually to develop calculator expertise. We have provided brief answers to these exercises below for your convenience; some teachers may be comfortable giving students the answers along with the exercises, so that they can check their own progress and seek help when necessary. This is your choice, of course.

Later Modules will also comprise some Activities for students to explore, but this introductory module is focussed on making sure that general calculator operations and settings are well understood, which will make later work with calculators more efficient. Nonetheless, some students may find aspects of mathematics to explore as a result of their introduction to calculator capabilities. We suggest that it is a good idea to allow them to do this, either by themselves or with other students, as many mathematical learning opportunities are offered by engaging with a classroom tool of this kind.

Answers to Exercises

1. self-checking 2. $4\sqrt{2}$, 5.657 3. 0.616 4. 13.038 5. 187/21, 8.905 6. 410 338 673 7. 4 8. ${}_{39}C_2 = 741$ 9. 5.230×10^{44} 10. 4.41 11. 3.146 12. 13 508.7714 13. Use $26939583 \times 1.015^{20}$ and edit the exponent to get the best approximation: about 35 years. So 2048 is the required year. 14. $-\sqrt{3}/2 \approx -0.866$ 15. 3.32 16. 374 165 738.7 17. 4032 18. 7.467

Module 2
Functions

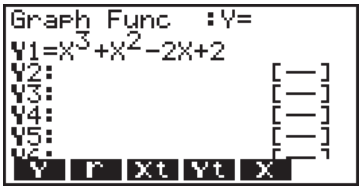
Note: This module is for the CASIO fx-9860GII series only. If you have the CASIO fx-CG 20 calculator, please refer instead to Module C1.

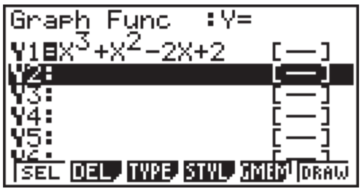
Functions are very important in mathematics, and are used to represent relationships in many ways and for many purposes. Graphics calculators have many capabilities helpful for understanding, representing and using functions. In this module, we focus especially on representing functions in three ways: symbolically, graphically and numerically, making use of the Graph and Table modes.

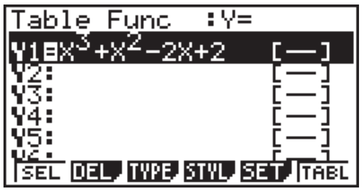
Symbols

Tap MENU 5 to access the symbolic menu for graphing and MENU 7 to access the same symbolic menu for tabulating. Use the ▲ and ▼ keys to select one of the 20 locations (Y1 to Y20) in which to store a function. The rule (or the symbolic formula) for a function can then be entered using the [X,θ,T] key for the variable, *X*, followed by the [EXE] key. (The calculator uses an upper case *X*, even though the normal representation uses a lower case *x*.)

If there is already a formula in the location you choose, you will replace it with any new formula you type. You can also delete any highlighted function by first tapping either DEL ([F2]) or the [DEL] key. The screen below shows the function $f(x) = x^3 + x^2 - 2x + 2$ stored in location Y1. Functions like this are also written as rules like $y = x^3 + x^2 - 2x + 2$. The calculator uses the same conventions as algebra: 2*X* means 2 x *X*, but it is not necessary to tap the multiplication key.



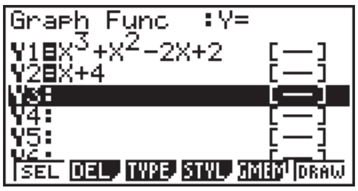




The calculator requires functions to be written as a function of *x* with *y* on the left of the equals sign. So a linear function such as $3x - y = 5$ must first be rearranged to $y = 3x - 5$ (and then represented as Y1 = 3X–5 on the calculator).

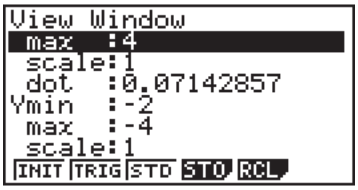
Tapping MENU 5 and MENU 7 allows you to jump from the symbolic menu in graph mode to the same menu in Table mode.

To deal with more than one function at a time, enter the two formulas in different places. The screen below shows two functions, $f(x) = x^3 + x^2 - 2x + 2$ and $g(x) = x + 4$. We will use these two functions to learn how to use the calculator.



Graphs

The graphics screen shows only a part of the coordinate plane, which extends infinitely both horizontally (left and right, like the horizon) and vertically (up and down). So you need to tell the calculator which part of the coordinate plane to use. This is called the *viewing window*. Like the calculator screen, the viewing window has the shape of a rectangle. Tap V-Window ([SHIFT] [F3]) to see what the current viewing window is. The settings will be whatever they last were, since the calculator keeps them the same until they are changed.

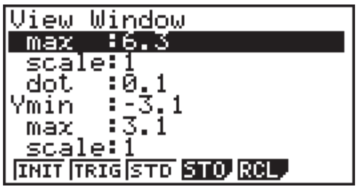


The minimum and maximum horizontal values (called ***Xmin*** and ***Xmax***) and the minimum and maximum vertical values (called ***Ymin*** and ***Ymax***) are enough to define the viewing window. The calculator will automatically draw the horizontal and vertical axes if you choose a window that includes them (although, as you will see later, you can prevent axes from being drawn if you wish).

The settings above show Xmin = -5, Xmax = 4, Ymin = -2 and Ymax = 4. That is, they define a viewing window for which $-5 \le x \le 4$ and $-2 \le y \le 4$.

Use the ▲ and ▼ keys to see the current settings on your screen. The ***scale*** values for *x* and *y* tell you what the tick marks on each axis represent. For the settings above, both scales have been set to 1. You can change any of the screen settings by typing a new value over a highlighted one, and tapping EXE to register your choice. To leave a setting unchanged, use the ▲ and ▼ keys to pass over it. Tapping the EXIT key or tapping the EXE key when no change has been made will result in returning from the view window screen to the function list.

For now, tap INIT (F1) to automatically choose the initial settings shown below. These have the advantage that every pixel across and down the screen will represent 0.1 units. Pixels are the little black rectangles that go to make up the screen – there are 127 pixels across and 63 pixels down on this calculator, so the screen is about twice as wide as it is high.

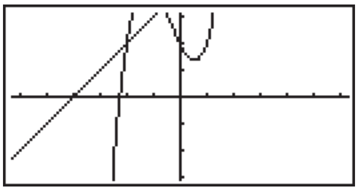
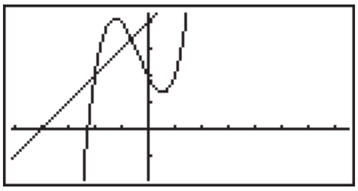


With these settings, the origin (where the two axes cross) will be in the centre of the screen. Another important advantage of this INIT screen is that the scales on the two axes are the same: a unit on the horizontal axis is the same length as a unit on the vertical axis, so that the graph does not distort the shape of the function.

To draw graphs of the stored functions in whatever the current viewing window happens to be, tap DRAW (F6).

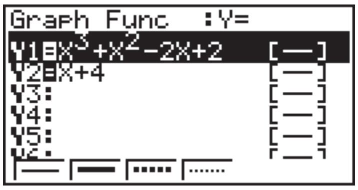
To change the viewing window, first display it with V-Window (SHIFT F3). Default settings are available with INIT (F1) and TRIG (F2) and STD (F3) or you can change any of the settings manually by typing over the present value and then tapping EXE. The INIT settings (shown above) are often useful to start with. The TRIG settings are especially useful for trigonometric functions. The STD (Standard) settings give $-10 \le x \le 10$ and $-10 \le y \le 10$, which is usually a poor choice since the screen shape is not square. After making choices, tap EXIT to return to the function list.

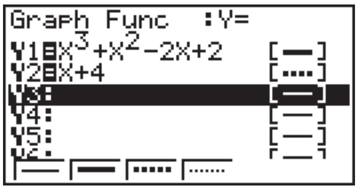
The left screen below shows the graphs of the above two functions, $f(x) = x^3 + x^2 - 2x + 2$ and $g(x) = x + 4$ drawn on the INIT window. Check that the tick marks seem to be about right.

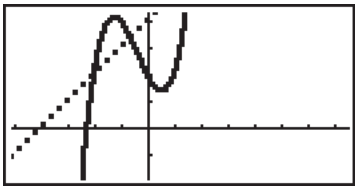



Immediately after graphs are drawn, you can tap the cursor keys ◀ ▶ ▲ ▼ to move the centre of the viewing window a little in any of the four directions. This is quite handy if the viewing window chosen is not quite convenient. In the case above, in which the top loop of the cubic graph is not showing, check the effect of tapping ▲ and then ▶ to get the graph screen on the right.

When more than one graph is drawn, as for the example being used here, it may be a good idea to use a different *style* for each graph to make it easy to tell them apart. On the calculator, four different pen styles are available for graphs: thin, thick, thin dots and thick dots. When you first draw a graph, the thin style is used – it is the default. If the graph is showing, tap [EXIT] to display the formulas again, and note below how the thin styles are shown in square brackets at the right end of the formulas. Tap STYL ([F4]) to display the alternative styles as shown in the middle screen below.







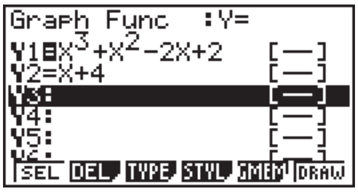
To choose a style for a graph, tap the corresponding choice, using the [F1] to [F4] keys. Tap [EXIT] when you have made the choices. The screens above show graphs using the styles of thick and thick dots on the same view window as above.

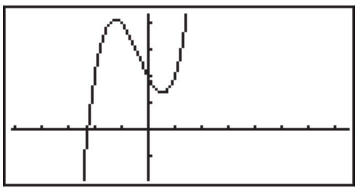
Although the dot styles are available to you, it's probably best to use them only for special purposes. The most common choice of styles when graphing is to use thick and thin, (or to use only thin styles, when there is little risk that you might confuse one graph with another).

You can toggle between the graphs and the symbols by tapping [F6], which activates the G-T command. (The 'T' stands for 'text'.) When the graphs are showing, you can also return to the function list with the [EXIT] key.

When the function list is showing, you can return to the graph screen with [F6](DRAW), but this will cause the graphs to be drawn again. The G-T command ([SHIFT] [F6]) is much quicker.

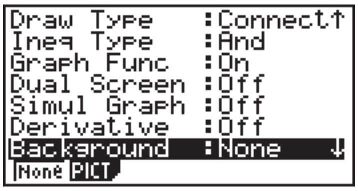
To temporarily turn off the graph of a function in the function list, highlight its formula and then tap SEL ([F1]). Notice that the shading on the equals sign is removed to show that the function will not be graphed. The following screens show how only the first function (*Y1*) is graphed. Tap SEL ([F1]) again to turn a graph back on.

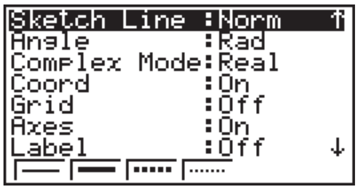




You can adjust various settings for graphs using the SET UP menu by first tapping [SHIFT] [MENU] while in Graph mode. You may have noticed that the SET UP menu is different in different modes of calculator use.

Make sure that you are still in graph mode to see the screens below, which show some of the most useful possibilities:





Draw Type Graphs can be drawn by connecting successive points or by just plotting points. *Connect* is usually the better choice; sometimes, however, it will lead to graphs being joined improperly.

Ineq Type When several inequalities are graphed (as described in the next module), you can choose which areas are shaded (by selecting *And* or *Or*)

Graph Func When this is turned on, the formula for a graph will be on the screen. This is usually helpful, especially if you want to know which graph you are tracing. It can be removed by turning *Graph Func* off.

Dual Screen When this is turned on, the screen is divided in half, with either a pair of graphs or a graph and a table showing.

Simul Graph When this is turned on, all graphs are drawn simultaneously, from left to right across the *x*-axis. Most people prefer to turn it off, so that they can see each graph drawn separately.

Derivative Controls whether or not the numerical derivative of a function is displayed when tracing or tabulating. (See Module 14 for more information about numerical derivatives.) It is best to leave this turned off for now.

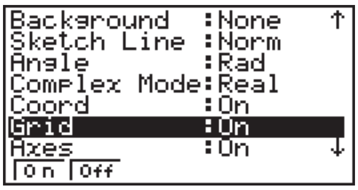
Coord This is usually left on, so that coordinates are showing on the screen. Turn this off if you don't want them displayed.

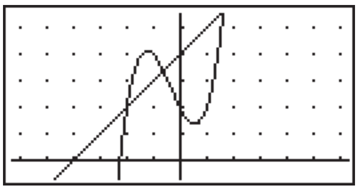
Grid Places a grid on the screen, with dots corresponding to tick marks.

Axes Controls whether or not the *x*- and *y*- axes are displayed.

Label Controls whether or not axis labels (i.e. *x* and *y*) are shown

As for other calculator settings, any changes you make will be retained by the calculator even after it is switched off. You will find it interesting to explore the use of these settings. For example, notice below the effect of including a grid on a graphics screen.

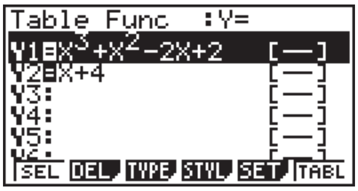


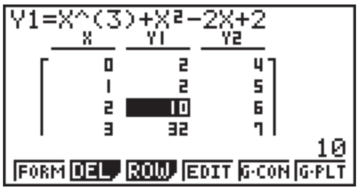


The grid places a dot at each integer point of the plane, so is most likely to be useful when the scales on the two axes are the same and also when the scales are neither too big nor too small. The screens above are good examples, since they are shifted only a few units from the INIT screen.

Tables

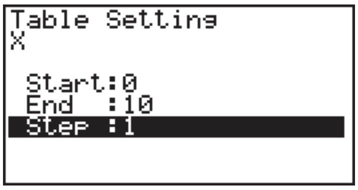
A table of function values is always a helpful way to represent a function and is available from MENU 7. Notice that the function list shown at left below is identical to that in graph mode. Tap TABL (F6) to show a table of values for the two functions defined above.

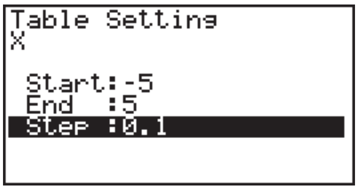


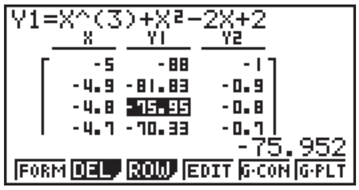


You can scroll tables vertically and horizontally using any of the four cursor keys. The function is shown in symbols at the top of the screen, and highlighted *y*-values are shown large at the bottom of the screen. You can scroll quickly by keeping your finger on a cursor key. Notice what happens if you scroll ◀ or ▶ repeatedly.

To set or change the values that are tabulated, tap EXIT and then SET (F5). Change the first (Start) and last (End) values for X, and the increment (Step) manually. Tap EXE after each change and then tap EXIT to return. The two tables above were obtained with the setting screen shown at left below:



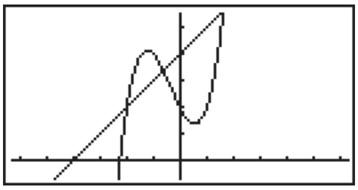


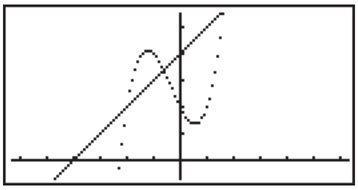


With the table setting screen changed to the middle screen above, the table below contains values of *X* from -5 to 5, going up in steps of 0.1. Notice that *Y1*(-4.8) = -75.952 is shown in full at the bottom of the screen, although the value in the table is approximated to save space.

Be careful not to make the *Step* too small, or the range from *Start* to *End* too large, or the calculator may run out of memory to store all the table values. To return to the formula screen from a table, tap F1(FORM) or EXIT.

You can draw a graph of table values in the current viewing window. Choose either G.Con (F5) for a continuous graph or G.PLT (F6)for a discrete plot of table values. The next two screens show what happens with each of these two options, using the same viewing window as previously:





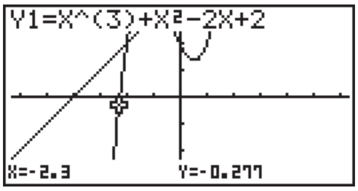
The continuous graph is the same as the one we drew in graph mode above. Notice that the graph on the right only plots individual points that are represented in the table (in this case, values of *x* going from -5 to 5 in steps of 0.1).

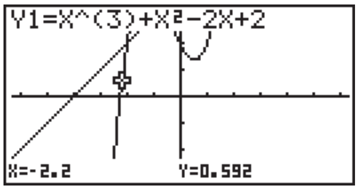
After you have drawn a graph, tap G-T (SHIFT F6) a few times to see how to toggle between a graph and the table. Tap EXIT or FORM (F1)from the table to return to the function list.

Tracing and zooming graphs

Once you have drawn some graphs, you will usually want to explore them to find out more about the functions concerned. The two main ways of doing this involve tracing and zooming. First, return to the graph screen with MENU 5 and DRAW (F6). Use the INIT screen for the moment.

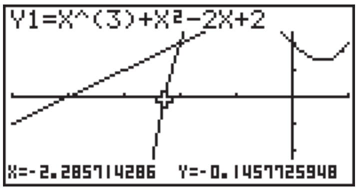
Tracing a graph is rather like running your finger along it, and seeing the coordinates of each point that has been plotted. Tap Trace (F1) to activate the graph trace. Use ◀ or ▶ to trace, and ▼ or ▲ to shift between graphs. Points traced are shown on a graph with a flashing plus sign.

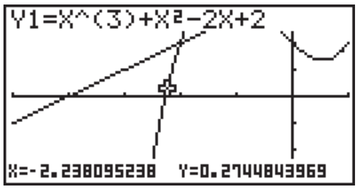




From these two screens, you can tell that the function $f(x) = x^3 + x^2 - 2x + 2$ has a value of zero for x close to -2.3. To trace quickly, press your finger down on ◀ or ▶. (Keep your eye on the coordinates if you do this, or you may miss the cursor flashing past.)

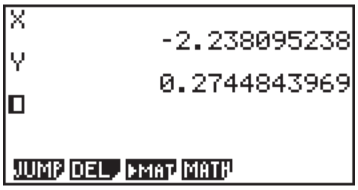
Notice that the x-values for the above traces go up in steps of 0.1 as you trace from left to right across the screen. This is a major advantage of the INIT screen (which takes advantage of the number of pixels on the screen, 127 × 63). Another advantage is that the scales on the two axes are the same. If you change the viewing window manually (say to $-5 \leq x \leq 1$), a less friendly step for x will be used by the calculator and the scales on the x and y axes will no longer be the same, as shown on the next two screens:





On the calculator, the coordinates of points on the screen are represented by (X,Y). It is important to understand how the calculator uses its memories for the two variables X and Y. Whenever you trace, the values of X and Y are changed automatically in the memories labeled X and Y, and you can access the most recent values in the Run-Mat mode. To see how to do this, first Tap MENU 1 to return to Run-Mat mode.

When you recall the contents of the X and Y memories, you will see that they contain the values from the most recent trace, as shown below.

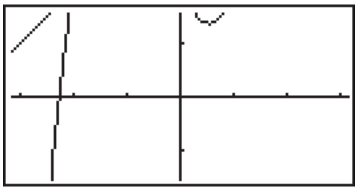


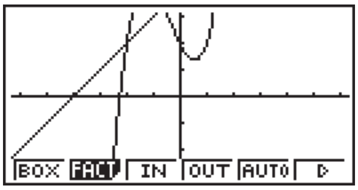
The Y value is recalled with ALPHA −; the X value can be recalled either with ALPHA + or merely with the X,θ,T key. Using the X,θ,T is easier, since it requires only one key.

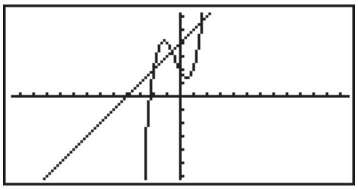
Return to graph mode with MENU 5 and draw the graphs again in the INIT window. Tap Zoom (F2) to access the *zoom* menu. Zooming allows you to change the viewing window very quickly, rather than having to change each of the *Xmin*, *Xmax*, *Ymin* and *Ymax* values separately.

Tap IN (F3) to zoom in; move the cursor to your preferred position for the middle of the screen and then tap EXE to conclude the zoom. 'Zooming in' is a bit like using a magnifying glass, to look more closely at some parts of graphs. (But notice that the 'thickness' of the graph and the axes don't change, so it's not exactly like a magnifying glass!). Similarly, tap Zoom (F2) and then OUT (F4) to zoom out. 'Zooming out' is a bit like looking at graphs from further away, so that you can see more of them. It is sometimes helpful to zoom out to see the overall shapes of graphs.

The next three screens show what happens after zooming in (the left screen) or zooming out (the right screen) from the middle of the INIT viewing window (the middle screen). Notice that the tick marks have not changed in the three cases (and continue to be at every one unit on each axis).

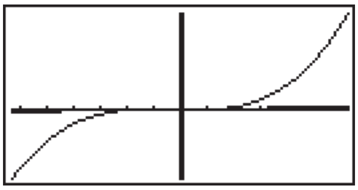


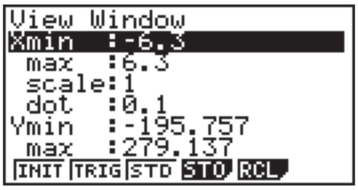




It's easy to get a bit lost after several zooms. If this happens, you can return to the Zoom menu with F2, tap the continuation key F6 and then tap ORIG (F1) to zoom back to the original viewing window. To undo a zoom, you can zoom back to the immediately previous screen by tapping the continuation key F6 and then PRE (F5). Another useful zoom is SQR (F2) which 'squares up' the axes, by giving the same scale to each.

The automatic scaling zoom, AUTO (F5) has the effect of adjusting the *y*-axis so that all the values associated with those on the *x*-axis are shown on the graph. This is sometimes useful to give an indication of the key features of a graph, but it can also lead to very distorted graphs. For example, the graph below shows the effect of using the automatic scaling zoom after drawing the sample graphs on an INIT window. Notice that there is a *y*-value plotted for each *x*-value on the screen.





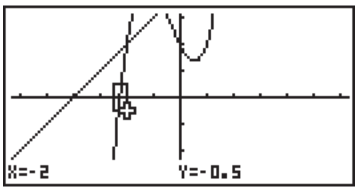
The viewing window that is produced by this zoom is also shown; this helps to make clear how distorted the graphs are in this case. The thickening on the *y*-axis is caused by putting a tick mark at every 1 unit (between -195.757 and 279.137), while the thickening on (part of) the *x*-axis shows the graph of $y = x + 4$.

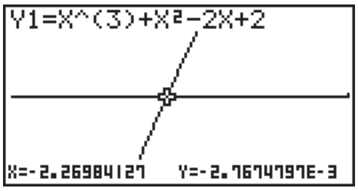
The factors for zooming in or out can be changed in the FACT (F2) menu after selecting Zoom (F2). The values selected are the values used to multiply or divide the values on the axes. The default value is for a factor of 2 on each axis. You can zoom very quickly and still keep some of the friendly aspects of the INIT screen by changing both zoom factors to 10. Tap EXE after each change.

The two factors do not have to be the same, and sometimes it is useful to use different factors on each axis. For example, to zoom in only one direction (vertically or horizontally), change the zoom factor for the *other* direction to 1.

The BOX (F1) zoom allows you to define a new screen by making a rectangular 'box'. First use the cursor keys to move to one corner of the box and then tap EXE. Then move the cursor to the *diagonally opposite* corner of the box and tap EXE again. (Notice you make a box on the screen as you move the keys.) This is a good way of quickly focusing on the parts of a graph of most interest.

For example, in the screen below on the left, a box zoom is being used to highlight the intercept of the cubic graph near the origin. The small rectangle outlined becomes the whole screen after EXE is tapped the second time.

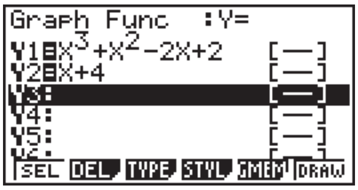


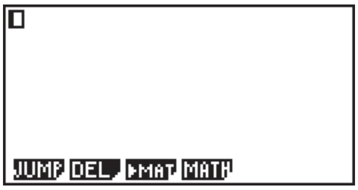


Usually, both tracing and zooming will be needed to answer important questions about functions from their graphs. For example, to find a good approximation to the intercept in the box, you will need to trace after the zoom has been completed. It seems from the graph at right that there is an *x*-intercept close to -2.27. Notice that tracing after a box zoom has been completed will usually *not* result in convenient steps in the *x*-direction. This is illustrated in the screen at the right above.

Evaluating functions

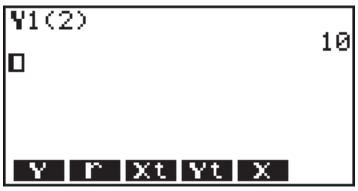
Sometimes you may want to find the value of a function for just one or two values of the variable (*x*), rather than for large set of values, in the form of a graph or a table. To do this, after the functions have been defined in Graph or Table mode, return to Run-Mat mode with MENU 1. In the screen below, the present functions are shown and it is clear that the home screen has been cleared.



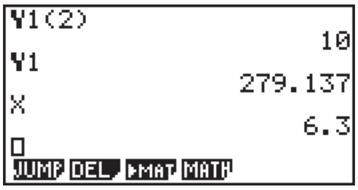


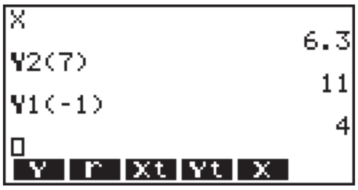
Consider the function $f(x) = x^3 + x^2 - 2x + 2$, which is represented in the calculator at the moment as *Y1*. In mathematics, we represent the value of the function when $x = 2$ as $f(2)$. The calculator uses a similar notation: the value of the function when $X = 2$ is represented as *Y1*(2).

To show this on the screen, and to obtain the numerical value of *Y1*(2), the **Y** symbol is needed. This is a different **Y** symbol from that for the Y memory: it is a function and not just a variable. Tap the VARS key (to access calculator variables) and then GRPH (F4) to access the familiar menu you have already seen when entering functions. The **Y** symbol is showing at F1. As the last screen below shows, this variable allows you to enter *Y1*(2) and obtain its value of 10 with the EXE key.



If you enter only **Y**1, not specifying a particular value for *X*, the calculator will evaluate the function for the most recent value of *X* to which you have traced. Recall that you can check this present value of *X* with the X,θ,T key.

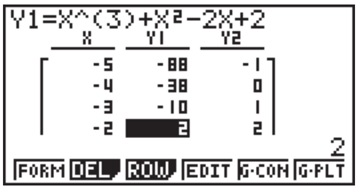




The first screen above shows the value of *X* immediately after drawing graphs on the INIT screen (as the calculator draws graphs from left to right). In the second screen, some other function values have been obtained, each using the **Y** (F1) key.

Tracing and zooming tables

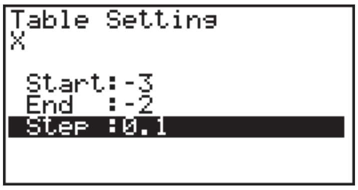
Just as you can trace and zoom a graph to find out more about the functions concerned, you can do something similar with a table. Return to Table mode with MENU 7. To trace a table, simply move the cursor keys ▲ and ▼. This is like running your finger up and down the values in the table to read them carefully.

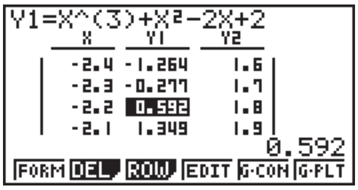


The screen shows that when *x* = -3, the value of the function is negative (*Y1*(-3) = -10) while for *x* = -2 the value is positive (*Y1(-2)* = 2). This suggests that there is a value of *x* between -3 and -2 for which $f(x) = x^3 + x^2 - 2x + 2$ has the value zero.

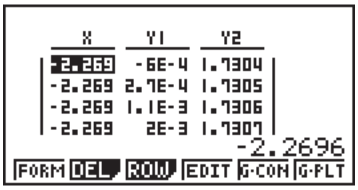
You can get a good approximation to this value by zooming in. To do this, tap FORM (F1) and then SET (F5) to make a table of values that goes up with a smaller step. A good choice is a step of 0.1 rather than 1, with *x*-values between -3 and -2.

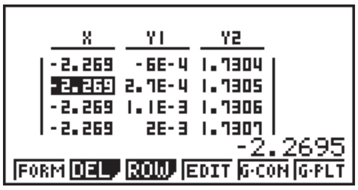
Now return to the table (by tapping EXIT and then TABL (F6). Trace the table of *Y1* values to see that the appropriate value of *x* now seems to be between -2.3 and -2.2, as shown below.





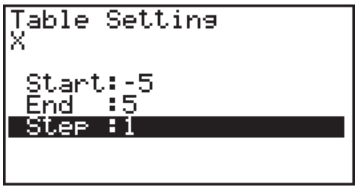
This process can be continued, and, if you decrease the step by a factor of 10 each time, each successive zoom will give another decimal place of accuracy. After using a step of 0.0001 below, we can see that the intercept is closer to *x* = -2.2695 than to *x* = -2.2696, since the corresponding *y*-value is closer to zero.

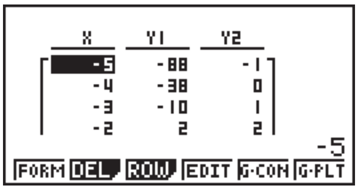




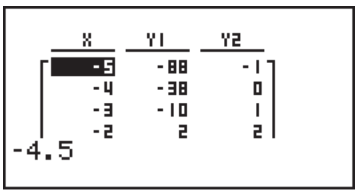
You can zoom out in a table in the same kind of way as zooming in, by making the interval and the steps bigger instead of smaller each time.

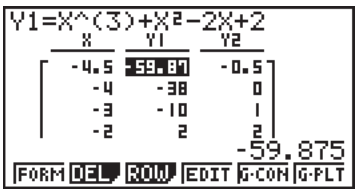
Another way of using tables to study the values of functions efficiently involves changing the X values directly in the table. To see how this works, start with the table settings below and draw a table in the usual way by tapping the EXE key twice.





To find the value of the functions when *x* = -4.5, simply enter -4.5 somewhere in the X column of the table and tap EXE, as shown below. Notice that the table values are automatically updated.

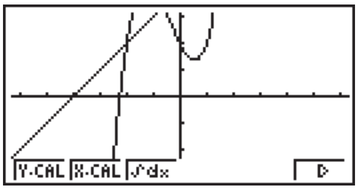




Change the *x*-value again, and watch the results.

Interpreting graphs automatically

Many useful things about functions can be found out by tracing and zooming graphs and tables. However, sometimes these processes can be a little tedious, and it is more efficient to use the calculator's powerful automatic capabilities. To see how these work, first use MENU 5 to draw the graphs again on the INIT screen. Then Tap G-Solv (F5) to bring the *graph solve* menu to the screen. The next two screens show the complete menu.

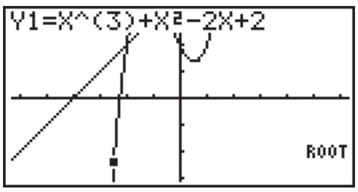


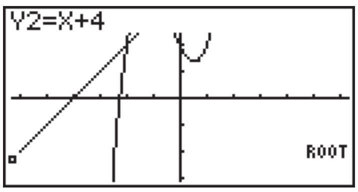
The immediate menu commands in this menu will automatically locate good numerical approximations to various values associated with graphs:

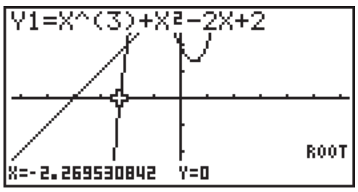
ROOT	An *x*-intercept, a point at which a graph crosses the *x*-axis gives a *root* of a function, a value of *x* for which the function has a value of zero.
MAX, MIN	The points at which a function has a *relative* or *local* maximum or minimum value. On a graph, these points are at the top or bottom of a curve. Local maxima and minima do not necessarily correspond with *global* maxima or minima, which are concerned with the entire set of values for the function.
Y·ICPT	The point where a graph crosses the *y*-axis (i.e. for which *x* = 0).
ISCT	A point of intersection of a *pair* of graphs.
Y·CAL	Calculates the *y*-value associated with a particular *x*-value for a function.
X·CAL	Calculates the *x*-values associated with a particular *y*-value for a function.
∫dx	A definite integral of a function, found as the area between a graph and the *x*-axis between two particular *x*-values. The area above the *x*-axis is regarded as positive, and that below the *x*-axis as negative.

The calculator will only provide ***approximations*** *to these important values. To find* ***exact*** *answers to questions of these kinds, you will need to use some mathematical analysis.*

We will use the menu to find a root of $f(x) = x^3 + x^2 - 2x + 2$. Tap ROOT (F1) as shown in the screen at left below, a small square cursor is located on the appropriate graph at the bottom, but the calculator needs to be informed which graph is the one to be used. (In this case, there is a choice, since two functions have been graphed.)





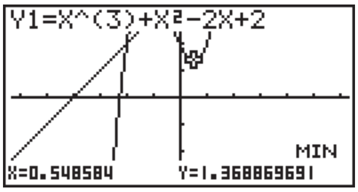


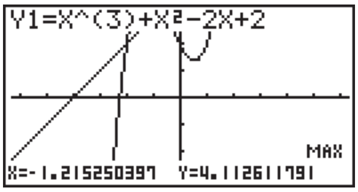
Move between graphs as in tracing with ▼ and ▲. Since *Graph Func* is turned on, the screen displays the formula for the graph which is currently selected. Make your choice by tapping EXE. After a few seconds, the calculator will find the first root of the cubic function, from left to right across the screen: $x \approx$ -2.269530842. The y-value given allows you to see how good the approximation is. Notice that the value is consistent with the value we obtained earlier by tracing and zooming both graphs and tables.

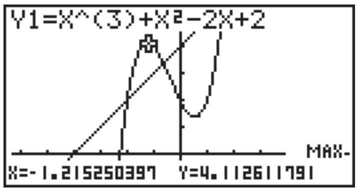
Only rarely would so many decimal places be needed in practice, so $x \approx$ -2.27 or even $x \approx$ -2.270 may be sufficiently accurate approximations to two and three decimal places respectively.

It is clear from the graph that there are no further roots of this function for these values of x. (It is clear, too, if you know about the shapes of graphs of cubic functions, that there will be no roots for other values of x.) Had the graph shown that there *were* other roots, however, the calculator would have found the succeeding ones after the ▶ key was tapped.

The other commands in the *graph solve* menu work similarly. The next two screens show the *MIN* and *MAX* commands for the cubic function. On the left screen below, notice that the minimum shown with the cursor is not the minimum value of the function overall, but merely the *relative* or *local minimum*near the region where the curve turns. That is, it is not a *global* minimum value. It is clear from the graph that the function has smaller values than this, negative values for example.







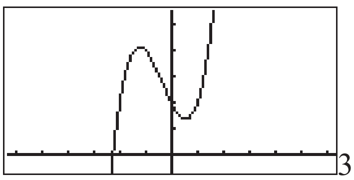
Notice above that the calculator finds a relative maximum point of $y \approx 4.11$, when $x \approx$ -1.22, even though the y-value is not showing on the middle screen. This is because it searches systematically the x-values from left to right across the screen, regardless of whether the associated y-values are visible. As we are using the INIT screen here, a better choice of screen might show the graph turning near (-1.22,4.11). The third screen above shows this.

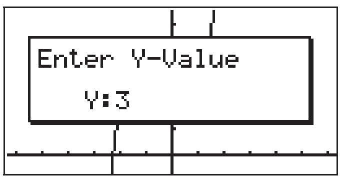
The *ISCT*command needs you to identify two functions (by tapping EXE for each one in turn) if there are more than two graphed. In this case, since there are only two graphs on the screen, this step is omitted. The calculator finds the points of intersection from left to right across the screen.

Tap ▶ to move to the next point of intersection to the right or ◀ to move to the next point to the left. Although the points on the screen are given to many decimal places, they are still not exact. In this case, mathematical analysis shows the *exact* three points of intersection are at

$$(-2,2),\ \left(\frac{1-\sqrt{5}}{2},\frac{9-\sqrt{5}}{2}\right) \text{ and } \left(\frac{1+\sqrt{5}}{2},\frac{9+\sqrt{5}}{2}\right).$$

You can tell from the graph at the left below that there are three values for which the function $f(x) = x^3 + x^2 - 2x + 2$ has the value 3. (Imagine a horizontal line at $y = 3$ to see this.)





To find these three values, the *X-CAL* command in the *graph solve* menu is very useful. As only one graph is now showing, it is unnecessary to select which graph to use. When you start *X-CAL*, you will need to insert the y-value (3), followed by EXE, as shown in the graph at right above. Three approximate solutions are produced ($x \approx$ -1.80, $x \approx$ -0.45, $x \approx$ 1.25) by tapping the ▶ key after each result.

Exercises

The main purpose of the exercises is to help you to develop your calculator skills.

1 Draw a graph of $y = x^2 - x - 3$ on the INIT screen. The bottom of the graph does not quite fit on the screen. Use the cursor keys (◀ ▶ ▲ ▼) to move the origin so that the graph does fit. Change the style of the graph from thin to thick.
  a Trace the graph to find y when x =-0.7.
  b Trace to find the two values of x for which y = 1.59.
  c Change to Run-Mat mode. Tap [X,θ,T] [EXE] and [ALPHA] [(−)] [EXE] to see traced values on the screen.

2 Draw a graph of $y = x^2 + 7$ on the INIT screen. Explain what you see.

3 Draw a graph of $f(x) = 15 \times 1.19^x$ on the interval $-1 \le x \le 10$ and $-10 \le y \le 100$. Put tick marks at every unit on the x-axis and at every ten units on the y-axis.
  a Trace to find the approximate value of x for which y = 40.
  b Use *X-CAL* in the *graph solve* menu to get a better approximation.

4 Make a table of values for the functions $f(x) = 4 - x^2$ and $g(x) = x^3 + x - 1$, for values of x from 1 to 1.5, going up in steps of 0.01.
  a For which value of x do the two functions have approximately the same value?
  b Use G.CON ([F5]) to draw graphs of the two functions on the INIT screen.
  c Use a box zoom to draw a box around the point of intersection of the two graphs near x = 1.3. Then trace the graphs to find the coordinates of the point of intersection to two decimal places.
  d Use *ISCT* in the *graph solve* menu (in Graph mode) to get a better approximation.
  e Turn off the graph of the parabola. Zoom back to the original screen and use *graph solve* to find the value of x for which $x^3 + x - 1 = 0$.

5 a Construct a table of values for $f(x) = 2.3^x$ for $0 \le x \le 10$ and *Step* of 1. In the viewing window, set *XMin* to 0 and *XMax* to 10.
  b Use G.PLOT ([F6]) to plot the table values. Use Zoom AUTO ([F5]) to automatically choose y-values to suit the x-values. Check the View Window to see the scales chosen.
  c Trace the plotted points. Between which two values of x is $2.3^x = 100$?
  d Zoom on the table twice by factors of 10 to get a better approximation to the value of x for which $2.3^x = 100$

6. a Draw graphs of $x - y = 3$ and $4x - y = 2$.
  b Draw a graph of $x - y = 4$ in bold style and of $8x + 10y = 5$ in thin style. Trace the graphs to find their point of intersection.

7 Draw graphs of $y = x - 3$ and $y = 2 - 4x$ on a standard screen. (Use STD ([F3]) in the viewing window.) Redraw the graphs on the INIT screen.

Notice the differences in scales and graphs.

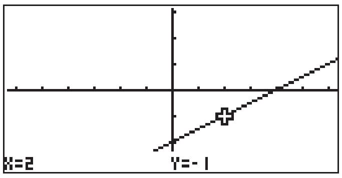
Activities

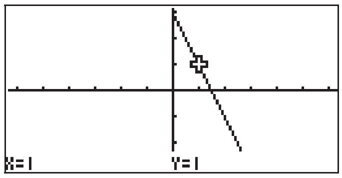
The main purpose of the activities is to help you to use your calculator to learn mathematics. You may find that some of them are too advanced for you. Ignore activities you don't yet understand.

1 a Draw graphs of $y = x + 2$, $y = 2x + 1$, $y = x - 1$, $y = 3 - x$ and $y = 1 - 0.5x$ all together on the INIT screen. Which graphs are parallel to each other? Which graphs are perpendicular to each other?
b Change the viewing window so that *Xmin* = -8 and *Xmax* = 8. What effect does this have on parallel and perpendicular pairs of graphs?
c Change to the STD screen. What effect does this have on parallel and perpendicular graphs?
d Change back to the INIT screen, and find some more functions whose graphs are lines that are either parallel or perpendicular to each other.

2 It's a good idea to imagine what a graph will look like before you draw it. Try to do this in each part of this activity.
a Find a pair of quadratic functions whose graphs cross each other exactly twice.
b Find a pair of quadratic functions whose graphs do not intersect.
c Find a quadratic function and a linear function whose graphs intersect at exactly one point.

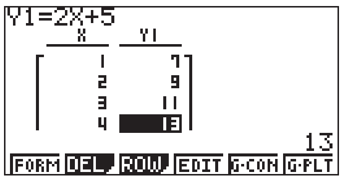
3 a Which linear functions have been graphed here on the INIT screen?





b Invent some questions like this to give to a partner. Make sure that they can see both intercepts on the screen and turn off the *Graph Func*. (Before you give them the calculator to look at, it might be advisable to define a function near the bottom of the list (such as Y_{20}), and then return with F2 to the top of the list, just in case they are tempted to peek!)

4 In Table mode, draw a table of values of a linear function for integer values of x from 1 to 10. Use the cursor in the table to scroll down the values of the function. The screen shows this for $y = 2x + 5$.



a Use the pattern of values of the function (in this case, 7, 9, 11, 13, …) to predict the next values of the function as you scroll down. Predict the remaining six values, i.e., those for x = 5, 6, 7, 8, 9 & 10.
b Now try a different linear function, like $y = 2x + 6$.
c Then try a different linear function, like $y = -3x + 6$.
d Try some others for yourself, until you can see how to make the predictions efficiently.

5 Draw a graph of $y = x + \frac{1}{x}$ on the default screen.
Describe and explain what happens if you zoom in and out several times.

6 Draw graphs of some absolute value functions, such as $y = |x|$, $y = |x - 2|$ and $y = |x - 2| - 4$. Predict what the graphs of $y = |x + 4|$ and $y = |x + 1| + 5$ will look like. Graph them to check your predictions. Compare the graphs of function pairs such as $y = 2 - x^2 - 3x$ and $y = |2 - x^2 - 3x|$. Make up some more examples.

Notes for teachers

This module is important for new users of the calculator, as the major advantage of graphics calculators over scientific calculators is the ability to represent, manipulate and explore functions on a graphics screen. The calculator provides many opportunities for learning about the behaviour of functions, and it is important that students develop sufficient expertise with the calculator to be able to realise these. Once basic calculator operations are mastered, there are many ways in which the calculator can be used to enhance learning about functions, some of which are reflected in the activities below. The text of the module is intended to be read by students and will help them to start to see how the calculator can be used efficiently for various kinds of calculations.

Answers to Exercises

1. (a) -1.81 (b) -1.7, 2.7 2. No graph is visible because of the screen settings. Tap ⏶ a few times. 3. (a) $x \approx 5.635$ (b) $x \approx 5.638$ 4. (a) $x = 1.28$ (c) (1.28,2.37) (d) (1.2278163073,2.366299159) (e) 0.682 5. (b) Note $1 \leq y \leq 4142$ (c) between $x = 5$ and $x = 6$ (d) $x = 5.53$ 6. Note that functions need to be written with y on the left of the equals sign, such as $y = x - 3$; (5/2,-3/2)
7. Graphs appear perpendicular (incorrectly) in STD screen.

Activities

1. This is an essential activity for students to explore both the relationships between slopes and linear graphs and also the effects of choosing different scales on each axis. [Answers: Graphs 1 and 3 are parallel and each perpendicular to graph 4; graphs 2 and 5 are perpendicular. Parallel lines remain parallel in STD, but perpendicular lines are no longer perpendicular.]

2.This activity is intended to encourage students to explore graphs of linear and quadratic functions, and to see how changing coefficients changes graphs. It will also provide students with lots of experience of graphing functions on the calculator. Encourage students to work in pairs and to compare their solutions with each other.

3. An activity of this kind can be used as a classroom activity or as an activity between two students. The purpose is to help them to understand the significance of both slopes and intercepts of linear functions. Tracing a graph will help students see how the slope determines the rate of change and the intercepts. Leaving a grid on the screen may also help students' thinking.

4. This activity focuses on tabulated values to help students see the idea of a rate of change, or slope, of a linear function. Working in pairs will encourage students to verbalise and understand the rate of change; for example, for $y = 2x + 5$, y increases by 2 when x increases by 1, which is the essential idea of a slope of 2.

5. Activities of this sort help students to appreciate that the appearance of a graph depends critically on the scales chosen. In this case, zooming in might show no graph at all, including a point of discontinuity at $x = 0$, while zooming out shows a graph that is close to the identity line $y = x$. Activities of this sort also help students with the idea of asymptotic behaviour, important in later studies of the calculus.

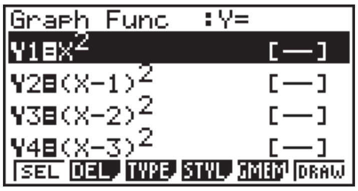
6. An absolute value command is available on the calculator via OPTN F5 and then F1. This activity illustrates how the calculator can be used to understand particular kinds of transformations by exploring several examples of them. It is helpful for students to undertake these explorations with a partner, in order to discuss their observations.

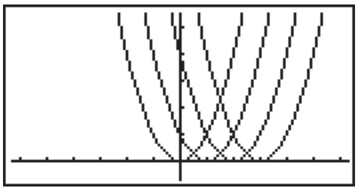
Module 3
Exploring functions

In this module, some of the many ways in which the calculator can be used to explore functions are briefly described and illustrated. It is often helpful to see several related polynomial graphs at the same time and an animation can help to see their similarities and differences. Various other kinds of graphs are also important, including graphs of inequalities, conic sections and polar graphs.

A family of functions

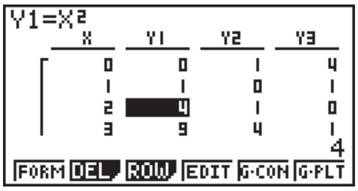
To understand the connections between functions and their graphs, it is helpful to draw some members of a 'family' of functions (i.e., a set of functions that are related closely to each other.) You can do this by defining each member of the family separately; e.g., the two screens below show some members of a family of parabolas.

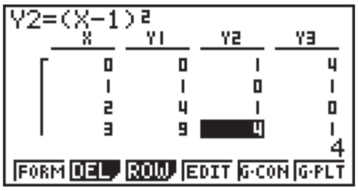


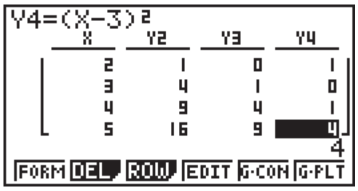


Notice that each of the four parabolas is the same size and shape. The difference among them is that they have been translated horizontally. (You may find it interesting to draw graphs like this with *Simul Graph* turned on in the SET UP menu, to see more clearly how the graphs are related.)

Now switch to Table mode with MENU 7. Use Table Setting to tabulate for $X = 0$ to 10, in steps of 1. Scroll down each column, *Y1, Y2, Y3, Y4*.

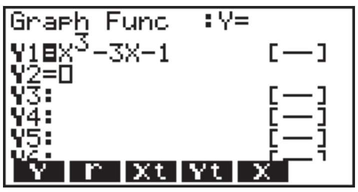


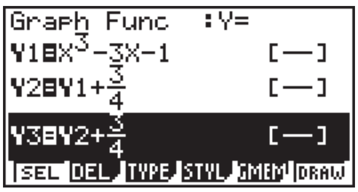


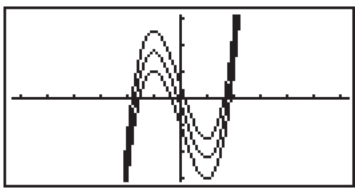


Notice how the values in the columns are closely related to each other. Each function has the same set of values, but they are moved 'down' the table. In the same way, each of the functions has the same graph, but they are translated to the right.

Another very useful way in which related functions can be graphed or tabulated together is to define functions in terms of one another. The screens below show two examples of this.



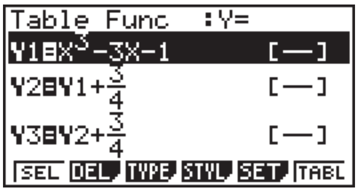


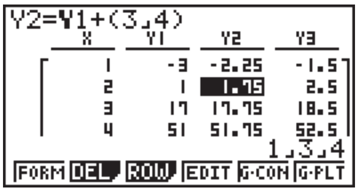


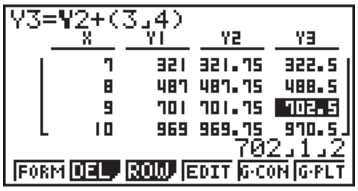
The second function in the list is defined to be the same as the first function in the list, with 3/4 added. The third function in the list is defined as ¾ more than the second function (and hence 3/2 more than the first function). (The fraction ¾ is made with [3] [a b/c] [4].)

In the formulas for *Y2* and *Y3*, the symbols ***Y1*** and ***Y2*** are made up of two characters, a ***Y*** and a number. But you cannot use the *Y* directly from the keyboard (i.e., you can't use [ALPHA] [–]). To access the ***Y*** symbol when defining a function, start by tapping the right cursor [▶] and you will have the menu shown in the first screen above. Insert ***Y*** with [F1].

As well as comparing the graphs of the two related functions, it is helpful to compare tables of values. Tap MENU 7 to switch to table mode. Check that each function is selected (with the equals sign shaded as below. Use SEL (F1) to change them if they are not selected. The screens below show table values for the three functions.







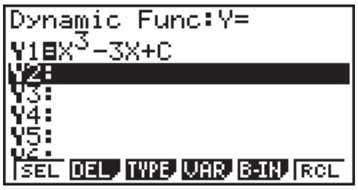
As might be expected, the tables make it easy to see that the values for *Y2* are 0.75 greater than the corresponding *Y1* values for each value of X, and that those for *Y3* are a further 0.75 greater. Notice, incidentally, that the table values are given at the bottom of the screen as fractions in this case, since the functions are defined using fractions.

Animation of graphs

As well as displaying several related graphs, the calculator is able to display them in quick succession, giving an effect of animation. Use MENU 6 to select the *Dynamic Graphing* mode for this purpose. To illustrate, we will use a similar family of functions to the one shown above:

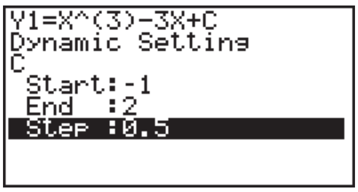
$$y = x^3 - 3x + variable$$

Only one family can be defined at once in Dynamic Graphing mode. A letter is used to represent the variable (in this case, *C*, which is obtained with ALPHA ln). The screen below shows the definition.



Tap VAR (F4) to identify the variable that will control the animation. If there are more than two variables in the formula, use ▼ and SEL (F1) to select the one you want.

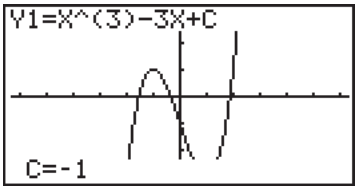
The variable can have several numerical values. Use SET (F2) to set these, in the same way as for a table. Make sure you don't have too many, or the calculator will take a long time producing them all (since each graph will take several seconds to construct). More graphs will give smoother animations, but will take more time to get started; about five to ten is best. The screen below at left shows seven values in all, to give seven graphs. Tap EXIT to return to the previous screen.

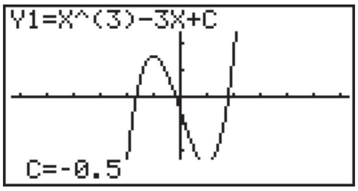


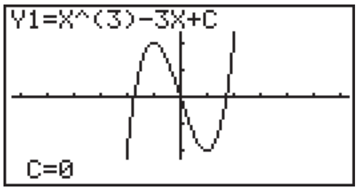
Next, select the animation speed with SPEED (F3) The first choice (*Stop & Go*) will mean that successive graphs will be produced when you tap EXE. This gives you a good chance to study each one, but loses the animation effect. Use F1 to F4 to make your choice, as shown in the previous screen. Try *Normal* speed to start with; it is fairly easy to change the speed again later if you wish. Notice that your choice is registered on the screen with the appropriate arrow symbol. Tap EXIT to return. Choose a suitable viewing window. For this example, INIT works quite well.

Now, all is ready for the animation. Tap DYNA (F6) to start. The calculator will ask you to wait while it constructs all the graphs and then will display them in the way you want. (If you watch the waiting screen carefully, you will see that seven different graphs are constructed in this case.)

Below are the first three graphs. Notice that the variable *C* is displayed for each (provided you have turned *on* the *Graph Func* setting in SET UP). Each graph is a vertical translation a distance 0.5 of the previous one, but the animation shows this much better than the printed page.

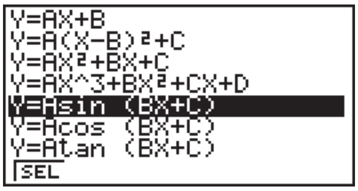


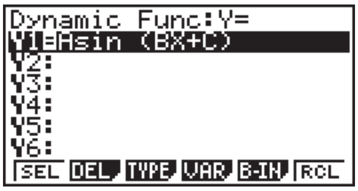




To change the speed, first stop the animations, by tapping AC/ON and EXIT. Then choose a new speed; your choice will take effect immediately. The animations will stop by themselves after ten full cycles, but you will have to tap DYNA (F6) to start again with a new speed, and then wait for all the graphs to be drawn again, so don't wait for this to happen if all you want to vary is the speed.

As well as defining your own families of functions in this way, the calculator has a number of built-in functions, from which you can choose. Tap B-IN (F5) to see the suite of choices shown below, and use ▼ and SEL (F1) to make a choice.



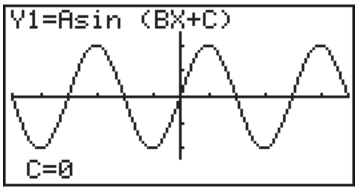


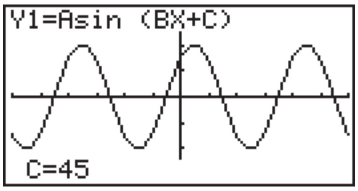
For example, consider a family of sine functions, as shown at left above. For these trigonometric functions, check that the SET UP is suitable by taping SHIFT MENU and make a choice of either degrees or radians. Then change the viewing window to TRIG, which will match your choice well.

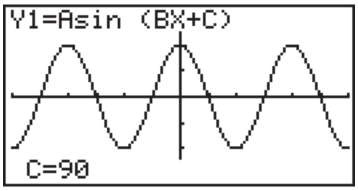
This family of functions has more than one variable, so you will need to identify which *one* is to control the animation, and give a value to each of the other variables. The screen above shows that both *A* and *B* have been given the value 1 in this case, while *C* has been chosen as the *dynamic variable.* These choices will allow for the animation of functions with formulae $y = \sin(x + C)$. (The value for *C* in this screen is irrelevant, since it will be set when you set the range.) Tap SET (F2) when you have finished (and *not* EXIT, which will undo your choice of variable) to move to the setting screen.

When degrees have been chosen, a good choice for the S*tep* is 45°, and a good choice for the animations will cover a complete circle from 0° to 360°.

The first few animations are shown below. The graphs appear to move horizontally this time, unlike the previous animation, which was in a vertical direction. Notice that the TRIG screen places tick marks on the x-axis at every 90^o.







Piecewise functions

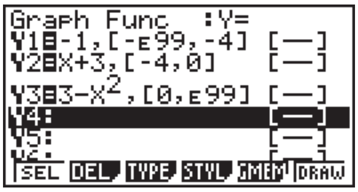
Some functions have two or more different definitions for different parts of their domain. Here is an example of a piecewise-defined function:

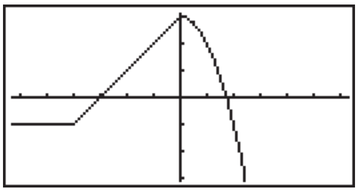
$$f(x) = -1 \text{ for } x < -4$$

$$f(x) = x + 3 \text{ for } -4 \leq x \leq 0$$

$$f(x) = 3 - x^2 \text{ for } x > 0$$

Tap [MENU] [5] to access Graph mode, and delete any remaining function definitions before you start.





Piecewise functions can be graphed by graphing each piece separately, using the special notation for a closed interval shown in the screen at left. There is a comma key on the keyboard and the square brackets are also on the keyboard, accessible with [SHIFT] [+] and [SHIFT] [−] respectively. Notice that the two open intervals ($x < -4$ and $x > 0$) are represented using very small and very large numbers: E99 refers to 10^{99}. The symbol for E is available on the calculator keyboard with [EXP].

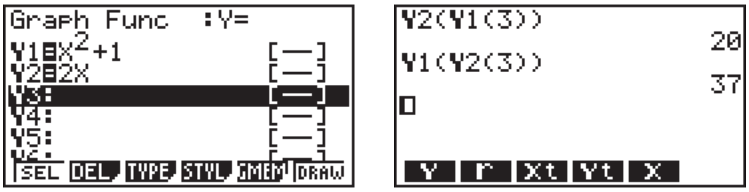
It is not possible on the calculator to tabulate piecewise functions, so you will have to restrict your analysis to graphing. Tracing and zooming are available, but each function must be dealt with separately.

Composite functions

Composite functions are composed from other functions. For example, consider the two functions $f(x) = x^2 + 1$ and $g(x) = 2x$. Function f involves 'squaring and adding one', while function g involves 'doubling'. If f is applied to a number and then g applied to the result, the composition of the two functions will have the effect of doubling of (squaring and adding one). For example, if we start with $x = 3$, the result after f is 10 and the result after then applying g will be 20. In mathematics, this is written as $g(f(3)) = 20$.

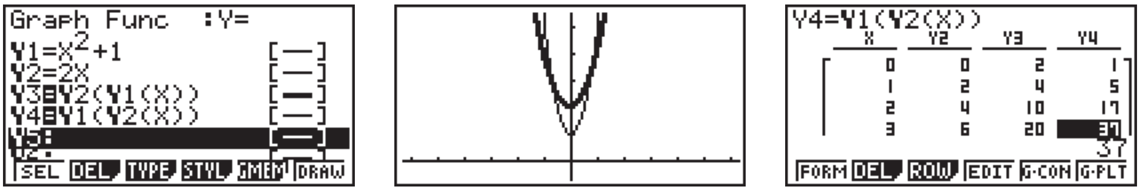
The composition of the two functions, $g(f(x))$ is sometimes called a composite function. In this case, $g(f(x)) = 2(x^2 + 1) = 2x^2 + 2$. Check for yourself that the composition of functions is not usually the same if the order of composition is changed. In this particular case, $g(f(x)) = (2x)^2 + 1 = 4x^2 + 1$, and $g(f(3)) = 37$. So $g(f(x)) \neq f(g(x))$ for these two functions.

On the calculator, composite functions are represented in similar ways to their representation in mathematics. The next screen on the left shows the two functions used as an example here. The right screen shows how the composite functions can be evaluated in Run-Mat mode, after obtaining the special **Y** symbol with VARS and then GRPH (F4).



Composite functions can be graphed and tabulated using the same mathematical notation, as shown in the following screens. (When defining the functions, make sure that you continue to use the X,θ,T key for the independent variable, X, not the X (F5) command in the menu.)

Both the graph and the table make it clear that the two composite functions *Y1*(*Y2*(*X*)) and *Y2*(*Y1*(*X*)) are not the same in this case.

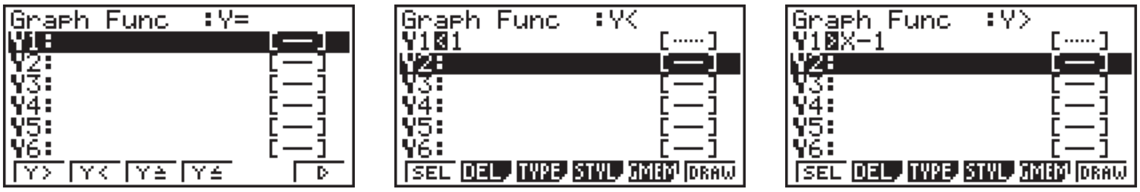


Notice that one of the composite functions (*Y3*) has been drawn in bold, to make it easy to distinguish it from the other and that graphs of the two original functions have been deselected (using SEL (F1)) to avoid cluttering the screen too much.)

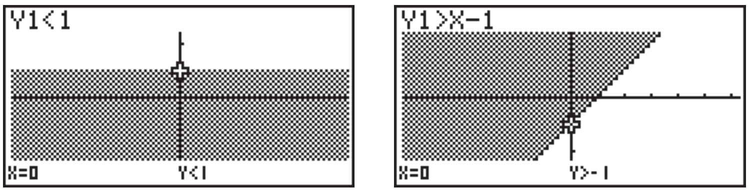
Inequalities

For these calculators, an inequality must have only the dependent variable (*Y*) on the left side of the inequality. Examples such as $y < 1$ and $y > x - 1$ are acceptable.

To graph these, first use TYPE (F3) to change the type of relationship to be graphed and the continuation key F6 to bring the inequality menu to the screen.



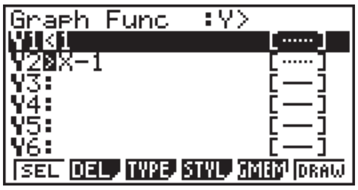
To define an inequality, *first* tap the appropriate key of F1 to F4. Then enter the right hand side of each inequality in the usual way. Notice that the top line of the screen shows you which inequality you are defining. To define more inequalities of the same kind, simply enter their rules. To define a different sort of inequality, however, you must first tap F3 F6 and then select the kind you want. The next two screens each show one of the inequalities above:

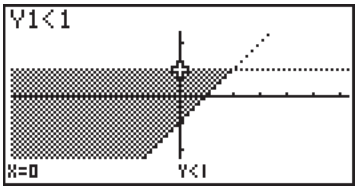


Notice that each of these two inequalities involves a line. Since the inequalities do not involve the lines themselves (i.e. they involve < and >, rather than ≤ and ≥), notice that the lines on the screen are shown as dotted; the calculator shows ≤ and ≥ cases using a solid line instead of a dotted line.

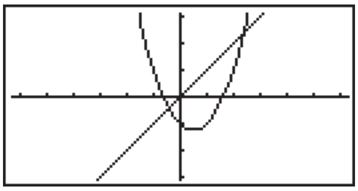
The next two screens show the two inequalities graphed together. (That is, the *compound inequality* is graphed.) The shaded area shows the points for which $y < 1$ and also $y > x - 1$ at the same time.

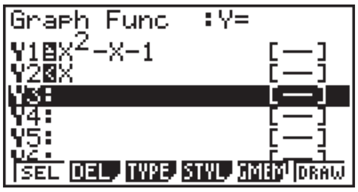
Before you do anything else, tap one of the cursor keys ◀ ▶ ▲ ▼ to see what happens if you move the origin a little.

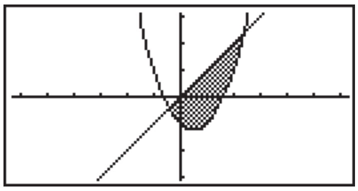




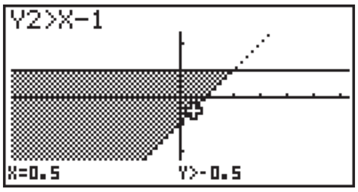
It is possible to convert easily from one kind of graph to another (a faster procedure than deleting an original inequality and then entering the new one.) To illustrate, the first graph below shows graphs of $Y1(x) = x^2 - x - 1$ and $Y2(x) = x$. In the middle screen, after tapping TYPE (F3) and CONV (F5), the F4 key allows the function $Y1(x) = x^2 - x - 1$ to be converted to the quadratic inequality $Y1(x) \geq x^2 - x - 1$. Similarly, $Y2(x) = x$ has been converted to $Y2(x) < x$.







Check for yourself that you can still use zooming and tracing on graphs involving inequalities. Graphs can be traced, using SHIFT F1. Only the functions that are traced, not the inequalities, although the coordinates at the bottom of the screen *do* show the inequalities correctly:



Various kinds of zooming are still possible with inequalities, in the same way as for graphs of functions. Try some zooming to see what happens.

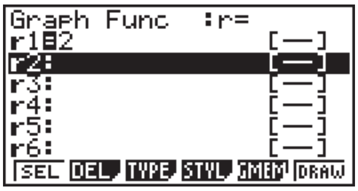
You can get the calculator to graph all defined inequalities at once, rather than one after the other, by turning on *Simul Graph* (simultaneous graph) in the SET UP menu.

Many inequalities are not expressed in an appropriate form for the calculator to be able to draw their graph. In these cases, you must first rearrange them so that there is only one variable (y) on the left side. For example, $2x + y < 5$ must be rearranged to give $y < 5 - 2x$, and $3x - 2y > 11$ must be rearranged to give $y < \frac{3x - 11}{2}$, before they can be graphed.

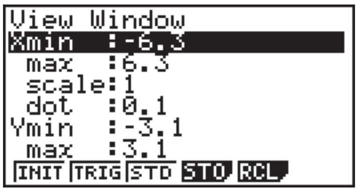
Polar graphs

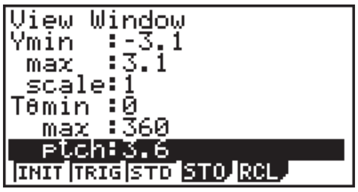
To draw graphs using polar coordinates, first tap MENU 5 to enter Graph mode, and then check that the angle setting in SET UP is the one you want. (The graphs drawn here are using degrees.) Tap F3 (TYPE)and the immediate command F2 (to change the graph type to polar graphs. Notice that the *Graph Func* notice on the top line of the screen changes, as do all the functions below the cursor in the function list.

Rather than a rule for defining y in terms of x, polar functions need a rule defining r in terms of θ. Consider first the constant function $r = 2$. Enter it in the same way as for previous graphs:

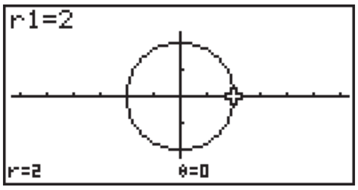


Tap V.Window (SHIFT F3) to set the viewing window. The best choice to start with is usually INIT (F1), which looks familiar as shown below. Use ▼ to go down past the bottom of the screen or ▲ to go above the screen, to see other entries shown at right below. This screen shows the INIT settings for θ: starting with $\theta = 0^{o}$ and going up to $\theta = 360^{o}$ in steps of 3.6^{o}. Each step is called the *pitch*. The calculator will graph 100 points with these settings.

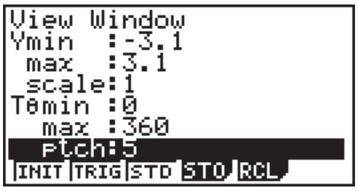


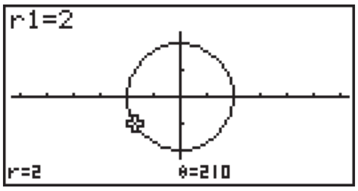


Return to the function list and tap F6 to draw the graph. You should not be surprised to get a circle with centre at the origin. (The choice of the INIT screen means that the horizontal and vertical scales are the same; were this not so, you would see an ellipse instead.)



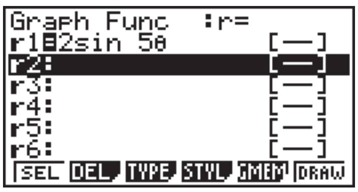
You can trace and zoom the graph in the same way as previously. However, notice that tracing now allows you to move through successive values of θ, rather than x, starting with 0^{o} and finishing with 360^{o}. Tap ► to go to a larger value for θ and ◄ to go to a smaller one. The coordinates now give values for r and θ, rather than x and y. To use a different step for tracing, you need to use a different pitch in the viewing window. For example, to trace every 5^{o}, rather than every 3.6^{o}, change the viewing window as follows, and draw the graph again:

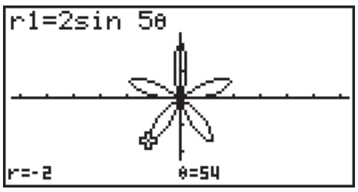




The graph is drawn a little more quickly (since only 72 points are drawn this time), but looks to have about the same resolution on the screen. Be careful with changes to the *pitch*. Notice the graphing speed if you change *pitch* to 1 and the shape of the graph if you change *pitch* to 60!

Most polar graphs are a bit more interesting than a circle. Examples include the 'petal curves', such as that given by $r = 2\sin 5\theta$. To define this function, notice that the X,θ,T key gives the variable θ instead of X when you have set the graph type to polar. Use INIT to set *pitch* to 3.6.





Conics

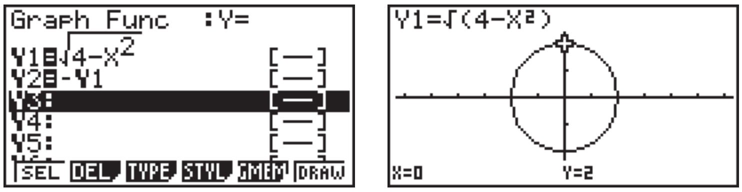
When you slice a cone in different ways, the cross sections take various shapes, which are called *conic sections*. Some of these, such as hyperbolas and parabolas, can be graphed with appropriate functions in Graph mode, while others, such as circles and ellipses, are more difficult to graph.

Consider, for example, a circle, with centre at the origin and radius 2, given by the relation:

$$x^2 + y^2 = 4$$

It is called a *relation,* rather than a function, because there is not a unique value of y associated with each value of x. The calculator will only allow you to graph functions in Graph mode, so you would need to graph the *pair* of functions: $y = \sqrt{4 - x^2}$ and $y = -\sqrt{4 - x^2}$.

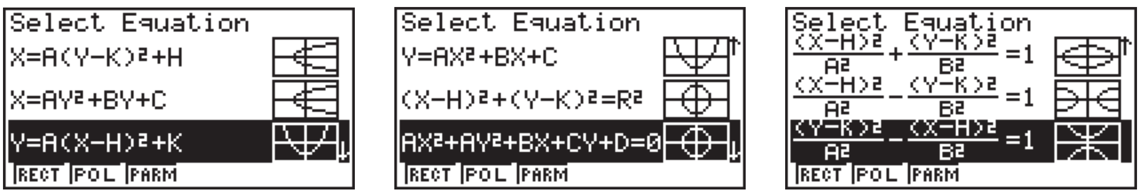
The screen shows the resulting Cartesian graph on the INIT window. (The INIT window has the same scale on each axis; otherwise, the circle will be shown as a non-circular ellipse):



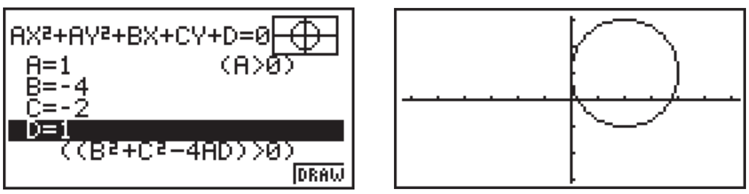
Some circles can also be drawn as polar graphs, as you have just seen above.

It would be much more complicated, however, to use these sorts of approaches to draw the graph of the circle of the same radius, but with centre at (2,1). The relation can be expressed as $x^2 + y^2 - 4x - 2y + 1 = 0$. Even if you could quickly rewrite this relationship into the more convenient form $(x - 2)^2 + (y - 1)^2 = 4$, it is still a complicated matter to enter it into the calculator to draw and study the graph.

The calculator has a Conics graphing mode to make it easier to graph conic sections. Tap MENU 9 to enter the mode and use ⓥ and ⓐ to see the suite of nine relationships available. All nine are shown in the three screens below:

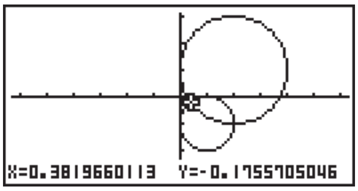


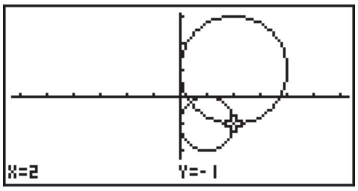
Select a relationship by tapping [EXE] when it is highlighted. Enter the values of any necessary parameters (*A, B, C, D, K, H* or *R*), followed by [EXE] after each. The screen on the left below shows the parameter values for the circle described above and the screen on the right below shows the resulting graph of a circle, after the viewing window was set to INIT.



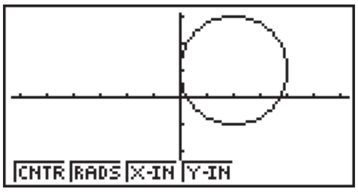
Only one graph can be drawn at a time in Conics mode, so you cannot use it for exploring in detail the intersections of pairs of graphs. Since you can also draw a circle with *Sketch* ([F4]), followed by [F6] and Crcl ([F3]) to get started), you can explore the relations between *some* graphs. For example, the circle with centre (1,-1) and radius 1 has been also drawn below at left, using the *Sketch* menu, making it easy to see that the two circles intersect in two points.

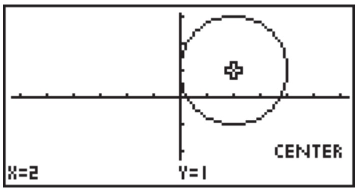
Although you can't use *graph solve* to find the points of intersection automatically, you can trace to find them *approximately*: (0.4,-0.2) and (2,-1), as shown below:

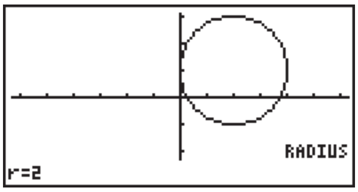




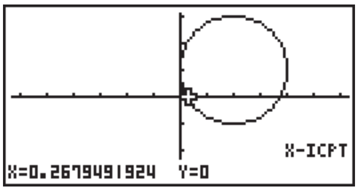
As well as tracing, a conic graph can be explored once it is drawn. The graph can be zoomed in the usual ways, and a *graph solve* facility is also available, but with a particular set of commands relevant to each kind of conic. For example, the screens below show the *graph solve* menu for the circle, allowing either its radius, centre and intercepts with the axes to be located automatically:

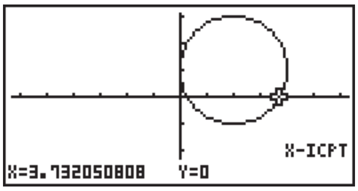




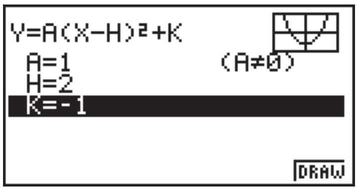


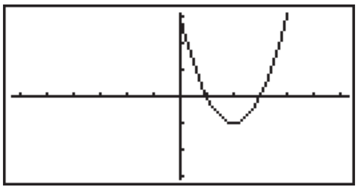
After the first x-intercept is found, the next one is found with the ▶ key:



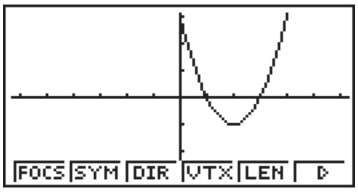


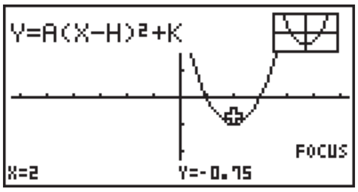
As a second example, the screens below show a graph of $y = (x - 2)^2 - 1$. The graph is a parabola, as expected. A parabola can be defined geometrically as the locus of points that are equidistant from a fixed point (the *focus*) and a fixed line (the directrix).

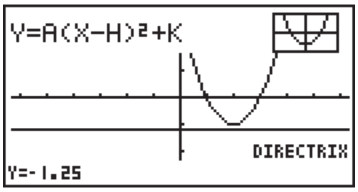




In this case, the Graph Solve menu lets you find the *focus, line of symmetry, directrix*, *vertex* and the *latus rectum* (the length of a line segment joining two points of the parabola, parallel to the directrix and passing through the focus), as well as intercepts with the two axes:







In this case of this parabola, the focus is at (2, $-\frac{3}{4}$) and the directrix is given by $y = -\frac{5}{4}$.

More detailed and complete information on graphing and studying conics is contained in the *Owner's Manual*.

Exercises

The main purpose of the exercises is to help you to develop your calculator skills.

1 Graph the function $f(x) = x^3 - 3x$ as $Y1$ on the INIT screen. Then draw the graphs of $Y2 = Y1 - 2$ and $Y3 = Y2 + 4$ on the same screen. Explain what you see.

2 Draw the graphs of $y = x^2$ and $y = x$ as Y1 and Y2 on the same screen. Then draw a graph of $Y3 = Y1 + Y2$ as well. How do the three graphs compare? Use a table to compare the three functions.

3 Draw a graph of $f(x) = 3x$ for $-1 \le x \le 4$ and with y-values to suit these x-values. Then use a Y variable to define a function whose values are half those of this function for every value of x.

4 Use the built-in function family $Y = Asin(BX+C)$. Select B as the dynamic variable rather than C. Graph the functions for $-2 \le B \le 2$.

5 Select the built-in function family $Y = AX^2+BX+C$ in Dyna mode. Define a family with $A = 1$, $B = 1$ and C varying from -3 to 3. Then change the dynamic variable to A varying from -2 to 2 and B and C both fixed at 1.

6 Imagine what a graph of the inequality $y < x + 1$ will look like. (Start by visualising $y = x + 1$.) Then draw the graph to check. Then, imagine the graph of $y > x + 1$. Draw a suitable graph to check.

7 Graph $2 > y$ on the INIT screen.

8 Rearrange the inequality $x + 2y < 4$ into a form suitable for entering into the calculator for graphing.

9 Draw a graph of $3y < 11$ on the INIT screen. Explain what you see.

10 If $f(x) = x - 2$ and $g(x) = x^3 - 1$, use the calculator to evaluate $f(g(4))$ and $g(f(4))$.

11 a Set the calculator to degrees. Draw a graph of the function $r = 3 \cos 5\theta$ on the INIT screen. It should have five petals.

b If you trace the graph, you will find that each tap of ▶ moves the cursor 3.6^o. Change the viewing window so that it moves 2^o instead. Trace to check that when $\theta = 60^o$, $r = 1.5$.

12 In Conics mode, draw the circle given by $x^2 + y^2 - 2x + y = 8$.

Use the *graph solve* menu to find the centre and radius of the circle.

13 Draw the parabola defined by $y = x - x^2 + 1$.

Find the focus, vertex and directrix of this parabola, and check that vertex is the same distance from the focus as from the directrix.

14 Draw the graph of a function $g(x)$ defined on the interval $-2 \le x \le 4$ as follows:

$g(x) = x + 2$, for $-2 \le x < 1$
$g(x) = 3$, for $1 \le x < 3$
$g(x) = 12 - 3x$, for $3 \le x < 4$

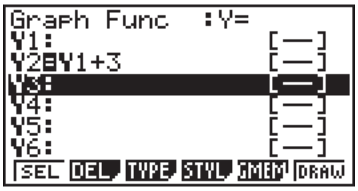
What shape is formed by the graph and the x-axis on the interval $-2 \le x \le 4$?

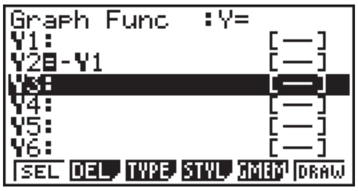
Activities

The main purpose of the activities is to help you to use your calculator to learn mathematics. You may find that some of them are too advanced for you. Ignore activities you don't yet understand.

1 Use the calculator to explore graphs of functions with rules like $y = (x - h)^2 + v$. (One example is for $h = 3$ and $v = -1$: $y = (x - 3)^2 - 1$.) Work with a partner to find ways of predicting what the graphs will be like. Test your predictions by imagining what the graph of $y = (x + 1)^2 + 8$ should look like. When you both agree, draw the graph on the calculator to check.

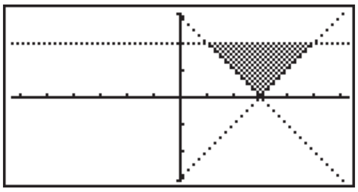
2 Make and use a function transformer by defining the second function in the function list in terms of the first function in the list. The screens below show two of the many possibilities:





Use your function transformer with an assortment of functions, to see what effects it has. Study the effects of one transformer on several different kinds of functions, until you know exactly what effect it has. Compare your observations and conclusions with someone else.

3 Here is a graph of the region for which $y < 2$, $y > x - 3$ and $y > 3 - x$.



Imagine and then investigate which regions will be shaded if you reverse some of the inequalities in this case (i.e. change < to > or change > to <).

4 Investigate petal curves with formulas like $r = 3 \sin k\theta$. Try integer values of k. A good way to do this is to use the Dyna mode.

What is the effect of replacing the constant (3 above) with 2? Why?

5 In Conics mode, there are two ways of drawing circles, with formulas: $(x - h)^2 + (y - k)^2 = r^2$ and $ax^2 + ay^2 + bx + cy + d = 0$. Draw some circles using these, and find their centres and radii. Find the connections between the formulas for the circles, their centres and their radii.

When you know the connections, check by predicting what the centre and the radius of the circles given by the following two formulas will be before you graph them:

$$(x - 2)^2 + (y + 1)2 = 16$$

$$x^2 + y^2 - 8x - 4y - 5 = 0$$

6 Investigate for different values of A and B the shapes of ellipses given by the formula:

$$\frac{(x-h)^2}{A^2} + \frac{(y-k)^2}{B^2} = 1$$

For example, what happens if $A = B$? What happens if $A = 2B$? What happens if you switch A and B for a particular ellipse? Explore also the effects of changing the values for h and k.

Notes for teachers

This module illustrates several ways in which the calculator can be used to explore various aspects of functions and graphs, using some of the many sophisticated capabilities of the calculators. The text of the module is intended to be read by students and will help them to start to see how the calculator can be used efficiently for various kinds of explorations in several different areas of mathematics. The Activities are appropriate for students to complete with a partner or in a small group, so that they can discuss their observations and justify their conclusions.

Answers to Exercises

1. The original graph is translated vertically downwards and upwards by 2 units.   2. Y3 is a parabola to the left of, and below, Y1.   3. Use Y2=Y1÷2   7. To obtain graph, define $y < 2$.
8. $y < 2 - x/2$ 9. Shading below a horizontal line at $y = 11/3$ 10. 61, 7   12. (1,-0.5), 3.04
13. (0.5,1), (0.5,1.25), $y = 1.5$ 14. A trapezium

Activities

1. Students can use any of Graph mode, Dyna mode or Conics mode to explore these parabolas, and will find it interesting to use all three of these. Encourage them to work together to appreciate that the value of *h* indicates the vertical line of symmetry, while the value of *v* helps to clarify the vertex of the parabola. Students should also be encouraged to try further examples for themselves.

2. There are many opportunities to understand the nature of functions through the use of transformers of these kinds, and students working together should be able to understand the two examples given as well as to try others for themselves. Understanding transformations will help them to interpret more sophisticated functions. [Answers: a vertical translation by 3 units; reflection about the *x*-axis.]

3. Students can change the inequalities efficiently using the conversion commands described in the module. This activity is intended to help them to visualise inequalities, so make sure that they imagine what they will see before using the calculator to check it. Activities of this kind are best attempted by students in pairs, so that hey can explain their predictions to each other. It is also a good idea to ask them to change from < to ≤ and from > to ≥ to see how these are represented by the calculators.

4. Students should enjoy the effects of making changes to both *k* and the constant here, to generate a variety of petal curves. An efficient way to explore this situation in Dyna mode is to graph Y1=$A\sin k\theta$ in polar mode. Remind the students to restrict their attention to integral values of *k*. [Answer: the constant before the $\sin k\theta$ affects the size of the graph while the value of *k* dictates the number of petals.]

5. Students can use the *graph solve* commands in Conics mode to find centres and radii of circles efficiently. Make sure that they begin their experimentation with small integral values of the coefficients to make it easier to see the patterns involved. Encourage them to routinely make predictions before they draw a graph to foster a thoughtful use of the calculators. Working in pairs is a good way to support such activity. After they have made the connections, they may seek to derive them algebraically. [Answers: centre (2,-1) and radius 4; centre (4,2) and radius 5.]

6. Even though ellipses are not often studied in school, students can learn about them in the manner suggested by this activity. [Answers: $A = B$ indicates a circle; $A = 2B$ gives a major axis twice as long as the minor axis. Switching *A* and *B* switches major and minor axes.]

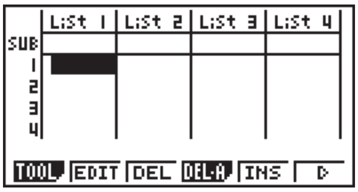
Module 4
Data analysis

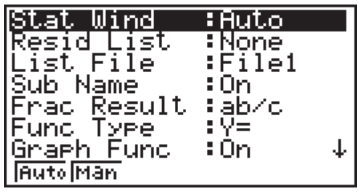
Note: This module is for the CASIO fx-9860GII graphics calculator only. If you have the CASIO fx-CG 20, please refer to Module C2.

For data analysis purposes, a graphics calculator like the CASIO fx-9860GII differs from a scientific calculator in two important ways. Firstly, substantial data sets can be stored, and so can be checked, edited and transformed. Secondly, graphical analyses are possible, as well as numerical analyses. In this module, you will see how to use the calculator to help you with data analysis.

Entering and editing data

Press MENU 2 to access Statistics mode. Data can be entered into any of 26 variables, called *List 1* to *List 26*. Use the cursor keys to see if there are any data already stored in your calculator. Press the continuation key, F6, to get the middle screen below. Note that F3 and F4 are used to DELete data. DEL (F3) deletes a single data point, while DEL.A (F4) deletes a whole column (list) of data. Use the DEL·A command, several times if necessary, to clear any data from your calculator before you start this module. Also, access the SET UP menu check that the *Stat Wind*(*ow*) option is set to *Auto*, as below.



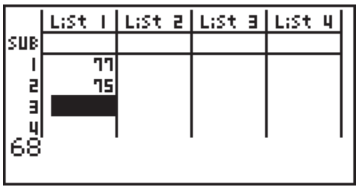


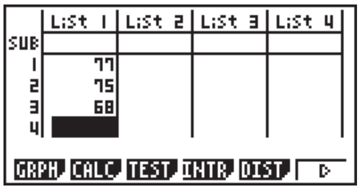
To explore some of the calculator's capabilities, consider the following data set, showing March rainfall (measured in millimetres) for a country town, collected annually for the past 30 years:

77, 75, 68, 81, 110, 90, 88, 42, 68, 88, 95, 62, 72, 120, 79, 80, 90, 81, 88, 77, 101, 91, 84, 85, 63, 62, 84, 82, 87, 76

Although this is not very large set of data, the ideas used here will apply for a set of up to 999 data points. It is hard to get a reliable impression of large sets of data without some form of analysis.

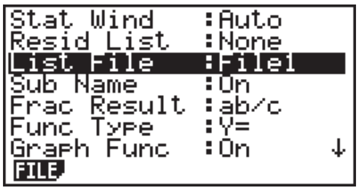
Enter Statistics mode with MENU 2 to enter the data into *List 1* of the calculator. Place the cursor in *List 1* and enter the data points one at a time, pressing EXE after each. As you type each data point, it is shown in normal size at the bottom left of the screen, and can be edited in the usual ways before pressing EXE. Notice that the cursor moves downward each time you press EXE. The screen after the first two entries in Statistics mode is shown below. Use the cursor to move around to correct any typing errors by retyping the correct value. Notice that the entry highlighted in a list is reproduced in normal size at the bottom of the screen.



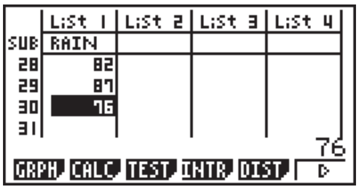


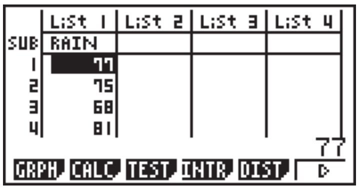
There are 26 data lists available, with each one permitted to have up to 999 elements. In addition, the calculator will allow you to enter and deal with even more data if you wish, using *data files*. A data file is a separate collection of 26 lists, *List 1* to *List 26*. The calculator has six data files, called *File 1* to *File 6*. (There are *overall* limitations on calculator memory of course, which may affect how much data you can enter, depending on what other material is stored in your calculator.)

To change from the present data file to another one, first enter SET UP to see the file which is being used at present, reported in *List File*, as shown below. Select your preferred file starting with FILE (F1) Choosing a new file does not delete any previous data you have entered in another file. Once data are stored in the calculator, in either lists or files, they will stay there until you delete or replace them, even when the calculator is turned off. Changing files allows you to store data in the calculator for later use, while still using the six lists for everyday use.



When entering data, it is easy to make a key pressing error, so you should be alert to ways of checking that the data have been correctly entered. One way is just to scroll up and down *List 1* and check against the original data. As you scroll, look out for entries that are obviously incorrect (such as those with only one digit or with more than three digits). An incorrect entry can be corrected by retyping it. The screen below shows the data after they were entered in STATISTICS mode. This gives a check that the correct number of data points have been entered as the 30th data point is 76, matching the original data; with so many data points, it is easy to miss one or enter one twice.

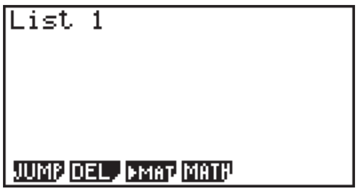


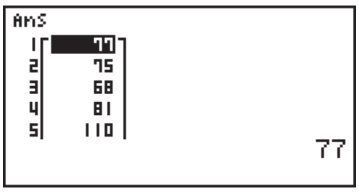


Errors and omissions can be corrected using the EDIT, DEL(ete) and INS(ert) commands shown above, which are available after tapping the continuation key F6 in the original screen. The DEL.A menu can be used to delete an entire list, if necessary. You can scroll in both directions using ▲ and ▼ or can use the TOOL (F1) menu to jump to the TOP (F3) or BTM (F4) of the list.

You can give a short name to each list, to help you remember what the data represent. Move the cursor to the SUB row and then use ALPHA and the letter keys to do this. In this case, the data have been labelled RAIN. You can turn off the *Sub Name* feature if you wish in the SET UP menu.

Once entered into the list, data can be recalled in Run-Mat mode also. The *List* command is available on the keyboard as SHIFT 1. Recalling a large list puts a scrollable version on the screen, as shown below. Tap EXIT to escape the list. Particular list elements can be recalled by number using the square brackets on the keyboard (SHIFT + and SHIFT −), as shown below on the right.



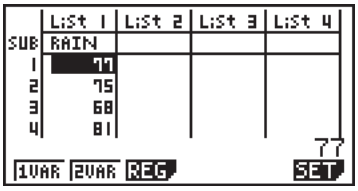


Summarising data numerically

Once data are entered, your calculator allows you to summarise them in various ways, to help you to interpret them. Two important ways are to calculate some statistics and to sort the data.

Statistics are numbers that can be used to help describe samples of data. Some statistics such as the *mean* and the *median* are associated with *central tendency*, since they summarise where the data are on average, in some sense. Other statistics are associated with *spread*; these include the *standard* *deviation*, the *quartiles*, and maximum and minimum data values. Other statistics such as the *mode* and the sum of the scores Σ*x* can also be used in various ways to understand a set of data better.

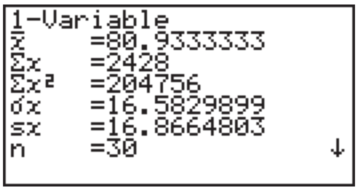
Calculations of statistics are controlled from the CALC menu in the main Statistics screen. Tap CALC (F2) to activate the menu and then SET (F6) to set the calculations to those you want.

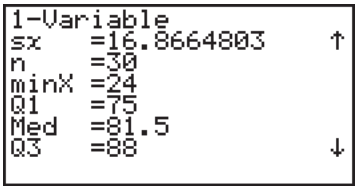


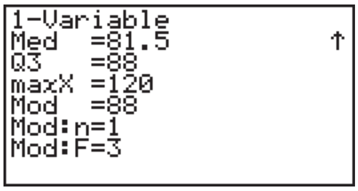
Calculations are available for one variable at a time (univariate, 1VAR) or two variables at a time (bivariate, 2VAR) as shown above. You need to set the calculator according to your data.

In this case, the choice of *List 1* above matches the data you stored in *List 1*. Notice that the 1Var Frequency is set to 1, as each data point represents a single year's rainfall. Press EXIT from the SET screen and then 1VAR (F1)to display the univariate statistics for the rainfall data.

There are many statistics provided, as shown below: you will need to scroll with ▽ and △ to see them all.

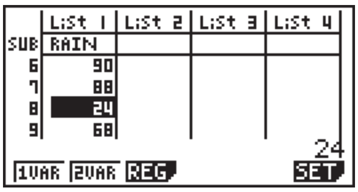


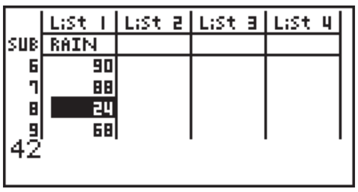


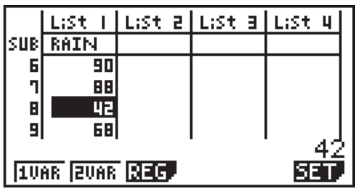


As well as helping to interpret the data, these calculations provide a few checks on your data entry. For example, the value for $n = 30$ is correct, the maximum value of 120 is correct, but the minimum value of 24 is *not* correct. (It is fairly easy in this case to scan the original data to find the largest and smallest values to check these.) The mode of 88 is correct. There is only one mode and it has a frequency of 3. There is at least one key press error here, as the minimum value is incorrect.

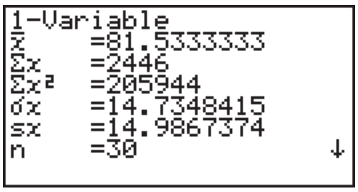
A very common error in typing is *transposition* – entering two characters in the wrong order. In this case, a scroll down to check the data shows that the 8th entry has indeed been transposed, as shown below. The error is easily fixed by typing over it with 42 EXE:

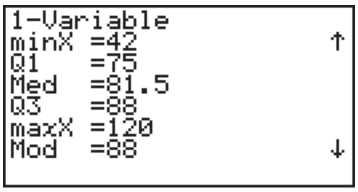




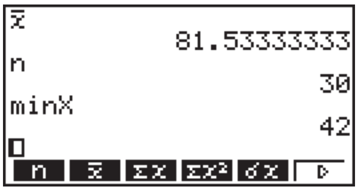


Now, if you return and calculate the univariate statistics again, they will be correct, as shown below. Check that yours match these, in case you have made any data entry errors.

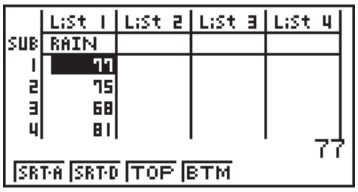




Conveniently, the more important statistics can be retrieved in the variables menu in RUN-MAT mode *after* data have been analysed. Press VARS and STAT (F3). You will recognise the symbols in X (F1) shown at the right below. Use the continuation key F6 to find what you want.



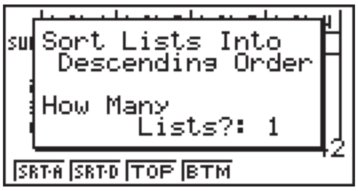
If the data are not in order of size when you enter them (as for the present example), it is a good idea to *sort* them from smallest to largest (*ascending* order) or from largest to smallest (*descending* order). A useful first analysis of the data, sorting may also reveal typing errors. Return to the main screen with EXIT and press the continuation key F6 and then TOOL (F1) to get the next screen. There are two sort commands, SRT.A (F1) to sort into ascending order and SRT.D (F2) to sort into descending order.

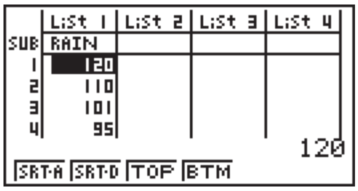


Be careful here! Once the data have been sorted, you cannot unsort them.

So it is usually not a good idea to *start* an analysis by sorting the data. Since the original order of data cannot be restored after sorting, it would also be unwise to conduct a sort if you later wanted to look at the annual trends in rainfall from year to year, for example. (It may be wise to make a copy of your data in an empty list, if one is available, before sorting. See the data transformations section later in this module for how to do this.)

In this case, we will sort the data from largest to smallest (i.e. into descending order) with F2. The calculator asks how many lists are to be sorted together. With only one list, tap 1 EXE.

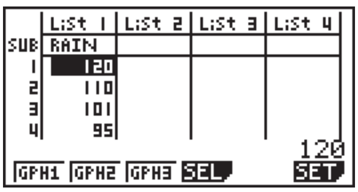


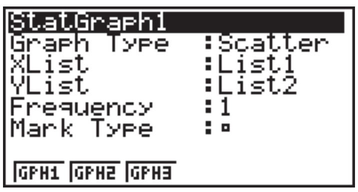


The list concerned is *List 1*, so tap 1 EXE in response to the next question. The data are then immediately sorted, as the third screen above shows. Scrolling this list will give you a better feel for the rainfall data than the original list. You can see, for example, that there are three rainfall readings of 88 mm, as suggested above with a single Mode of 88, for which the frequency is 3.

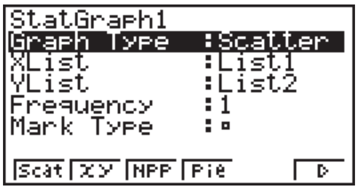
Displaying data graphically

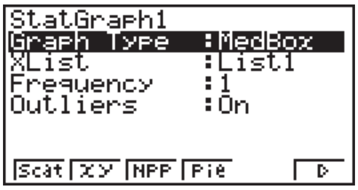
A good reason for using a graphics calculator to handle data is to deal with large amounts of data, for which a quick visual summary is hard to produce or interpret. A common way to represent larger amounts of univariate data is with a box plot or a histogram, each of which is accessible on this calculator. Graphical displays are controlled from the GRPH menu in the main statistics screen. Press GRPH (F1) to activate this menu and then SET (F6) your choices, as for the CALC menu in the previous section.



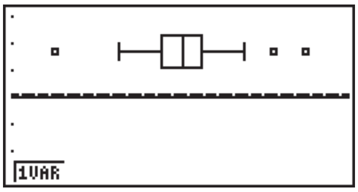


There are three graphs available at any time, labelled GPH1, GPH2 and GPH3. For each graph, move the cursor to see the settings and their choices. The previous screen shows that GPH1 is set to draw a scatter plot of *List 1* and *List 2* data, which is not useful here. Scroll down to highlight *Graph Type* and note the choices at the bottom.



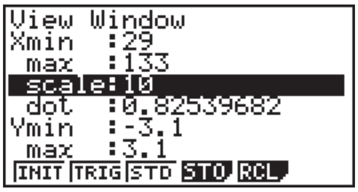


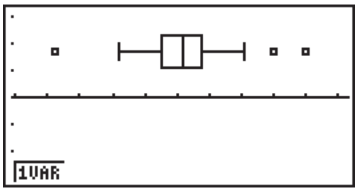
Use the continuation key F6 several times to see more choices. We will choose Box (F2) to choose a box plot for the rainfall data. A box plot can also be directed to display *outliers*, unusually large or small data points, less than the maximum and greater than the minimum scores; these may need closer inspection and explanation. (They may represent unusual data or they may simply be typing errors at some stage.) Set your screen to match the choices shown above on the right, including turning the outliers *on*. Then press EXIT to return and GPH1 (F1) to get the graph below.

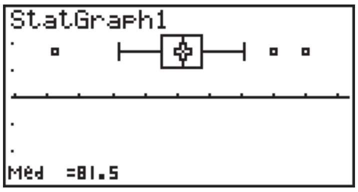


The box plot makes it clear that the rainfall is generally quite similar from year to year, with occasional very high and low years, as the 'box' is quite small in relation to the 'whiskers'.

Since the *Stat Wind*(*ow*) was set to *Auto* at the start of this chapter, the calculator chooses settings for the *x*-axis to suit the data. This is useful, as it will also show any data entry errors not yet detected. However, the choice of *scale* for the *x*-axis is not done automatically, so it's a good idea to adjust this for yourself. The result of changing the *x*-scale from 1 to 10 is shown below.

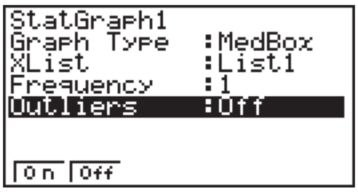


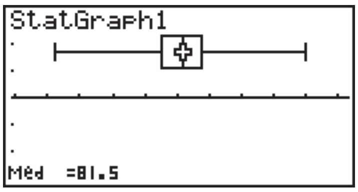




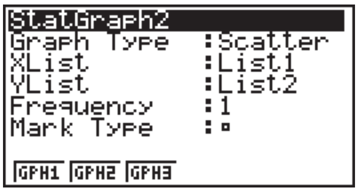
The changed scale makes the graph easier to read. The graph can also be traced with SHIFT F1 to show the five numbers used to construct it (The minimum and maximum scores, the median and the first and third quartiles) as well as the outliers; e.g., the median of 81.5 is shown above at the right.

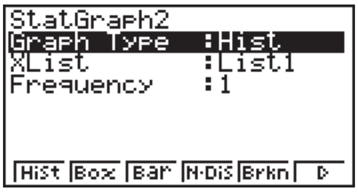
The original definition selected outliers to be displayed. Had you not chosen this option, a different box plot would result as shown below, where outliers are included in the 'whiskers'. This box plot gives a different impression of the data, since it doesn't recognise that three of the extreme points are outliers, suggesting that March rainfall is more variable than it actually seems to be.

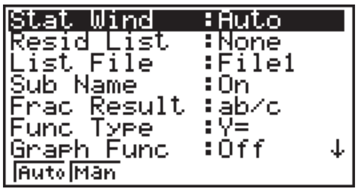




As well as a box plot of these data, a histogram may give a useful visual summary. A histogram has data values on the horizontal axis and frequencies on the vertical axis. It is a convenient form of summary to give a visual impression of the 'big picture' of a set of data. Return to the GRPH menu and tap SET (F6). Choose the second graph with GPH2 (F2) to be a histogram as shown below:



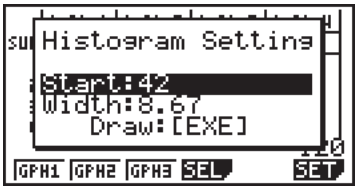


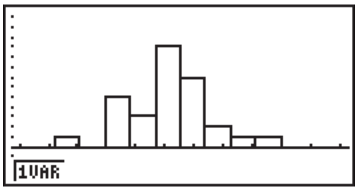


The advantage of allowing the calculator to automatically choose ranges for the histogram is that it will graph all the data, even those entered in error, allowing you to detect any data entry errors. But it is better to make some decisions yourself, in order to control what happens.

It is usually best to leave the *Stat Wind*(*ow*) on Auto, as the calculator will allow you to make changes. (On earlier models, it may be better to *not* do this.) Use SHIFT MENU to access the SET UP menu, and adjust the *Stat Wind*(*ow*) if necessary. It is also a good idea to turn the *Graph Func* to *off*, so that writing on the screen does not obscure the picture. Press EXIT to finish.

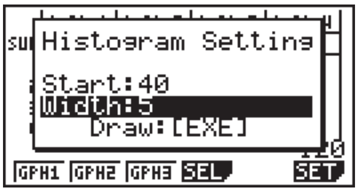
When you graph the histogram using F2 (GPH2), you will be presented with a choice of where to start the histogram and how wide to make each interval (or bin) for the data. The automatic calculator choice will *rarely* be appropriate, as below. (It starts at the minimum point and chooses a non-integral bin width to suit the data, which is quite hard to interpret when graphed.)

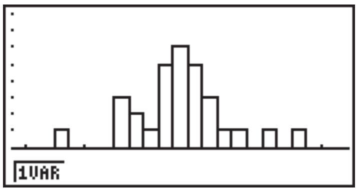




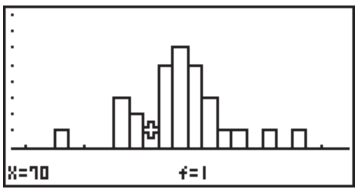
Notice also that the horizontal scale has many tick marks, because the calculator does not automatically change the *x*-scale: you are better able to decide this for yourself to suit your choice of histogram, as for the box plot. In this case, a better choice is to use an *x*-scale of 10, changed through the Viewing Window.

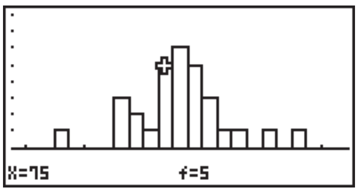
Better choices here for the histogram intervals might be to start at 40 mm and group the data into intervals of 5 mm (the *Width* on the calculator). Redraw the histogram, making these changes. Make sure you press EXE after each of the Start and Width entries. The result is shown below.

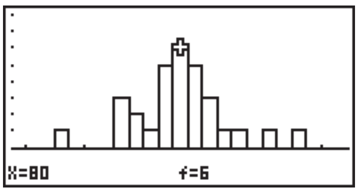




The histogram shows a strong clustering of annual rainfalls around 75 mm – 95 mm, with some rather wet years, a few dry years and a bad drought year. You can trace this histogram with TRACE (SHIFT F1) as shown below. Use ▶ and ◀ to trace. For each interval, the calculator shows the *left* endpoint as *X* and the interval frequency as *f* at the bottom of the screen.







These screens show that there is only one measurement in the interval $70 \leq x < 75$, five in the interval $75 \le x < 80$ and six in the interval $80 \le x < 85$, all consistent with the original data. Notice in particular that the rainfall of 75 mm is placed in the middle of these three intervals and the 80 mm rainfall is in the third one.

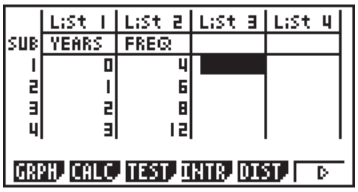
You can easily produce a different histogram by starting at a different point or (more importantly) changing the width of the intervals. There are few hard and fast rules on how to deal with data of these kinds, and the calculator will help you to explore the data in a number of ways in order to reach plausible conclusions. Because the shapes of histograms depend on the choices you make, you should always be cautious in interpreting histograms, both your own and those made by others.

Frequency data

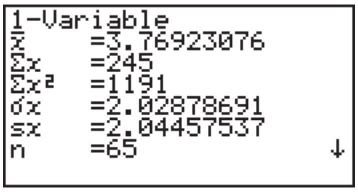
Some data have already been sorted into frequency distributions, so that it is convenient to analyse them taking the frequencies into account. E.g., the following data show the number of years (rounded to the nearest whole number) for which the 65 houses in a certain street had been occupied by the same people.

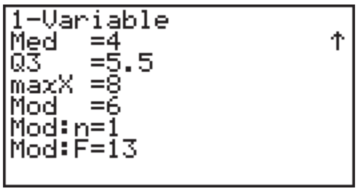
Number of years	0	1	2	3	4	5	6	7	8
Frequency	4	6	8	12	10	9	13	0	3

Although it would be possible to enter these data into a large list in the calculator (with four 0's, six 1's, eight 2's, and so on), there is an easier way. Enter the number of years into *List 1* and the frequencies into *List 2,* as shown below.

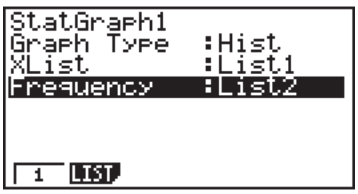


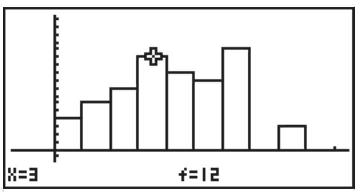
When setting the (univariate) calculations, identify *List 2* as the frequencies (*1VAR Freq*).





The *n* value shows that the calculator recognises that there are 65 data points in the data set, not just 8. Residents in the street have been in their houses about 3.8 years, on average. The median length of stay is 4 years, and the most common (the mode, with 13 households) is 6 years. Similarly, when setting graphical displays, identify *List 2* as the frequency associated with *List 1* data, as shown below, to create a histogram of the data. Set the *Width* = 1 so that each column represents one year.



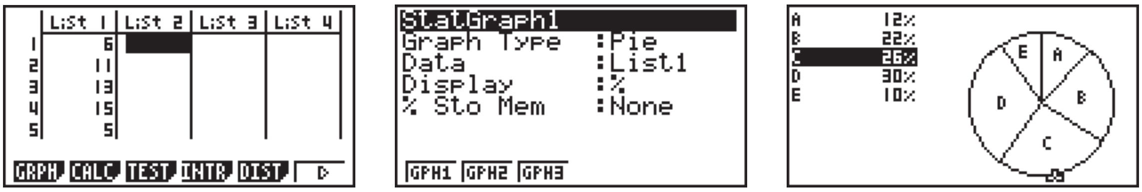


Categorical data

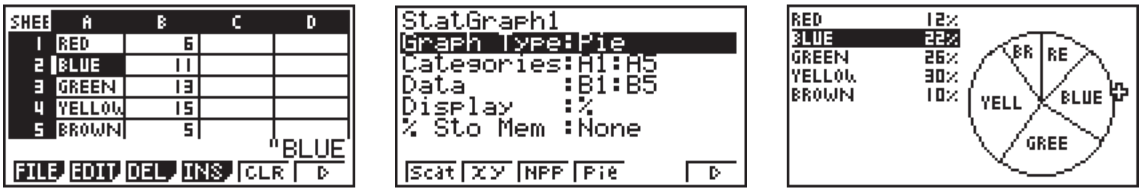
Some statistical data are arranged into distinct categories, and we are interested in comparing the categories in some way. The calculator is helpful for offering graphical representations for this purpose, especially pie charts and bar graphs. For example, consider the following table, showing the numbers of sweets of various colours in a box.

Colour	Red	Blue	Green	Yellow	Brown
Number	6	11	13	15	5

Enter the numerical data into List 1 as shown below. Select a Pie graph and choose to represent the *Display* as a percentage (which allows for good comparisons with other pie graphs), as shown below. The resulting Pie graph is shown below. Notice that the graph can be traced to show which sector of the graph corresponds with which category. The categories are described as A, B, C, D, etc. according to the order in which the data were listed.

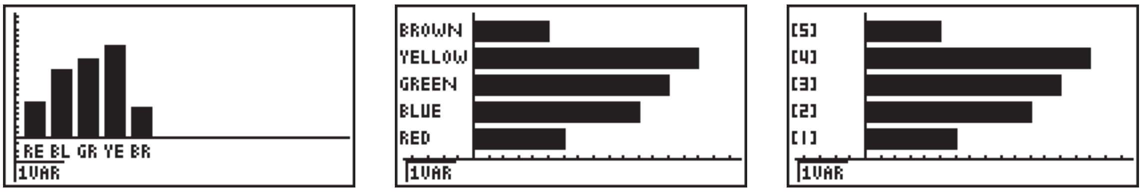


While this is a helpful graph, as better version is available if the data are stored in Spreadsheet mode, as the category names can be included as well. Use MENU 4 to access a spreadsheet. The screens below show the equivalents of those shown above for Statistics mode.

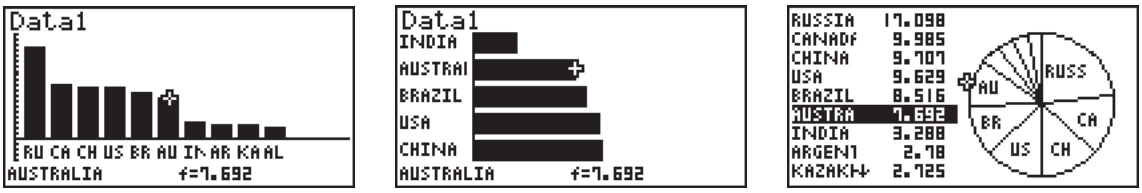


Notice that the data are referenced in cells, rather than lists, with A1:A5 referring to the column containing A1 down to A5. Notice importantly that the category names are included; to enter these, it is necessary to precede the text with inverted commas (SHIFT EXP). It is a good idea to use the Alpha-Shift Lock command (SHIFT ALPHA) to make entering names more efficient; shorter names are better than longer names. The GRAPH menu is available after tapping the continuation key, F6.

Data of this kind can also be represented with a bar chart, in both Statistics and Spreadsheet modes. The screens below show the choice of horizontal or vertical and the effects of choosing to use the Spreadsheet mode to include category labels (showing in the first two screens).



Sometimes, categorical data include numerical information, not frequencies. A bar graph is a suitable way of representing such data graphically, but care is taken to choose a suitable graph and label it suitably. The example below concerns the areas of the ten largest countries, according to *Wikipedia*. Data have been entered into the spreadsheet with full country names and areas (in millions of square kilometres). Some possible graphs from the spreadsheet are shown below:



The first bar graph is perhaps the best of these three representations, as all ten country populations are seen at once, although country names are abbreviated. The second bar graph requires that the

▲ and ▼ keys are needed as soon as the graph appears to choose which countries are shown, as there is insufficient screen space for ten categories. The pie chart shows many country names better (on the left of the screen) but not all fit on the screen and some are not shown at all on the graph.

Bivariate data

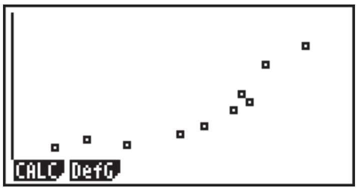
To explore some of the bivariate data analysis capabilities of this calculator, enter the following data set into the *List 1* and *List 2* columns, pressing [EXE] after entering each value.

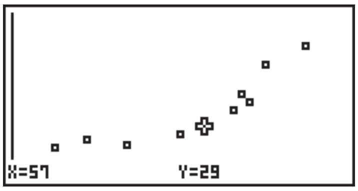
X (year after 1900)	20	28	38	51	57	64	66	68	72	82
Y (number of hotels)	15	20	17	25	29	42	53	47	75	88

The data refer to the number of hotels (*Y*) in a small growing city over some years (*X*) in the twentieth century. The years have been abbreviated to their last two digits, so in 1968 ($X = 68$), there were 47 hotels. Before graphing the data, go to GRAPH mode with MENU 5 and either delete with DEL ([F2]) or turn off with SEL ([F1]) any functions listed.

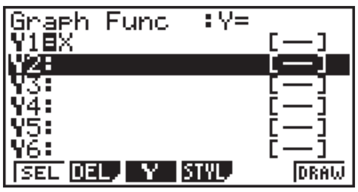
In Statistics mode tap GRPH ([F1]) to access the graph menu. Tap SET ([F6]) to define the first graph (GPH1) as shown below, a scatter plot with *List 1* on the horizontal axis and *List 2* on the vertical axis. The frequency of each pair is 1. Press [EXIT] when finished.

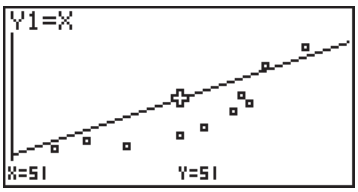
As for univariate data, it's best to use the *Auto* Stat Window (in SET UP) at first, to make sure all data are plotted. Choose GPH1 ([F1]) to draw the graph you have just defined, as below. Notice that the scatter plot can be traced, using [SHIFT] [F1]. This is a good way of checking your data entry.



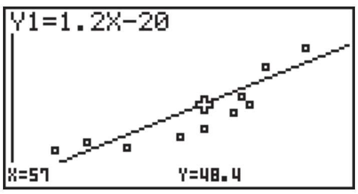


One way of exploring the relationship between *X* and *Y* here is to look for functions that fit the data approximately. In order to do this manually, tap [EXIT] and then [F1] to redraw the graph. Then select DefG ([F2]). The calculator is now in GRAPH mode. Delete or turn off with SEL ([F1]) any existing graphs. Enter a guess for a function that might fit the hotels data. For example, a first rough guess might be that the number of hotels is approximately the same as the year, $Y = X$.



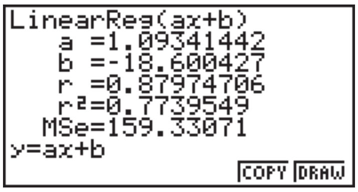


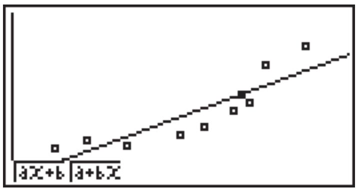
Use DRAW ([F6]) to draw the graph on top of the scatter plot. You can trace either the data or the graph to compare them and move between one and the other with ▲ and ▼. In the screens here, (*Graph Func* has been turned *on* in SET UP.) The guess $Y = X$ does not seem a very good one, as most points are below the line. A better guess may be for a slightly steeper line, dropped a little. The graph below shows such a guess with $Y = 1.2X - 20$.



To calculate and draw a regression line called *a line of best fit* through the data points, tap [EXIT], redraw the scatter plot and then tap CALC (F1). Choose one of the immediate menu items at the bottom of the screen, or one of the others shown by pressing the continuation key (F6). Various kinds of relationships between the two variables, *List 1* and *List 2*, can be represented.

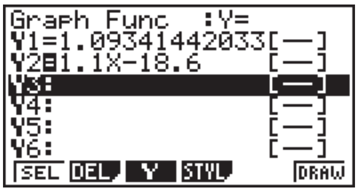
After the regression coefficients are displayed, tap DRAW (F6) to draw the line on the scatter plot. (Use COPY(F5) to copy the regression function to the Graph and Table function list for later analysis. It is not essential to do this, but it is a good idea to allow for options later such as those described later in this section.) The screens below show a regression line with X (F2), and then *ax+b*, to obtain a linear regression function through the data in the form $y = ax + b$.

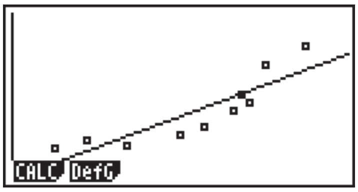




The regression coefficients *a* and *b* show that the best fitting line is given by $y = 1.0934x - 18.600$, close to the earlier guess, but the visual evidence still suggests the line is a poor fit to these data.

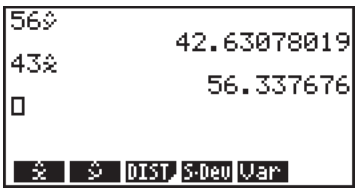
In this case, especially since the original data are all whole numbers, the calculator probably provides more precision than necessary with the regression coefficients. A more suitable line for you to use may be $y = 1.09x - 18.60$ or even $y = 1.1x - 18.6$. To enter this function, first tap EXIT and select DefG (F2), as shown below in *Y*2, having temporarily turned off *Y1* with SEL (F1).

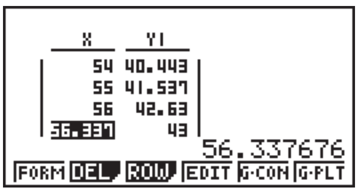




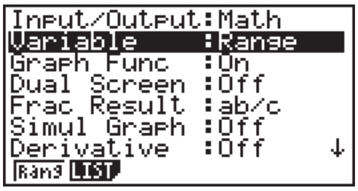
The line is a *model* for the data; it is an attempt to provide a simplified description of the relationship between the two variables. Of course, a linear model like this one is only one kind of model; there are many other possibilities to model relationships of different kinds. The *correlation coefficient* in this case $r \approx 0.88$, gives a measure of how well the data can be modelled by this linear function. The possible values for *r* are $-1 \leq r \leq 1$, with the positive value indicating that the larger values of *y* tend to be associated with the larger values of *x* (i.e., the slope of the line is positive). The square of the correlation coefficient, $r^2 \approx 0.77$ is also useful to indicate how well the model accounts for the data. In this case, it can be interpreted as 77% of the variation in hotel numbers can be accounted for by the year in the linear model.

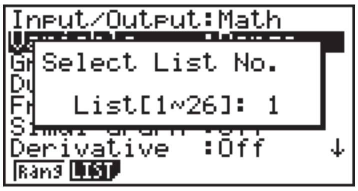
The regression line just found can be used to make predictions. The calculator will work these out automatically, using the STAT menu under OPTN, but you will first need to return to Run-Mat mode with MENU 1. The two menu choices in STAT are shown below. To predict the *y*-value associated with $x = 56$, enter 56 followed by $\hat{y}$ (F2) EXE. To find the *x*-value associated with a *y*-value of 43, enter 43 followed by $\hat{x}$ (F1) EXE. You can check manually that these are the values found by substituting in the linear regression function above. Be careful when making predictions that go beyond the data you started with (this is called *extrapolation.* You can also explore the model in Table mode (provided you copied the function previously), as shown on the right below.



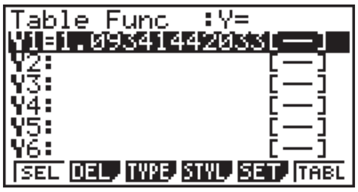


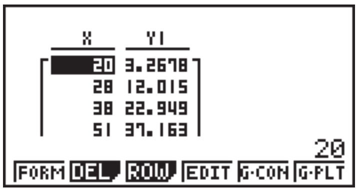
There is a third, different, way of using the table functions of the calculator, especially useful in this situation. The screen below shows that the SET UP for Table mode includes *Variable*, which can be set to *Range* or *List*. Change the setting to *List* and select *List 1* as the list, as shown below.





Now, when the table is constructed with TABL (F6), the calculator uses only the *List 1* values for the independent variable, *x*, as shown below. This method allows you to check the predictions from the linear model more efficiently. The predictions do not match the data very well in this case.

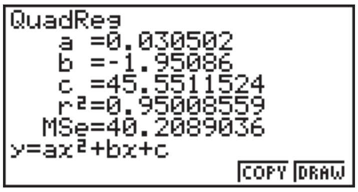


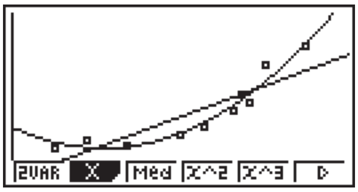


Make sure you return the *Variable* setting to *Range*, the usual preference in Table mode, after you have finished with your statistical analyses.

Return to Statistics mode with MENU 2 and draw the regression line again. As noted earlier, the line doesn't seem to fit the data particularly well, since there are several points that are not close to the line, and the general shape of the scatter plot does not closely resemble a line anyway. That is, the linear model is not a particularly good model for these data. In fact, it appears that the relationship between the two variables is *curvilinear* (i.e., a shape that isn't a line).

You can make more than one choice, to look for a good fit to the data. In this case, the data suggest that a quadratic model obtained with X^2 (F4), may be a better choice than a line. The screens at the top of the next page confirm that this seems to be the case, with both a line and a quadratic curve fitted to the data:





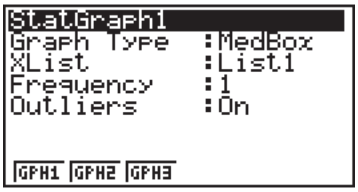
The quadratic model $y = 0.03x^2 - 1.95x + 45.5$ can also be explored in the same way as the linear model in Graph, Table and Run-Matrix modes. In general, you should be wary of choosing models that are very sophisticated, unless there are good historical or other reasons for doing so, especially with very small data sets of this kind.

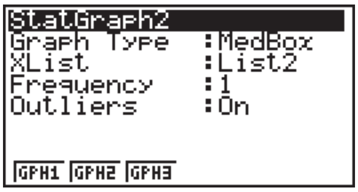
Comparing groups

Sometimes, data are gathered on several groups, but there is no pairing of the data, as there was with the hotels data. The intention of the data collection is often to make comparisons between different groups. For example, the data following show the heights in centimetres of the boys and girls separately in a class of fourteen-year olds.

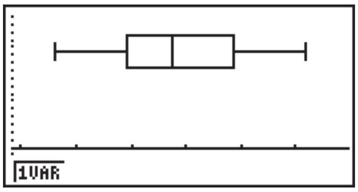
Girls	Boys
140, 165, 156, 156, 126, 139, 145, 131, 145, 147, 135, 157, 172, 162, 160, 148	136, 158,166, 173, 128, 132, 158, 157, 182, 162, 169, 148, 145, 129, 151, 181, 150

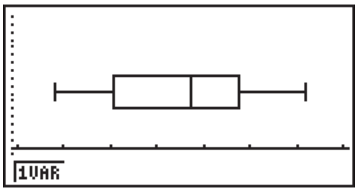
Enter the data into *List 1* and *List 2* for girls and boys respectively. A graphical comparison is often the most effective; the following screens show how to set up a box plot for each group separately.



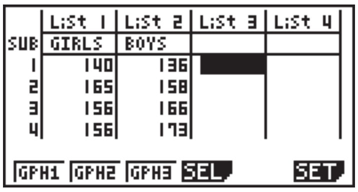


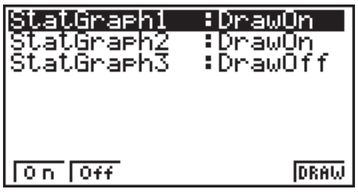
When the box plots are drawn (using, as usual, the *Auto Stat Window*), they both seem to be similar, with the box plot for boys (on the right) a bit lower on the screen than that for girls. These do not provide a useful basis for comparison, partly as each box plot occupies the entire screen width.

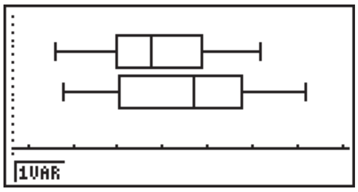




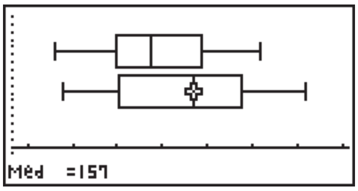
A better alternative is to draw both graphs on the screen at once. To do this, enter the GRPH menu and choose SEL (F4)to select graphs to be drawn, as shown below.

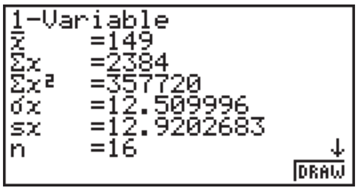


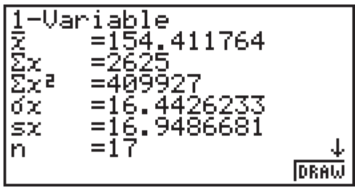




Use the cursor and F1 keys to turn *both* GPH1 and GPH2 on, as shown on the screen above in the middle. Then, when you tap DRAW (F6), both graphs are drawn together.







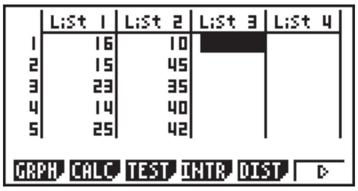
Notice that the graph for boys is still below the graph for the girls, as originally. It is now clear that there are substantial differences between the two groups. You can trace the graphs to see the differences in the five numbers used to create them.

If you redraw the graph and choose the one variable statistics with 1VAR (F1), the calculator will require you to select which of the two groups you want to obtain statistics for. Use the cursor keys to select and press EXE to enter your choice. Return to the box plots with DRAW (F6).

The previous screens show that the mean height for girls is about 5.4 cm less than that for boys, but the boys' heights are more varied, reflected in the higher standard deviation. These observations are consistent with the box plots.

Data transformations

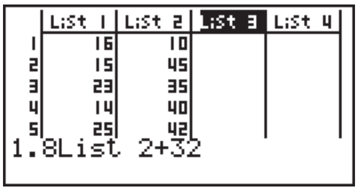
As well as entering data manually, as we have done so far in this module, lists of data can be *transformed* to make new variables. To illustrate the procedure, consider the small data set below, consisting of five data points in each of *List 1* and *List 2*. (In this case, the list names have been turned off in SET UP, with *Sub Name*.)

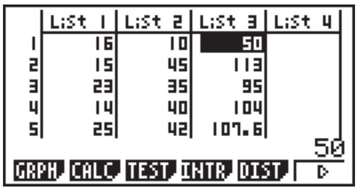


Suppose that the data in *List 2* were all temperatures in degrees Celsius, and you wanted them in degrees Fahrenheit instead. The relationship between these two temperature scales is:

$$Fahrenheit = 1.8 \times Celsius + 32$$

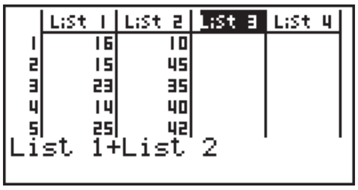
To transform the data in *List 2* in this way, we will make a new variable, *List 3*. First move the cursor to the heading at the top of the *List 3* column, highlighting the name, *List 3*, as shown below.

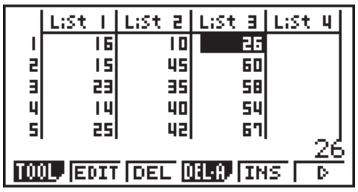




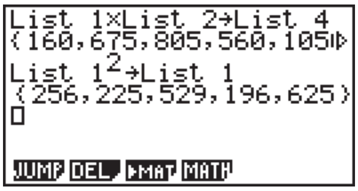
Then enter the transformation, 1.8 List 2 + 32, as shown above. For the List command, you will need to tap [SHIFT] [1]. When the transformation is entered, press [EXE] and it will be carried out immediately. The screen on the right shows the results. Notice that the entire set of *List 2* data is transformed in the same way. Check some of these for yourself to see that the data points in *List 3* are the Fahrenheit temperatures associated with Celsius temperatures shown in *List 2.*

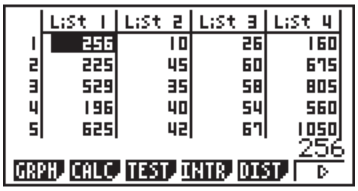
Transformations can involve several variables, too. For example, suppose you wanted to create a new variable that was the sum of the first two variables above. Using the transformation List 1 + List 2 in the *List 3* header will have this effect, as shown below.





Data transformations are usually best carried out in Statistics mode, as you can see the contents of the lists on the screen, and ensure that new variables created do not overwrite existing variables unless intended. They can also be conducted in Run-Mat mode, using the [→] key, as shown below.





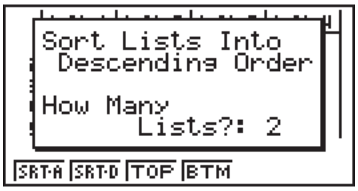
Return to Statistics mode and check the final results carefully, to understand the importance of the order in which these transformations were carried out.

Sorting bivariate data

In the previous section, the data were sorted to help make sense of them and as a kind of check on data entry. You should always take care when sorting data, but be especially careful when sorting bivariate data, represented by *pairs* of lists. Consider people's weights and heights shown below. Each row shows one person. It is important that the weights (*List 1*) and heights (*List 2*) for each person are kept together when sorting, or the data will become nonsensical.

Name	Weight	Height
Ong	46	124
Kym	57	161
Anil	43	138

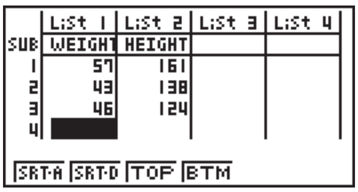
To sort the data (person's pairs of measurements) in descending order of height, first enter the data into *List 1* and *List 2*, tap F6 and TOOL (F1) and then SRT.D (F2). Enter 2 EXE to indicate that you have two lists altogether, as shown below.



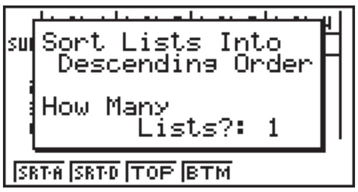
In this case, the *Base List* is height, since we want the data sorted by height. As heights are stored in *List 2*, enter 2 EXE to indicate this choice:

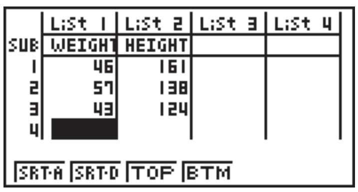
The calculator will then ask you for the other list to be sorted. In this case, it is *List 1*, so enter a 1 and then tap EXE to complete the process, as shown below:

The result of the sorting is shown at the right below. Notice that the heights and weights are still paired correctly with each other, and the data have been sorted into descending order of the heights in *List 2*.



The next screens show what happens if you sort the second list into descending order, *without* informing the calculator that there are two lists to be considered at once.





The connections between height and weight have now been ***permanently*** lost, since the weights have retained their original order, while the heights have been sorted.

Exercises

The main purpose of the exercises is to help you to develop your calculator skills.

1 a A record was kept in a golf tournament of the approximate distance (in metres) from a certain hole of the first shots of players. Here are the data:

Distance (m)	1	2	3	4	5	6	7	8	9	10
Frequency	7	10	18	16	14	12	12	39	13	11

Enter these data into *List 1* and *List 2* of your calculator. Name the lists DIST and FREQ respectively. Use CALC to set the 1Variable statistics so that the calculator knows which list refers to the frequency. Find the mean distance of the golf shots from the hole.

b Draw a histogram of the golf data. Trace the histogram to check that the data have been correctly entered. Which observation appears to be an outlier? Explain why you think it may be an outlier.

c In fact, there were only nine golf shots that finished eight metres from the hole. Change the data accordingly, and recalculate the mean distance.

d Now draw a box plot of the corrected golf data. Trace to find the median distance.

e Use a transformation to change the distances of the golf shots from metres into feet. (1 metre is about 3.28 feet). Store the transformed data in *List 3*. Draw a box plot of the transformed data, and find the mean distance of the golf shots in feet.

2 a The data in the table show the height of a stone thrown into the air from the roof of a house at various times after it was thrown until it hit the ground. The heights and times were estimated from a video replay of the incident.

Time (seconds)	0	0.2	0.4	0.7	1	1.2	1.6	1.9	2
Height (metres)	15	17	20	22	24	23	18	8	1

Enter these data into your calculator. How high was the house?

b Draw a scatter plot of the stone data. What sort of relationship (linear or curvilinear) appears to be involved?

c Change the plotting symbol on your scatter plot to a small cross.

d Find the regression coefficients for the line that best fits these data. Draw the line on the scatter plot. Does it seem to provide a good fit?

e Return to Run-Mat mode, and use the STAT menu to predict where the stone was after 1.5 seconds and the time at which the stone will return to its original height of 15 m. Then use the VARS menu to retrieve the correlation coefficient.

f Return to Statistics mode. Draw a quadratic curve through the data. Find the equation of the curve and the correlation coefficient. Which fits the data better – the line or the curve?

3 Children in a school were asked in a survey to nominate their favourite lunch food for a class picnic. Sandwiches were preferred by 53 children, 62 preferred fruit, 105 preferred rice, 44 preferred salad and 34 preferred pies.

a Use a spreadsheet to represent these data with a pie chart; then use the pie chart to find the percentage of children preferring fruit.

b Represent the data with a suitable bar graph.

Activities

The main purpose of the activities is to help you to use your calculator to learn mathematics. You may find that some of them are too advanced for you. Ignore activities you don't yet understand.

1 Choose a random sample of ten students in your class by selecting ten names from a complete list in a hat. Find out how many hours each student watches TV each week, and then use the calculator to find the mean, median and mode of these data. Represent the data using a box plot and histogram.

If one student changes their mind about the number of hours of TV watching each week, what difference will this make to the results?

2. Roll a standard six-sided die 50 times and make a frequency table of the results. Summarise the results in various ways, including finding the mean, mode and median and drawing suitable graphs such as a box plot or histogram. Compare your findings with your partner. Repeat these processes.

3. A group of students had their mathematics and science test results as follows:

Mathematics	20	28	38	51	57	64	66	68	72	82
Science	15	20	17	25	29	42	33	47	75	88

Plot these data and find an appropriate equation to predict the science result of a student in the class with a test mark of 65 for mathematics, but who was absent for the science test. Use box plots to help make some statements about the differences and similarities of the marks.

Try to obtain some marks on a pair of different tests in your own class, and use methods like these to compare them.

4. Two groups of patients were randomly selected and given medication to lower their blood pressure. One group was given a drug for this purpose, while the other group was given a placebo (a pretend drug); patients were unaware whether their drug was real or a placebo. The following results were recorded.

Drug

Before	100	106	105	103	100	70
After	100	90	90	95	75	80

Placebo

Before	110	110	105	95	85	82
After	100	105	95	100	90	90

Plot these data and use statistics to comment on the effectiveness of the drug.

5 The table shows the mean heights of 12 sets of parents, all of whom have at least one son, and the height of their eldest son at age 21.

Parent mean height (cm)	168	179	175	179	161	183	170	172	174	186	175	163
Son mean height (cm)	170	186	172	186	162	191	181	195	179	199	176	171

Plot the data and determine the best equation to predict the heights of sons from their parents. use your equation to make some predictions. Determine some heights for parents and their sons or daughters in your class and make some suitable comparisons.

6 Find out the population in your country over the past five decades. Find the best-fitting equation for these data over time and use this to predict the population in 2030.

What reservations might you have about predictions of this kind?

Notes for teachers

This module illustrates several ways in which the calculator can be used to explore various aspects of statistics, to help students understand the elements of univariate and bivariate data analysis. The text of the module is intended to be read by students and will help them to start to see how the calculator can be used efficiently to summarise and represent data in various ways, and become a tool for analysing their own data. The Activities are appropriate for students to complete with a partner or in a small group, so that they can discuss their observations and justify their conclusions.

Answers to Exercises

1. (a) 5.99 m (b) 8^{th}, entered as 39 instead of 9 (c) 5.49 m (d) 5.5 m (e) 15.01 ft 2. (a) 15 m (b) curvilinear (d) $y = -5.49x + 21.93$, not a good fit (e) 13.7 m, 1.26 seconds, $r \approx -0.53$ (f) $y = -15.49x^2 + 26.20x + 13.13$, $r \approx 0.97$ 3. (a) 20.8%

Activities

1. Activities of this kind are important, as they allow students to deal with real data of local interest to them. As the data are stored, and can be edited, they should be able to see for themselves the effects of making changes to the data, noting particularly the fragility of the mode and the possible effects of outliers on the mean.

2. Activities involving random data lend themselves to statistical analysis of the kinds suggested, and it is a good idea for students to work together, to see the variations between students and between trials. As well as comparing results, you might encourage students to amalgamate their results, to produce a more stable outcome.

3. The information provided is intended to provide a context for students to examine bivariate data, although a more substantial sample would normally be expected for work of this kind. This activity is intended to help the students refine their data analysis skills and to use regression results for a practical purpose. Many contexts will provide good data for this purpose, including the suggestion to obtain school test data. While this is frequently confidential, other bivariate data would serve a similar purpose and you should encourage students, or pairs of students, to obtain some real bivariate data for analysis in the manner suggested.

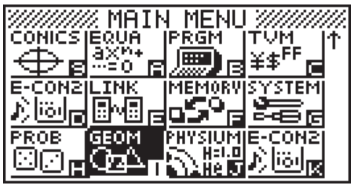
4. This activity offers an informal introduction to an important use of statistics in practical and research settings, which often involve small samples. More formal consideration of issues of this kind, and the comparisons involved, require statistical hypothesis testing, addressed briefly in a later module.

5. An activity of this kind will involve students refining their bivariate data analysis skills, making use of suitable plots and regression equations. It may be relatively easy for a class of students to obtain data of this kind from students in the class (together with their parents), so that real data are being used, rather than the data provided.

6. Students will usually be able to locate some suitable bivariate data (year, population) from government bodies (such as a bureau of statistics) or from Internet sources, although advice for your particular country may be needed. Over short periods, natural population growth is often linear, although over longer periods, different models are more appropriate. The most likely models to suit population natural population growth are exponential, which will require students to have some experience with exponential functions. Students should appreciate that population growth is rarely entirely natural, but is influenced by government policies (e.g. for limiting or encouraging family growth, warfare, or for migration), as well as improvements in health practices and reductions in child mortality, so that care is needed not to extrapolate models carelessly.

Module 5
Geometry

When you load the Geometry Add-in application to your calculator, a variety of geometric capabilities will be available to you. Some of these will be explored in this module. You will know that the Geometry Add-in is available to you if the GEOM icon is showing on the Main menu, as shown below on the fx-9860GII. (On the fx-CG 20, the icon is called 'Geometry')

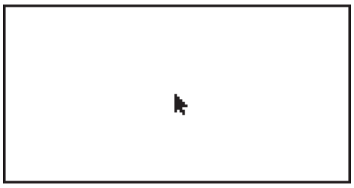


If the icon is not showing, you will need to download it and install it via your computer or transfer it from a calculator that already has it installed. Instructions for downloading are available on the web. The official international site which includes the Geometry Add-in and many other resources is *http://world.casio.com/edu/*.

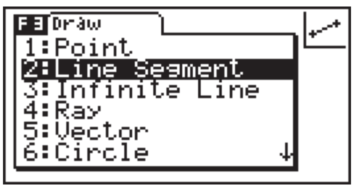
In this module, the fx-9860GII calculator has been used. If you are using the fx-CG 20, you will find that most commands are the same, although small differences will be noticed at times.

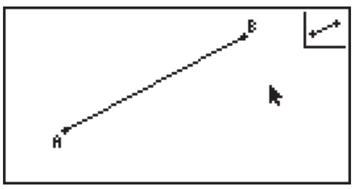
Drawing and changing objects

When you first select the Geometry mode, you will be presented with a drawing screen and a cursor, as shown below. If there is already a drawing on the screen, you can clear it with F2 6.



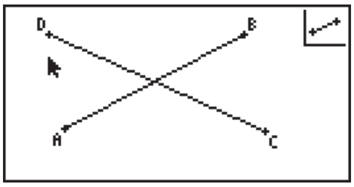
Commands are available in menus via the function keys, F1 to F6 and the OPTN key; you can also move between menus using the cursor keys ◀ and ▶. Start with Draw (F3), which allows for basic geometric objects to be drawn. The arrow at the bottom of the screen shows that further objects are available. (On the fx-CG 20, there is also a Draw Special menu showing, with commands in different places.) For example, select Line Segment (using a cursor and EXE or just by tapping 2) and notice that the top right corner of the screen shows your selection with an icon for a line segment.



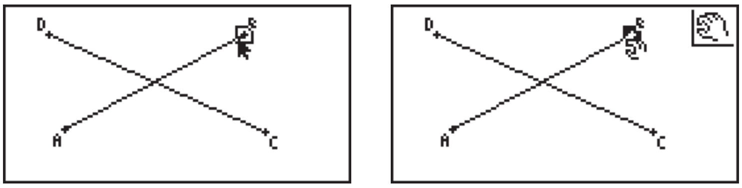


Move the cursor to one end of the intended line segment and tap EXE. Then move to the other end and tap EXE again. The result is a line segment, labelled AB, as shown above. The line segment icon remains on the screen until the EXIT key is pressed or another menu choice is made.

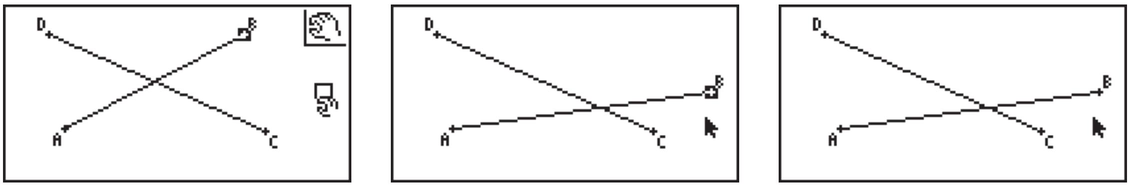
While the line segment icon remains in the top right of the screen, a second line segment can be drawn by selecting the two endpoints, each followed by EXE. The following screen shows that the next two points selected in this way result in a line segment CD.



Once an object is drawn, it can be changed. For example, to change the line segment AB by moving point B to a new location, start by tapping EXIT to exit from the Draw menu. Notice that the line segment icon disappears from the screen. Move the cursor to point B until a small selection box appears around it, as shown in the left screen below.

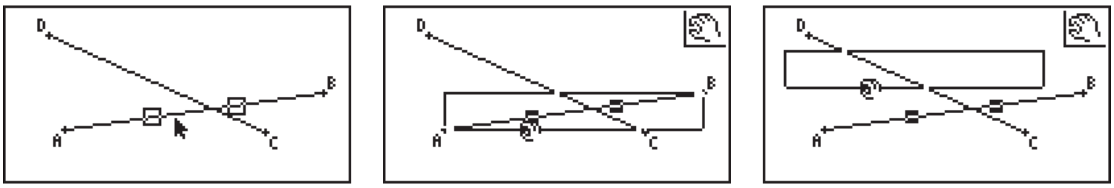


To grasp the point once it is selected, tap the X,θ,T key. Notice that the cursor changes to a small hand and the grasp icon appears in the top right of the screen, as shown in the right screen above. Now move point B with the cursor to a new location, as shown at left below. To finish the change, tap the EXE key and the new line segment will be drawn. When you move the cursor again, notice that the grasp icon is no longer showing, and the point B is still selected (shown by a dark square) as shown in the middle screen below.

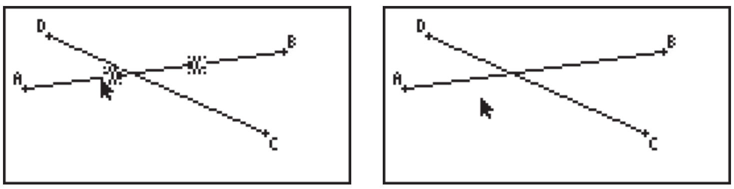


To deselect any selected elements, in this case the selection of point B, tap the AC/ON key. The result is shown in the third screen above. The line segment is longer than originally, and inclined at a different angle from CD.

As well as changing a point, you can move an entire object, once it is selected. To move line segment AB, move the cursor to the line segment itself, rather than to an endpoint, as shown in the first screen below; the rectangles show when the line segment can be grasped. Tap the X,θ,T key to grasp line segment AB, as shown in the right screen below. A *selection rectangle* is shown.



Use the cursor to move the selection rectangle, as shown above at right. Tap EXE when finished to get the result below at left. After moving the cursor and tapping AC/ON to deselect AB, the final result is shown at right below. The line segment has been moved to a new position, parallel to its earlier position, and has retained the same length.

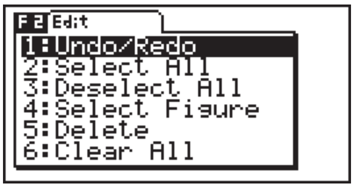


To delete an object, first select the object by moving the cursor to the object and tapping the [EXE] key. Then tap the [DEL] key. Note that tapping the [EXE] key a second time will deselect the object. You can also use the [tan] key to select and deselect an object.

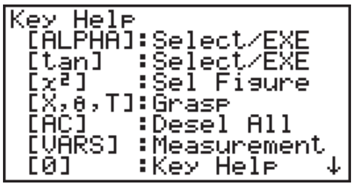
If you delete an object accidentally, you can undo the command by immediately pressing the [×] (Undo/Redo) key.

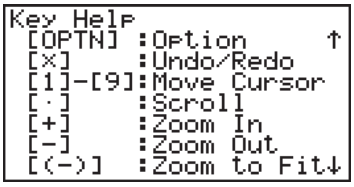
To delete all objects on the screen, tap the [AC/ON] key twice in succession. *Be careful: you cannot undo this command.*

Various editing commands are available in the Edit ([F2]) menu, as shown below:



Shortcut keys are usually easier to use than menu keys (as less keystrokes are needed). There is a key help screen available by tapping the [0] key, as shown below.

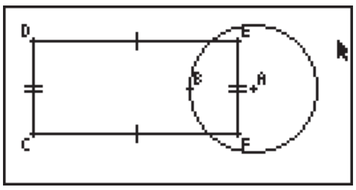




Notice that the cursor can be moved with the number keys [1] to [9], as well as the cursor keys. The number keys are useful for moving the cursor quickly to a new location on the screen. To see how this works, start with a blank screen (with [F2] [6] or by tapping the [AC/ON] key twice in succession) and then tap the number keys in turn to see their effects. The same number keys can be used with the grasp tool to move an object efficiently to a new location. You can combine the two kinds of cursor movements (e.g., by first using a number key and then a cursor key for finer movements).

Changing the view

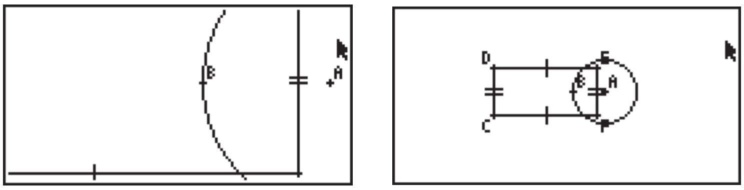
You can change the view of the plane showing in the calculator screen by zooming in or out, by scrolling horizontally or vertically or by panning in a chosen direction. The View menu shows these various options, for most of which there are also shortcuts available. To illustrate, the screen below shows a drawing of a circle and a rectangle.



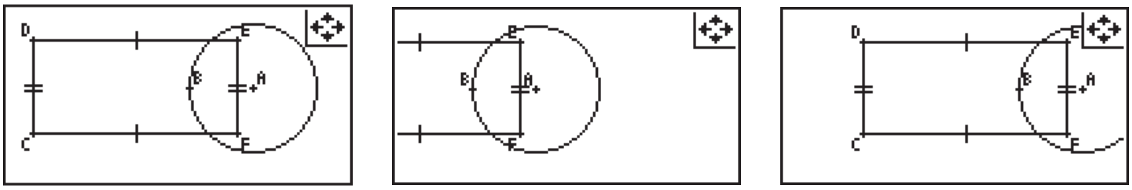
The circle was drawn with a command in the Draw ([F3]) menu. After selecting the circle command, first a centre A was chosen with the [EXE] key. As the cursor is moved, the circle is formed and you can select a point B on the circumference with the [EXE] key.

The rectangle was also drawn with a command in the Draw ([F3]) menu. After selecting the rectangle command, first a vertex C was chosen with the [EXE] key and then the opposite vertex E was chosen, also with the [EXE] key. (Notice that the opposite sides of the rectangle are marked to show that CD = EF and CF = DE.)

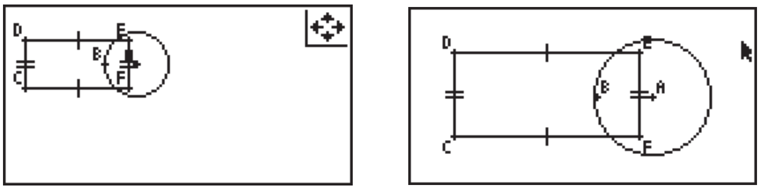
Zoom in and out, by tapping the [+] and [−] keys, to see their effects, as shown below.



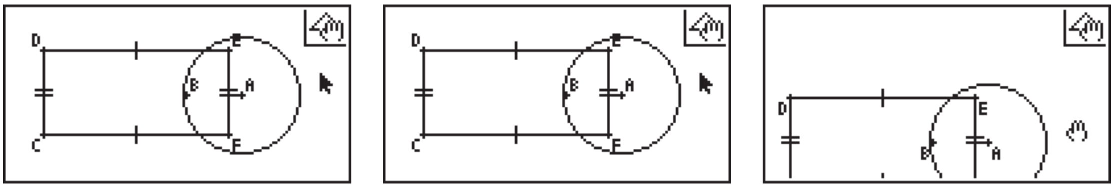
To scroll the screen, first tap the [•] key, and notice the scroll icon in the top right of the screen. Use the cursor keys to scroll the screen vertically with [▲] and [▼] or horizontally with [◀] and [▶]. Notice that the cursors move the background screen, not the objects themselves.



Tap the [EXIT] key or a zoom key to stop scrolling. Scrolling and zooming can be combined, as the screen on the left below shows. Notice the effect at the right below of the Zoom to Fit command, which can be activated from the View ([F1] [▶]) menu or directly with the [(−)] shortcut key.



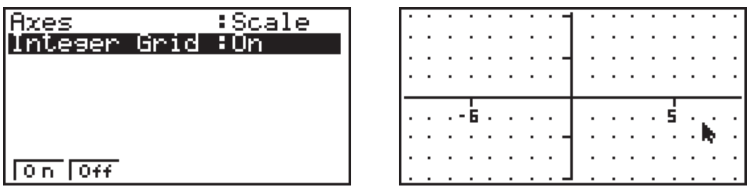
Finally, to see how panning works, activate the View menu with View ([F1] [▶]) and select Pan ([2]). The Pan icon is shown in the top right corner. Tap [EXE] to change the cursor to the pan hand, as shown in the right screen below.



Now, moving the cursor in any direction will move the plane in that same direction, as shown in the third screen above. Tap the [EXIT] key to finish panning.

Showing the coordinate plane

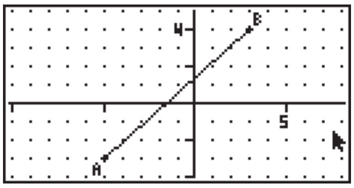
To show the coordinate plane, use SET UP ([SHIFT] [MENU]) to show the axes and to turn on the integer grid, as shown below for the fx-9860GII series. (Other settings are also available on the fx-CG 20.)



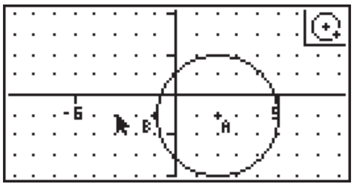
The choice of Scale for the Axes setting will display some numbers on the axes.

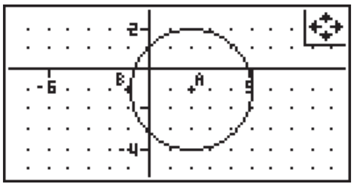
To centre the origin, use Zoom to Fit ([(−)]) before you draw anything.

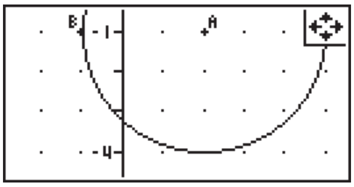
When the integer grid has been chosen, points with coordinates that are integers are readily chosen, as the next screen shows, where A is the point (-5,-3) and B is (3,4).



The screen below shows a circle drawn with centre A(2,-1) and radius 3. To draw the circle, the circle command was chosen in the F3 (Draw) menu and then the centre chosen with the cursor followed by EXE. Then the cursor was moved three units to the left and the EXE key tapped again.



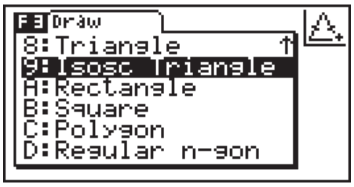


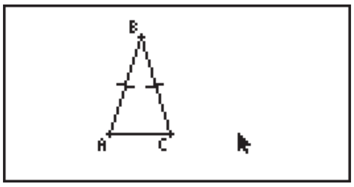


After scrolling (with ⊙), a different view can be shown, as in the middle screen above. The screen shows that (2,2) is on the circle, while zooming (with ⊞) shows that (3,-4) is outside the circle.

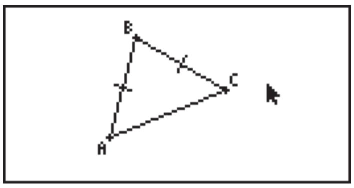
Constraints

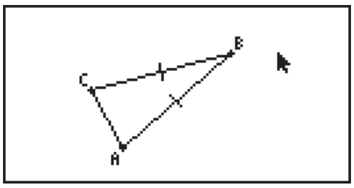
The calculator relies on *constraints* to retain the essential mathematical properties of figures. To see how this works, use the Draw (F3) menu to draw an isosceles triangle on a blank screen. (On the fx-CG 20, you will need to use the Draw Special (F3 ▶) menu for this and some other shapes such as the Rectangle.) After selecting the command, tap EXE twice to draw a large isosceles triangle. Alternatively, tap EXE and then move the cursor to another position and tap EXE again. In either case, the result will be an isosceles triangle, with the two equal sides AB and BC marked, as shown below.





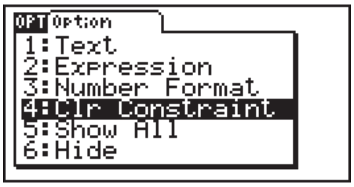
To see how the triangle has been constrained, use the grasp tool (via the X,θ,T key) to move a vertex. The two screens below show two of the possibilities.

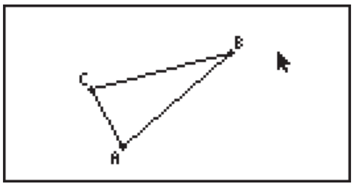




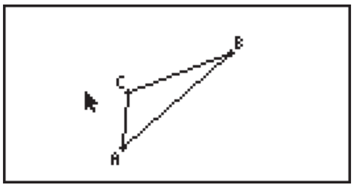
Notice that, in each case, AB = BC, so the triangle continues to be isosceles, even though it is shifted to a new position or size or is rotated or reflected. Movements of any of the vertices will not change the fact that the triangle is an isosceles triangle: it is constrained to be an isosceles triangle.

Constraints themselves can be removed from drawings, using the *Clear Constraint* command in the Option menu, obtained with the OPTN key, as shown in the following screens. Tap the EXE key to activate the command and remove the constraints.



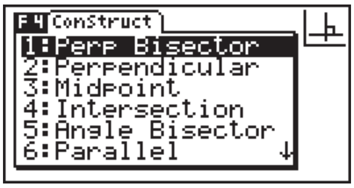


The screen above shows that the object no longer shows the marks indicating that AB = BC. Moving a vertex now will allow a triangle that is not isosceles to be made, as shown below.



Constructions

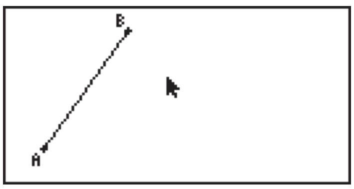
A powerful feature of the geometry module is the ability to undertake geometric constructions, which are included in the Construct (F4) menu.

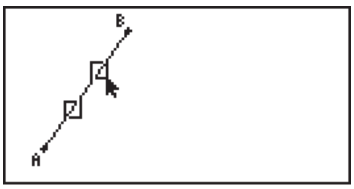


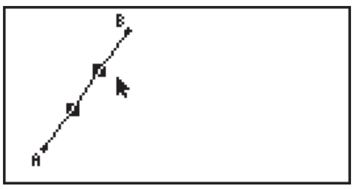
Without a calculator, geometric constructions are completed with a compass and straightedge (such as a ruler), and allow for important geometric ideas to be studied theoretically. The constructions available in the geometry module are the main elementary ones studied today and also long ago by the ancient Greeks. In each case, a construction requires that you first select an object and then activate the construction. The 'object' selected depends on the construction, as the table shows.

Construction	*Object(s) to be selected*
1. Perpendicular Bisector	Line segment
2. Perpendicular	Point and line segment
3. Midpoint	Line segment
4. Intersection	Two intersecting lines, line segments or arcs
5. Angle Bisector	Two line segments or rays forming the angle
6. Parallel	Point and line segment
7. Tangent	Point on circle or arc

To see how constructions work, we will use the perpendicular bisector as an example. Start with a line segment AB. To construct the perpendicular bisector of AB, first move the cursor to AB and select it (using either the EXE key or the tan key) when the blank squares appear on the line). The sequence of screens below shows the whole process.



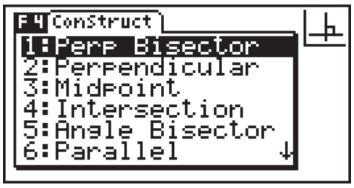


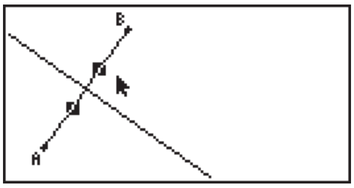


Now that AB is selected, use F4 to select the construction menu. Notice that an icon appears in the top right of the calculator screen for each construction. Tap 1 to select the perpendicular bisector construction and the construction will be completed as shown below. The perpendicular bisector of

AB is shown: a line that is perpendicular to AB, passing through the midpoint of AB.

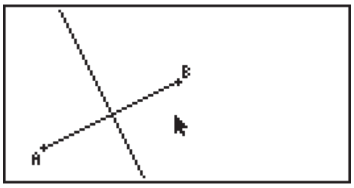
Tap the [AC/ON] key to deselect the line segment AB.

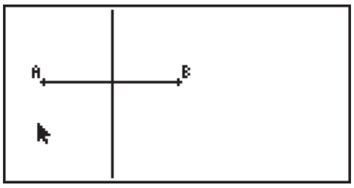




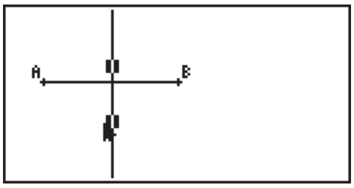
Following this construction, the perpendicular bisector of AB will be shown, *even if AB is changed.*

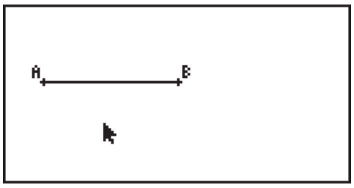
To see this in action, select point A or B with the [X,θ,T] key and use the grasp tool to move the point to a new location, giving a new line segment AB. The perpendicular bisector will move accordingly. The screens below show two examples of this.



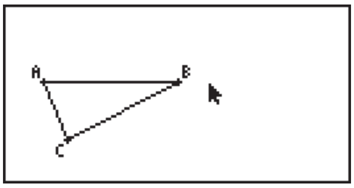


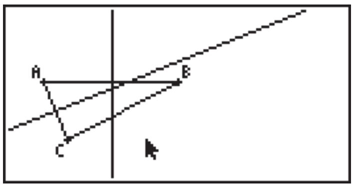
To remove the construction, select the perpendicular bisector itself and tap the [DEL] key to delete it. The line segment will remain, but the construction of the perpendicular bisector will be removed. The sequence of actions is shown below.





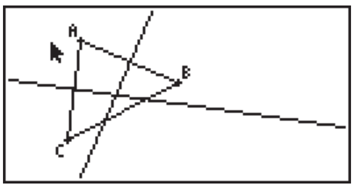
Multiple constructions can be used at once on the calculator, to explore relationships. For example, line segments AC and CB have been added to line segment AB below to draw triangle ABC.





The second screen shows the perpendicular bisectors of both AB and AC, which were constructed one after the other.

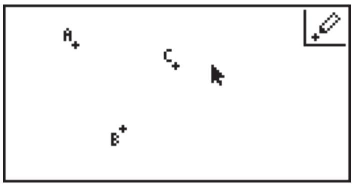
If point A is now moved, both the perpendicular bisectors will be moved. One example is shown below. Try this for yourself.



Now add the perpendicular bisector of line segment BC as well as the other two. What do you notice about the three perpendicular bisectors? See for yourself what happens if one of the vertices of the triangle is moved.

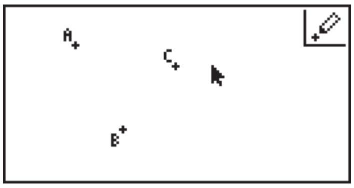
Sequences of constructions

A sequence of constructions can be used to make more complex constructions, and thus to study properties of figures in the plane. For example, start with three points A, B and C on the screen. (To clear previous constructions, you can either use Clear All (F2 6) or tap the AC/ON key twice.)

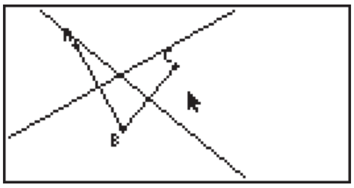


We will construct a circle through these three points.

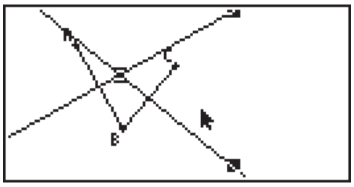
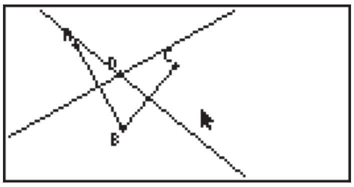
First draw line segments AB and BC, using the Draw (F3) menu:



Then construct the perpendicular bisectors of AB and BC, using the F4 (Construct) menu:



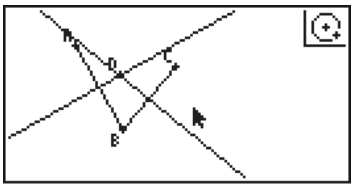
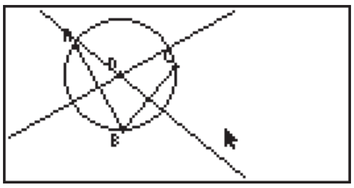
Now construct the intersection point of the two perpendicular bisectors. To do this, first select each of the two bisectors and then use Intersection (F4 4) to select the Intersection command from the Construct menu. The screens below show this process.

Notice that the point of intersection has been labelled D. (Before the point of intersection is constructed, it is not labelled.)

Point D is the centre of the circle passing through A, B and C. You will notice that D moves if you move any of the three original points.

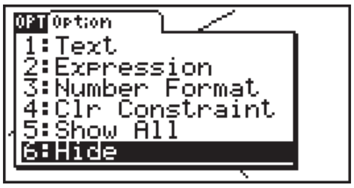
To complete the construction, use Circle (F3 6) to select the Circle drawing tool from the Draw menu. Then select firstly D (the centre of the circle) and then any one of A, B or C (to show the radius). Tap EXIT to exit from the circle tool.

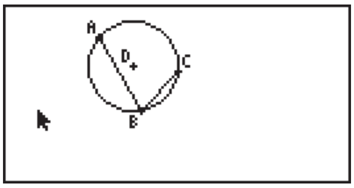
Notice that the circle passes through each of A, B and C.

In addition, if you move any of the three points, the circle will still be constructed. Similarly, if you use the [+] or [−] keys to zoom, the construction will still be shown.

While it is a good idea to leave the intermediate constructions (i.e., the perpendicular bisectors) on the screen, so that you can see how the circle was constructed, it is not necessary to do so, and sometimes they may even be a distraction. Any screen objects can be hidden from view, using the Option menu shown below. Tap [OPTN] to show the menu.



To hide objects, first select them and then use the Hide ([OPTN] [6]) command. The screen below shows the results of hiding both of the perpendicular bisectors in this case.

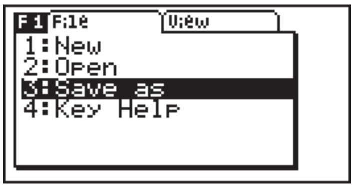


The constructions are still on the calculator screen, as you can see by moving points A, B and C around and noticing that the circle continues to be constructed. They are merely hidden from view. To see them again, use the Show All ([OPTN] [5]) command.

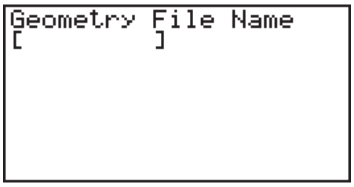
Saving and retrieving your work

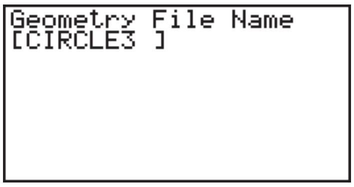
Especially if you have made a fairly complicated construction, you may wish to save your work as a file so that you can retrieve it later. As an example, we will use the construction of a circle through three points that you have just completed.

Tap File ([F1]) to access the File menu, shown below.



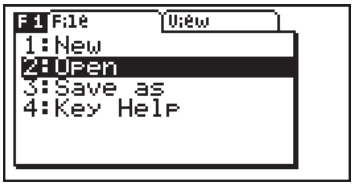
Save the work, including all drawings and constructions, using Save as ([3]). You will need to name the file suitably, with a name of your own choice, as shown below. Use the [ALPHA] key as necessary to select characters. Tap [EXE] to complete the process.

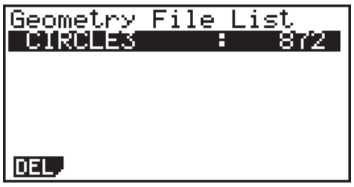




The file has now been saved. You can safely clear the screen (using [F2] [6] or tapping [AC/ON] twice) to begin a new construction.

To retrieve your construction at a later point, use the File menu again and select Open ([2]). Select the file from the list (the screen below shows only one file in the list) and tap [EXE] to open it again.

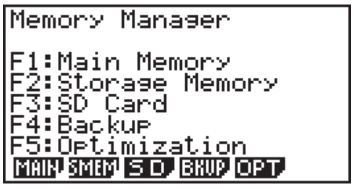


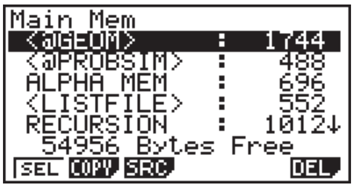


The same screen allows you to delete files that you no longer wish to keep, using the DEL (F1) command. It is a good idea to do this, to use your calculator memory efficiently.

The number on the right of the screen (872 in this case) tells you how much space a file consumes, measured in bytes. You can see how much free space is left in your calculator in Memory mode, accessed with MEMORY (MENU tan). Select MAIN (F1), since Geometry files are stored in the Main memory. Add-in modules (such as the geometry module) are stored in the Storage Memory.

The screen below shows in this case that the calculator had 54 956 bytes free (after the above file was stored).



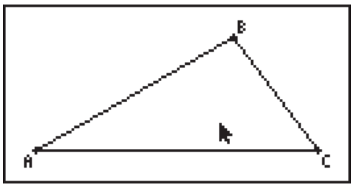


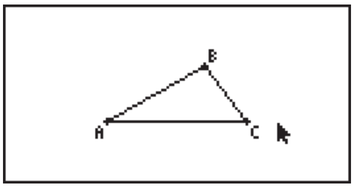
To build upon an existing file to make a new file, first open the file and then use the Save as (F1 3) command to save it with a different name. Be careful with your choice of names to make sure you can efficiently retrieve your work and know which files to delete in order to make space for other work.

Making measurements

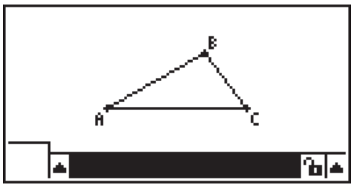
Screen objects such as line segments, polygons and angles can be measured in various ways, which will help you to understand the relationships between them.

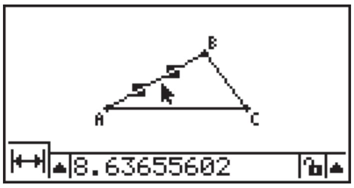
Consider, for example, triangle ABC shown at right below. This triangle was drawn with the Triangle command by tapping EXE twice in the Draw (F3) menu (or Draw Special (F3 ▶) menu on the fx-CG 20). The resulting triangle, shown in the first screen, is quite large. Tap the − key to zoom out, as shown in the second screen.





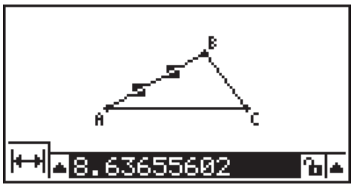
Various attributes of any object can be measured by first selecting the object and then tapping VARS to access the Measurement Box, shown darkened below. (If you move the cursor onto the screen, the box will become blank, indicating that nothing is presently selected for measurement.)

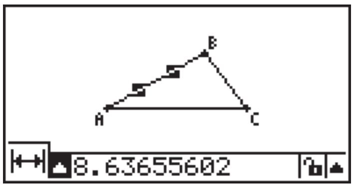


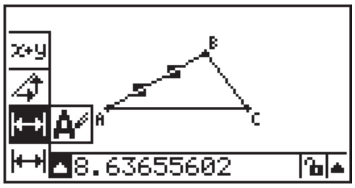


To measure the length of AB when the measurement box is on the screen, use the cursor and then EXE to select AB. The measurement of 8.637 is shown in the box, while the icon at the bottom left of the screen shows that a measurement of length is involved. Note that measurements on your own calculator may differ from those shown in this module, as different objects are involved. Note also that *measurements* do not change when you zoom in or out; it is only the view that changes.

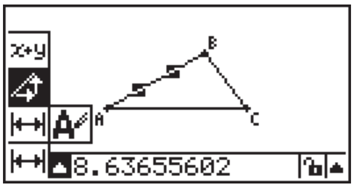
Now, tap [VARS] to select the Measurement Box again, which places the cursor into the Measurement Box, as shown by the darkened box in the first screen below. Move the cursor to the left to highlight the vertical arrow, as shown in the middle box below and then move the cursor up to display other possible measurements for the highlighted object, which in this case is AB. The other possibilities are shown in the third screen below.

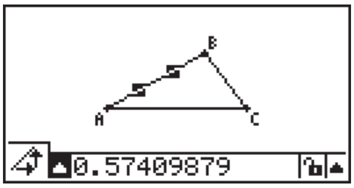




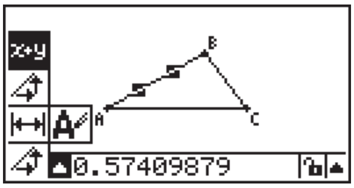


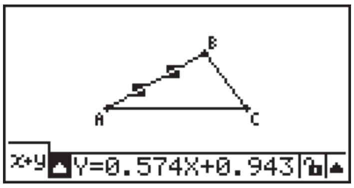
Three icons are shown vertically. The first of these is the one already used: the length of the line segment. The second icon shows the gradient of the line segment, while the third shows the equation of the line containing AB. The screens below show the result of highlighting the gradient icon and then tapping [EXE]. Notice in the second screen that the gradient icon is now showing at the bottom left and the gradient itself (about 0.574) is shown in the Measurement Box.





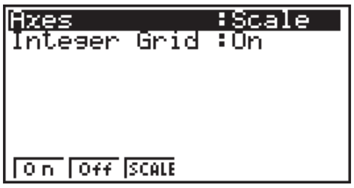
The two screens below show a similar process for the equation.

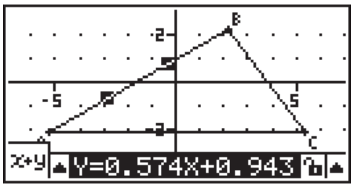


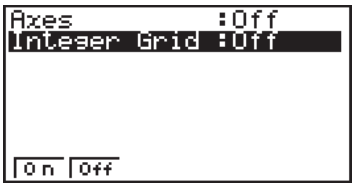


Notice that the equation $y = 0.574x + 0.943$ has the same gradient as shown above.

The equation of course needs to be referred to a set of coordinate axes, which are not showing at present on the calculator screen. These can be displayed, as noted earlier in this chapter, using the SET UP menu, as shown below at left. The effects of adding the axes, a scale and an integer grid as well as zooming to fit with [(-)] are shown in the middle screen below.

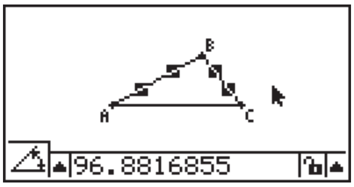




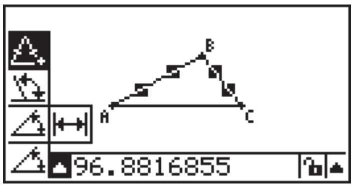


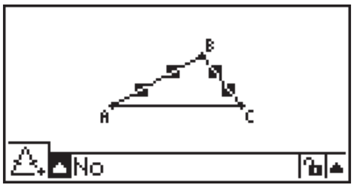
For our present purposes, however, the coordinate settings are kept off, as shown at right above.

Different attributes of the triangle can be measured by selecting them with the cursor. To do this, firstly move the cursor back to the Measurement Box and then up to the screen. The following screen shows the result of measuring angle B, defined by the two line segments AB and BC. In this case, the angle size is almost 97°.

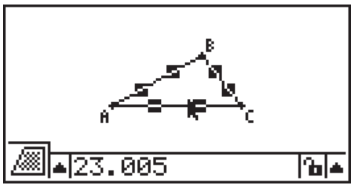


Tapping VARS and moving the cursor to the arrow indicates that there are other possibilities for a pair of line segments. The screen below shows whether or not the two line segments are congruent: that is, have the same length. In this case, the answer is ‘No’ … they are not congruent.

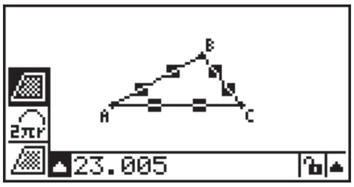


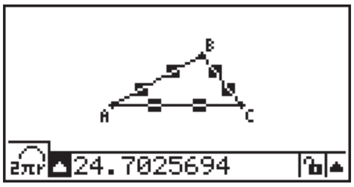


Further measurements of the triangle are possible. Move the cursor back to the Measurement Box and then up to the screen. Select the third side, AC, in addition to the other two sides. These three sides together define triangle ABC. The next screen shows the Measurement Box displaying the area of the triangle (about 23.005 square units), shown by the polygon area icon at the bottom left.



Again, there are alternative measurements. Tap VARS to put the cursor into the Measurement Box and then move left and up to see them. In this case, the other alternative is a perimeter measurement (using the icon 2πr). The perimeter of about 24.703 units is shown below.



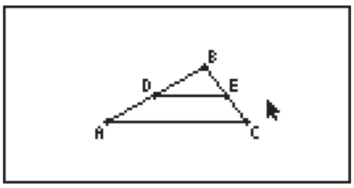


These examples illustrate some of the very many possible measurements that can be made in Geometry mode. There is a complete description of other possibilities and their icons in the *Owner’s Guide* for your calculator, but you may be able to work out many of the possibilities for yourself by selecting different objects.

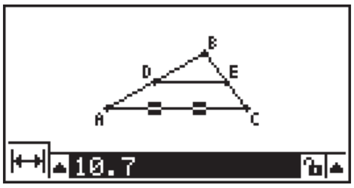
To stop making measurements, tap the EXIT key twice in succession.

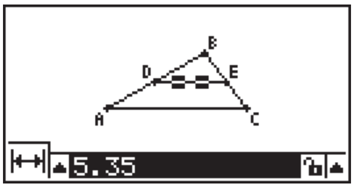
Displaying measurements

Measurements of objects can be displayed on the screen rather than in the Measurement Box, which allows you to easily see how they are changing when points are changed. Consider the diagram below, in which the midpoints of sides AB and BC have been joined with a line segment DE. (To do this, the midpoints were first constructed and then joined with a line segment.)



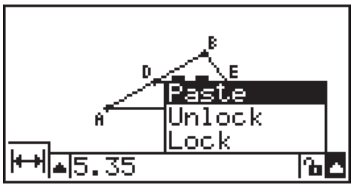
Select AC and tap VARS to see the length in the Measurement Box, in this case 10.7 units.

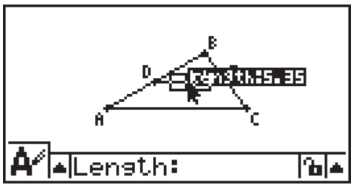




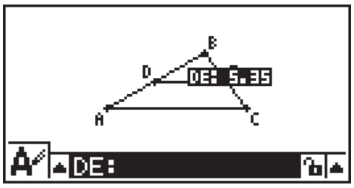
Then use the cursor to select DE instead of AC. (To do this, first tap AC/ON to deselect AC and then select DE.) The length is 5.35 units, as shown at right above.

In this case, the length of DE is half the length of AC, rather surprisingly. To see if this relationship holds more generally, we can move point B to new locations and check the measurement of DE each time. The easiest way to do this is to firstly transfer the measurement of DE to the screen. To do so, firstly tap VARS to return the cursor to the Measurement Box and then move to the arrow on the right and then upwards. The result of this is shown at left below. Highlight the Paste command and tap the EXE key to confirm that the length of DE is to be pasted to the screen. The result is shown at right.

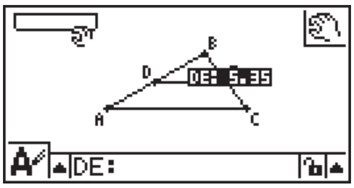


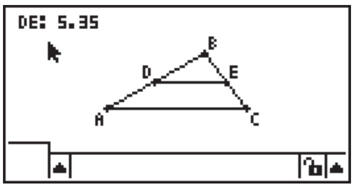


Although it is not necessary to do so, you can edit the label used in the measurement. At present, the word ‘Length:’ is used. You can, if you wish, change it to something else, such as ‘DE’. To do this, tap VARS to highlight the word ‘Length:’ and enter an alternative, followed by EXE. The screens below shows ‘DE: ’ instead of ‘Length:’. (A space was typed after the colon for ease of reading.)

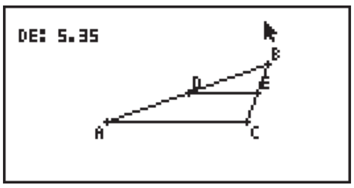


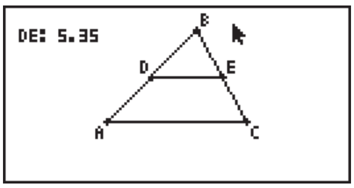
Regardless of whether you have changed the label or left it as ‘Length:’, use the cursor to highlight the measurement box that is on the screen, select it with the X,θ,T key and move it to somewhere away from the triangle, as shown below.

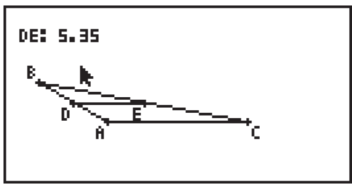




Tap the EXIT key to remove the Measurement Box from the bottom of the screen, but notice that the measurement of the length DE stays on the screen. If you move point B to some different locations, the measurement for DE stays the same, as the next three screens show.

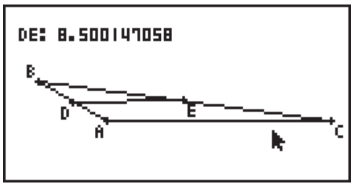


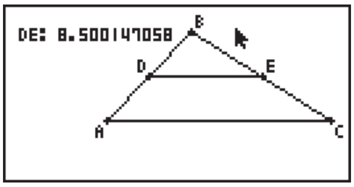


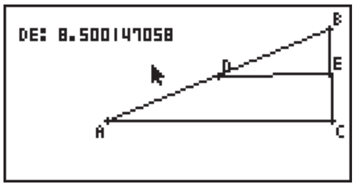


In each case, the length of DE is half the length of AC (which was 10.7).

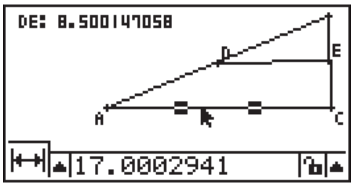
If the length of AC is changed (e.g,, by moving point C), then the length of DE changes, to again seem to be half of the length of AC. The three screens below suggest that the length of AC seems to have increased to about 17 units (since DE is now about 8.5 units).

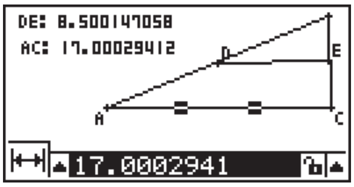






This prediction can be tested using the Measurement Box, as shown below. (The second screen shows that it is possible to include both measurements on the screen at once, if desired.)



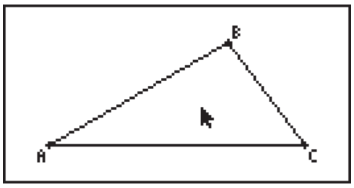


The prediction seems correct, suggesting strongly that the length of DE is half the length of AC. Of course, further testing and then a good mathematical argument is needed to *guarantee* or *prove* that this is always the case. The calculator can merely help to show that a consistent result is obtained.

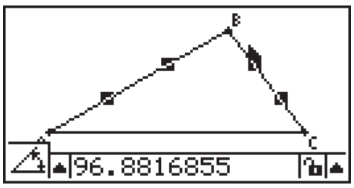
Constraining measurements

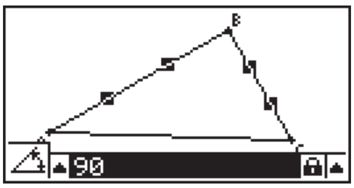
As noted earlier, using the example of an isosceles triangle, this geometry application relies on objects being constrained to have certain properties. The Measurement Box may also be used to *constrain* measurements in various ways: that is to force measurements to be of a certain size. This can be useful to draw objects with particular specifications.

For example, to draw a right triangle ABC with perpendicular sides of length AB = 8 and BC = 9, start with a Triangle from the Draw menu of your calculator. Tap the EXE key twice to produce a triangle ABC, as shown below.

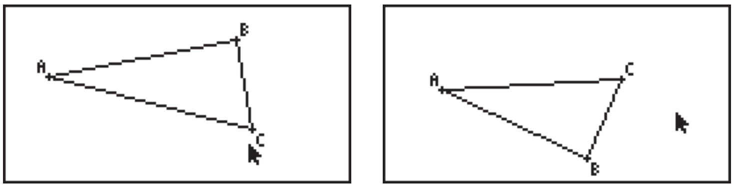


Select AB and BC and then select VARS to show the Measurement Box. Clearly, the angle at B is not a right angle. However, it can be constrained to be a right angle by entering 90 into the Measurement Box and tapping the EXE key. Notice that, as soon as you do this, the angle changes to 90^{o} and the padlock icon at the bottom right of the screen is shown as closed (where it was previously open). The screens below show all of this.

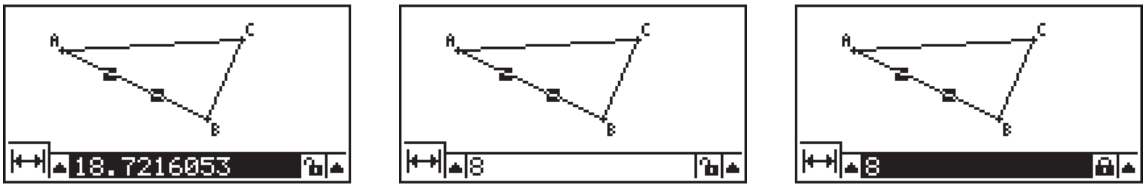




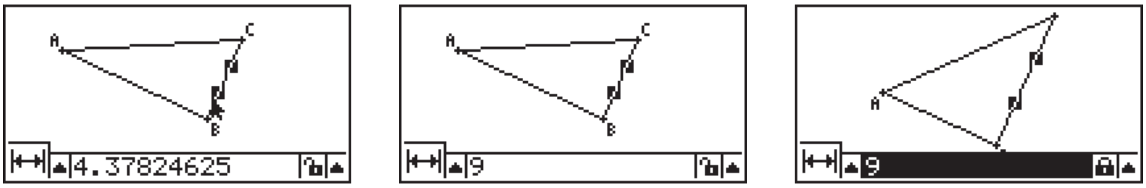
To see how the angle has been constrained, tap the EXIT key twice to exit from the Measurement box, tap the AC/ON key to deselect AB and BC, and move the triangle's vertices using the Grasp tool in the usual way. Regardless of how you move the vertices, the angle at B will remain a right angle: it has been *constrained* to be a right angle. The screens below show two examples.



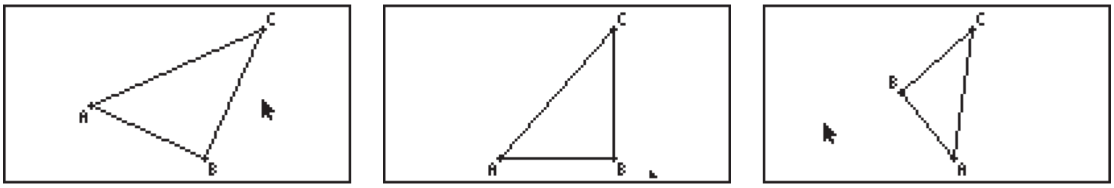
The lengths of AB and BC can be constrained in similar ways. The case of AB is shown below, constrained to 8 units, using VARS to highlight the Measurement Box.



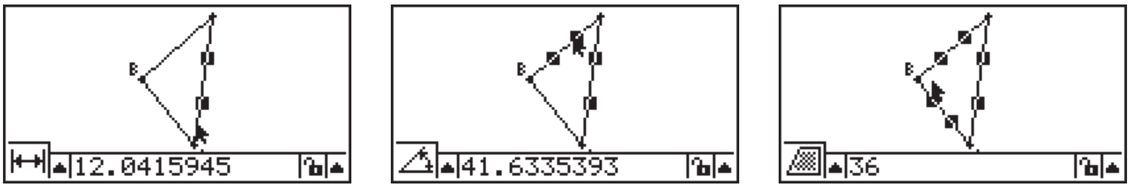
These screens show the process of constraining BC to be 9 units long:



You may have to zoom (with [+] or [−]) or zoom to fit (with [(−)]) to show triangle ABC satisfactorily on the screen. You will notice that, however you now move the triangle around, by dragging either vertices or sides, the shape does not change: the triangle is completely constrained so that AB = 8, BC = 9 and angle B is a right angle. Here are three examples:



Now that the triangle is constrained, it is easy to measure directly various attributes. For example, the screens below show that AC ≈ 12.04 and angle C ≈ 41.63°. As expected, with AB = 8 and BC = 9 perpendicular to each other, the area is shown as 36 (exactly).



Again, however large or small the triangle looks on the screen, the dimensions remain the same.

Other geometric activities

Other useful features of the Geometry application are available through the Transformation menu ([F5]), which allows you to produce and study the standard transformations of objects, and the Animation menu ([F6]), which allows for drawings to be animated. Details of these features are available in the *Owner's Guide* for your calculator.

Exercises

The main purpose of the exercises is to help you to develop your calculator skills.

1 a Draw a pair of points.
b Join them with a line segment.
c Move one end of the line segment.
d Measure the length of the segment.
e Constrain the line segment to have length 7.
f Zoom out and in a few times.

2 a Draw two lines that intersect.
b Construct the intersection point of the two lines.
c Measure the acute angle between the two lines.
d Measure the obtuse angle between the two lines.

3 a Draw a regular pentagon, using the *Regular n-gon* command in [F3] (Draw) menu.
b Drag a vertex to make the pentagon larger than the screen.
c Zoom in to make the pentagon fit on the screen.
d Check to see how the measurements are affected by zooming.
e Use [F1] [▶] (View) to select [3] (Scroll) and use the cursor to scroll in all four directions.

4 a Draw a circle.
b Construct a tangent to the circle at a point, using the [F4] (Construct) menu.

5 a Draw a regular hexagon.
b Save it as a file named HEX.
c Clear the screen.
d Open the file to retrieve the hexagon.

6 Use the *Polygon* tool in the Draw ([F3]) menu to draw a trapezium ABCD for which both AB and CD are horizontal, and CD > AB. (To use this tool, use [EXE] to select vertices of the polygon in succession, and make sure that the first vertex is selected again as the last vertex.)

7 a Draw a triangle ABC and use the Measurement Box to constrain AB = 4 and BC = 7.
b Constrain the angle at B to be 108^{o}.
c Measure the length of AC.

8 a Use the Draw ([F3]) menu to draw a rectangle, using the Rectangle command.
b Move the rectangle's vertices several times to check that it remains a rectangle. (use Zoom to Fit ([(−)]) if necessary, if the rectangle is larger than the screen)
c Use Clr Constraint ([OPTN] [4]) to remove the constraints, and then move the vertices of the rectangle again.
d Explain what has now changed.

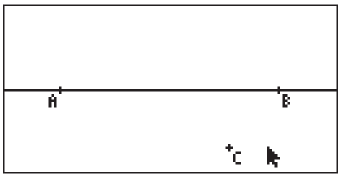
9 a Draw a triangle ABC.
b Measure each of the angles and paste all three measurements onto the screen.
c Check that the angles add to approximately 180 degrees.
d Move one of the vertices of the triangle, and note that all three angle measurements change.
e Check that the new angles also add to approximately 180 degrees.

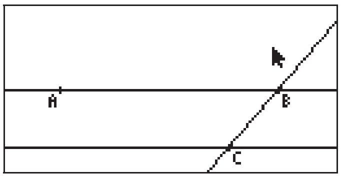
Activities

The main purpose of the activities is to help you to use your calculator to learn mathematics.
You may find that some of them are too advanced for you. Ignore activities you don't yet understand.

1 Draw an isosceles triangle ABC, using the Draw menu. Move the vertices to check that the triangle remains isosceles. Measure the sizes of the angles at each vertex. What consistent relationship do you notice between the angles?

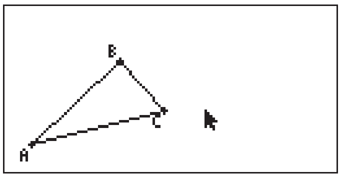
2 Draw a line AB and a point C not on the line, as shown in the first screen below.

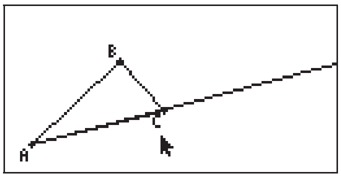




Construct a line through C parallel to AB, using the Construct (F4) menu and draw line BC, as shown in the second screen. Measure the angles between the lines on the screen and look for relationships between them. Check any relationships you notice by moving B or C.

3 Draw a triangle like ABC shown below. Then draw a ray AC, as shown in the second screen.



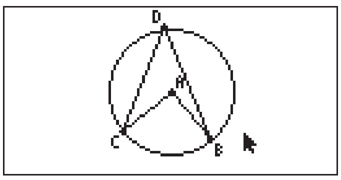


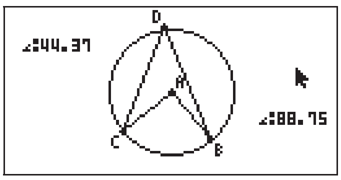
Compare the size of the external angle at C with those of the internal angles at A and B. Move points A, B and C to look for a pattern among the three angle sizes.

4 Use the *Regular n-gon* command to draw a regular hexagon. First specify the number of sides, and then tap EXE twice. Select a pair of adjacent sides and tap VARS to measure an angle of the hexagon. How is the number of sides related to the angle size at each vertex of the hexagon? Try some other polygons. Look for a mathematical explanation of the relationship.

5 Draw a triangle ABC and then use the Measurement Box to constrain the lengths AB = 5 and BC = 4. With these two side lengths constrained, notice that you can make many different shaped triangles by moving vertex C around. Now use the Measurement Box to constrain angle A to be 30^o. Notice that it is now possible to make only two different triangles with the three constraints (AB, BC and angle A). Make sure you can find both triangles. How many different triangles can you make if you constrain angle B and not angle A?

6 Draw a circle with centre A and radius AB, and draw two more points, C and D, on the circle. Join the points to make line segments, as shown below.





Use the Measurement Box to measure angles BDC and BAC, as shown at right above. What happens to the angle sizes if point D is moved? What happens to the angle sizes if point B is moved? Look for a mathematical explanation of your observations. How could you *prove* them?

Notes for teachers

This module illustrates several ways in which the calculators can be used to explore various aspects of geometry, using the constraint-based geometry add-in application, which may need to be downloaded. The application allows students to draw geometric objects, manipulate them and measure them, in order to understand many different kinds of geometric relationships. The module is unavoidably longer than other modules even though some capabilities are not addressed. The text of the module is intended to be read by students and will help them to start to see how the calculator can be used efficiently for geometric explorations. The Activities are appropriate for students to complete with a partner or in a small group, so that they can discuss their observations and justify their conclusions. While the activities are presented for students to explore, they might also be used in class for a class discussion, led by the teacher, if a projected version of the calculator is available, using CASIO's Manager software.

Answers to Exercises

Most of these exercises are self-checking. 7. c) approximately 9.072

Activities

1. This activity serves two purposes: to help students understand the nature of constraints and also to recognise that the angles at the base of an isosceles triangle are equal, by examining many different isosceles triangle. At a later stage, without using the calculators, they might be encouraged to look for a mathematical proof of this famous relationship, rather than being satisfied by their observations that it continues to hold.

2. In this activity, as in the previous activity, students can find many relationships between angles and parallel lines by changing their diagram, and be encouraged to look for reasons for the relationships. Corresponding, alternate, vertically opposite and cointerior angles can all be measured and compared in order to understand the relationships.

3. Students have not previously been shown how to draw a ray, but should be able to do so in similar ways as for drawing other objects. The relationship that he external angle of a triangle is the sum of the two opposite angles is both powerful and interesting, and once again, students should be encouraged to look for reasons for the relationship.

4. Encourage students to work together with this activity, and to make a table of results for different polygons in order to see the patterns involved. (Note that the *Regular n-gon* tool is in the Draw Special menu for the fx-CG 20 calculator).

5. This activity begins to address the key concept of congruent triangles, as the first example suggested has insufficient information for triangles to be congruent (and, in fact, is what is sometimes called an ambiguous case). However, when two sides and the included angle of a pair if triangles are congruent, then the triangles themselves are congruent, one of the key Euclidean examples. Students might also be encouraged to explore the case of three sides being constrained, to see that a triangle is uniquely determined by its three sides, accounting for the common use of triangles to make objects (like scaffolding) rigid.

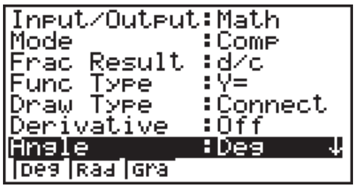
6. This activity is included as a further example of how standard Euclidean results can be explored by students on the calculators, once a basic mastery of the geometry application is obtained. In this case, because the angle at the centre is twice the angle at the circumference, angles at the circumference on the same arc are congruent.

Module 6
Trigonometry

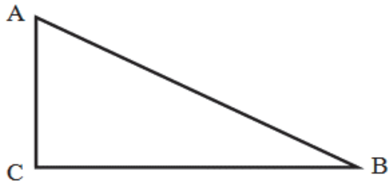
A calculator is a useful tool for many aspects of trigonometry, both for solving problems involving measurement and understanding relationships among angles. Graphics calculators also allow for graphs to be used for these purposes.

Trigonometry and right triangles

Definitions of trigonometric functions can be based on right triangles, such as the one shown below, for which C is a right angle. We will use Run-Mat mode here; when dealing with triangles, it is best to set your calculator into degrees, using SET UP, as shown below. (The fx-CG 20 shows the setting on the top of the screen.).



For this triangle, sine B = $\frac{AC}{AB}$, cosine B = $\frac{BC}{AB}$ and tangent B = $\frac{AC}{BC}$.



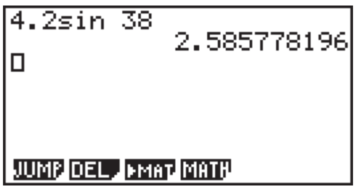
In addition, because the triangle is right angled, the Theorem of Pythagoras allows us to see the relationship between the lengths of the two sides (AC and BC) and the hypotenuse (AB):

$$AC^2 + BC^2 = AB^2$$

Together, these relationships allow us to determine all the sides and the angles of a right triangle, even when only some of the information is known. For example, if we know that angle B is 38^o and that AB = 4.2 m, we can find the length of AC using the definition of sine above:

$$\sin B = \frac{AC}{AB} = \frac{AC}{4.2}, \text{ so } AC = 4.2 \sin B.$$

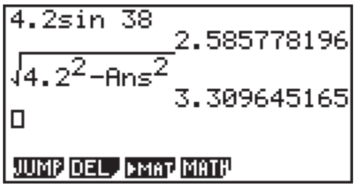
Notice that it is not necessary on the calculator to use a multiplication sign in this case or to use parentheses for the sine of the angle. Although the calculator gives the result to many places of decimals, care is needed in deciding the level of accuracy of calculations like this. Since the original measurement of AB, which is used in the calculation, is given to only one place of decimals, it would be consistent with this to give AC to only one place of decimals too: AC ≈ 2.6 m.

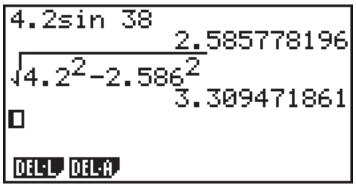


To find the length of the other side, BC, a similar process could be employed, using the cosine of B. Alternatively, the Pythagorean Theorem can be used to see that

$$BC^2 = AB^2 - AC^2 \text{ and so } BC = \sqrt{AB^2 - AC^2}$$

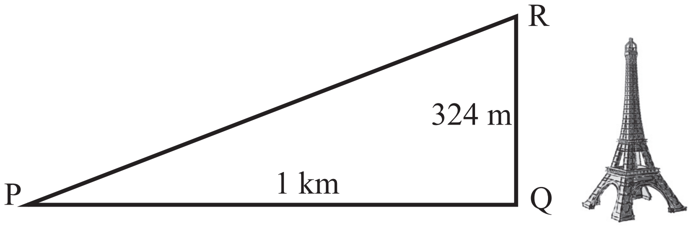
This can be readily determined from the above calculator result (which shows AC) as follows:





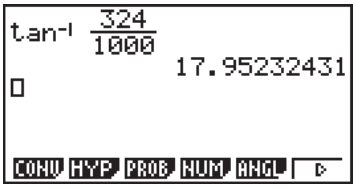
So BC ≈ 3.3 m. Notice that it has not been necessary to record any intermediate results here, and nor has it been necessary to use approximations (such as AC ≈ 2.586), which is likely to introduce small errors. In this case, use of an approximation for AC results in a (slightly) different and slightly less accurate result for BC, as shown in the right screen above. In general, it is better to not round results until the final step in any calculations on calculators or computers.

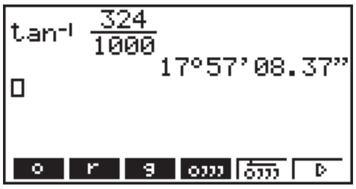
The calculator can also be used to determine angles in a right triangle, using inverse trigonometric relationships. For example the inverse tangent ($\tan^{-1}$) of an angle can be accessed with [SHIFT] [tan]. [Note that it is not possible on the calculator to use either [^] [−] [1] or (x^{-1}) for this purpose.] If you know a perpendicular height and a distance, you can use this to find an angle of elevation. In the diagram below. QR represents the height of the Eiffel Tower in Paris. The tower is 324 metres high. P is a point one kilometre away from the base, measured on level ground. What is the angle of elevation from P to the top of the tower?



$$\text{Since } \tan P = \frac{RQ}{PQ}, \text{ Angle } P = \tan^{-1}\frac{RQ}{PQ}.$$

The calculator gives the result using decimal degrees, as shown below on the left.

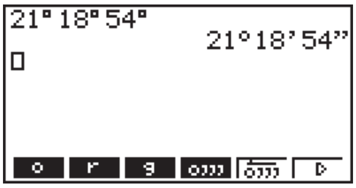




To represent this angle using degrees, minutes and seconds, tap the [OPTN] key, followed by [F6] [F5] to get the ANGLE menu, and then [F5] to give the result shown at right above. Once again, care is needed to express results to a defensible accuracy. In this case, one kilometre from the Eiffel Tower, the angle of elevation from the ground is almost 18°.

When using the calculator for trigonometry, it is sometimes necessary to enter angles in degrees, minutes and seconds. However, the calculator works with decimal degrees. To see the relationship between these two ways of using sexagesimal measures, enter an angle in degrees, minutes (and possibly seconds also) and tap [EXE] to see the decimal result.

To enter the angle, first make sure that the ANGLE menu is showing, using tap the [OPTN] key, followed by [F6] [F5]. Tap the [F4] key after each of the degrees, minutes and seconds are entered; although a degree symbol is shown each time (not symbols for minutes and seconds) the calculator interprets the angle correctly, as the screen below on the left shows.

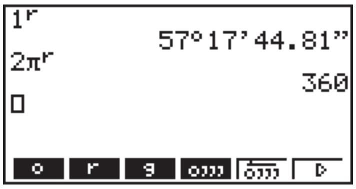


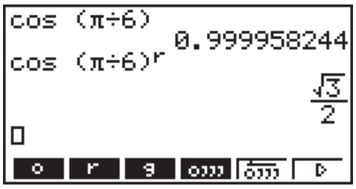
Tap F5 to see the angle in degrees, minutes and seconds again, as shown on the right screen.

Radian measure

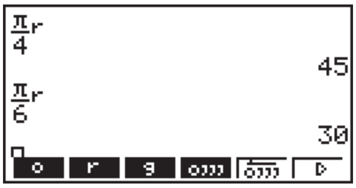
Radian measure provides a different way of measuring angles from the use of degrees, by measuring along a circle. An angle has a measure of 1 radian if it cuts an arc of a circle that is one radius long. As there are 2π radius measures in the circumference of a circle, the measure of a full rotation in a circle is 2π (radians), the same angle as 360^o in degrees (or sexagesimal measure.)

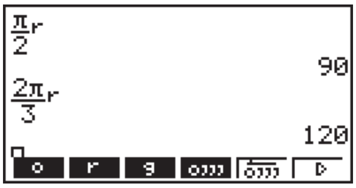
When the calculator is set to degrees (via the SET UP menu), angle measurements are assumed to be in degrees. However, you can convert from radians to degrees, using the ANGLE menu (with OPTN F6 F5)and tapping F2 to indicate that a measurement is in radians. (The small r symbol is used to indicate radians). The left screen below shows this, confirming that a radian is a little more than 57^o, and that a full circle of 360^o has a measure of 2π radians.





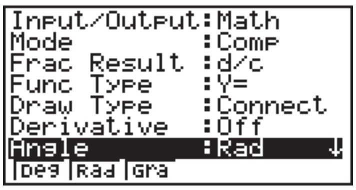
The second screen shows that it is not necessary to convert firstly from radians to degrees in order to find a trigonometric ratio, but that the radian symbol can be used directly. Notice in the first line that an error is caused if the radian symbol is not included. The screens below show some common equivalents between radians and degrees.

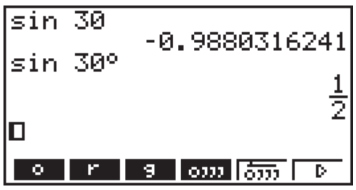




Conversions from degrees to radians can be made by multiplying by $\frac{\pi}{180}$.

If you will be using radians often, it is better to set the calculator to Radian mode in SET UP, as shown below. Then measures will be assumed to be in radians and a similar process to the above can be used for converting from degrees to radians. Some examples are shown below.

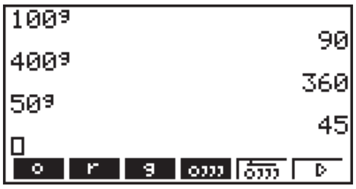




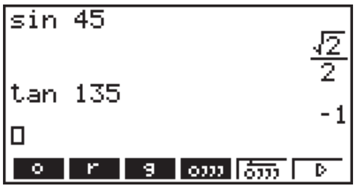
Notice in the third screen that omitting the degree symbol leads to an error of 30 being regarded as 30 radians, instead of 30 degrees.

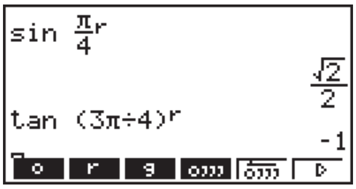
Gradian measure

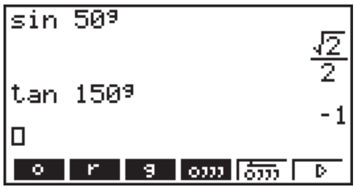
The idea of gradian measure was invented to provide a way of measuring angles that would be consistent with the metric system, which relies on powers and multiples of 10 for both measures and numbers. It is not widely used today, except for some surveying purposes. There are 100 gradians in a right angle and so there are 400 gradians in a full circle. You can use similar processes as above to convert between measures. For example, in the screens below, the calculator is set in degrees. The equivalents to some measures in gradians are shown.



The same values and relationships apply, however, regardless of the way an angle is measured, as the following screens show, with the calculator set to degrees.

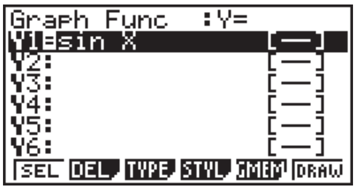


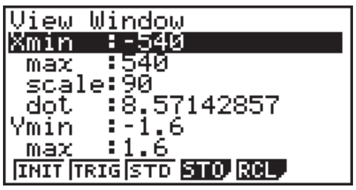




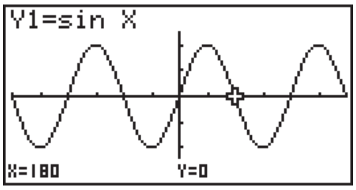
Exploring trigonometric functions

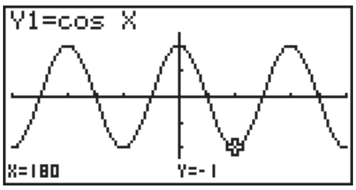
It is instructive to examine graphs and tables of trigonometric functions. With the calculator still set to degrees, a good Viewing Window involves choosing the TRIG window (F2), as shown below:

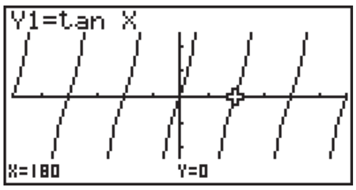




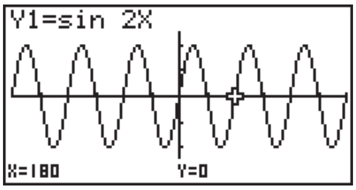
Graphs of the sine, cosine and tangent functions show their periodic nature.

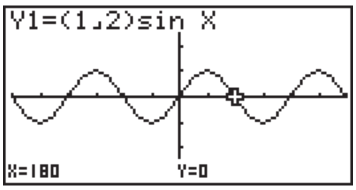


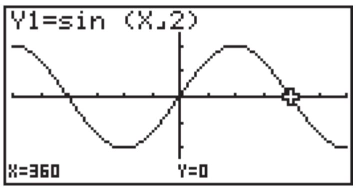




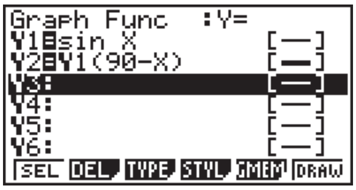
The functions can be transformed in various ways, as hown below for the sine function. Each of the graphs is drawn on the same TRIG scale.

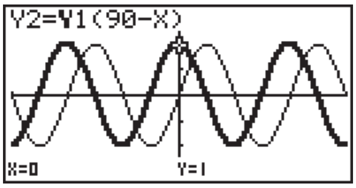






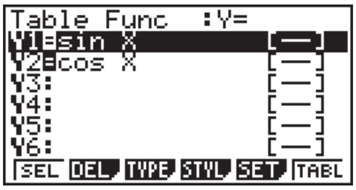
Connections between sine and cosine can be seen from the graphs of each above. It seems that the graphs are the same, but have been horizontally moved. To explore this kind of phenomenon, a function transformer, as shown below is helpful.

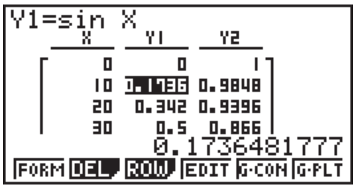


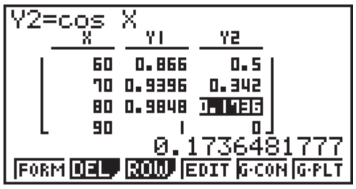


The graph of the transformed function, shown in bold, is the same as the cosine function, graphed earlier. So it is clear that sin (90$^{\text{o}}$ – x) = cos x.

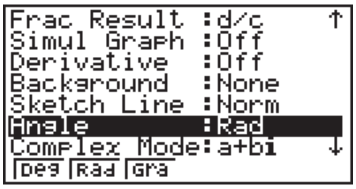
Tables of values provide similar understandings of the functions. For example, the tables below of the sine and cosine functions show that the sine function increases from 0 to 1 as the variable increases from 0 to 90, while the cosine function decreases from 1 to 0 over the same interval. The tables also show that sin (90$^{\text{o}}$ – x) = cos x.

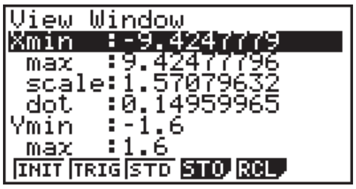


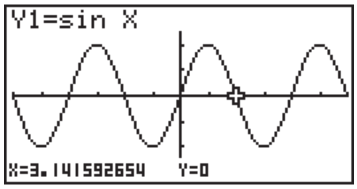




When radian measures are used, the same relationships are evident in both graphs and tables. The calculator will automatically adjust the TRIG scales to suit the measure, as can be seen below.



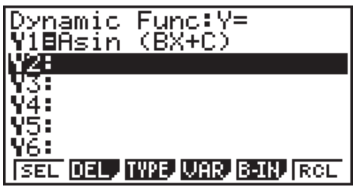


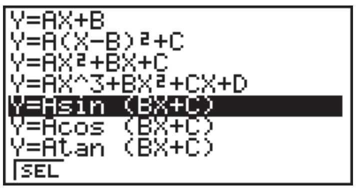


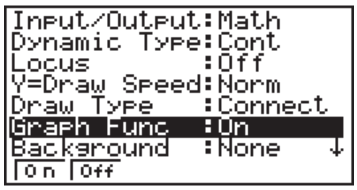
The middle screen shows that the default TRIG scales are -3π < x < 3 π, with tick marks every π /2.

Dynamic graphing

A good way to examine graphs of trigonometric functions is to use Dynamic graphing mode, with MENU 6. Although you can enter your own function definitions, there are some convenient dynamic functions built in for trigonometry, as the second screen below shows after tapping B-IN (F5). We have chosen to explore sine functions. Leave the calculator in radian mode and set the View Window to TRIG. Make sure that the Graph *Func* is turned on.



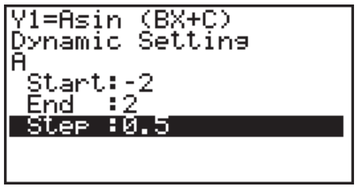




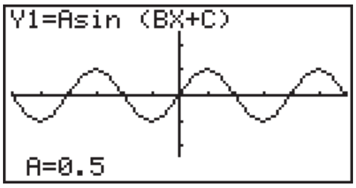
In Dynamic mode, you can choose one of the three values, *A*, *B* or *C* to be the dynamic variable, and need to set values for the other two. Firstly, select the variable by tapping VAR (F3). Highlight A and tap SEL (F1) to select it. Highlight B and give it the value 1, and give C the value 0, as shown in the middle screen below. (The value for the dynamic variable does not need to be set yet.)

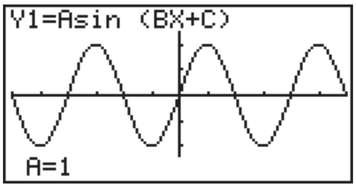
Tap SPEED (F3) to choose a dynamic speed. The *Stop and Go* setting will require you to tap EXE to change the graph each time. Tap EXIT when finished.

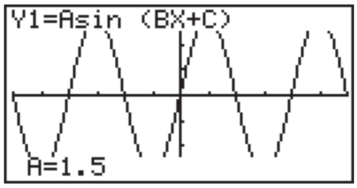
Finally, tap SET (F2) to choose suitable values for the dynamic variable. Make sure that you do not set too many values (with a very small step) as each value will take both time and memory to generate a graph: about five to ten values is a good choice usually. The screen at right below shows nine values in all (A = -2, -1.5, -1, -0.5, 0, 0.5, 1, 1.5, 2). Tap EXIT when finished.



Now that the dynamic graphs are defined, you can activate them with DYNA (F6). The calculator will first draw all of the graphs and then present them according to the requested speed. The graphing will stop automatically, but you can stop it before it has has completed, and adjust the speed, by tapping AC/ON.







The screens above suggest that the value for A affects the amplitude of the graph, but does not affect its basic shape or the points at which it crosses the *x*-axis; the graphs show that A has a vertical stretching or shrinking effect.

Experiment for yourself with using other variables as the dynamic variable and with other trigonometric functions.

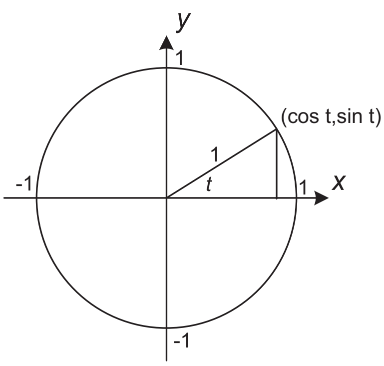
The unit circle

Parametric graphs can also be used to define relations that are awkward to draw using the usual function graphing mode of the calculator. A good example involves drawing a graph of a circle, but in a different way from the use of the Conics mode. A circle with centre at the origin and radius 1 can be defined by the relation

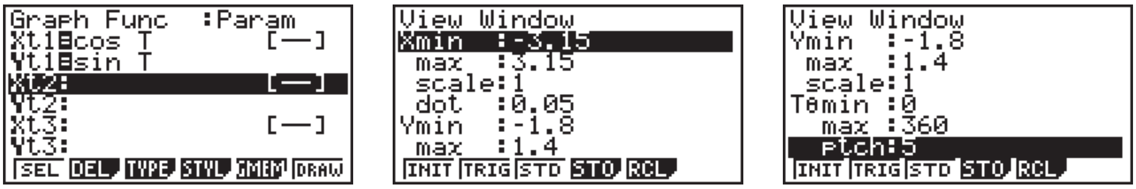
$$x^2 + y^2 = 1.$$

This particular circle is usually called a *unit circle* and it is very useful for learning about trigonometry. To graph this relation in normal function graphing mode, you have to express it as two separate functions as noted earlier in this chapter.

With the calculator in parametric mode, it is possible to graph the same relation in a quite different way. In this case, it is helpful to think of the circle from the perspective of circular functions, as shown on the next page. The coordinates of the point shown are $x = \cos t$ and $y = \sin t$, with t referring to the angle of rotation.

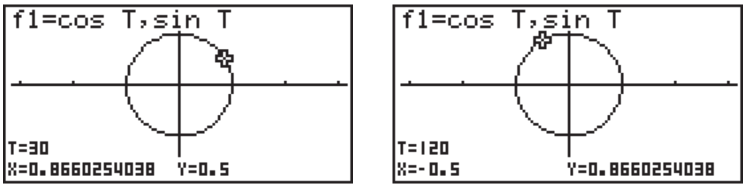


Enter Graph mode and select TYPE (F3) to select Parametric mode with F3. To draw a graph of this circle in parametric mode, define the *x* and *y* components accordingly, as shown below. In Parametric mode, the X,θ,T key will automatically enter *T* (not *t*). Set the viewing window suitably, but make sure that the *x*- and *y*-axes have the same scale, or the circle will be distorted.



The example above is suitable when the calculator is set to degree mode; you will need to choose different *T* values for radian mode. The scales were chosen to suit the calculator screen and to show values of the sine and cosine functions every 5° when tracing.

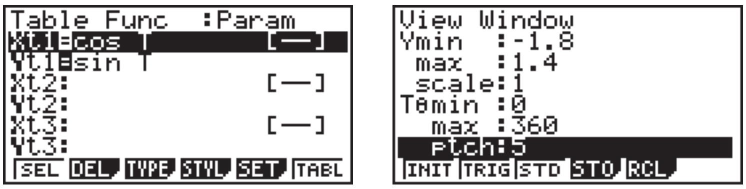
You can trace the circle in the usual way, but you will notice that you can't trace to values of *T* outside those defined in the viewing window. In this case, the trace starts at the point (1,0) and proceeds anticlockwise around the circle. (i.e., tracing to the right moves anticlockwise around the screen, which will seem strange at first.) The screens show how tracing this unit circle will allow you to see the sine and cosine values of angles between 0° and 360°.



The left screen shows that sin 30° ≈ 0.5 and cos 30° ≈ 0.866.

The screen on the right shows that sin 120° ≈ 0.866 and cos 120° ≈ -0.5.

You can also explore the unit circle using a table. The tables below show values for the cosine and sine functions, with the angle *T* increasing in steps of one degree. The tables show that for small positive angles, the cosine function has a value close to one while the sine function is close to zero.



The unit circle drawn on the calculator is not very smooth, because of the screen resolution. You can change the *T* values in the viewing window to draw unit circles with different tracing characteristics. If you choose a small value for the pitch, the circle will take longer to draw, however.

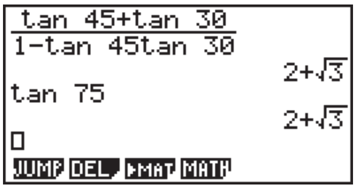
Exact values

Although practical measurement always involves approximations (as no measurement is ever exact), it is also interesting to study the exact values of some trigonometric relationships. You may have noticed that the calculator provides some of these, two of which are shown at the top of the next page.

It is often possible to see the origins of values like these, by drawing suitable triangles. For example, an isosceles right triangle includes angles of 45°, while an equilateral triangle has angles of 60°, and also 30° when a perpendicular bisector is drawn. In addition, you can use these and other exact values to find the exact values of the sine, cosine and tangent of other angles, using various formulae for combinations of angles. To illustrate, consider the formula for the tangent of a sum of two angles:

$$\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}$$

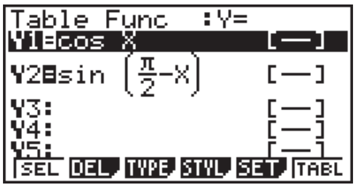
You can use the exact values of tan 45° = 1 and tan 30° = 1/√3 to calculate tan 75° for yourself. You should be able to obtain the same exact result as the calculator, shown below.

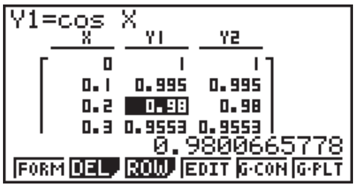


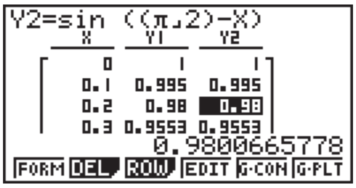
Experiment in this way with other addition and subtraction trigonometric formulae for sine, cosine and tangent.

Trigonometric identities

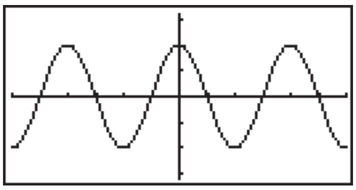
In trigonometry, we are frequently interested in relationships among the various ratios, such as the example referred to earlier (in degrees): sin (π/2 – *x*) = cos *x*. To see that the two expressions are in fact always equal, it's a good idea to tabulate them or graph them. The screens below show that the tables are the same:

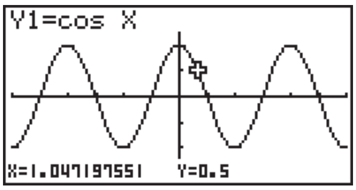


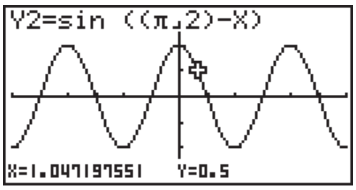




Similarly, the graphs of these two functions are also the same for all values. The first screen below shows that only one graph is provided. In the second screen the graph for y = cos *x* has been traced, while in the second screen the graph of *y* = sin (π/2 – *x*) has been traced, by tapping the ▼ key. If you tap the ▲ and ▼ keys, you can see that the two graphs are identical.

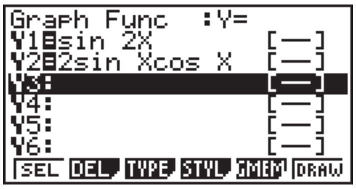


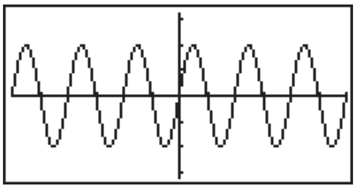
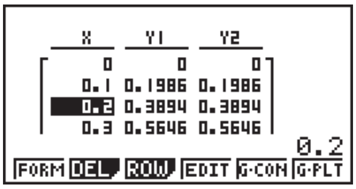




(When two expressions have the same value for all values of the variable like this, the relationship is generally called an *identity*.

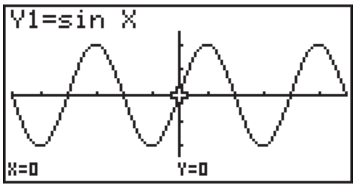
Another example of a trigonometric identity is sin 2*x* = 2 sin *x* cos *x*. The screens below show how the calculator can be used to represent this identity.

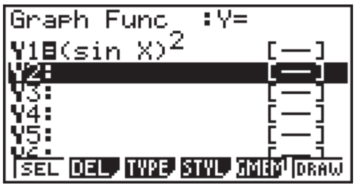


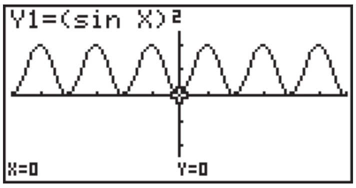



Again, only a single graph is produced, although two separate functions are defined. Similarly, the values of the two functions in the table are identical on every row: i.e., for all values of the variable, the two expressions refer to the same number, so the relationship is an identity.

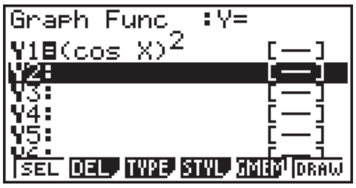
A calculator can provide a different perspective on an identity. Consider the Pythagorean identity, $\sin^2 x + \cos^2 x = 1$. To explore this relationship, note firstly that (like the inverse trigonometric function notations) the notation of $\sin^2 x$ cannot be entered into the calculator directly, as it is normally written, but must be entered differently, as shown in the middle screen below.

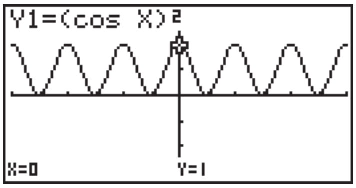




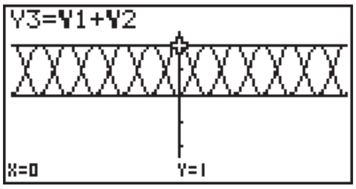


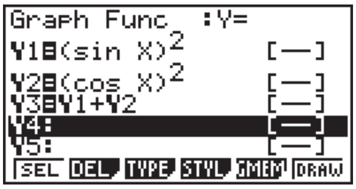
The same is the case for $\cos^2 x$:

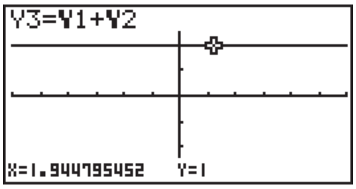




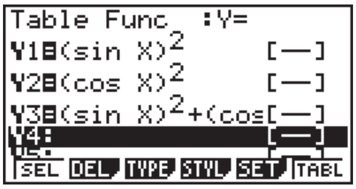
It is surprising that the sum of these two expressions should give the same result each time, as shown in the first screen below. In the third screen, notice that only the sum of the two expressions is graphed, for clarity. The sum appears to have the constant value of 1 for all values of *x*.

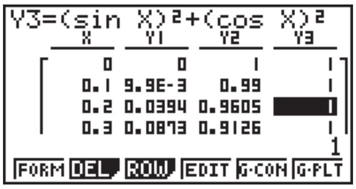






Similarly, a table of values suggests that the sum is a constant: $\sin^2 x + \cos^2 x = 1$ for all values of *x*.





Demonstrations like these do not *prove* that a relationship is an identity: you need a formal matehmatical argument for that purpose. However, they show that an identity seems to be present by demonstrating it for a large number of values.

Exercises

The main purpose of the exercises is to help you to develop your calculator skills.

1 Given a right triangle ABC, use your calculator to complete the missing values in each row of the table below:

AB	BC	AC	Angle BAC	Angle ACB	Angle ABC
2.5		5.2	90^o		
		10.5	75^o	90^o	
	10			48^o	90^o

2 With your calculator set to degrees, plot a unit circle in parametric mode using $y = \sin t$ and $x = \cos t$. Use the unit circle to find the values of the following to two decimal places:

(i) $\sin 70^o$ (ii) $\cos 95^o$ (iii) $\sin 125^o$ (iv) $\sin (-20^o)$

3 Use your calculator to check that $\sin 18^o = \sin (\pi/10)^R = \sin 20^g$.

4 (i) Use your calculator to check that $\sin 40^o = \sin 400^o = \sin 760^o$.

(ii) Use a unit circle to explain why the ratios in part (i) are all equal.

(iii) Write down two other angles that have the same sine as $\sin 40^o$; check with your calculator that you are correct.

5 The angle of elevation of a plane as it flies over the coastline 75 km away is 10^o. To the nearest metre, at what height is the plane flying?

6 The observation deck on the Burj Khalifa building in Dubai is 452 m high. Find the angle of depression for an object that is one kilometre from the building, giving your answer to the nearest degree.

7 Find to the nearest second two angles between 0^o and 360^o that have a cosine of 0.6. Give your answers in degrees, minutes and seconds.

8 Find $\tan 19^o17'24''$, correct to two decimal places.

9 Use your calculator to make the following conversions:

(i) to radians: 30^o, 126^o, 250^o, 290^o

(ii) to degrees: 3^R, 10^R, 20^R, 1^R

10 (i) Draw dynamic graphs of $y = a \cos kx$ for values of k between 1 and 5 and with $a = 1$.

(ii) Change the speed setting to be Stop and Go and redraw the graphs

(ii) Draw dynamic graphs of $y = a \cos kx$ for values of a between 0.25 and 2 and with $k = 1$.

(iii) Use your graphs to describe the effect of the values of a and k on the shape of the graph.

11 Use your calculator in Graph mode to find the amplitude and period of $f(x) = 3 \sin (2x + \pi/4)$

12 Use a table and a graph on your calculator to show that $\sin 2x = 2 \sin x \cos x$

Activities

The main purpose of the activities is to help you to use your calculator to learn mathematics. You may find that some of them are too advanced for you. Ignore activities you don't yet understand.

1 According to *Google Maps*, the position of India's famous Taj Mahal is latitude 27.17799 North and longitude 78.042111 East.

Use your calculator to convert these two angles into degrees, minutes and seconds.

The circumference of the Earth at the equator is 40 000 km and covers a span of latitude of 360^o. If locations are given to the nearest one hundredth of a second (0.01 seconds), determine what distance this involves on ground on the circumference.

Find the latitude and longitude of your school from the Internet. Use these to make a sketch of your position on the earth.

2 The cosine law $c^2 = a^2 + b^2 - 2ab \cos C$, describes the relationship between the lengths of the sides of a triangle ABC and one of its angles. For a triangle ABC that does not include a right angle, this rule can be used to calculate side lengths and angles. Use this rule to find the angles of a triangle for which AC = 10.3 cm, BC = 7.2 cm and AB = 5.5 cm.

Draw a triangle on paper and check to see that he rule describes the measurements correctly.

What sorts of triangles have $a^2 + b^2 > c^2$?

What sorts of triangles have $a^2 + b^2 < c^2$?

3. Obtain the average monthly maximum and minimum temperatures for your city for a year. Set your calculator to radians before entering data. Enter your data into List 2 and List 3 of your calculator, with List 1 containing the month (1 for January, 2 for February, etc.). Use the List menu to put the mean of the maximum and minimum temperatures into List 4. Explore graphs of these data. You should find that a function such as $f(x) = a \sin (bx + c) + d$ will fit the data reasonably well for the mean temperatures. Find suitable values for a, b, c and d.

Explore this approach to model temperatures in other cities.

4 With your calculator in radian mode, use a graph to see how many solutions there are to the equation $2 \sin x = 0.4$, and how the solutions are related to each other.

How many solutions are there to the equation $1.5 \sin 3x = 0.2x$? How are the solutions differently related to each other than they were for the previous equation?

5 A student measured the height of a tree by using a clinometer (made with a straw and a protractor) and a trundle wheel. She found the distance to the tree to be 25 m (to the nearest metre) and the angle of elevation of the tree to be 40^o (to the nearest 5^o) on a windy day.

She used this information to determine the height of the tree, using trigonometry.

How accurate is her answer?

6 Use the Dynamic graphing mode of your calculator and the built-in functions with sin, cos and tan to explore some periodic functions. Make sure that you predict the shape of the graphs before you begin.

Work with a partner. Vary the values of the constants in the functions and keep careful notes of your results. Discuss your observations with other pairs of students.

Notes for teachers

This module illustrates several ways in which the calculator can be used to explore various aspects of trigonometry. Various kinds of calculations are enabled, including those with different angle measures: degrees, radians and gradians. Trigonometry graphs show the periodic nature of trigonometric functions and also allow identities to be understood. The text of the module is intended to be read by students and will help them to start to see how the calculator can be used efficiently for various kinds of explorations. The Activities are appropriate for students to complete with a partner or in a small group, so that they can explore the ideas together.

Answers to Exercises

1. 5.77, 25.68°, 64.32°; 40.57, 39.19, 15°; 11.11, 14-94, 42° 2. (i) 0.94 (ii) -0.09 (iii) 0.82 (iv) 0.94 3. Use OPTN F6 F5 for ANGLE menu. All are 0.309 4. (i) All are 0.643 (ii) Same point on the circle is involved (iii) 140° and 1120° 5. 13 225 m 6. 24° 7. 53°7′48″ and 306°52′12″ 8. 0.35 9. (i) π/6, 7π/10, 25π/18, 29π/18 (ii) 171.89°, 572.96°, 1145.92°, 57.30° 10. (iii) *a* describes the amplitude – the vertical stretch, while *k* affects the period, with larger values of *k* having a smaller period. 11. Amplitude is 3, period is π 12. Draw or tabulate $y1 = \sin 2x$ and $y2 = 2 \sin x \cos x$ to see values of the two functions are equal at each point.

Activities

1. This activity is intended to encourage students to think carefully about the accuracy of everyday measurements and understand the significance of using excessive decimal places; they will find online variations in measurements such as those given for the Taj Mahal. Encourage students to use available resources to determine local latitude and longitude. [Answers: 27°10′41″ N and 78°02′32″ E. At the equator, 0.01 seconds is about 31 cm.]

2. Students will find that activities of this kind that require successive solution of equations are easier once they have completed Module 9, in which the equations solver is described. Encourage careful measurements of their own triangles, attending to appropriate accuracy on the calculator. [Answers: 107.6°, 41.8°, 30.6°; acute and obtuse triangle respectively.]

3. Trigonometric functions are powerful for modelling periodic phenomena, so this activity is intended to help students see this in practice. In Statistics mode, the choice of sinusoidal regression allows for models like those suggested in the activity to be obtained, and thus interpreted by students. Alternatives to temperatures might be used to suit available local data.

4. Trigonometric equations often have an infinite number of periodic solutions, as illustrated by the first equation, which students can explore efficiently using G.Solve in Graph mode. The second equation has only three solutions, however, again well understood using graphs.

5. This activity allows students to explore an apparently simple situation using the power of the calculator to make repeated calculations efficiently. If measurements are assumed (unreasonably) to be exact, the height is 25 tan 40 = 20.9775 m. When inaccuracy is accommodated, the tree height can be determined to be somewhere within the (wide) interval of (17.2 m, 25.5 m). This activity might be used for a productive whole-class discussion.

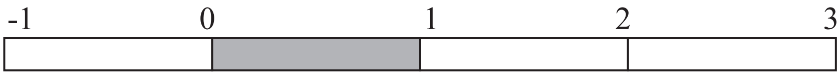
6. The Dynamic graphing features of calculators are especially useful for students to explore trigonometric functions. This activity is written in a rather general form to encourage students to make some choices for themselves. You might prefer to give more specific advice for some students to constrain their investigations, or to focus on a particular trigonometric ratio. Students should be able to explore issues of amplitude, period and phases in interesting ways through the use of Dynamic mode.

Module 7
Probability

In this module, some of the many ways in which the calculators can be used to explore probability are briefly described and illustrated. The calculator's random number generation capabilities can be used to simulate random events. You can use these capabilities to understand how randomness works, but it is easier to use the add-in application, *Probability Simulation*, to efficiently study random events in the everyday world. Important probability distributions are available through the calculators, in particular the normal and binomial distributions.

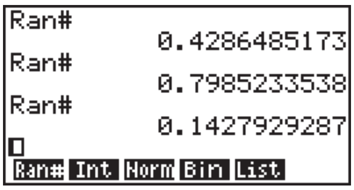
Pseudo-random numbers

The calculators have a capability to simulate random numbers between 0 and 1, which is used in many different ways. The Ran# command is obtained from the PROB menu, accessible after pressing OPTN F6 and then F3. The command generates a random number larger than zero and less than one; in the long run, the random numbers produced are spread evenly over the whole interval from zero to one, as shaded on the number line representation below:



In fact, the numbers are not *really* random, as the calculator itself is a predictable device. They only seem to be random, and have many of the useful properties of random numbers. Technically, they are *pseudo*-random numbers. But they are still useful for simulating random events.

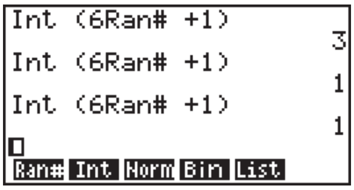
After the Ran# command is executed (with EXE), a new random number will be produced with every press of the EXE key. This is because the calculator repeats the previous command line if no new command line is entered. This is a handy feature, since it saves you entering the command each time. The screen below shows three random numbers. After the Ran# command was entered, EXE was pressed three times. By their nature, random numbers are unpredictable, so you will not get the same three random numbers on your calculator as shown here, and you will get a different set of numbers next time you do this.

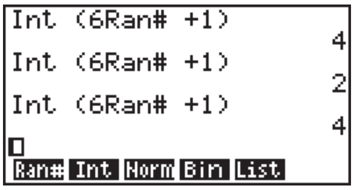


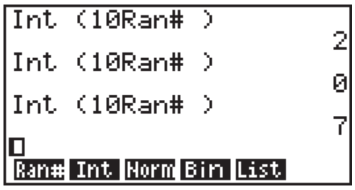
In most cases, you will need to transform random numbers to make them more useful. For example, to simulate the rolling of a standard six-sided die, the random numbers between 0 and 1 need to be transformed into random integers between 1 and 6. Each of the six integers should be equally likely to occur, since each of the six faces of a standard die is equally likely. The table below shows a sequence of steps to do this:

Command	Random number result
Ran#	between 0 and 1
6Ran#	between 0 and 6
6Ran#+1	between 1 and 7
Int(6Ran#+1)	integers from 1 to 6

The *Int*command finds the integer part of a number. So the integer part of 1.345 is 1 and that of 6.8975 is 6. *Int* is obtained with F2 from the NUMmenu, accessible after pressing OPTN F6 and then [F4]. After entering the command, Int(6Ran#+1), each time [EXE] is pressed, an integer in the set {1,2,3,4,5,6} is chosen with equal probability. The command is a little awkward to construct, so it is fortunate you don't need to construct it again for *each* simulated roll of the die. To simulate six rolls, just press [EXE] six times. The screens below show this:



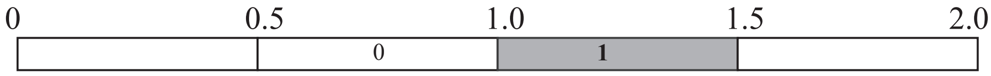




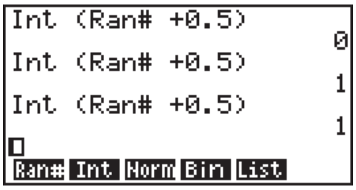
The same idea is easily adapted to other situations involving chance. For example, to simulate random digits uniformly distributed in the set {0,1,2,3,4,5,6,7,8,9}, change the above command line to Int (10Ran#). Some results are shown in the third screen above.

To simulate tossing a fair coin (1 for heads, 0 for tails), use instead the command line, Int (Ran#+0.5), which will give a 1 half the time and a 0 half the time.

In the diagram below, the shaded parts shows possible values of Int (Ran#+0.5). The diagram shows that half of the time, the command will give a zero, and the other half of the time it will give a one. You can think of the result each time as the number of heads on each toss.

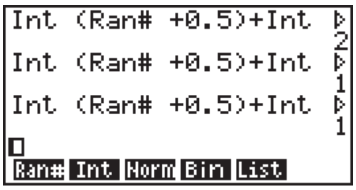


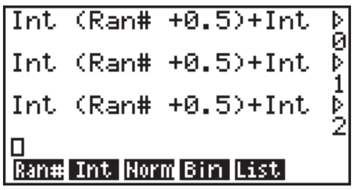
The next screen shows some results of using this command:

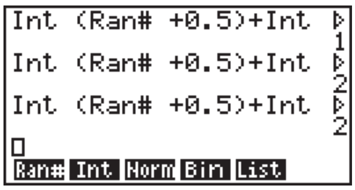


For these three simulations, there were two heads and only one tail produced. To use a biased coin, say one for which Prob(Head) = 0.7, you can change the command line to Int (Ran#+0.7).

To simulate tossing a *pair* of fair coins, use the more complex command line, Int (Ran#+0.5) + Int (Ran#+0.5), which will simulate the number of heads on each toss of the pair of coins. Here are some results of using this command nine times:



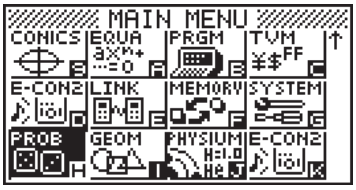


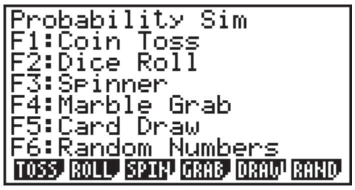


In this case, there was a pair of heads four times, a pair of tails (i.e., no heads) once, and the other four tosses produced a head and a tail (i.e., one head).

Conducting experiments with *Prob Sim*

So far, the simulation capabilities of the calculator have been used to collect just a few observations. This is a good idea to make sure that the data generated seem to be appropriate, and that you understand its origins, but usually you will need to collect *many* observations to get a good idea of what is happening in the long run. One way to do this is to use the Table mode of the calculator to collect data and the Statistics mode of the calculator to analyse the data. However, an easier way is to use the *Probability Simulation* Add-in application, available for free download from CASIO's Worldwide Education site . (On the fx-CG 20, the icon is labelled *Prob Sim*, while on the fx-9860GII, shown below, it is labelled as *PROB*.





If the icon is not showing, you will need to download it and install it via your computer or transfer it from a calculator that already has it installed. Instructions for downloading are available on the web. The official international site which includes the Probability Simulation Add-in and many other resources is *http://world.casio.com/edu/*.

Six different applications are provided in Prob Sim, as shown in the screen above. Here is a brief description of each simulation:

Coin Toss Tosses up to three coins at once and records the numbers of heads on each toss.

Dice Roll Rolls up to three dice at once, with various numbers of sides and records the result of each dice roll as well as the total each time.

Spinner Spins a spinner with up to eight equal sectors and records the result each time.

Marble Grab Grabs one of up to five marbles from a jar, with or without replacement, and records the results of each grab.

Card Draw Draws a set of playing cards from up to three standard decks (or a reduced deck), with or without replacement and records details of the set.

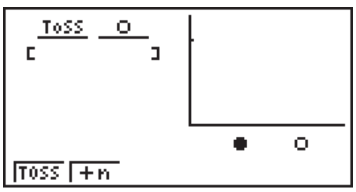
Random Numbers Selects 1 to 6 integers from a set of integers, with or without replacement, and records the details of each selection.

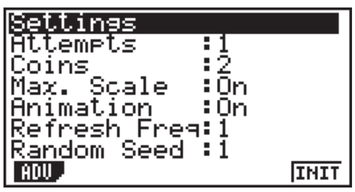
These six applications are very versatile, and allow you to simulate many experiments in probability, to see long term results. Both tables and graphs are provided in each case and the Statistics module can be used to analyse results more carefully. In each case, the SET UP menu allows you to configure the application to suit your purpose and interests.

Coin Toss

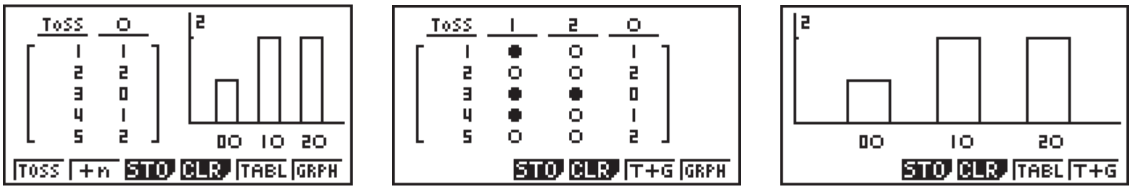
We will consider just one of these applications, in order to see how it can be used, as there are many similarities in the way in which the suite of six *Probability Simulations* applications can be used.

Coin Toss can be used to understand what is likely to happen if a pair of coins is tossed many times. How many heads can you expect? What is the probability of obtaining two heads? The screens below show the opening screen, with the SET UP screen showing that two coins will be tossed.

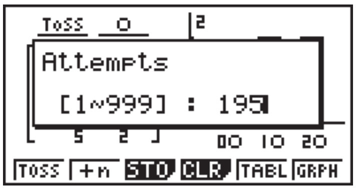




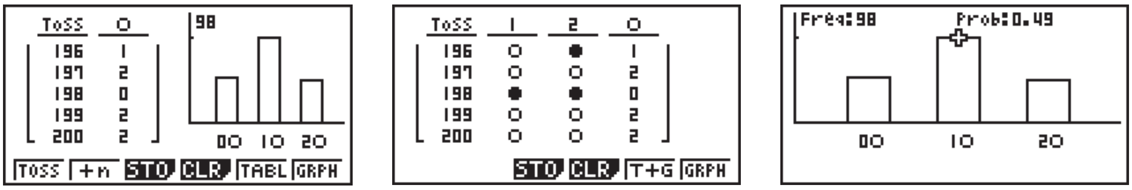
To toss a pair of coins once, tap TOSS (F1). Because Animation is turned on, an image of coins being tossed is shown each time. The screens below show one example, after five tosses, although your example will probably be different. The number of heads obtained each time is shown in both the table and the graph in the left screen. The middle screen shows a table of results in more detail, after tapping TABL (F5). The black circles refer to a tail and the white circles refer to a head, while the total number of heads each time is also recorded. A full-screen graph of results is shown in the third screen after tapping GRPH (F6). Return to the first screen by tapping T+G (F6).



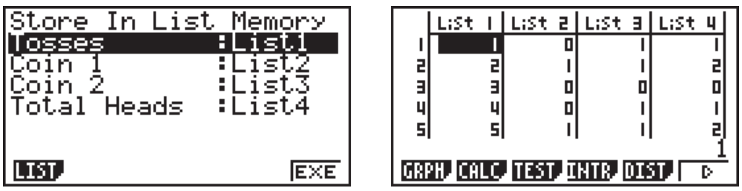
It is unwise to reach conclusions about random events with only a few pieces of data. More stable results will be available if you simulate a large number of tosses. To get 200 attempts, for example, you need another 195 attempts. Tap + n (F2) to add a further 195 attempts, as shown below:



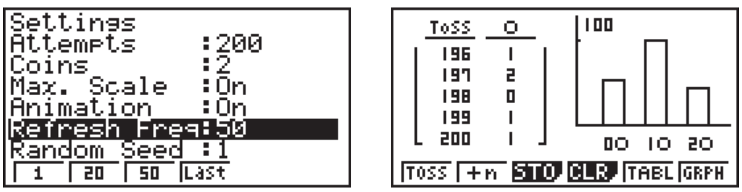
The results are now based on 200 attempts, and strongly suggest that the probability of getting a head and a tail (that is, one head) is close to 0.5, while the probability of obtaining either two tails or two heads is close to 0.25. Notice that you can scroll through the table of results with the cursor and can trace the graph with the cursor after tapping F1; tap EXIT to return to the original screen.



You could now add further attempts if you wished, by using + n (F2) again. You can store the data into lists for analysis in Statistics mode by tapping STO (F3), as shown below. Look carefully to see how the first five rows match the first five attempts. Note that storing data will over-ride existing data in the lists, so check before you start that lists do not contain data you wish to keep.

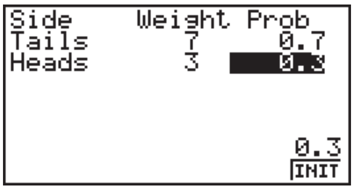


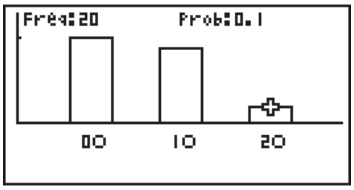
You can repeat a simulation of 200 tosses of a pair of coins. It is a good idea to do this to see if the results are similar each time, which will be a signal that good approximations to the probabilities of interest have been obtained. In the SET UP screen below, note that the number of attempts is now 200 (instead of 1 previously), so that each Toss will produce 200 results. In addition the *Refresh Frequency* is set to 50 (previously 1) so that the graph only changes after every 50 attempts.

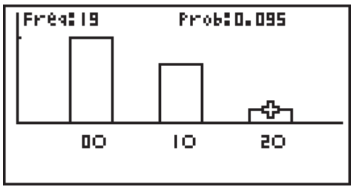


The result this time is very similar to the first result, but not exactly so. Random events like this will rarely be precisely the same, but these results are close enough to reinforce the earlier interpretation that the probability of getting a head and a tail in tossing two coins is ½.

Finally, this application (and others) can also be used to simulate events for which the probabilities are *not* equal, such as tossing a pair of biased coins, for which the probability of a head is 0.3 instead of 0.5. To make changes of this kind, use the advanced commands, with ADV (F1) shown in the first SET UP screen above. Change either the probability or the weight; the calculator will automatically adjust settings where necessary.







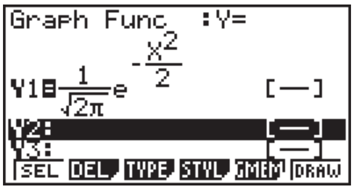
When 200 attempts of tossing a pair of coins biased in this way were simulated, in two separate runs, the results shown in the middle and at right above are rather different from those obtained with fair coins, with only 10% of the attempts showing two heads the first time and 9.5% the second time (comparing well with the theoretical probability of 0.3^2 = 9%.).

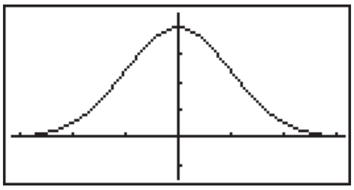
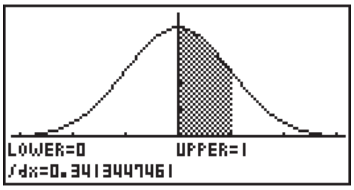
The normal distribution

The normal distribution is a theoretical probability distribution that can be described by the following rather complicated function:

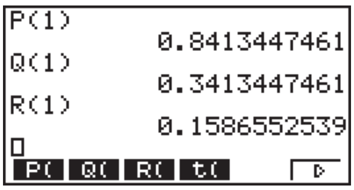
$$y = \frac{e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}}{\sigma\sqrt{2\pi}}$$

This function is sometimes referred to as the *normal density function* and can be graphed in Graph mode, for various values of the mean (μ) and the standard deviation (σ). The particular case for a mean of zero and standard deviation of 1 is known as the *standard normal distribution*, and is shown below for -3.15 ≤ *x* ≤ 3.15 and for -0.15 ≤ *y* ≤ 0.45.



The third screen above shows the approximate area under this curve using the definite integral command in *Graph Solve*. The area from *x* = 0 to *x* = 1 is about 0.34134. The total area under the curve (which never crosses the *x*-axis) is 1, and can be interpreted as a probability. This screen shows that, for a normally distributed random variable, with mean zero and standard deviation one, the probability that a randomly chosen value will be between 0 and 1 is approximately 0.34134. The normal distribution is so important that it appears elsewhere in the calculator. For example, in Run-Mat mode, tap OPTN F6 and then F3 F6 to see the commands shown below, all of which relate to the normal distribution.



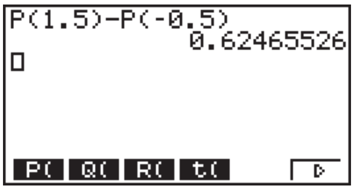
These show the following probabilities, with z referring to the standard normal random variable:

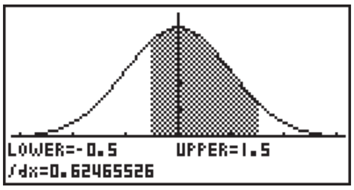
$$P(t) = \text{Prob}\ (z \le t)$$
$$Q(t) = \text{Prob}\ (0 \le z \le t)$$
$$R(t) = \text{Prob}\ (z \ge t)$$

The normal distribution functions allow you to determine various normal probabilities. For example, the probability that a randomly chosen z value will be between -0.5 and 1.5 can be determined using printed tables of values. Using the calculator makes this task much easier than otherwise possible, using the relationship:

$$\text{Prob}\ (-0.5 \le z \le 1.5) = \text{Prob}\ (z \le 1.5) - \text{Prob}\ (z \le -0.5) = P(1.5) - P(-0.5)$$

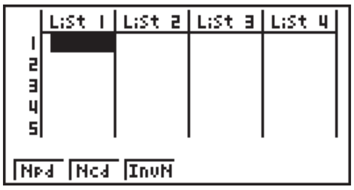
The screen on the left below shows how this computation can be accomplished with a single command line in Run-Mat mode. The second screen shows the same computation in Graph mode.



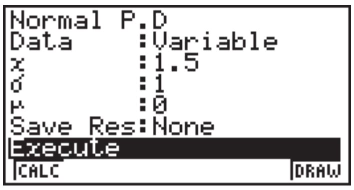


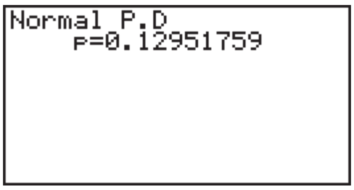
Using Statistics mode

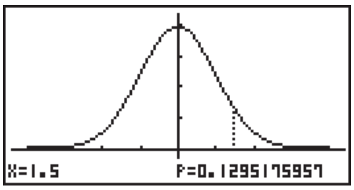
The normal distribution is also accessible in Statistics mode. Select DIST (F5) and then NORM (F1) to see the three normal distribution options shown below.



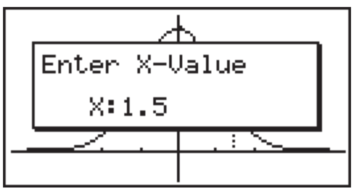
The first of these options, *Npd* allows you to evaluate the normal probability density function for various values of x and even to graph the entire function. First you must change *Data* from *List* to *Variable*, then set the appropriate mean (μ) and standard deviation (σ) values. The *Save Res* command allows you to save the result into a list, if you wish, but it is not necessary to do so. Use ▼ to highlight the *Execute* command. The middle screen below shows the result of a calculation with CALC (F1); it shows that the standard normal density function has a value of 0.12951 when $x = 1.5$. (You can compare this value with that obtained by tracing the graph in Graph mode.)

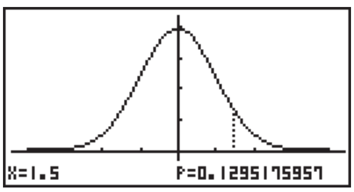






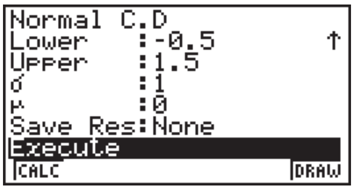
The DRAW (F6) option is more useful, showing the familiar shape of the distribution, on scales automatically chosen by the calculator. Again, it is clear that the standard normal distribution is mostly contained within values of ± 3 on the horizontal axis. The G.Solve menu via SHIFT F5 allows you to find the normal density at any x-value, as shown below.

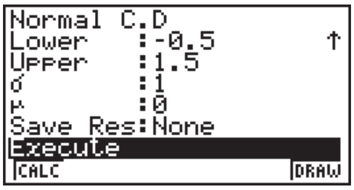


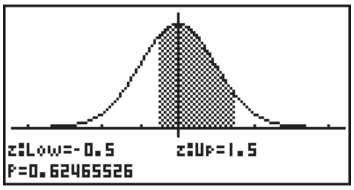


Different normal distributions can be represented graphically and explored by changing the mean and standard deviation values in the original screen.

The main use of the normal distribution involves finding the area under the curve between two horizontal values, since this can be interpreted as a probability. These values are readily obtained using the second DIST option here, *Ncd* which stands for the *cumulative normal distribution*. The screen below shows the same case as that shown above for the standard normal distribution (i.e., $\mu = 0$, $\sigma = 1$).



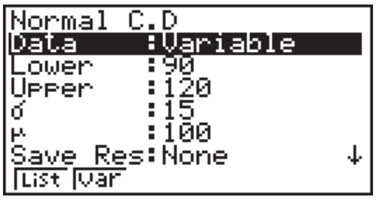


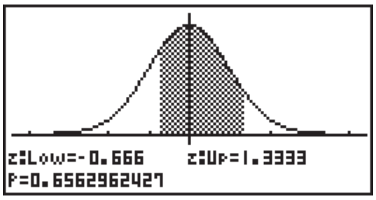


Tap CALC (F1) to evaluate Prob ($-0.5 \leq z \leq 1.5$) = 0.62465, the same result as before, or DRAW (F6) to show the result visually. Different normal distributions can be dealt with by first changing the mean μ and standard deviation σ since all normal distributions have a similar symmetrical shape. You can compare any distribution with the standard normal distribution, using the *z*–transformation:

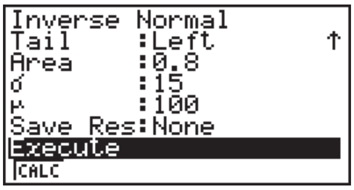
$$z = \frac{x - \mu}{\sigma}.$$

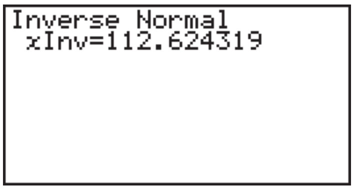
That is, if the random variable *x* is normally distributed with a mean of μ and a standard deviation of σ, then the *z*-transformation above will produce a standard normally distributed random variable (i.e., with a mean of zero and a standard deviation of one). For example, some standardised tests are normally distributed with a mean of 100 and a standard deviation of 15. The screens below show that the probability of obtaining a score between 90 and 120 is about 65.6%. Notice that the calculator shows the two transformed values of $z = -0.66$ and $z = 1.33$ respectively.



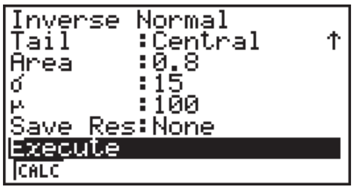


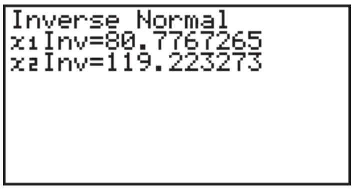
The final normal distribution function provided by the calculator, *InvN* is to permit you to determine the horizontal value associated with a particular probability. This is the inverse problem of finding the probability associated with the horizontal values. The screens below show how to find the value below which 80% of the standardised test scores fall.





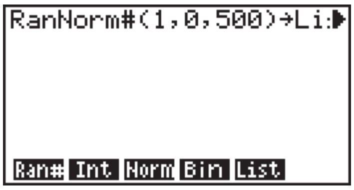
Notice that the Tail command allows you to find the value above which 80% of the scores fall or the (central) interval within which 80% of the scores fall; this is between 80.78 and 119.22.

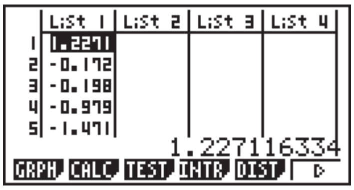




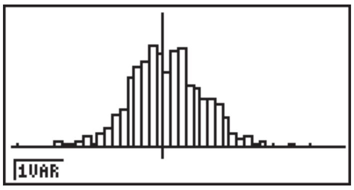
Simulating normal distributions

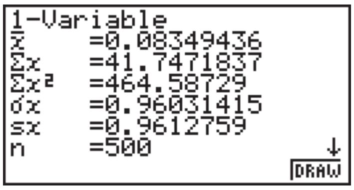
While earlier simulations using *Ran#* used a *uniform* distribution, you can use the OPTN menu in Run-Mat mode to *simulate* a random normal variable. This is useful in practice to simulate real world phenomena that have a normal distribution of occurrence. The screens below show the use of Norm (F3) to store 500 values from a standard normal distribution (i.e., $\mu = 0$, $\sigma = 1$).in List 1:





In Statistics mode, a histogram of the simulated data shows a roughly normal shape, although not perfectly so, of course. The actual mean and standard deviation are also not exactly 0 and 1, although they are close to these values.

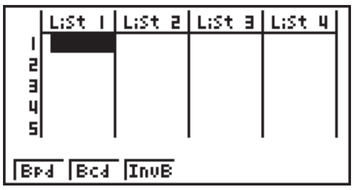


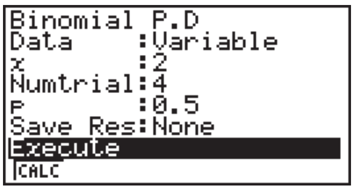


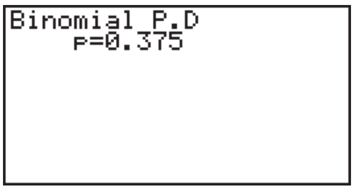
The binomial distribution

The calculator DIST function in Statistics mode can be used to determine probabilities associated with different probability distributions. Another important theoretical distribution in statistics accessible via this function is the *binomial distribution*. This is relevant to a situation in which an event with only two possible outcomes and a particular fixed probability of occurring is repeated several times, and the number of times it occurs is counted. A simple example involves repeated tossing of a fair coin, with an assumed probability of $p = 0.5$ of obtaining a head on each of the tosses; if the result of each toss is independent of the other tosses, and there are n tosses altogether, the resulting number of heads is said to be binomially distributed with parameters n and p.

Consider using the binomial distribution to model the case of families with four children. Let us assume that the probability of a girl being born is 0.5. So, in this case, $n = 4$ and $p = 0.5$. In Statistics mode, tap DIST (F5)and then BINM (F5) to access the choices for the binomial probability distribution (*Bpd*) or the cumulative binomial distribution (*Bcd*).

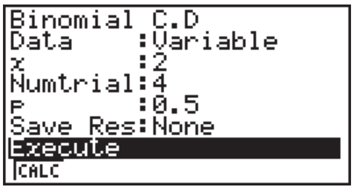


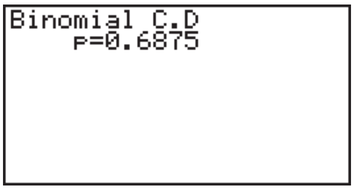




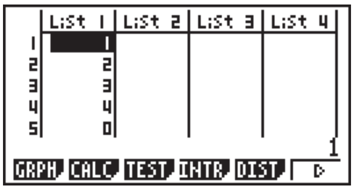
Notice there is a choice of entering a single variable value or a list of values. The middle screen above shows the choice of a single value. To find the probability that exactly two of the four children will be girls, enter the appropriate values as shown, move the cursor to the *Execute* command and press F1(CALC). The result shows that the probability of getting exactly 2 girls (and thus also two boys) out of four children is 0.375. Notice that this means that the probability that a 4-child family will *not* comprise two of each gender is 0.625, much more likely than having an 'even' mix of genders.

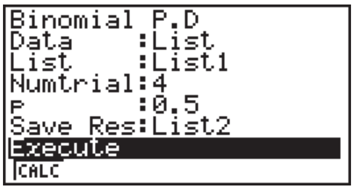
The calculator will similarly evaluate a cumulative probability. The following screens show that the probability of obtaining zero, one or two girls in four children is 0.6875.



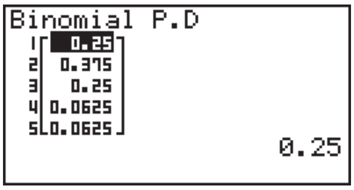


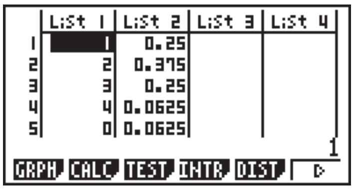
Binomial probabilities can be found for a several values of a variable, rather than just one. Enter the values into a list, and choose the *List* instead of *Variable* for *Data*. The calculator will evaluate the probabilities for each value in the list, giving results in the form of a list. For example, to find the probability of getting exactly 1, 2, 3, 4 or 0 girls in a family of four children, under the assumption that boys and girls are equally likely, the screen below shows a suitable list in *List 1*. The middle screen shows that a list has been selected instead of a variable. Although it is not necessary to do so, the results themselves can be saved into a list; in this case, they are set to be saved to *List 2*.





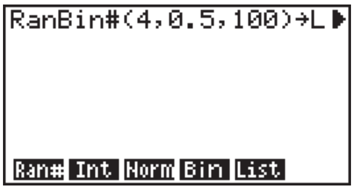
The resulting probabilities are shown below. Notice that the list above starts with 1 instead of zero, in order to make the final result easier to interpret: each row of the probability list is labelled in the same way as the row in the original list, and it may be confusing to start the list with a zero. Use EXIT to return to the data screen and see that the probabilities below have been saved to *List 2*:

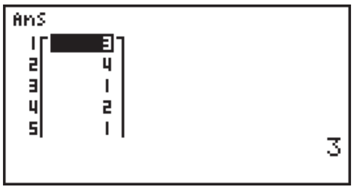


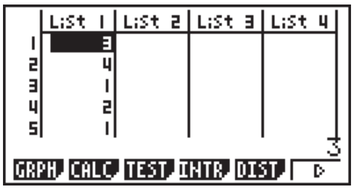


Simulating binomial distributions

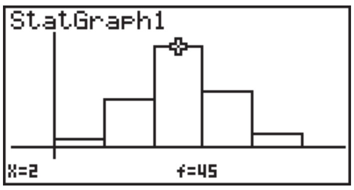
If data are simulated according to a binomial distribution, the results will not match the theoretical probabilities. To see this, you can use a random command in the OPTN menu, noted earlier. The binomial random command Bin (F4) shown at left below generates a list of 100 observations from a binomial distribution with n = 4 and p = 0.5, representing the number of boys in 100 separate families and stores it into List 1 for statistical analysis. The middle screen shows the first few simulations and the third screen, in Statistics mode, shows the transferred data.

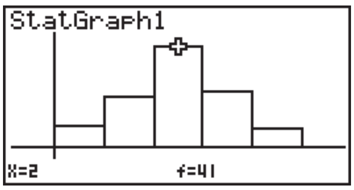


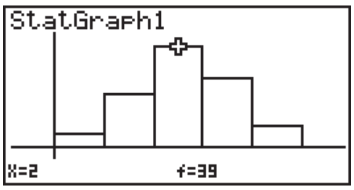




A histogram of the results shows that the actual distribution is similar to, but not the same as the theoretical distribution seen earlier. The other two histograms show the effect of repeating the simulation of 100 families a further two times, with similar, but slightly different results each time.







Exercises

The main purpose of the exercises is to help you to develop your calculator skills.

1 a Which calculator command will allow you to simulate rolling an 8-sided die in Run-Mat mode?

b Test your answer to part (a) by simulating a number of tosses. Check to make sure that all of the results are in the set {1,2,3,4,5,6,7,8}

c Simulate rolling a pair of 8-sided dice. Check that your command only produces scores in the set {2,3,4,5, ... ,15,16}

2 Days of the week are to be chosen at random from a calendar, and checked to see whether or not they are weekdays (5 out of 7) or weekend days (2 out of 7). What command will allow you to simulate an event with probability 2/7?

3 Anna used a calculator to simulate events using the command Int(Ran#+0.05) with a result of 1 signifying success. Was the event being simulated very common or was it very rare?

4 a Use *Prob Sim* to simulate rolling a pair of standard fair six-sided dice 500 times.

b Make a histogram of your results on the calculator in Statistics mode.

c What is the most likely result for the sum of the two dice?

5 a Use *Prob Sim* to simulate a spinner with four sections, three of which are the same size and the fourth section is twice as large as each of the others.

b What results are obtained when the spinner is spun ten times?

6 a Use *Prob Sim* to draw sets of 5 cards without replacement from a standard deck of 52 cards. Tap CLEAR (F4) after each set.

b About how often do you get at least a pair (two cards the same value, but different suits)?

c What would be the main effect of choosing a set of 13 cards with replacement, tapping CLEAR (F4) after each set?

7 Use the OPTN menu of the calculator to find the following standard normal probabilities:
a Prob ($z \leq 1.3$) b Prob ($-2 \leq z \leq 1$) c Prob ($z \geq 1.2$)

8 The lengths of leaves of a certain large bush are normally distributed with mean 8 cm and standard deviation 2 cm. What is the probability that a randomly chosen leaf will be more than 5 cm and less than 10 cm in length?

9 For a standard normal distribution, what z value is exceeded 8% of the time?

10 A standard fair coin is tossed 16 times.

a What is the probability of obtaining exactly 7 heads?

b What is the probability of obtaining no more than 8 heads?

11 Simulate 100 values from a binomial distribution with $n = 20$ and $p = 0.2$, storing the results in List 2 of your calculator for analysis. Check that the mean of the list is close to 4.

Activities

The main purpose of the activities is to help you to use your calculator to learn mathematics. You may find that some of them are too advanced for you. Ignore activities you don't yet understand.

1 Some children's games require that you roll a six on a die before you can start. Simulate a dice roll on your calculator using *Dice Roll*. How many times does it take you to get a six? Repeat this activity a few times, and record the results into a list so that you can analyse them.

Compare your results with a partner. What is the largest number of rolls needed to get a six? What is the smallest number? What is a *typical* number?

2 Another game requires that you roll a die until you get at least one *each* of the numbers from 1 to 6. How many rolls do you think this is likely to take? (Write down your guess before you start.)

Now simulate rolling the die, until you have the required result. Record how many rolls it takes. Compare your results with a partner.

What is the largest number of rolls needed to get all six? What is the smallest number? What is a typical number?

3 What are the possible numbers of heads when you toss three fair coins?

Use *Coin Toss* to simulate some tosses of the three coins.

What happens as you conduct an increasing number of simulations? Compare your experiences with a partner.

4 A certain procedure is described as "95% safe", meaning that it “works” 95% of the time. Such a description might be used for various procedures, such as a mechanical device, a vaccination, a medical operation or a form of contraception. Do you regard 95% as “very safe”?

Use the *Coin Toss* to simulate a procedure with this level of safety.

See how long it takes until you get a failure. Does the result surprise you? Compare your results with those of other people.

Then investigate procedures that are 99% safe.

5 Zoë is a goal shooter in a netball team. She has found in the past that she scores goals from about 70% of her shots during netball games.

If a typical game includes 50 shots on goal, how many goals do you expect her to score?

Simulate some suitable data for this binomial situation and compare it with the theoretical expectations. Compare your results with those of other people.

6 The eventual heights of a species of bean plant are known to be normally distributed with a mean of 180 cm and a standard deviation of 15 cm.

A nursery guaranteed customers that the bean plants they sold will be at least 160 cm high or customers’ money will be refunded. What percentage of plant sales will be refunded?

Simulate some sets of 100 plants of this kind and compare the simulated results with the theoretical expectations.

Compare your answers with others.

Notes for teachers

This module illustrates several ways in which the calculator can be used to explore various aspects of probability, especially the use of simulation, in order to understand the nature of randomness and theoretical probability. Make sure that students have the *Probability Simulation* Add-in available to them. The text of the module is intended to be read by students and will help them to start to see how the calculator can be used efficiently for various kinds of explorations. The Activities are appropriate for students to complete with a partner or in a small group, so that they can discuss their observations and see the range of things that can happen at random, and the important distinction between theoretical probability and actual data. Some of the material on normal and binomial distributions is more suitable for older students and can be safely omitted for younger students..

Answers to Exercises

1. (a) Int(8Ran# + 1)   (c) Use Int(8Ran# + 1) + Int(8Ran# + 1)   2. Int(Ran# + 2÷7)   3. Very rare, as the probability is only 0.05. Most trials will give 0.   4. (a) Use *Dice Roll*, Attempts = 500 with 2 dice and STO results in a list   (c) The most likely sum is 7, which will probably be clear in the histogram in Statistics mode or the Graph in *Dice Roll*   5. (a) Use *Spinner* with 4 sections and use ADV to give probabilities of 0.2, 0.2, 0.2 and 0.4 or weights of 1, 1, 1 and 2.   (b) Results will vary, but the fourth section is likely to be hit more often than the other three   6. (a) Use *Card Draw* with Attempts = 5   (b) About 40% of the time   (c) Usually at least one card will be repeated
7. (a) 0.903   (b) 0.819   (c) 0.115   8. Prob ($-1.5 < z < 1$) = 0.775   9. 1.405   10. (a) 0.175
(b) 0.598   11. Use the command RanBin#(20,0.2,100) ➡ List 2 and then analyse List 2 in Statistics mode. The mean will be close to, but not exactly, 4.

Activities

1. .Activities of this kind are easily conducted in *Prob Sim*, and give students a chance to see that random results become more predictable if more information is gathered. They will need to record results on paper systematically and may need help doing so. While the largest number is theoretically infinite, it is rare that more than about 12 rolls will be needed and about half the time only 4 or 5 will be needed. Organise the class to compare results to get a better picture.

2. Students can use Dice Roll to simulate the rolls, one attempt at a time, but will need to record results separately. Many people grossly overestimate how long this will take, although the average is around 14 or 15 rolls. A whole class activity is appropriate, to help students see how a random event can be simulated on the calculator, although formal analysis is more difficult.

3. This is an important kind of activity for students to see the consistency that results from an increasing number of observations. A *Refresh Frequency* setting here of 50 with a large number of attempts will demonstrate this well.

4. Help students use the ADV menu to set probabilities of 0.95 and 0.05 here to facilitate simulations. Most will be surprised at the results and this is a good activity for the whole class to do at once.

5. Provided they are sufficiently sophisticated, help students to use the Binomial distribution commands in Statistics mode here. The simulations in Run-Mat mode will vary, of course, so it's a good idea to have students compare results. [Answer: 35 goals.]

6. Provided they are sufficiently sophisticated, help students to use the normal probability commands in and generate simulations in the OPTN menu. Again, arrange students to compare their simulations tohelp appreciate differences between theory and practice.
[Answer: Prob ($z \leq -1.333$) = 9.1%]

Module 8
Advanced data analysis

When dealing with data, a *population* consists of all the values of interest, while a *sample* comprises only some of them. In practice, it is not normally possible to study a population of data, so we are forced to rely on samples. A key element of statistics involves using a *sample* of information in order to make *inferences* about a population. An inference can be thought of as an informed guess. Studying the various processes of statistical inference and undertaking advanced data analysis does not usually happen until you are confident with some elements of descriptive statistics, to summarise, represent and interpret data, some aspects of which were dealt with in Module 4, and probability, provided in Module 7.

An example of a population is the set of voting preferences of electors at the next national election. All of these are tallied on election night, a long and very expensive exercise. Often, people want to gauge the likely popularity of a particular candidate or political party before then, however, so a sample of voting preferences might be obtained for this purpose. The interest is not just in the sample, but also in what it will allow us to conclude about the population.

As a second example, consider how long the battery for your mobile phone might last. The population of the duration of all battery lives for your phone is not accessible: after all, you could keep testing batteries essentially for ever. Instead, you might test a smaller number of batteries and use the sample of battery lives to infer things about the population (of all battery lives).

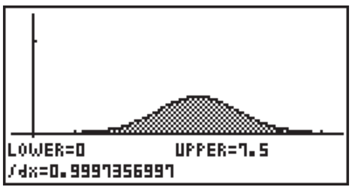
There are many ways of taking a sample from a population. For example, with the election results in mind, you might ask the next fifty people you meet how they intend to vote. But such a haphazard sample is unlikely to be a reliable source of information. In fact, the only samples of importance to statistics are *random samples* which are chosen in such a way that all possible samples have the same chance of being chosen. Putting names of all electors in a hat, shuffling them carefully and then choosing fifty of them is one (clumsy!) way of selecting a random sample in this case. In this module, we assume that the data being analysed arose from a random sample.

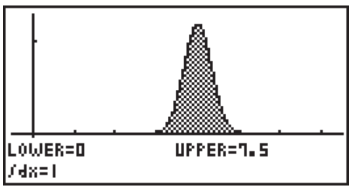
Sampling distributions

An important theoretical idea here is that of a *sampling distribution* of a statistic. We illustrate this with the idea of a very important statistic, the mean. If random samples are taken repeatedly and independently from a particular population, the mean of the sample will almost certainly be different each time. Remarkably, however, the means of samples obtained in this way will be approximately normally distributed, with a mean which is the same as the population mean and a variance which is related to the population variance and the sample size:

Population:	Mean μ	Variance σ^2
Sample means:	Mean μ	Variance $\dfrac{\sigma^2}{n}$

To illustrate this idea, the two screens below show a *population* (on the left) and the theoretical *sampling distribution of means* of samples of size 9 taken from that population (on the right). The two graphs have the same scale, making it clear that the distributions have the same mean (4) and the area under each curve is essentially the same (1, although it is approximated in the first graph).





Confidence intervals for a mean

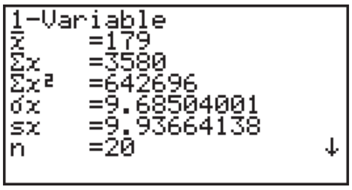
As noted earlier, samples are used in practice to infer information about a population. Provided a sample is taken at random, statistics offers some powerful procedures to allow us to state results with some precision. One of these concerns the idea of a confidence interval. You will be able to use the calculator directly to construct and use these.

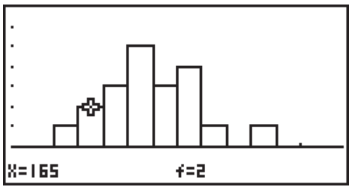
We will illustrate the ideas and procedures with an example. Suppose a plant nursery was concerned with how high a particular bush will grow from the seedlings that it sells, so that they can advise their customers accordingly. They selected at random a sample of 20 seedlings and grew them carefully according to the directions and under controlled conditions. They measured the final heights of the bushes (in centimetres) and obtained the following results:

163, 165, 166, 170, 171, 174, 175, 175, 175, 177, 178, 180, 182, 184, 185, 188, 189, 189, 193, 201

How might this information be used to inform their customers?

Enter the data into *List 1* of your calculator. One possibility is to consider the mean and standard deviation of the heights, as shown below.





The sample mean is 179 cm, and the histogram shows that there was a spread of bush heights. (The width of the histogram intervals is 5 cm). The standard deviation of the bush heights was about 9.69 cm. What information might these convey about the *population* of bush heights (grown under ideal conditions)? The population here comprises the bush heights to which *all* their seedlings are likely to grow, not just the 20 in their sample. It is clear that the seedlings will grow to different heights (not all the same height), so our interest is in the mean height of the bushes, which is probably described to customers as the *average* height.

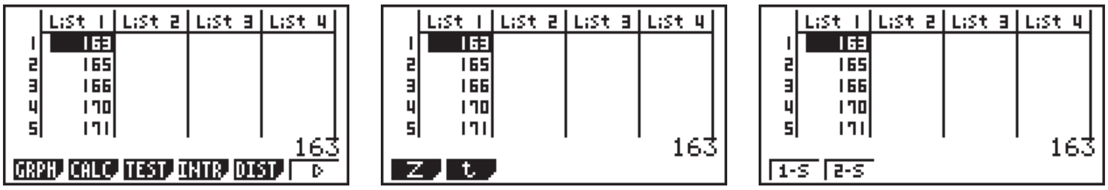
Remarkably, when samples are chosen at random, statistical theory tells us about the *distribution of sample means*, noted in the previous section. There is not space in this module to describe the details, but a statistician will be able to make a prediction for the population from just one sample, and also suggest how likely the prediction is to be correct!

The distribution is more varied (and thus less accurate) for a smaller sample than a larger sample. But here we have only *one* sample mean, since the nursery has been able to obtain only one sample (of size 20). Presumably, if they were to take another sample, the results would not be the same.

One way of making an inference about the population is to describe a *confidence interval*, suggesting that the mean *of the population* is likely to be within a certain interval, based on the sample information. The calculator has been programmed in advance to allow you to obtain a confidence interval, provided you know enough about the data concerned.

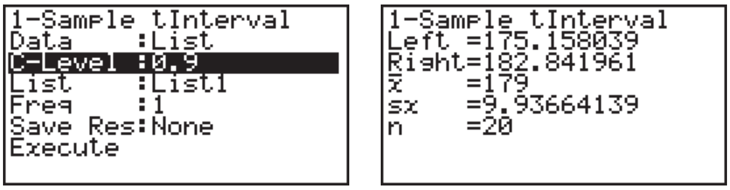
In this case, the single best estimate for the population mean is the sample mean of 179 cm. But, when random sampling is involved, another sample of size 20 is likely to produce a different sample mean, so having only a single value of the mean is of limited use.

The critical idea here is that the sample has been chosen at random and that we don't know anything about the population except the sample information. The first of the following calculator screens shows that INTR is one of the menu choices.



Tap INTR (F4) to select the confidence interval commands to see that there are two kinds of confidence interval available (*z* and *t*). To construct a confidence interval in this case, statistical theory tells us to use the *t*-distribution, since we do not know the population standard deviation. (It is very rare to know the population standard deviation, in fact, so you will almost always use *t* instead of *z*. Choose *z* only if you *do* know the population standard deviation.)

The final choice involves whether you have one sample (1-S) or two samples (2-S); in this case, clearly there is only one sample, so tap 1-S (F1) to see the screen below.



Set up your calculator with the choices shown here. The screen records that the data are in the form of a list and they are located in *List 1*. The confidence level has been entered as 0.9 to give a 90% confidence interval for the mean. Then move the cursor to the *Execute* command and choose CALC (F1) to construct the confidence interval, as shown above on the right.

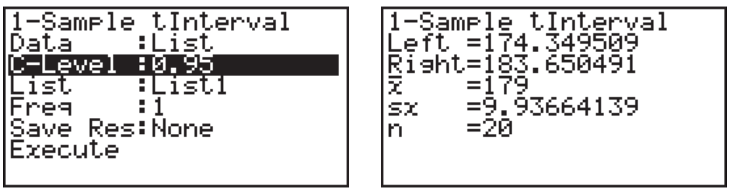
The 90% confidence interval is described in the screen above as (175.16, 182.84). This is suggesting that the *population* mean is between the two values of 175.16 cm and 182.84 cm.

It is called a 90% confidence interval because statistical theory tells us that if we were to repeat this entire process many times, 90% of the intervals constructed in this way would actually include the population mean (and, of course, 10% of the intervals would not include the population mean).

We do not know what the population mean is, and it may or may not be within the confidence interval (175.16, 182.84), but the theory and the calculator together have given us more information than we had from just the sample.

The choice of a 90% confidence interval is quite arbitrary. You can choose any value you wish between 0% and 100%. A higher value is sensible, since of course the process is then more likely to capture the population mean, but will result in a larger interval.

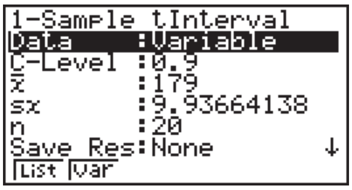
For comparison, a 95% confidence interval is shown below.

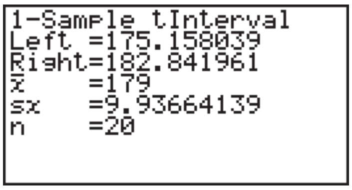


Notice that the interval this time is (174.35, 183.65), a bit wider than before, with a process that is a bit more likely to capture the population mean.

Incidentally, you might also notice that the screens above show the standard deviation calculated from the random sample as 9.9366; this is the so-called *unbiased* standard deviation (reported also above in the original sample statistics). It is the single best estimate of the population standard deviation. The sample standard deviation of 9.69 is sometimes called the *biased* standard deviation, as it is not the best available estimate of the population standard deviation, as it a bit too small.

In fact, the only information needed to construct a confidence interval about a population mean in this way is the mean, (unbiased) standard deviation and size of the sample. So, if the actual data are not available, you can use the calculator to construct the confidence interval by entering these values directly. The screens below show this for the case of the 90% confidence interval here, when *Variable* is selected rather than *List*.





Of course, the results for the confidence interval are the same.

Hypothesis testing

Another way of using a sample to make inferences about a population involves testing an *hypothesi*. An hypothesis is like a careful guess, that is specially made so that its likelihood of being correct can be investigated.

In the case of the seedlings described above, suppose that the nursery staff had been informed by the original seed suppliers that the bushes would grow on average to 182 cm in height. The sample data can be used to test whether this seems to be a reasonable description of the seedlings.

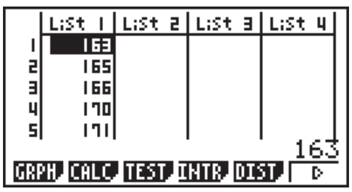
Our hypothesis in this case is that the mean height to which the bushes grow is 182 cm. Our interest is in whether or not this seems reasonable; we can test this hypothesis by taking our sample data. The hypothesis we are testing is called the *null hypothesis*, represented by the symbol H_0. We have seen already that the mean of a sample of data is not necessarily the same as the population mean, so we are not surprised that it is not *exactly* 182 cm. The statistical question is whether it is reasonable to get a sample like the one above from such a population, or, alternatively, whether the population mean seems *not* to be 182 cm. Our hypothesis test will result in either rejecting the null hypothesis (in favour of its alternative) or in not rejecting it (i.e., accepting that it does not seem unreasonable).

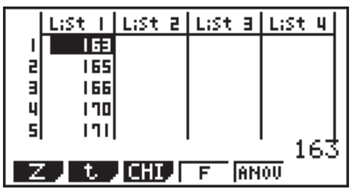
Symbolically, our null hypothesis is: H_0: $\mu = 182$.

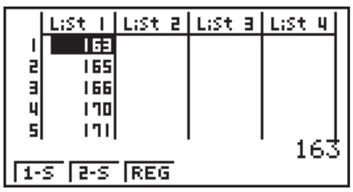
Our alternative hypothesis is H_1: $\mu \neq 182$.

Again, further details of the statistical theory are omitted here, assuming you will study them elsewhere. We consider only how the calculator performs the necessary test.

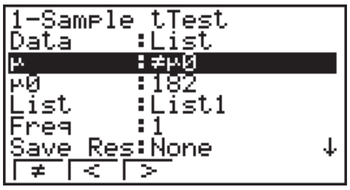
In the window shown below, note that one of the menu items is TEST (F3) Selecting this menu reveals that several statistical tests are available, as shown. In this case, we choose the *t*-tests and, specifically, the one-sample *t*-test, similar to our choices above for confidence intervals. (The choice of *t* instead of *z* is because we do not have assumed information about the population standard deviation.)



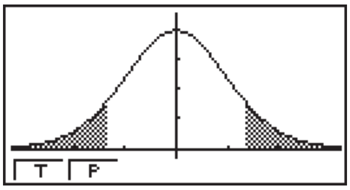


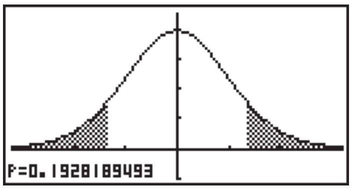


The next screen shows how the test is communicated to the calculator, using 1-S (F1). The null hypothesis value for the population mean is 182 (cm) and the data are in *List 1*. The screen below also shows that there is are one-tailed tests available as well as the two-tailed test used here. (The details of these one-tailed tests are left to you, after you have learned the necessary theory.)

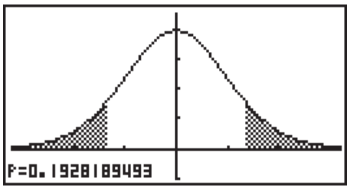


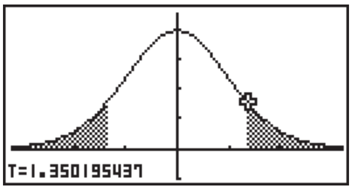
When the test is executed, you will have a choice of a calculation or a drawing (or both) to show the details of the test. The drawings below give the shape of the *t*-distribution, in this case very similar to that of a normal distribution. Selecting P (F2) shows the probability associated with the sample mean, assuming the hypothesised mean of 182 is correct.





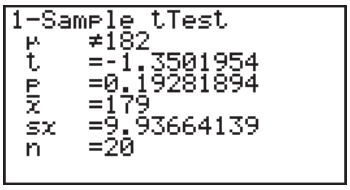
Selecting T (F1) shows the associated *t*-values instead of the probability. Use ▶ and ◀ to move from one to the other.





The sampling distribution graphshows *t*-statistics associated with samples of size 20 (for which the number of degrees of freedom are 20 – 1 = 19) *when the null hypothesis is true.* The mean of the theoretical distribution is zero, but non-zero values are likely in practice, due to random variation. In the present case, the actual sample mean (of 179 cm) is 3 cm away from the hypothesised mean (of 182 cm); the shaded parts of the graph indicate that values this far away from the mean (in either direction) *or even further* can occur at random 0.19281 or 19.3% of the time.

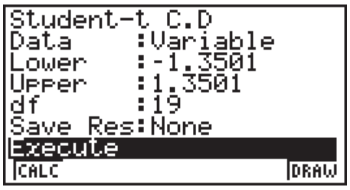
If you choose the CALC (F1) option instead of DRAW (F6), the *t*-statistics are shown. These are the same as those on the graphs, of course. It is usually a good idea to look at the graph as well as the statistics, to help interpret the results.

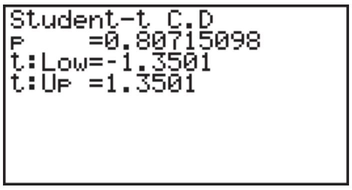


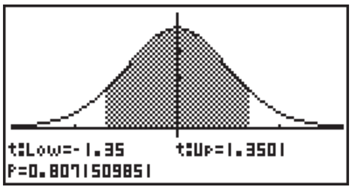
Many people would say that something that happens around 19% of the time is not especially unusual, and is not strong enough evidence to reject the null hypothesis. If the percentage were quite small (say 5% or less), we may be more inclined to reject the null hypothesis as an unlikely description of reality and instead accept its alternative. But this is not the case here. So the result of our test is to *accept* the null hypothesis, μ = 182 cm (i.e., there is insufficient evidence to reject the null hypothesis). So the claim of the seed suppliers seems reasonable, given these sample data.

The numerical information shown above should be interpreted in a similar way. The *t*-statistic of -1.3501 provides the means of conducting the test. The number $p = 0.19281$ expresses the probability of getting a value this far from the mean *t*-value (which is zero) when the null hypothesis is true. This number is sometimes referred to as a *p*-value.

You can also evaluate the probability of *t*-statistics between ±1.3501 using the *t*-distribution directly. Return to the data screen and choose DIST (F5) and then t (F2). The screens below show that 80.72% of the time, the *t*-values with 19 degrees of freedom are between the two values, so that 19.18% of the time they are outside this range, consistent with the above calculations.

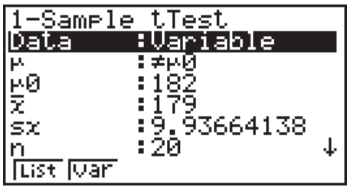


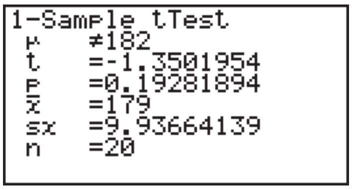




It is common practice to decide in advance of looking at any data what will be regarded as too unusual to accept the null hypothesis and thus will be grounds for rejecting the null hypothesis. A common choice is $p \leq 5\%$. Another common choice is $p \leq 1\%$. When the null hypothesis is rejected on the grounds that $p < 5\%$, we say that the result is *statistically significant* (or just *significant*) at the 5% level. One meaning of this is that there is still a 5% chance that the null hypothesis is actually true and we merely have an unusually deviant sample from the population.

The hypothesis test described here has been conducted using the actual data in a list. As for confidence intervals, you can also use the calculator to conduct the test based on the summary information from the random sample. You need to know only the sample size, the mean and the unbiased standard deviation (which are commonly reported in statistical research reports, while the actual data are frequently not available). The screen at left below shows how the information is entered into the calculator, after first selecting *Variable* rather than *List* as the data source.





The results are the same as before, as expected. A graphical representation is also available.

Comparing two populations

Another common form of inference involves comparing two populations using a sample from each of the two populations to do so. The basic ideas are similar to those for a single population.

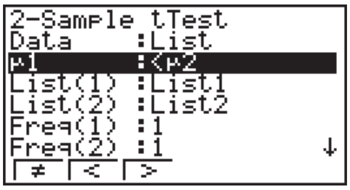
To illustrate, suppose that the nursery mentioned above recorded heights on a second random sample of the same sort of seedlings, but grew the twenty bushes in a location near the ocean, where there was more wind and a slightly salty atmosphere. They expected that the bushes, native plants originally from a coastal environment, may grow better when close to the salt air. The heights of the second sample are shown below:

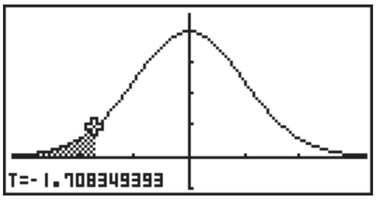
167, 169, 171, 176, 177, 179, 180, 180, 181, 182, 185, 185, 188, 192, 192, 193, 194, 195, 197, 205

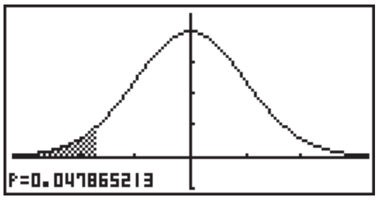
This time, the relevant null hypothesis (ie, the hypothesis of 'no difference') is that the two samples have been drawn from populations with the same mean ($\mu_1 = \mu_2$).

The relevant alternative hypothesis is that the oceanic population may be *higher*, a one-tailed test (ie $\mu_1 < \mu_2$). Our test will accept this hypothesis if we reject the null hypothesis.

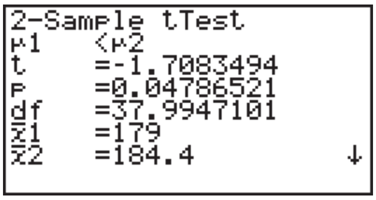
We will decide in advance that we will regard anything that happens less than 5% of the time as 'significant' and reject the null hypothesis accordingly. Enter the data into the calculator in *List 2* and choose a two-sample *t*-testas shown below.

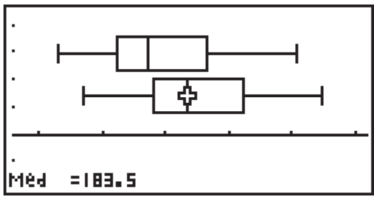






This time, the hypothesis of no difference seems unlikely in view of the data, which suggest that a *t*-value this far from zero occurs less than 5% of the time, our previously decided limit. So we choose to *reject* the null hypothesis in favour of its alternative, and conclude that the seedlings grow into taller bushes when grown near the ocean. As for the one-sample test, the statistical calculations are available by selecting CALC (F1) instead of DRAW (F6), as shown below:





It is usually wise to evaluate data from several perspectives. In this case, the two-sample 90% confidence interval for the difference between the population means (first – second) suggests that the difference is likely to be negative. The resulting interval of (-10.73,-0.07) does not include zero. Similarly, the pair of box plots above suggests that the second sample has been drawn from a population with mostly larger heights.

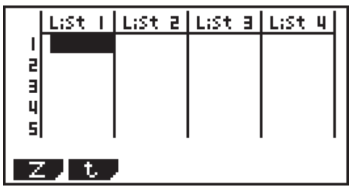
Finally, notice that if we had decided that only outcomes that happened less than 1% of the time would be regarded as significant, then we would have *accepted* the null hypothesis in this case.

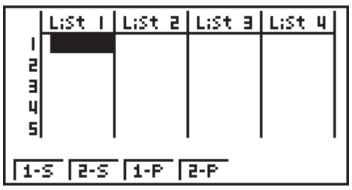
Making inferences about proportions

When data comprise proportions, different tests are used and different confidence intervals are constructed, but the essential ideas remain the same. A common example of this involves political opinion polls in advance of elections, when a random sample of voters is asked to indicate their support for a particular candidate or political party.

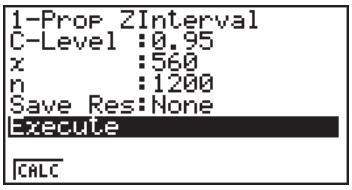
For example, suppose a random sample of 1200 people is asked whether they intend to vote for the present government at the next election. If 560 people in the *sample* indicate that they *do* intend to vote for the government, what can we infer about the *population* of voters?

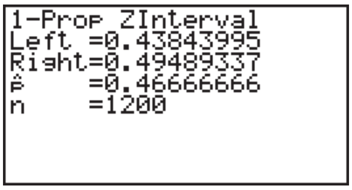
In Statistics mode, select INTR (F4) and then Z (F1) to see that confidence intervals are available for proportions with one sample (as in this case) and for two samples (where the intention is to compare proportions in two populations).





In this case, our interest is in a single proportion, so select 1-P (F3). The next screen shows the settings to construct a 95% confidence interval for the population proportion in favour of the government. Tap CALC (F1) to obtain the confidence interval, shown at right below.



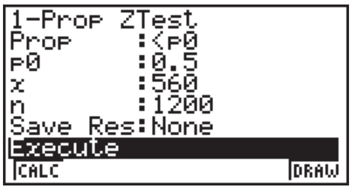


Notice that the 95% confidence interval is from about 44% to 49%, which is suggesting that the government will not attract the required 50% of voter support to remain in office. (Since the confidence interval does not contain the figure of 0.50 within it). Even though the sample seems to be quite small (only 1200 voters out of a possible voter population of millions), the process allows for useful predictions regarding the state of the election.

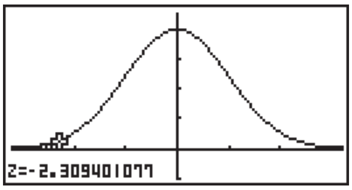
Again, remember the meaning of a 95% confidence interval: *if you continued to take random samples of this size, 95% of the time the confidence interval would actually contain the population proportion (so 5% of the time it would not).*

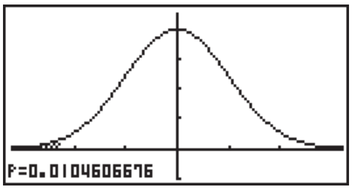
This accounts for the remarkable success enjoyed by opinion polling prior to elections, provided samples are taken appropriately, even though many people still think that a relatively small sample is inadequate to describe a much larger population.

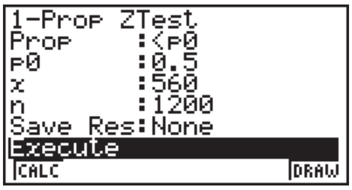
In a similar way, an hypothesis test can be conducted regarding proportions. In this case, a suitable null hypothesis is that the population proportion is 50%, while the alternative hypothesis is that it is *less* than 50% (i.e., a one-tailed test):



To set up the calculator to conduct this test, select TEST (F3) and Z (F1) to access the appropriate test, which is a one-sample test of a proportion, 1-P (F3). The data entry is shown in the previous screen. The results for the test, both graphically and numerically are shown below:







As you can see from the normal distribution data here, it is very unlikely that a sample of 1200 would contain such a low proportion (46.7% or even less) of support for the government if in fact the null hypothesis were true and 50% of the voters were in favour of them.

Although it is not impossible for such a low proportion to occur at random, the analysis shows that it will only happen about 1% of the time ($p = 0.01046$), and a better conclusion is that the null hypothesis is false. So we *reject* the null hypothesis and *accept* its alternative, concluding that less than 50% of the voters will vote for the government.

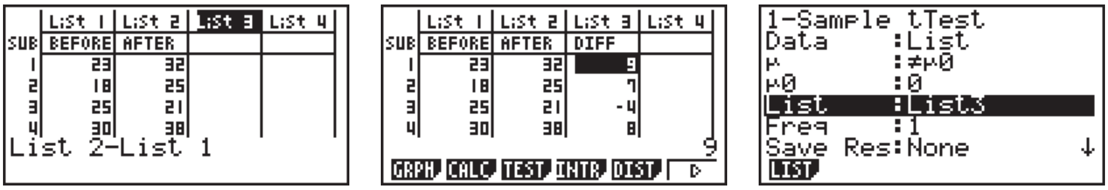
Paired data

A special case of looking at means of samples involves paired data, where there is a connection between a pair of elements, one from each sample. For example, the data may involve repeated measures (where the same thing is measured under two different conditions) or the data may have included some form of matching, where two elements are paired because they are the same in many respects. A good example of this involves a controlled experiment, in which there is a control group and a treatment group. Another example is when something is measured on two occasions in time. When data are paired in this way, the groups are described as *dependent*, unlike the earlier situation with seedlings, where they were *independent* samples.

Consider for example a psychological experiment in which ten children were chosen at random from a large school and given a questionnaire to determine their attitude to violence. A few days later, they were shown a series of violent cartoons and given another questionnaire of the same kind to complete. The researchers' interest is in whether or not the showing of cartoons affects children's attitudes to violence. Some people have speculated that violent cartoons will encourage violence, others have argued that they may even have an opposite effect, while still others suggest that they will not make any difference. The data below show the children's paired attitude scores on a scale from 0 to 100 before and after the video was shown:

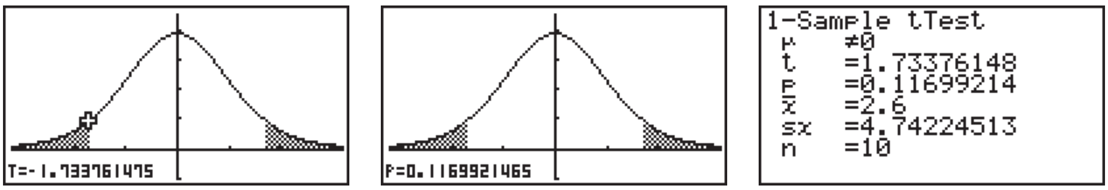
Child	**A**	**B**	**C**	**D**	**E**	**F**	**G**	**H**	**I**	**J**
Attitude before	23	18	25	30	12	19	24	7	21	16
Attitude after	32	25	21	38	14	18	27	6	18	22

Enter the paired data into the calculator in *List 1* and *List 2*, keeping the paired scores together. Because the data are paired, it is not appropriate to test whether the means of the two samples are different using a 2-sample *t*-test. Instead, we look at the *differences* in attitude scores for each person. Transform the data to produce these differences as List 3 = List 2 – List 1, shown below.



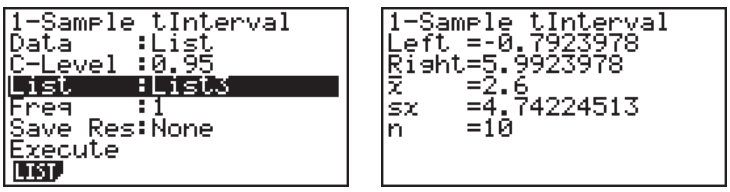
Now we will conduct a *t*-test for a single sample, with the difference data in *List 3* being the sample. Our null hypothesis (of no difference) is that the population mean from which *List 3* data have been randomly drawn is zero; in other words, there is no difference between the attitudes on the two occasions. The alternative hypothesis is that the population mean is *not* zero.

We will decide in advance that we will describe as statistically significant a result that is likely to occur 5% of the time or less. The test results are shown below:



In this case, while the *t*-value may seem large, it is not large enough to reach the 5% criterion that we set, so we conclude that there is insufficient evidence here to suggest that the null hypothesis is false: the cartoons do not have a statistically significant effect on children's attitudes to violence.

A suitable 95% confidence interval for the difference between the means is consistent with this conclusion, as shown below. Notice that the confidence interval (-0.79, 5.99) *includes* zero, suggesting that the data are consistent with the possibility that there is no difference between the children's attitudes before and after the cartoons.



Exercises

The main purpose of the exercises is to help you to develop your calculator skills.

1. When an unbiased coin is tossed, it should result in heads half the time in the long run. Clare believed that she had an unbiased coin. She tossed it 2000 times and obtained heads 1050 times. Is this evidence that the coin is unbiased? (Assume that 2000 tosses is sufficient.)

2. A random sample of size 12 has a mean of 25 and unbiased standard deviation of 6. Find a 90% confidence interval for the mean of the population from which the sample was drawn.

3. A survey on a random sample of 20 Grade 10 students asked them to record to the nearest half hour how many out of class hours per week they spent on a computer, laptop or smartphone. Enter these responses shown below into List 1 of your calculator.

   27, 30, 12.5, 20, 5, 16, 21, 26, 15.5, 19, 18, 12, 25, 19, 13.5, 16, 15, 16, 19, 21.5

   (i) Find the 90% confidence interval for the mean time the students spent in these ways;
   (ii) Find the 95% confidence interval for the mean time the students spent in these ways.

4. A similar survey to that in Exercise 3 was undertaken with a random sample of Grade 11 students. Enter these responses shown below into List 2 of your calculator.

   10, 9.5, 20, 25, 32, 36, 25, 20.5, 15, 22.5, 18, 19, 20, 19.5, 21.5, 22, 27, 32, 31.5, 20

   (i) Find the 90% confidence interval for the mean time the students spent in these ways;
   (ii) Find the 95% confidence interval for the mean time the students spent in these ways.

5. For each of the groups of students in Exercises 3 and 4, construct box plots on the same screen. For each group, give the five values (minimum, quartiles and maximum). Compare the two box plots, and comment on what you notice.

6. Teachers at the school expected that Grade 10 students would spend an average of 20 hours per week on these activities. Use the data in Exercise 3 to test their hypothesis at the 5% level with a single-sample t-test, and report both the t-value and the associated probability. What conclusion do you draw about the teachers' hypothesis?

7 Students at the school had predicted that the Grade 11 students would spend more time on these activities out of school than the Grade 10 students. Use the data in Exercises 3 and 4 to test this hypothesis, using a suitable two-sample t-test at the 5% level. What conclusion do you reach?

8 An agricultural research report describes a random sample of 25 wheat stalks with a mean of 64 cm and an unbiased standard deviation of 12 cm. Test the hypothesis that the population mean height is 60 cm against the alternative that it is not 60 cm.

  Would you accept or reject this hypothesis at the 5% level of significance?

9 Prior to a referendum, a random sample of 800 voters were asked their opinions; 450 claimed to be in support of the proposal. Find the 95% confidence interval for the proportion of the population in favour of the referendum proposal.

10. Two groups of adults, one older than 60 (> 60) and the other aged 60 or younger (≤ 60) were surveyed. The survey asked them whether they preferred comedy or documentary TV shows. Of the 500 people over 60, 260 preferred comedy. Of the 500 people 60 or younger, 233 preferred comedy. Use a z-test at the 5% level to check whether the two proportions are significantly different.

Activities

The main purpose of the activities is to help you to use your calculator to learn mathematics. You may find that some of them are too advanced for you. Ignore activities you don't yet understand.

1 Interview some students chosen at random from your class. Ask them which kinds of lessons they like best out of maths, science, etc. or arts, music, etc. Use their responses to make two groups.

Also ask them also how many hours per week they spend on the Internet out of class with a computer, tablet or smartphone.

Use the information to compare the two groups.

2 Use an official Internet site or another credible source to find the average (mean) heights of students of your age in your country. (Consider boys and girls separately if both genders are in your school.)

Obtain the heights of a sample of your classmates.

Use the data to construct suitable confidence intervals and to test whether your classmates can reasonably be regarded as a random sample from the respective national populations.

3 Construct a small data set of about ten numbers between 20 and 30.

Use your calculator to explore how changing only one of the numbers can lead to changes in the result of a one-sample *t*-test. Discuss the implications of this for practical work with data.

4. Generate two separate data sets, either by using the random integer capabilities of the calculator or by putting whole numbers between 1 and 50 on slips of paper into a container and then selecting them at random (with replacement).

Conduct an appropriate significance test to determine if the two sets are significantly different from each other.

After your test is concluded, select some numbers in one set to change so that the result of the significance test is reversed. Explore how changing the numbers affects the results.

5. Consider changes over time for individuals by choosing a topic that interests you. For example, you might choose to look at improvements in a sport, such as long jump, or in academic achievement. You might expect individuals to improve in these areas after a period of training or learning.

Invent some data for 10 imaginary students so that (i) there is no significant change or (ii) there is a significant change after training or learning for a number of weeks.

Compare your results with those of others.

6. A market research company wants to choose a random sample of voters to predict the results of a forthcoming state election, known to be quite close.

Experiment with confidence intervals on your calculator to work out the sample size they will need to take to be 90% confident of being close to the actual result.

Notes for teachers

This module illustrates some of the ways in which the calculator can be used to undertake advanced data analysis, involving inferential statistics. It is assumed that readers are already familiar with Module 4 on Data analysis and Module 7 on Probability. The text of the module is intended to be read by students and will help them to start to see how the calculator can be used efficiently to undertake some inferential statistics and to understand the ideas involved. The Activities are appropriate for students to complete with a partner or in a small group, so that they can discuss their observations. Much of this module is more suitable for older students and can be safely omitted for younger students.

Answers to Exercises

1. Do not reject the null hypothesis. 2. (21.889, 28.111), using t 3. (i) (16.095, 20.605) 3. (ii) (15.620, 21.080) Both intervals using t 4. (i) (19.039, 25.561) (ii) (19.606, 24.994) 5. Grade 10: Min = 5, Q_1 = 15.25, Mdn = 18.5, Q_3 = 21.25, Max = 30. Grade 11: min = 9.5, Q_1 = 19.25, Mdn = 21, Q_3 = 26, Max = 36 5. The box plots overlap with Grade 10 median ≈ Q_3 for Grade 11. Outlier of 5 for Grade 10, but no outlier for Grade 11. Grade 11 has a larger range and higher maximum value. 6. H_0: $\mu = 20$, H_1: $\mu \neq 20$. t = -1.265, p = 0.221, mean = 18.35. Do not reject the null hypothesis at 5% level. 7. H_0: $\mu_{10} = \mu_{11}$, H_1: $\mu_{11} > \mu_{10}$. t = 1.944, p = 0.0297. Reject the null hypothesis at the 5% level. Conclude that Grade 11 students spend more time out of class on the Internet than Grade 10 students. 8. t = 1.667 and p = 0.109 so the hypothesis should be accepted at the 5% level. 9. (52.81%, 59.69%), using z 10. z = 1.708, p = 0.044. Reject the null hypothesis at the 5% level and conclude that more people aged over 60 prefer comedy.

Activities

1. The intention of this activity is to encourage students to use some real data to compare two groups on some attribute of interest. You may prefer to choose a different context that suits your class, however we suggest that you resist providing students with the data, as they are potentially more interested in data they have collected themselves. Help them to appreciate the need for cautious interpretation of self-report data (in case students are likely to report exaggerated data).

2. You may need to advise students in advance of this activity to allow them some time for purposeful searching for national data on the Internet. (Alternatively, you might seek the information from local personnel such as a school nurse, physical education teachers or local health officials.). Encourage students to work together to measure heights with care and encourage them to take second measurements to check their data before processing it.

3. This activity will highlight for students how even small changes in a small data set can contribute to substantial differences in results, and help them to appreciate the need for care in data collection.

4. Students might need help in constructing the data sets for this activity. Again, as for Activity 3, the activity provides them with an opportunity to explore the consequences of changing data.

5. In this activity, students need to consider what kinds of values are sensible and realistic, and then compare them appropriately. It would be appropriate for such work to be conducted in groups. Help students appreciate the need for a suitable paired test, if necessary, and encourage them to also consider comparisons of mean improvement. As for Activity 1, you may prefer students to obtain some actual data for this activity, rather than inventing data.

6. Students can readily explore questions of this kind by changing values for x and n for a 1-proportion confidence interval using z. They may be surprised that a value of n as low as about 1600 will yield results within about 2% of the actual result.

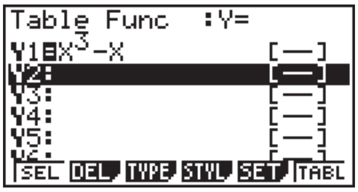
Module 9
Equations

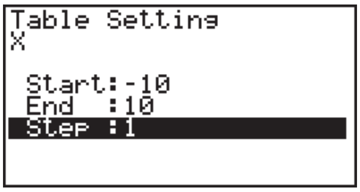
An equation is a statement of equality involving variables, such as $x^3 - x = 3$. To *solve* the equation means that you need to determine whether there are any values of the variable that make the equation true and, if there are some values, what the values are. These values are called the *solutions* to the equation.

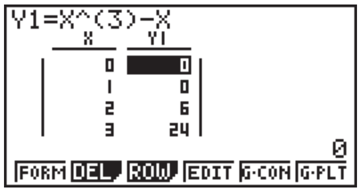
In this module, we explore a number of ways of finding approximate solutions to an equation using the graphics calculators, based on using graphs and tables. There are also some built in mechanisms in the calculators for solving equations *without* using a graph or table. The calculators are only able to provide numerical approximations to solutions, and usually cannot give *exact* results. Nor are they capable of providing a general (symbolic) solution to an equation. In most cases, however, the numerical approximations are good ones, and adequate for most practical purposes.

Zooming in a table

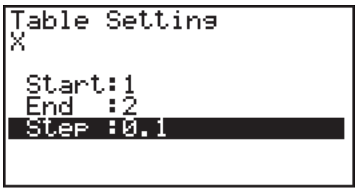
As explained in Module 2, a table of values can be manipulated efficiently in order to find particular values, rather like zooming on a graph. To illustrate, consider the equation $x^3 - x = 3$. One way of thinking of this is to ask if there are any values of x that result in $x^3 - x$ having the value of 3. One way to examine this question is to tabulate values of the function $y = x^3 - x$ for several values of x and check to see whether any of them give the required value. The screens below produce a table with a wide range of x-values ($-10 \le x \le 10$):

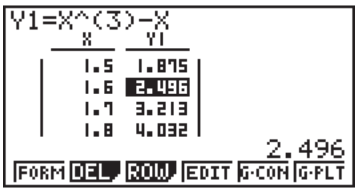




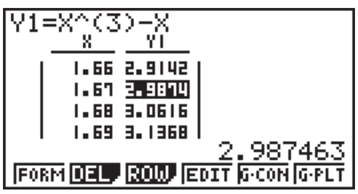


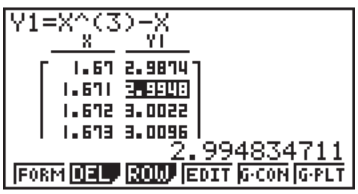
Scrolling the table suggests that there may be a solution (where the *Y1* value is 3) between $x = 1$ and $x = 2$. So, suitable settings are chosen to check below. The *Step* is chosen to be 0.1 for efficiency. You could also choose a *Step* of 0.01, but the table will then have 101 elements in it and will take longer to display and also to scroll.

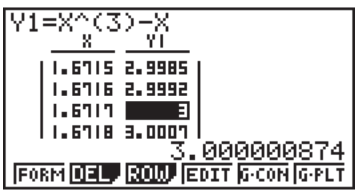




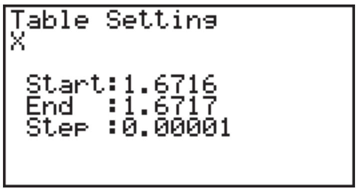
The table now shows that there seems to be a solution between $x = 1.6$ and $x = 1.7$, so we can continue zooming like this to get closer and closer to a solution. Notice that each screen improves the approximation by one decimal place, so the process is quite fast.

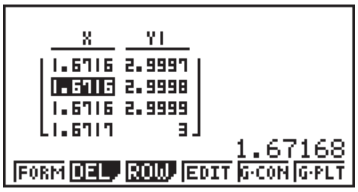


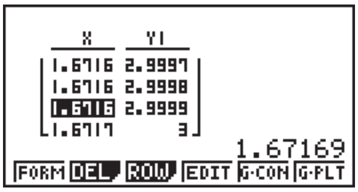




Notice also that the calculator displays only a few decimal places in the table itself, so that the value of 3.000000874 is actually displayed as if it were exactly 3. But there is a solution *between* 1.6716 and 1.6717. You need to be observant, using the calculator in this way, as the next screens show.

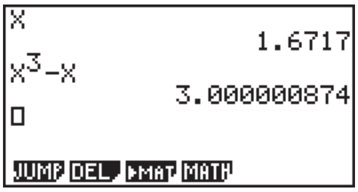






In this case, because the table only displays a small number of decimal places, several of the (different) *x*-values seem to be the same (1.6716). Moving the cursor over the values shows their value in the bottom right of the screen correctly, however.

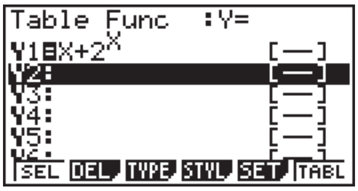
It seems that $x = 1.67170$ is a very good approximation to a solution of this equation. You can check the solution by scrolling in the table to this value and then evaluating the expression in Run-Mat mode. Notice in the screen below that X has the same value (1.67170) as was scrolled to:

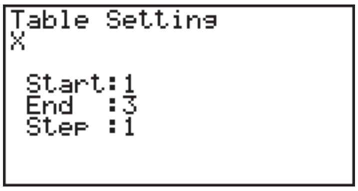


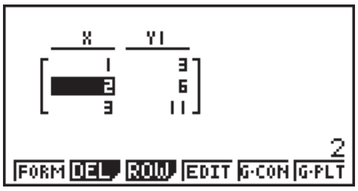
It is always a good idea to check your solutions to an equation, although in this case, the check is continuously available with the table as you zoom.

Guess, check and improve

A table can be used in a slightly different way from zooming, in order to find solutions to an equation. To see this, consider the equation $x + 2^x = 7$. As before, we start with the associated function $y = x + 2^x$ and ask which values of *x*, if any, will give this function the value of 7. A bit of mental arithmetic suggests that $x = 1$ and $x = 2$ are too small while $x = 3$ is too large. We will check the guesses on the calculator and then improve them. The table has been made deliberately small in this case, so that the entire table can be seen at once:

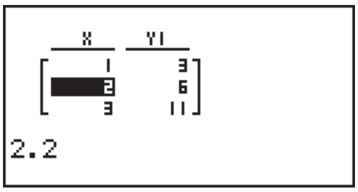


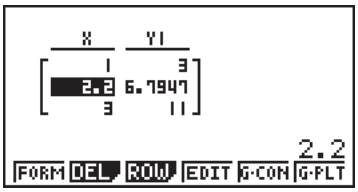




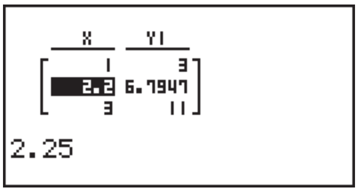
To improve the guesses, notice that $x = 2$ is a bit too small, but a solution close to 2 seems likely than one close to 3. We need to improve the guess.

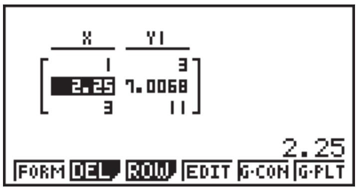
A good next guess may be to choose a value one fifth of the way between 2 and 3, since 7 is one fifth of the way between 6 and 11. So an improved guess is $x = 2.2$. To check the guess, move the cursor to 2 in the *x* column and use the keyboard to change it to 2.2. As soon as you press EXE, the associated *y*-value appears in the *y*-column, as shown in the following screens.

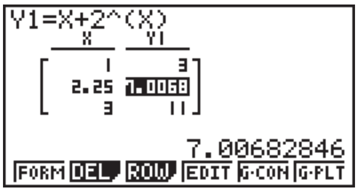




The value of 6.7947 is still too small. To improve the guess again, try $x = 2.25$, which is this time a little too large:







Notice that moving the cursor to the *y*-column gives a more detailed version of the value.

By continuing to guess, check and improve in this way, you can approximately solve many equations quite efficiently to a desired level of accuracy.

The *Solve* command

The approximate roots of a function can be found in Run-Mat mode using the *Solve* command, which is in the OPTN menu, located with CALC (OPTN F4). The syntax for the command is

Solve(*equation,estimate,left,right*)

The *equation* must use *x* as the variable. An initial *estimate* is needed, to start the calculator search for roots, while *left* and *right* allow you to specify an interval on which to look for roots. The interval and the closing bracket are optional; the command will also work with the restricted syntax:

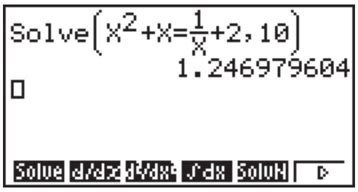
Solve(*equation,estimate*)

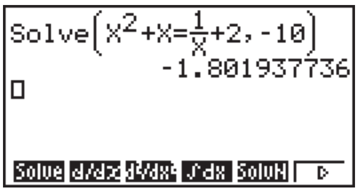
This command works a bit like the *ROOT* command in the *Graph Solve* menu, except that no graph is involved. You need to be careful to check that you have found all the solutions. Different estimates are needed to find different solutions.

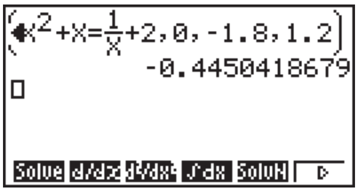
For example, consider the equation $x^2 + x = \frac{1}{x} + 2$.

To enter this equation into the *solve* command, notice that there is an equals sign on the keyboard, accessed with SHIFT •.

The screens below show three estimates (10, -10 and 0 respectively) used to find three solutions of the equation: $x \approx 1.24$, $x \approx -1.80$ and $x \approx -0.45$. In this case, new estimates were provided by editing the previous command (using ▲ ▲ ◀ after each result). Notice in this case that the third solution used a command that specified the interval (-1.8, 1.2) between the first two solutions.







To use this procedure efficiently, you need to have an approximate idea of how many solutions the equation has, and their approximate values, in order to choose estimates. If you have no idea about either the number of solutions, or their location, it may be better to use a graphical method involving the *Graph Solve* menu, as described in the next section.

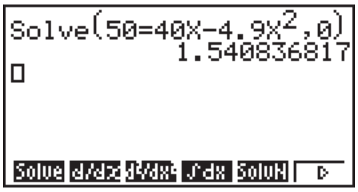
In practical situations, however, you will often have enough information about a context to use the *Solve* command to solve an equation. Consider, for example, a formula used in physics to describe the location *s* of an object with initial velocity *u* and acceleration *a* after an elapsed time of *t*:

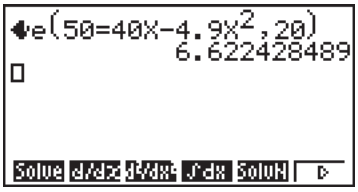
$$s = ut + \frac{1}{2}at^2$$

Suppose you want to find out how long after a stone is thrown vertically, with an initial velocity of 40 metres per second, it will reach a height of 50 metres above the Earth's surface. For this particular situation, $s = 50$, $u = 40$ and a is known to be about -9.8 metres per second per second (the acceleration due to gravity, which is negative because it is acting in an opposite direction to the initial velocity). For simplicity, we assume that there is negligible resistance to the stone because of the air through which it passes. So the equation to be solved is

$$50 = 40t + \frac{1}{2}(-9.8)t^2$$

Now the equation can be solved with the *Solve* command. You need to use x for the variable instead of t. In this case, only values for $t > 0$ are sensible. To find both solutions to the equation, start with an estimate of 0 (which will produce the first solution greater than 0, $t \approx 1.54$) and then use an estimate a good deal larger than the first solution, to make sure that you get the second solution:





In the screens above, the estimate $t = 20$ is clearly too large, but was used to ensure that the second solution was found. These two solutions suggest that the stone will reach a height of 50 m about 1.54 seconds after being thrown. It will also be 50 m high (on the way down again) about 6.62 seconds after being thrown.

The equation in this case is a quadratic equation, and so you can tell that it has two solutions. In other cases, you may have to rely on your knowledge of the context of the equation to decide how many solutions are of interest.

Later in this module, you will see that there are special procedures in the calculator for solving quadratic equations. The stone example here is used to illustrate how to use the *Solve* command. If you recognise the equation as quadratic, you may prefer to use the later procedure for solving it.

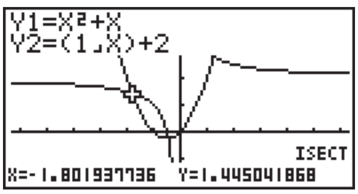
Using the *Graph Solve* menu

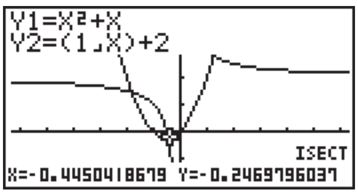
The *Graph Solve* menu is especially useful for finding automatic solutions to equations, using the *ROOT*, *ISCT* and *X-CAL* procedures described earlier in Module 2, once you have drawn appropriate graphs. An advantage of this method over the use of the *Solve* command (described above) is that you can usually tell from the graph how many solutions to the equation there are and their approximate values. For example, consider again the equation solved in the previous section:

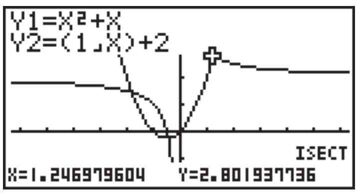
$$x^2 + x = \frac{1}{x} + 2 .$$

There are at least three ways to solve it, using the *Graph Solve* menu.

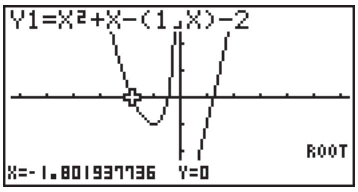
1. Draw the graphs of $y = x^2 + x$ and $y = 1/x + 2$. Then use the *ISCT* command to find the points of intersection. The x-values (-1.802, -0.445, 1.247) provide three solutions to the equation:

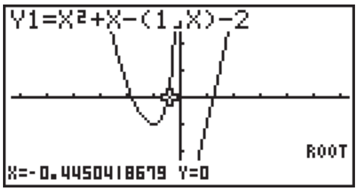


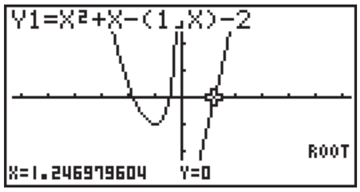




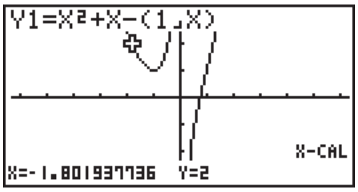
2. First rearrange the equation to have zero on the right side: $x^2 + x - 1/x - 2 = 0$. Then draw a graph of $y = x^2 + x - 1/x - 2$ and use the *ROOT* command to find three x-intercepts, -1.802, -0.445, 1.247:

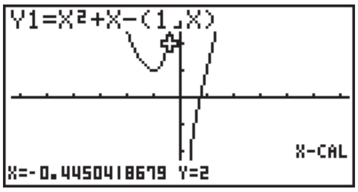


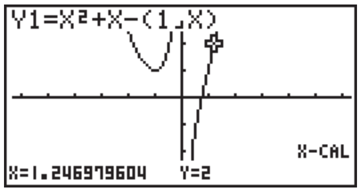




3. First rearrange the equation to put the number on the right side: $x^2 + x - 1/x = 2$. Then draw a graph of $y = x^2 + x - 1/x$ and use the *X-CAL* command to find the x-values for which the function has a y-value of 2. Once again, the three approximate solutions are $x \approx$ -1.802, -0.445, or 1.247.



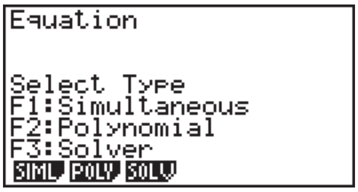


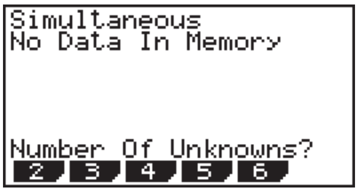


Notice that each of these three methods of solution involves graphing *different* functions. When using the *graph solve* menu to solve equations, you will often have to decide for yourself which of several ways is best in a particular situation.

Simultaneous linear equations

As well as these various ways of interpreting and dealing with equations in general, the Equation mode of the calculator contains inbuilt functions for solving some particular kinds of equations. In this section, we look at the solution of systems of simultaneous linear equations. Tap MENU A (use the [X,θ,T] key for A) to enter Equation mode, and tap SIML ([F1]) to select simultaneous (linear) equations. Systems of up to six linear equations can be solved.



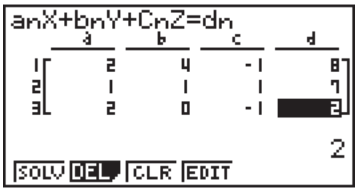


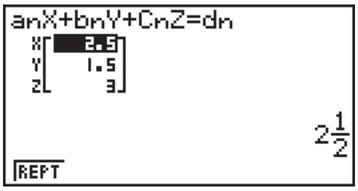
After you choose the number of equations, you need to enter the *coefficients* of the system into a *matrix* (a rectangular arrangement). Make sure that the equation is expressed in the appropriate way, with the variables *in the same order* for each equation, and a constant on the right hand side of each, as shown at the top of the screen. The coefficients are the numbers in each equation, including the number on the right hand side. As an example, consider the 3 x 3 system below.

$$2r + 4s - t = 8$$
$$r + s + t = 7$$
$$2r - t = 2$$

It is called a 3 x 3 system because there are 3 equations with three unknowns (variables). In this case, the three variables are *r, s* and *t*. (The variables do not have to be the same as those on the calculator screen.) Notice that one of the equations does not include all three variables, so that it has a zero coefficient for one of them.

Enter the coefficients of the system into the matrix, tapping [EXE] after each one (including the zero coefficient for *s* in the third row). The easiest way to enter them is one row at a time, since the cursor jumps to the right after each tap of the [EXE] key. The complete matrix is shown below.





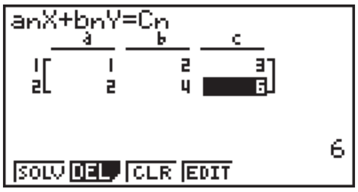
Check your typing. If you make an error, use the cursor keys to move around and then replace it with the correct value. When you are confident that the coefficients are correct, tap SOLV (F1) to get the solution list, if there is one. (Not all systems have a solution.) The solution is shown above.

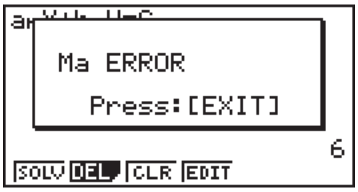
When the coefficients are all integers or fractions (with no decimal points), the solutions will be given as both decimals and fractions. You can then use the ▲ and ▼ keys to get fraction equivalents to the solutions, as shown above.

The three variables in this case are *r*, *s* and *t*, not *x*, *y* and *z*, as shown on the calculator screen, but you cannot expect the calculator to know that. The solutions are in the same order as the original variables. So $r = 2½$, $s = 1½$ and $t = 3$ is the (unique) solution to the system. If possible, you should check your solutions mentally, as one way of detecting a possible typing error. In this case, $2½ + 1½ + 3 = 7$, verifying the second equation above.

Tap EXIT EXIT to return to the original simultaneous equation screen to select a different system.

When there is no solution to a system, the calculator will give a Ma ERROR (i.e., a mathematical error); this usually means that it has attempted to divide by zero at some point. If this happens, check that you entered the coefficients correctly and, if you did, check the equations to find out *why* there is no solution. The next screen shows an example of a system that does not have solutions.

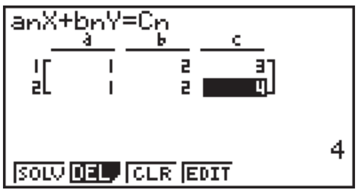


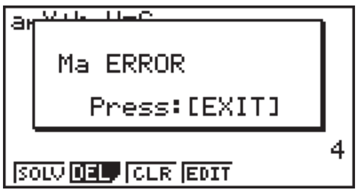


The two equations represented here are $x + 2y = 3$ and $2x + 4y = 6$. In fact, these two equations are not different equations, since they describe the same relationship between the two variables, since if $x + 2y$ is equal to 3, then $2x + 4y$ must be twice as much, or 6. That is, the second equation $2x + 4y = 6$ doesn't add any further information to the first equation; in mathematics, the equations are said to be *dependent* on each other.

In this case, any solution that satisfies the first equation (e.g. $x = 1$ and $y = 1$) will also satisfy the second equation, so there are an infinite number of solutions to the system. A system of *n* equations in *n* unknowns will only have a *unique* solution if each of the *n* equations is independent of each of the others. In Module 13, a method of dealing with dependent equations using matrices is described.

Here is another example of a system that has no solutions:





This is a different case from the previous one. If $x + 2y$ is equal to 3, as the first equation demands, then we cannot at the same time have $x + 2y$ equal to 4, which the second equation demands. These two equations are said to be *inconsistent* with each other.

Polynomial equations

In Equation mode, the calculator will allow you to get the solutions of polynomial equations up to the sixth degree, after you have entered the coefficients of each term. At the opening screen, tap POLY (F2) to select the *degree* of the polynomial – the highest power of the variable used. Polynomials of the second degree are called *quadratic* and those of the third degree are *cubic*.

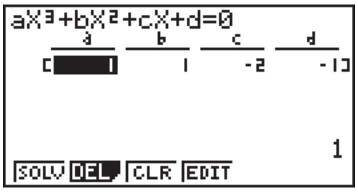
The coefficient screen shows you the meaning of the coefficients; notice especially that the equation must be arranged to have a zero on the right hand side, unlike the case for linear systems.

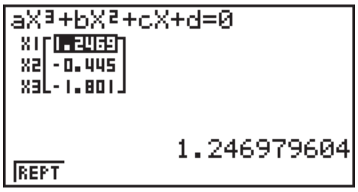
For example, consider the equation examined earlier in this chapter: $x^2 + x = \frac{1}{x} + 2$.

To remove the fraction, multiply each side by x, assuming that $x \neq 0$: $x^3 + x^2 = 1 + 2x$. This can now be seen to be a polynomial equation, which is cubic, since the highest power of the variable is three. So choose F2 (3) to solve the equation. To get the coefficients into the correct order for the calculator, rearrange it into descending order of powers of x, with a zero on the right hand side:

$$x^3 + x^2 - 2x - 1 = 0$$

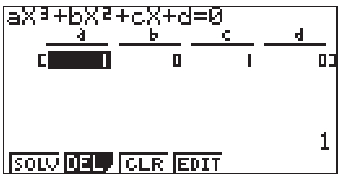
So the four coefficients are 1, 1, -2 and -1. Enter these carefully into the coefficient row. Errors can be corrected as for simultaneous equations. Finally, tap SOLV (F1) to get the solutions, shown as a list on the screen below at right. The three solutions match those obtained earlier.

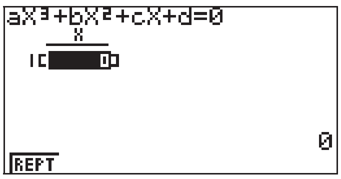




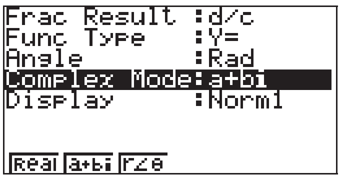
Polynomial equations have as many solutions as the degree of the equation. So cubic equations will always have three solutions, and quadratic equations will always have two, but sometimes the solutions will be *complex* numbers, rather than *real* numbers. The best known complex number is ***i*** which is used in mathematics to represent one of the square roots of -1. That is, $\boldsymbol{i}^2 = -1$. Complex numbers are described in detail in Module 12.

Sometimes, polynomial equations have 'missing' terms. However, you must still enter a zero coefficient for them. An example is the cubic equation: $x^3 + x = 0$, for which there is no x^2 term. The screens below show zeroes entered for both the quadratic and constant terms.

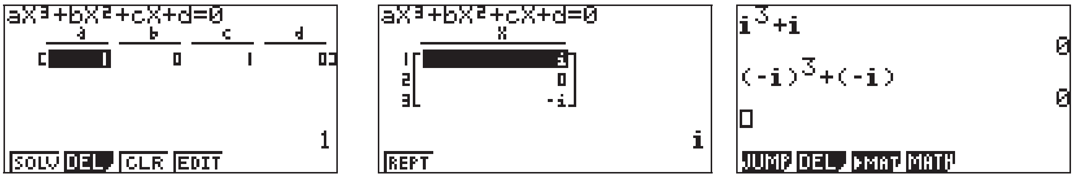




The only solution provided by the calculator in this case is $x = 0$. This is because there is only one *real* solution to this equation. In general, solutions to polynomial equations will not always be real numbers. To obtain all solutions, tap EXIT and then use the SET UP menu to change *Complex Mode* from the default of *Real* to *a* + *b**i*** one of the two complex number choices, as shown below.

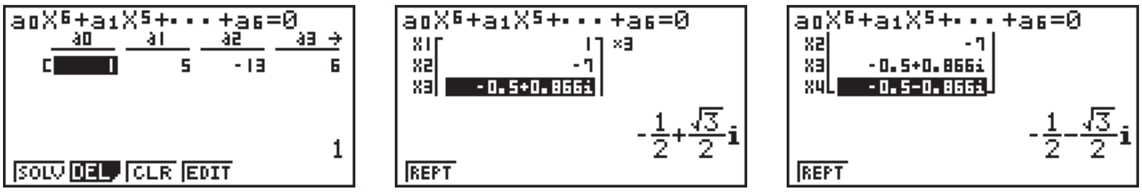


Solve the equation again, and you will see all solutions.



In the case of the cubic equation above, two of the three solutions above are complex numbers. These three solutions are fairly easy to check. It is very easy to see that $x = 0$ is a solution. To check the complex solutions satisfy the equation $x^3 + x = 0$, substitute $x = \boldsymbol{i}$ and $x = \boldsymbol{-i}$ respectively in Run-Mat mode. Use SHIFT 0 on the keyboard to obtain $\boldsymbol{i}$. The third screen above verifies that the two complex solutions, $x = \boldsymbol{i}$ and $x = \boldsymbol{-i}$, also satisfy the equation.

When this equation is solved with the polynomial solver, the calculator indicates repeated solutions with the x3 symbol (in the middle screen below). Notice in the following screens that only the first four coefficients are shown in the first screen, while the six solutions are shown in the next two screens. Note also that the two complex solutions are shown exactly when they are highlighted.



So, the six solutions to this equation are $x = 1$, $x = 1$, $x = 1$, $x = -7$, $x = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$ and $x = -\frac{1}{2} - \frac{\sqrt{3}}{2} i$.

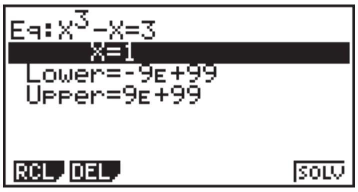
Using the Solver

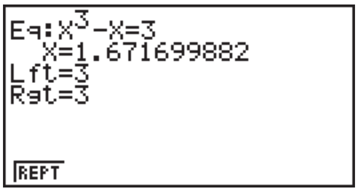
Numerical solutions to equations can be readily obtained with the Solver facility in Equation mode. Access the Solver with SOLV (F3). Then an equation can be entered in the equation line at the top of the screen, shown with *Eq*:

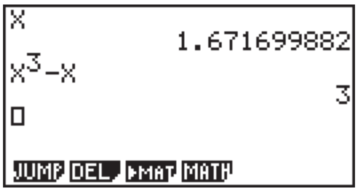
If there is already an equation in the Solver, you can type a new one over it, as for functions in Graph mode. In the right screen above, the equation $x^3 - x = 3$ has been entered, followed by EXE. You will need to use the 'equals' sign on the calculator keyboard, accessed with SHIFT •.

The calculator adds three lines under the equation. The last two lines describe an interval, (Lower,Upper) within which a solution will be sought. The values showing on the screen above represent (-9×10^{99},9×10^{99}), which for all practical purposes is the entire real number line. (There are certainly numbers larger than 9×10^{99} and numbers smaller than -9×10^{99}, but they are unlikely to arise in everyday practical settings.)

The calculator has added a line immediately under the equation in order to give the variable a value. In this case, the variable is *X*, but other variables may be used for equations in the Solver, unlike the *Solve* command. (The calculator will automatically assign to variables here the most recent value in the calculator memory, so your screen will not be the same as the one above.) The best way to think of this is to assume the calculator is using a guess, check and improve strategy like the one you used earlier in Table mode. So it needs a starting value for the variable. You can enter any starting value. Use the cursor to highlight the variable (*X* in this case) and then tap SOLV (F6) to start the calculation process. The example below starts with $x = 1$.



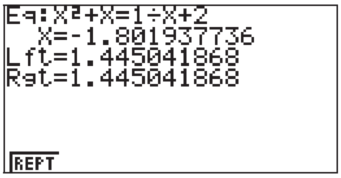


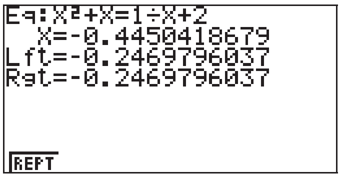


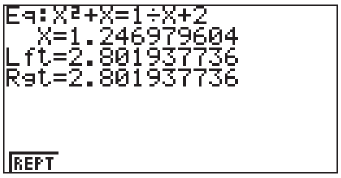
The calculator will usually stop quite quickly and report the value of the variable, together with the value of the left side of the equation and the right side of the equation. In this case, the approximate solution, $x \approx 1.671699882$ gives a value of 3 to the accuracy of the calculator (and more quickly than the procedure at the start of this module.) To check the solution directly in the original equation, switch to Run-Mat mode as in the third screen above. The calculator retains the value of the variable (in this case, *X*) when you change modes, as shown by the first line. The screen verifies that $x^3 - x \approx 3$ when $x \approx 1.671699882$.

Although you can enter any value you wish as your first guess in the Solver, a close value will give the solution more quickly, and a value too far from a solution may not give a satisfactory result at all. So you still need to think about the equation, how many solutions you expect and what you think might be a close approximation.

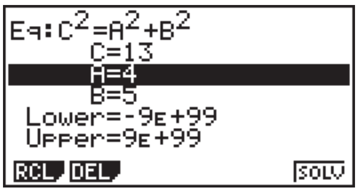
When using the solver, you need to be careful to obtain *all* the solutions, when there are more than one. This will require you to use various starting points and not be satisfied with merely finding a single correct solution; of course, you will need to think about how many solutions are to be expected for a particular equation. To illustrate, below are the three solutions to the equation used earlier in this module; the three starting points used were $x = -5$, $x = -1$ and $x = 1$. You can tap REPT (F1) to repeat the process with a fresh starting value after the solution is displayed.

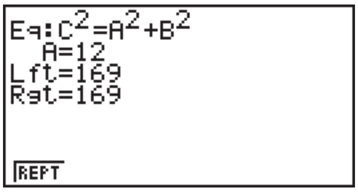


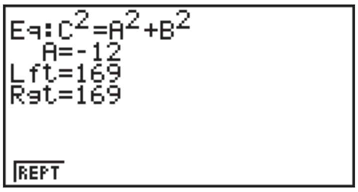




Although all the equations so far have used *x* as the (single) variable, any of the algebraic variables in the calculator can be used. The example below shows how the Solver might be used to deal with a familiar relationship, the Pythagorean Theorem, in the form of the equation $C^2 = A^2 + B^2$.







The variables are entered in the usual way, using the ALPHA key. Only upper-case versions are available, as usual. The Solver works by allowing you to give a value for all of the variables except one. To solve an equation, highlight the variable whose value is sought, make sure it has a sensible starting value and tap SOLV (F6). In this example $C = 13$ and $B = 5$ have been entered and the value of *A* highlighted in the first screen. A starting value of 4 for *A* is sufficient in this case.

The Solver generates $A = 12$, as shown in the middle screen. Notice that, if you had started with a negative value for *A*, a negative solution would be given (shown in the third screen), although this would usually be unacceptable for solving right triangles.

Exercises

The main purpose of the exercises is to help you to develop your calculator skills.

1 Solve $3x^2 - x - 5 = 0$.

2 Solve $x^2 - 11 = 7x$. (Hint: rearrange the equation first to make sure that you get the coefficients in the correct order.)

3 Use a graph to explain why the solutions to $x^2 + 3 = 0$ are complex numbers.

4 Solve $5t^2 - 13t = 0$ *without* using the calculator.

5 Use the 'guess, check and improve' method in a table to solve $x + 2^x = 13$.

6 Find the roots of the function $f(x) = x^3 - 2x - 1$.

7 Use your answer to Exercise 6 to solve $x^3 = 2x + 1$.

8 Solve $x + \log x = 2$.

9 Find how many points of intersection there are of the graphs of $g(x) = x + 2$ and $f(x) = |x^2 - 2|$.

10 Use your answer to Exercise 9 to solve simultaneously the pair of equations:

$$y = |x^2 - 2|$$
$$y = x + 2$$

11 Solve the following systems of equations, if possible:

a $2x + 3y = 7$, $3x - y = 9$

b $a - b = 2$, $2a = b - 5$

c $0.50x + 0.20y = 0.25$, $0.60x - 0.30y = 0.20$

d $5x - y = 4$, $y = 5x - 1$

e $3x + y + z = 4$, $x - y + z = 2$, $x + 4y - z = 2$

f $p + r + 1 = 0$, $p + q + r = 4$, $4q + r + 3 = 4p$

12 The Theorem of Pythagoras states that for a right triangle ABC, with the right angle at C, $c^2 = a^2 + b^2$. Use the Solver to:

a find c when $a = 23$ and $b = 11$
b find b when $c = 16.5$ and $a = 9.5$
c find the length of the hypotenuse of a right triangle with other side lengths 12 cm and 3 cm. Draw a triangle on paper to check your answer.

13 The total resistance R of a pair of resistors in parallel, with individual resistances r_1 and r_2, is given by $\frac{1}{R} = \frac{1}{r_1} + \frac{1}{r_2}$.

a Solve for R if $r_1 = 60$ and $r_2 = 40$.
b Solve for r_1 if $R = 5$ and $r_2 = 8$.

Activities

The main purpose of the activities is to help you to use your calculator to learn mathematics. You may find that some of them are too advanced for you. Ignore activities you don't yet understand.

1 Solve $2x^2 – 5x = 12$. Then use the solutions to the equation to write down the factors of the expression $2x^2 – 5x – 12$. Graph the function $f(x) = 2x^2 – 5x – 12$. Compare your graph with the factors of the expression and the solutions to the equation. How are all of these related?

Use this approach to help you find the factors of the cubic expression $-6x^3 + 25x^2 – 16x – 15$.

2 Another way to solve a system of equations is to use a table of values. In Table mode, construct a table for the following two functions:

$$y = 3x – 9 \qquad\qquad y = \frac{7 – 2x}{3}$$

Scroll the table to find a value for x for which the two functions have the same y-value.

Compare your results with the graph of the two functions and with your answers to question 11a in the Exercises above.

3 Use the *solve* command to find solutions of $2x^3 + 36 = 57x – 19x^2$. Notice that two of the solutions are quite easy to obtain, but the third one is a bit harder. Then solve the equation using a graph. Does this explain the possible difficulties with the *solve* command? Finally, solve the equation in Equation mode.

For which initial estimates does the *solve* command locate the smallest solution to the equation? Use the graph to help you decide.

Test your solutions by using the *solve* command with various estimates.

Which method of solution do you prefer? Why? Discuss this with somebody else.

4 The volume V of a cylinder is given by $V = \pi r^2 h$, where r is the radius and h is the height.

a How high must a cylindrical juice container be if it has a radius of 5 cm and is required to hold 500 mL?

b A fuel drum of height 120 cm holds 400 L of fuel. What is its radius?

5 Solve the equation $x^2 + 2^x = 7$ in as many different ways as you can.

Compare your methods of solution with somebody else. What distinct advantages are provided by the various methods? Which are the *best* methods, in your view?

6 The following system of equations contains coefficients (based on physical measurements) that have been rounded correct to one decimal place; for example, the coefficient of 2.4 in the first equation represents a number in the interval from 2.35 to 2.45.

$$2.4x + 5.7y = 4.2$$
$$3.4x + 8.3y = 3.2$$

Investigate the effects on the solution of these equations of using rounded coefficients like those above. (For example, try 2.37 and 2.42 for the first coefficient: how do the solutions compare?) You will find it useful to draw some graphs for the system. Comment on the practical implications of your observations.

Systems of equations like this are described as *ill-conditioned*

Notes for teachers

This module illustrates several ways in which the calculator can be used to explore various aspects of equations, from a variety of perspectives. The text of the module is intended to be read by students and will help them to start to see how the calculator can be used efficiently for various kinds of explorations in several different areas of mathematics. The Activities are appropriate for students to complete with a partner or in a small group, so that they can discuss their observations and justify their conclusions.

Answers to Exercises

1. $x = \frac{1+\sqrt{61}}{6}, \frac{1-\sqrt{61}}{6}$ or $x \approx 1.468, -1.135$ 2. $x = \frac{7 \pm \sqrt{93}}{2}$ or $x \approx 8.322, -1.322$ 3. The graph of $y = x^2 + 3$ does not cross the x-axis, so there are no real values of x for which $x^2 + 3 = 0$.
4. $t(5t - 13) = 0$, so $t = 0$ or $t = 13/5$. 5. $x \approx 3.281$ 6. $x = -1, x \approx -0.618, x \approx 1.618$ 7. $x = -1$, $x \approx -0.618, x \approx 1.618$ 8. $x \approx 1.756$ 9. Four 10. (-1.561,0.439), (-1.000,1.000), (0.000,2.000), (2.561,4.561) 11. (a) $x = \frac{34}{11}, y = \frac{3}{11}$ (b.) $a = -7, b = -9$ (c) $x = \frac{23}{54}, y = \frac{5}{27}$ (d) No solution, as the equations are inconsistent (e) $x = -1, y = 2, z = 5$ (f) $p = 4\frac{2}{5}, q = 5, r = -5\frac{2}{5}$
12. (a) 25.495 (b) 13.490 13. (a) $R = 24$ (b) $r_1 \approx 13.333$

Activities

1. Links between functions, graphs and equations are of critical importance, so that activities of this kind are especially important for students to undertake. In Equation mode, solutions will be given in exact forms, making factors identifiable. Encouraging students to work together will assist with correct use of terminologies of factor, solution, root, function, graph, equation. [Answers: $x = 4, -3/2$ give factors $(x - 4)(2x + 3)$. Factors of cubic can be found similarly to be $(x - 3)(3x - 5)(2x + 1)$.]

2. Solving equations in this fashion allows students to see various approximations to solutions, although it is difficult to get exact non-integer results in a table. Encourage students to explore the solutions in other ways as well, such as Graph-solve with graphs. The graphs make clear that there is only one solution. [Answers: $x = 34/11$ and $y = 3/11$.]

3. Activities of this kind allow students to understand the significance of the initial guess for a *solve* command. Two roots close together are hard to see on a graph, but relatively easy to find in Equation mode. Discussion of alternative methods will give students an opportunity to appreciate their various strengths. [Answers: $x = -12$, 1 3/2]

4. Problems of these kinds, common in elementary mathematics, can be well handled using the Solver in Equation mode. Use the equation $V=\pi R^2 H$, taking care with the units of L and mL; remind students that 1 cm^3 = 1 mL and 1000 mL = 1 L. [Answers: a) 6.366 cm b) 32.57 cm]

5. Activities like this offer students a chance to explore many possibilities and can stimulate a good classroom discussion regarding several different methods. [Answers: $x \approx -2.615, 1.846$]

6. The solution $x = 277/9$, $y = -110/9$ of the linear system assumes that the coefficients are exact, which is clearly not the case when they have been rounded. Small changes of the kinds suggested produce dramatically different solutions, and should help students to understand the significance of exact and approximate values and to be cautious when dealing with real data. In this case, the two graphs associated with the two equations are almost parallel, with very similar slopes, so that a very slight change in either affects the intersection point dramatically.

Module 10
Iteration, recursion and spreadsheets

One of the most important properties of a graphics calculator or a computer is the capacity to carry out instructions repeatedly and quickly. Many practical matters involve situations in which one thing depends on the previous thing. Growth and decay processes are good examples. For example, how heavy (or wealthy, or tall, ...) you are today depends to an extent on how heavy (or wealthy, or tall, ...) you were last week.

In Run-Mat mode, you have already seen that pressing [EXE] without entering a command line repeats the previous command. In this way, a *sequence* of results can be obtained. (A sequence is a list in a certain order.) This property of the calculator can be used to good effect to produce successive terms of a *recursively* defined sequenc, for which each term is defined by its relationship with the previous term.

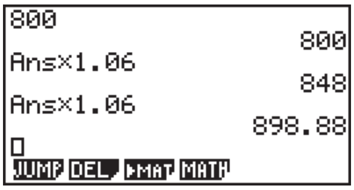
For example, suppose you deposit $800 in a bank, where it earns interest at a rate of 6% p.a., compounded annually. So, in this case, the amount of money in the bank each year is 106% of the amount the previous year. The amounts at the beginning of each year form a sequence.

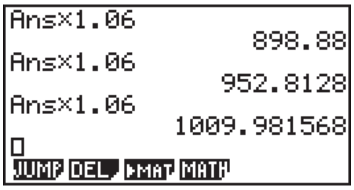
Note the effects of the following steps, which produce this sequence of terms, showing the amounts of money in the bank at the beginning of each year.

800 [EXE]	enters the first term
[×] 1.06 [EXE]	enters the command Ans x 1.06, where Ans refers to the last answer
[EXE]	enters the next term, since it repeats the command Ans x 1.06
[EXE]	enters the next term, since it repeats the command Ans x 1.06
[EXE]	enters the next term, since it repeats the command Ans x 1.06
etc.	

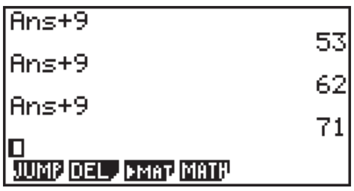
This is an *iterative* process, since the results of each calculation feed into the next one.

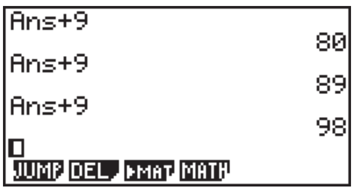
The screens below show that there will be $1009.98 in the bank after four years,





This example shows a *geometric* sequence since each term is a constant multiple of the previous term. An *arithmetic* sequence is one with a constant difference between successive terms. The screens below show the first 9 terms of an arithmetic sequence, with first term 26 and constant difference 9:





This time, each press of the [EXE] key repeats the recursive command to add 9 to the previous value.

Some recursive sequences are very useful in practice. For example, suppose that you borrow $9000 to buy a car. The terms of the loan are that you must make a payment of $350 each month, and interest of 1.5% of the remaining balance of the loan is added each month.

At first, you owe $9000.

After one month, you owe 1.015 x $9000 (with 1.5% interest added), less your payment of $350. So the amount owing at the end of the first month is:

$$\$(9000 \times 1.015 - 350) = \$8785$$

Notice that almost a third of your payment was absorbed by the interest added for the month!

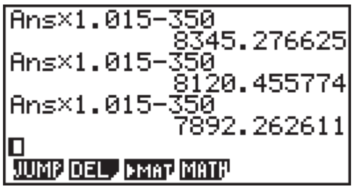
In the next month, you must add 1.5% (of $8785) interest and deduct your next payment of $350. So the amount owing at the end of the second month is:

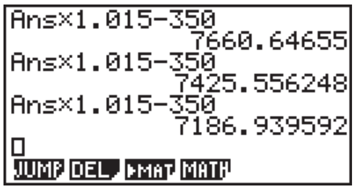
$$\$(8785 \times 1.015 - 350) = \$8566.78$$

It can get rather tedious working out how much is owing each month in this way. It is much more efficient to use the calculator to perform the iterations for you.

Each month, the amount owing is 1.015 times the previous amount owing less $350.

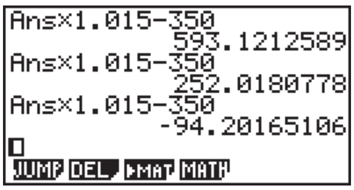
The command Ans x 1.015 – 350 will perform all of the arithmetic each month with a single press of the EXE key: To enter this command the first time, press ✕ 1.015 − 350 EXE. After the first time, just tap the EXE key to repeat the command.





After 8 months, there is $7186.94 owing on the loan.

To see how long it takes for the loan to be paid off this way is still a little tedious, but a lot less so than performing each month's calculations separately. Count the EXE presses carefully as you proceed, to see that the loan will be paid off after 33 months. Here are the last few amounts owing:



From this screen, you can see that the last payment needs only to be $252.02. Notice too that near the end of the loan, most of the $350 monthly payment is being used to pay off the loan rather than the interest. For example, the second last payment of $350 reduces the amount owing by just over $341 – much more than the early payments, a good part of which were absorbed by interest.

More sophisticated recursions may require the use of the *Ans* variable (SHIFT (−)). For example, to generate successive terms of a logistic function of interest in the study of chaosgiven by

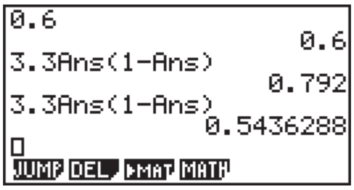
$$X_n = 3.3X_{n-1}(1 - X_{n-1})$$

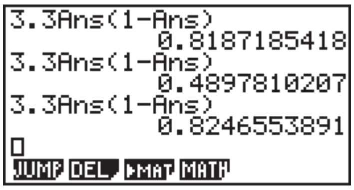
the following command line is needed:

3.3*Ans*(1–*Ans*)

after a suitable initial value is entered. X_n refers to the nth term of the sequence, and X_{n-1} refers to the previous term, (the $(n-1)$th term), so the sequence is also recursively defined.

The screens below show a starting term (X_0) of 0.6, and the next few terms.

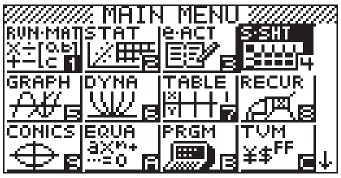


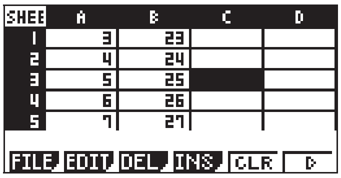


Unlike the car payments, after a short while, the terms of this sequence seem to be oscillating between two values (near 0.82 and 0.48). If you continue to generate terms in this iterative way (by pressing EXE many times), you will see this pattern continue. Before the invention of computers and calculators, it was more difficult to study phenomena like this, because the arithmetic took too long.

Using the Spreadsheet

A spreadsheet is an ideal tool for studying recursive situations. To see how the above examples can be explored using the spreadsheet, enter Spreadsheet mode with MENU 4.



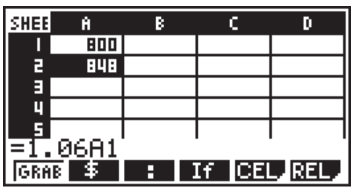


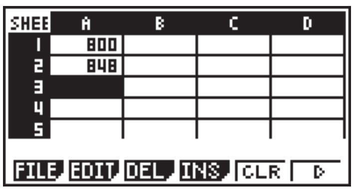
The spreadsheet comprises a grid of cells, referenced horizontally with a letter (A, B, C, …) and vertically with a positive number (1, 2, 3, …). In the spreadsheet shown above, cell A2 contains a 4, cell B4 contains 26 and cell D3 is blank. When you open the spreadsheet, the contents may be blank or they may contain whatever was inserted the last time it was used.

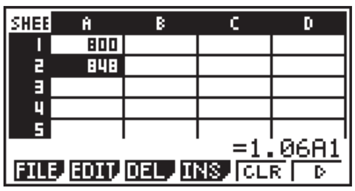
You can move around with the cursor, as for lists. You can also change the contents of a cell by entering a new value. To delete the contents of a cell, use CLR (F5). You can delete the contents of an entire row or column or delete the contents of the whole spreadsheet with DEL (F3). It's usually a good idea to start afresh with each situation you wish to explore.

To study the first situation described above, depositing $800 in a bank account at 6% interest per annum, start by entering 800 in cell A1. Press EXE to complete the process.

The amount after one year can be computed in cell A2 as =1.06xA1 (or just 1.06A1), as shown below. The equals sign is available on the keyboard with SHIFT • and *must be included* to indicate that a *formula* is involved. Use the ALPHA key to obtain the symbol A for the cell.

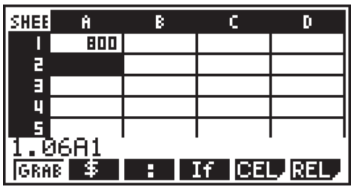


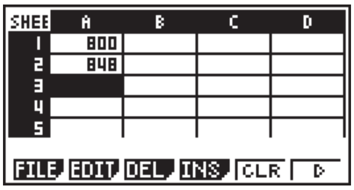


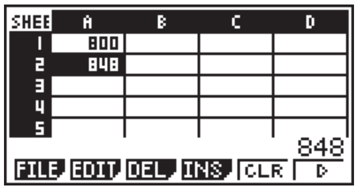


Notice that, although the result of 848 is showing, if you move the cursor back to A2, the formula used to compute the value is shown at the bottom right.

Incidentally, if a computation for a cell is entered *without* the equals sign, then the numerical result is the same, but the formula will be permanently lost, as the following screens show. Make sure you understand the differences between the first and third screens in the two cases. In most situations, it is best to use formulas, to take advantage of the power of a spreadsheet.

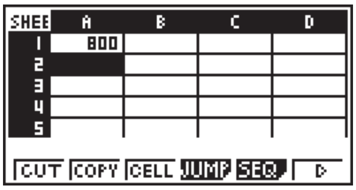


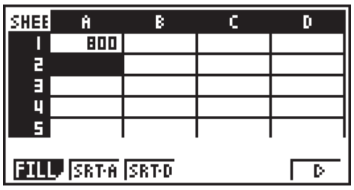




Instead of computing just one value, however, we want to compute many values. To do this we will use a formula to fill a range of cells, instead of just one cell. To do this, first use F5 (CLR) to clear cell A2. This will leave the cursor in cell A2.

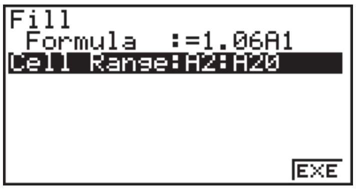
To access the Fill command, start with EDIT (F2) then tap F6 and FILL (F1)

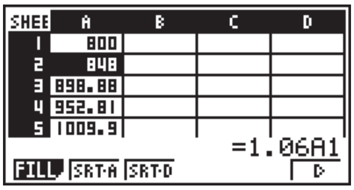




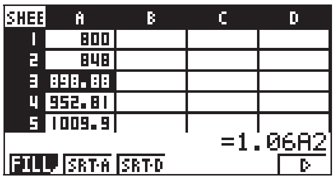
In the Formula line of the Fill screen, enter the formula to be used for the present cell (i.e. for A2), making sure that you include the equals sign. Press EXE when finished.

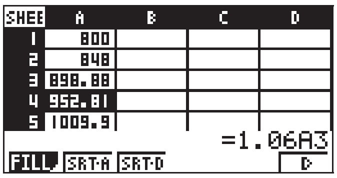
Then in the Cell Range, edit the existing range to show A2:A20, which represents all the cells from A2 down to A20, as shown below. Tap F6 (EXE) to complete the fill command.

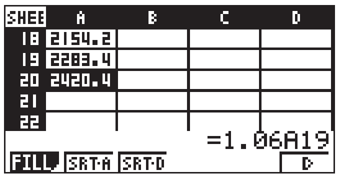




To see how the fill process works, use the ▼ cursor to move down, and notice the formulas in each cell. To scroll quickly, hold your finger down on the cursor key. In each case, the cell contents are 106% of the previous cell's contents, as shown below.

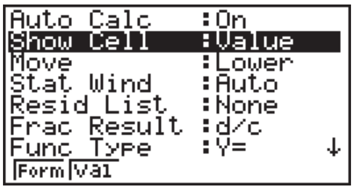


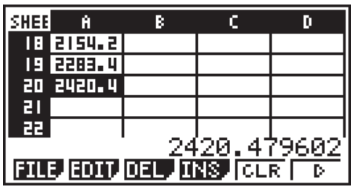




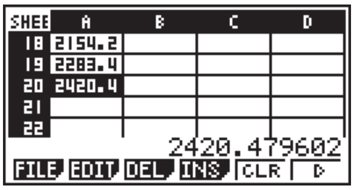
The spreadsheet shows the same results as previously, but they are now easier to access.

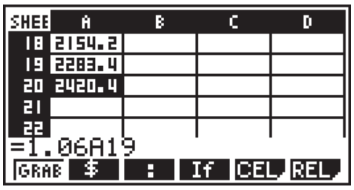
Notice in the screens above that the formula for each cell is showing in the bottom right of the screen. If you want to see the value in the cell instead of the formula used to generate it, use SET UP to adjust the *Show Cell* variable, as shown below.





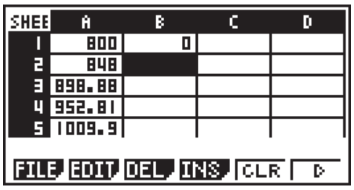
This changes only what is displayed, not what is contained in the cell. If the contents of the cell are edited with EDIT (F2) and then CELL (F3), the formula will still be shown, as the next screens show.

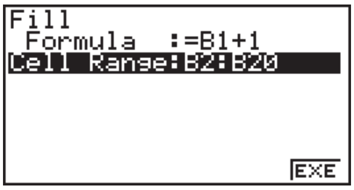


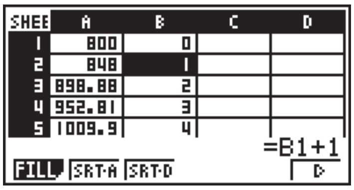


The cell will always show the first few digits of a numerical result; if you need to see the number in full, you will need to choose to display the values in this way. The spreadsheet itself is unaffected by this, and you can return to display the formula if you wish.

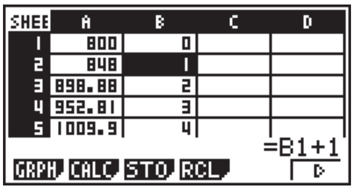
It is instructive to graph the results of the growing bank balances. To do so, use column B to add a number corresponding to the number of years the money has been in the bank. Start with 0 in cell B1 and use a formula to make each cell down to B20 one more than the previous cell, as shown below. Tap EXIT to return to the original menu.

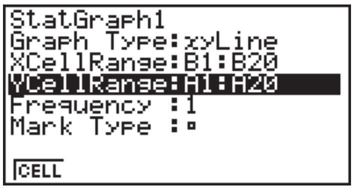




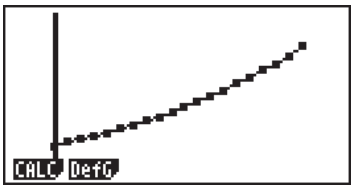


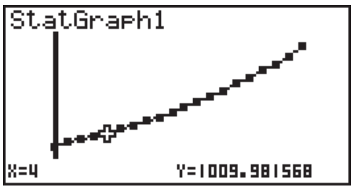
To access the graphing capabilities (which are exactly the same as those in Statistics mode, described in detail in Module 4), tap F6 and then GRPH (F1). Use F6 to set up a line graph, making sure that the *x*-values are in column B and the *y*-values in column A, to match the spreadsheet construction. Tap EXE when finished.

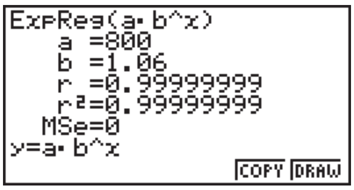




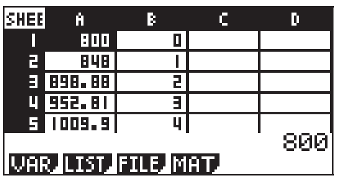
When you draw the graph with GPH1 (F1), the characteristic exponential shape is readily seen. The same graphing and calculating capabilities as in Statistics mode are available for data in the spreadsheet. For example, the graph can be traced and a regression model examined with CALC (F1), as shown in the following screens. In this case, the exponential model $y = 800 \times 1.06^x$ is an excellent fit, because of the exponential nature of the growth of the bank balance.

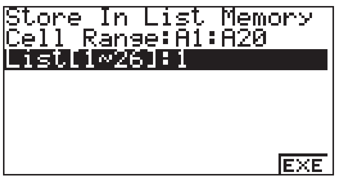


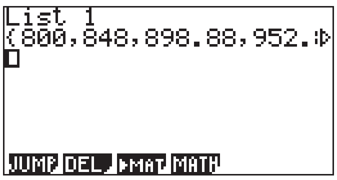




Data in a spreadsheet can be transferred to the calculator lists for analysis. Start with STO (F3) and then select LIST (F2). The screen below shows how to save the first 20 cells of column A into List 1. (Make sure that List 1 does not already contain data you do not want to lose, as the command will overwrite any existing List 1 data.)





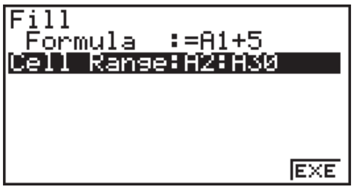


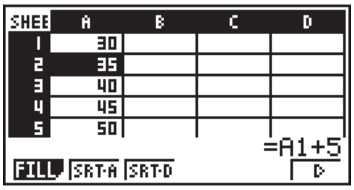
Once data are saved to a list, the list can be used in Statistics mode or in Run-Mat mode.

Absolute and relative addressing

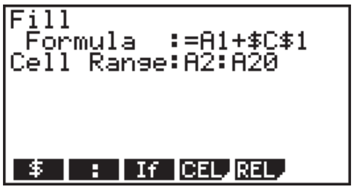
The fill command in the previous section used *relative addressing*. That is, in each cell, the command worked differently, interpreting '=1.06A1' to mean 'multiply the previous cell by 1.06' each time. If you want the spreadsheet to use precisely the contents of a *particular* cell, then *absolute addressing* is needed.

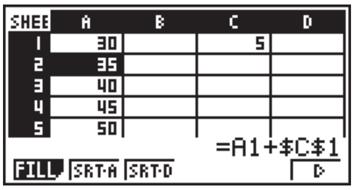
Here is an example to illustrate this important idea. A girl decided to increase how long she exercised each day by 5 minutes, starting with 30 minutes on the first day. This spreadsheet below uses relative addressing. If she wanted to consider a different daily increase, a different spreadsheet would be needed, using a different formula (replacing the 5 with another value).



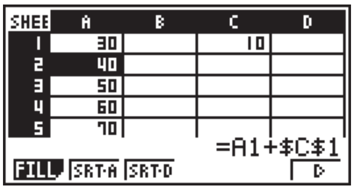


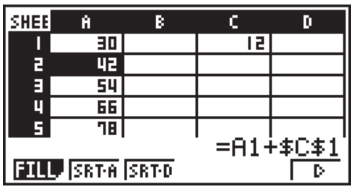
The following alternative spreadsheet has the daily increase stored in cell C1 and referred to in the formula as C1, with the dollar signs showing that absolute addressing is being used, and the value stored in C1 is to be used each time. The dollar sign is available with F1 when entering the formula. The result seems at first to be the same as the first spreadsheet.





However, if the value in C1 is now changed, then the spreadsheet will update automatically and be different each time, as shown below. In the first case, she increases the exercise by 10 minutes each day, and in the second case by 12 minutes each day.



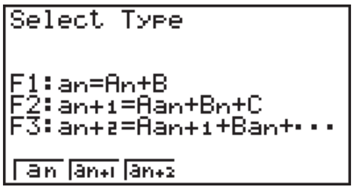


This spreadsheet (which uses both relative and absolute addressing) is much more versatile than the first spreadsheet.

Recursive functions

For many practical problems, the methods of the first part of this chapter work fine, even though it can get a bit tedious pressing the EXE key lots of times. On the calculator, you can instead define a recursive *function* in a similar way to defining other functions, and then study it graphically or in a table. Press MENU 8 to move to Recursion mode to see how to do so. As previously in Graph and Table modes, use DEL (F2)to delete any of the three functions (*a*, *b* or *c*) showing in your calculator before you start.

Then press TYPE (F3) to select a type of recursive function. The screen below shows the three possibilities:



The first choice, a_n (F1), is not actually recursive at all. It involves functions for which each successive term (such as a_n) is related to the sequence number of the term (n). An example is

$$a_n = 9n + 17$$

This example (with n = 1, 2, 3, ...) leads to the arithmetic sequence shown on the first page of this module, where it *was* defined recursively.

The second choice, a_{n+1} (F2), involves recursive functions, in which each successive term depends on the previous term (and possibly *also* on the term's position in the sequence). As well as the relationship between terms, you need a starting value to define a function in this way. An example involves the car payments described earlier. These can be described in symbols as

$$a_{n+1} = 1.015a_n - 350; \;\; a_0 = 9000$$

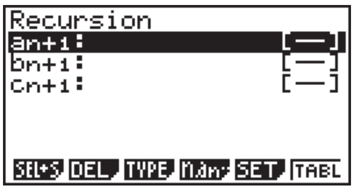
The nth term of the sequence (a_n) is the amount owing after n payments have been made. This term is used to find the following term (the ($n + 1$)th term, a_{n+1}). The first value is 9000.

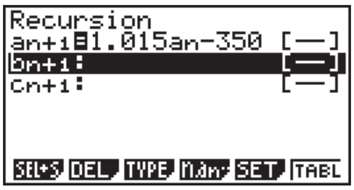
The remaining choice, a_{n+2} (F3), involves more complicated recursive functions in which each successive term depends on the previous *two* terms (and possibly also on the term's position in the sequence). Again, a starting value of the function is needed.

The most famous example of this is the *Fibonacci sequence*, named after the medieval Italian mathematician, Leonardo Fibonacci. This is a sequence of numbers, starting with 1 and 1, in which each term is the sum of the previous two terms: 1, 1, 2, 3, 5, 8, 13, ... This sequence can be described as a recursive function:

$$a_{n+2} = a_{n+1} + a_n; \;\; a_1 = 1; \;\; a_2 = 1$$

To illustrate the use of recursive functions, we will use the car payment example. The screen below shows how the recursive part of the function is defined (after choosing the second type). Three functions (a, b and c) can be defined in the calculator; for this example, we need only one. Notice that, when na_n (F4)is pressed, the relevant variables are shown at the bottom of the screen.

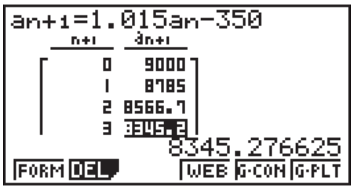


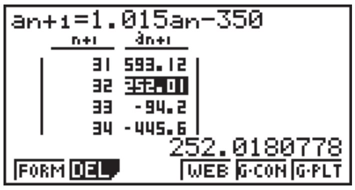


Press EXIT to return to the main screen and SET (F5) to define the starting point for the function and to specify which values are to be generated. A recursive sequence can start with a first term with a subscript of either zero or one:

$a_0, a_1, a_2, a_3, a_4, ...$ or $a_1, a_2, a_3, a_4, a_5 ...$

Either one of these can be used, provided you know what the terms stand for. In this case, a choice of zero is a good one, since that will mean that a subscript of n refers to the amount of money owing after n payments have been made. Choose to start with a_0 by pressing a_0 (F1). To finish defining the function, you need to give a value for a_0, the initial amount owing, and specify with *Start* and *End*which terms of the sequence you want to generate. Enter your choices, pressing EXE after each. These choices are shown below.

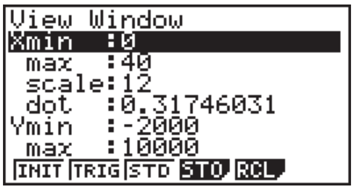




Return to the main screen with EXE and press F6 (TABL) to generate the table of values of the function shown above. Scroll down the table to see that the values are the same as those found earlier. After 32 payments, only \$252.02 is owing, so the car will be paid off after 33 months.

Obvious differences between the various ways of studying the problem are that all the results can be seen at once in the table and the spreadsheet, and you don't need to press the EXE key so many times. There are some other advantages of this way of studying recursion, however. One is that it is possible to draw a graph of the values, to get a better sense of what is happening.

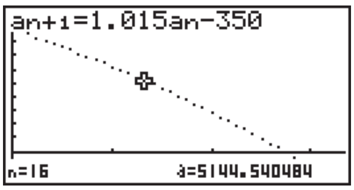
In this case, you need first to define a suitable viewing window such as the following.



The horizontal axis now represents the variable n, the term number, even though the viewing window still uses the symbol X. The choice of 12 for the tick marks on the horizontal gives a tick mark every year, which will help interpret the graph. The values of the function are given on the vertical axis (y); starting with a negative value provides space for text and the x-axis on the screen.

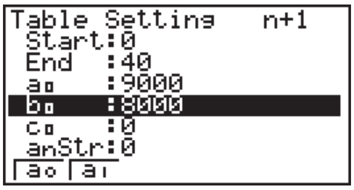
Return to the table screen and plot the table of values with G.PLT (F6). Although it is possible to plot a connected graph with G.CON (F5), it doesn't make sense in this case to join the points. The values of the function are *discrete*initially \$9000 is owed, and after one month \$8785 is owed, but at no stage is there an amount between these two owed. Nor does n = 0.3 or 0.57 make any sense here, since only whole payments are made.

The graph below shows that the amount owing decreases fairly steadily, and that the loan will be paid off in a little under three years. The screen below shows that you can trace the graph points using SHIFT F1 to avoid having to return to the table.

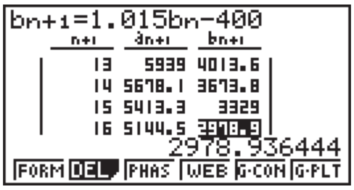


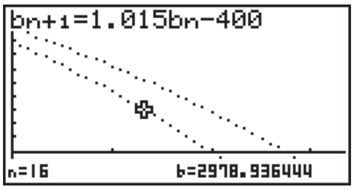
Another advantage of dealing with the car payment problem in this way is that changes can easily be explored. For example, when the table is showing, press FORM (F1) to return to the formula, which can be edited to consider other options for repayment. To see the effect of paying \$400 a month rather than \$350 a month, all you need to do is to change the 350 in the formula to 400. To consider the effect of a higher interest rate (such as 1.8% per month) change 1.015 to 1.018.

You can compare the alternatives *directly* by using the other recursive function at the same time. The screens below shows how to set up the calculator to compare the original repayment plan with an alternative in which $1000 deposit was paid initially (so that only $8000 is owing at the beginning of the loan) and monthly payments are $400 rather than $350.



Now both the table and the graph show the two plans together:





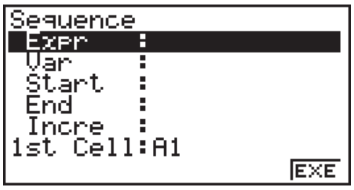
The graphs show that the new plan will lead to the loan being paid off in about nine months less time than the original.

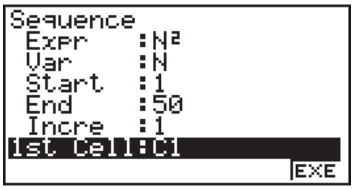
Another way of dealing with the data is to store table values into a list memory, and then to analyse them in Statistics mode.

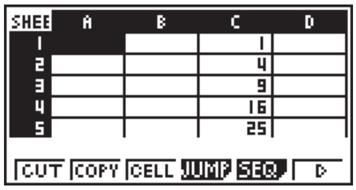
Sequences and series on spreadsheets

A very useful spreadsheet function is for the easy creation of sequences and series, when an expression for a general term of the sequence is known. (So the sequence is not recursively defined). We will consider an example to see how this works.

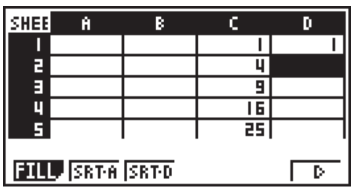
Tap EDIT (F2) and then SEQ (F5) to access the sequence generator shown at left below. Define a sequence in a spreadsheet column by giving an expression for its general term, using the ALPHA key for any variable. Enter the start, end and increment values (usually 1) for the terms, as well as the desired location of the first term of the sequence. The example below shows a sequence for the squares of the first 50 integers, starting with 1, and located in the column starting with C1.

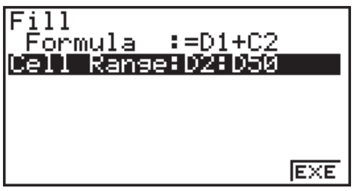


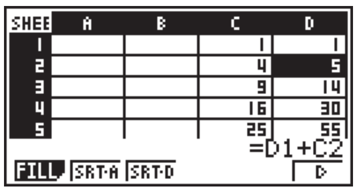




To examine the associated series, obtained by adding successive terms of the sequence, you need to define a suitable spreadsheet formula. For example, to construct the series in column D, start by entering the first term in D1. Then the screen in the middle below shows the formula in D2 to make each term the sum of the previous terms, increased by the next term in column C.







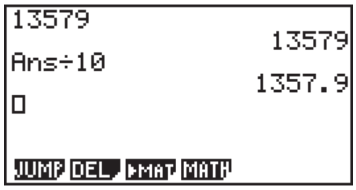
The resulting series is shown in the right screen. Check that $1^2 + 2^2 + 3^2 = 1 + 4 + 9 = 14$.

Sequences and series are described more extensively in Module 11.

Exercises

The main purpose of the exercises is to help you to develop your calculator skills.

1 What will you expect the next few numbers to be if the EXE key is pressed a few times, after the following has been completed?



Check your expectations by completing the iterations on your calculator.

2 What calculator steps will produce the arithmetic sequence below (sometimes called an *arithmetic progression*)?

$$34, 40, 46, 52, 58, 64, ...$$

Test your answer by trying it on the calculator.

3 What calculator steps will reproduce this geometric sequence below (sometimes called a *geometric progression*)?

$$12, 18, 27, 40.5, ...$$

Test your answer by trying it on the calculator.

4 A loan of $2000 is being repaid monthly with payments of $124. If interest of 1.8% per month is added, how long will it take for the loan to be repaid?

5 Set up a recursive sequence on your calculator, starting with 15, and with each term the square root of the previous term. Find the 20th term of the sequence.

6 Use the Spreadsheet to analyse the car loan described in the module.

a Start with 9000 in cell A1 and then use an appropriate formula to fill cells A2 to A40.

b Verify that the loan is paid off after 33 payments.

c Edit cell A1 to become 10000, to study a loan of $10 000. Notice that all other cells change automatically. How long does it now take to pay off the loan?

7 Find the first ten terms of a recursive sequence starting with 7 and with the recursive rule:

$$T_{n+1} = 5 - T_n$$

(T_n refers to the nth term of the sequence.)

8. Find the first five terms of the sequence starting with 3 and defined by

$$T_n = 1.5(T_{n-1} - 1)^2$$

9. Use the sequence command in a spreadsheet to generate a sequence of the first 20 cubes of the natural numbers.

Then generate the series associated with that sequence to find the sum of the first 10 cubes.

Activities

The main purpose of the activities is to help you to use your calculator to learn mathematics. You may find that some of them are too advanced for you. Ignore activities you don't yet understand.

1 A model for the growth of some virus cells was derived from some data. The model concerned was $S = 241(1.15)^t$ where S is the number of virus cells and t the number of hours of growth.

The first term of the sequence of virus cells is 241, and each term is 1.15 times the previous term. Use the calculator to generate the first five terms of this sequence.

What is the percentage rate of growth each hour, according to this model?

Describe what happens eventually for the growth of the virus cells.

2 Find the population (in millions) of your own country now and the annual growth rate for your country. Use these data and your calculator to study possible future population growth.

If the population continued to grow in the same way, what will it reach in 20 years from now?

How long will it take for the population to double, if it continues to grow at the current rate?

Suggest some reasons why a (human) population may *not* continue to grow at a steady rate for a long period of time.

3 A radioactive substance is decaying at a steady rate; each year it loses 2% of its mass. (So, after one year, it will have only 98% of the mass it started with.)

After how many years will it be half gone? (This is called the *half-life*.)

Would you expect the half-life to be more or less than this, for a substance decaying at 1% per year? Check your guess by finding the half-life.

4 Investigate what happens eventually with the recursive sequence given by $X_{n+1} = kX_n^2 - 1$, for different values of k, $1 \le k \le 2$.

Start with $X_0 = 0.4$, and with $k = 1$.

Once you have found what happens with $k = 1$, try a larger value of k.

5 The Fibonacci sequence 1, 1, 2, 3, 5, 7, 12, … begins with 1, 1, ... Each term is the sum of the previous two terms.

Investigate sequences constructed in the same way (i.e., for which each term is the sum of the previous two terms), but starting with a different pair of numbers.

For example, try starting with 1, 3, ... or with 1, 4, ... or with 2, 5, ...
(Don't start with 1, 2, ... , however. Why not?)

Examine the ratio of successive terms of the sequences.

Compare your observations with someone else's.

6 When people invest money, they earn interest, usually described as an annual percentage rate, such as 5% per annum. Many banks compound interest on investments on a daily or monthly basis, rather than annually, however. How much difference to these alternatives make in practice to an investment? Try some data for yourself, using your calculator to explore various possibilities. Discuss your observations and conclusions with others.

Notes for teachers

This module illustrates several ways in which the calculator can be used to explore basic ideas in iteration and recursion, including the use of spreadsheets to study these. A variety of ways of generating recursive sequences is explored, although it is expected that, in time, students may prefer one procedures over others in practice. The text of the module is intended to be read by students and will help them to start to see how the calculator can be used efficiently for various kinds of explorations related to iteration and recursion. The Activities are appropriate for students to complete with a partner or in a small group, so that they can discuss their observations and justify their conclusions and explore how numerical calculators can be powerful tools for addressing these mathematical ideas.

Answers to Exercises

1. 51, 40, 29, … 20, 40, 80, … 135.79, 13.579, 1.3579, … 2. [3] [4] [EXE] [+] [6] [EXE] [EXE] [EXE]
3. [1] [2] [EXE] [×] [1] [•] [5] [EXE] [EXE] [EXE] [EXE] 4. 20 months 5. 20^{th} term is 1.000005165
6. 39 payments 7. 7, -2, 7, -2, 7, -2, 7, -2, 7, -2 8. 6, 37.5, 1998.3, 5.98E6, 5.3E13 9. Use Sequence command with expression N^3 and variable N for N = 1 to 10. The sum is 3025.

Activities

1. Once students have followed this structured activity, they can explore other growth and decay models for themselves. [Answers: 241, 217.15, 318.72, 366.53, 421.51; growth is 15% per hour.]

2. This activity requires students to research data that applies to their own country and to then apply them to their calculations. Encourage students to explore the growth using various methods, including Recur mode and a spreadsheet. They should be cautious about assuming population growth rates remain constant and a good class discussion can be arranged to consider the reasons for their particular country. [Answers: growth rates are affected by health conditions, family size preferences, government regulations, immigration, military events, as well as other influences.]

3. Questions concerning growth and decay will help students to realise that substances decay at different rates. Although formal methods for finding half lives are available, these require sophisticated mathematical analysis, using logarithms and exponential functions. Good approximations can be obtained by generating a sequence of values in a table or a spreadsheet column and looking to see when the amount has approximately halved.

4. This activity provides students with an opportunity to study chaotic behaviour. A good way to do this is to plot successive values on a graph, after generating a table in Recur mode. A connected graph will make it easier to see the effects of small changes in initial conditions, or k, or both. Encourage students to consult information about chaos in popular sources to understand its modern significance.

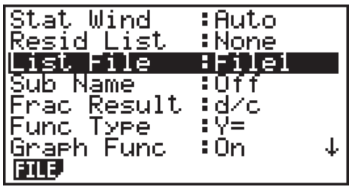
5 This activity is designed for students to address in Recur mode with a sequence defined as $a_{n+2} = a_{n+1} + a_n$, although it is also possible to explore it in a spreadsheet. Students (and others) may be surprised to find that the ratio of successive terms of the sequences approaches the golden ratio of 1.618 … or $(1 + \sqrt{5})/2$. The easiest way to see this is to create a second sequence, consisting of a_{n+2}/a_n and to scroll down to see how it converges, regardless of the two starting terms.

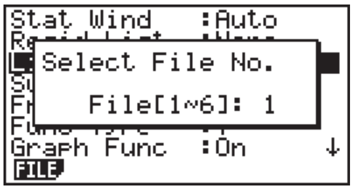
6. Compound interest is difficult to calculate by hand, but much easier with a spreadsheet and suitable formulas. It is much easier to multiply by suitable terms, so help students to find, say, 6% annual interest as 106% of the original amount. If interest is compounded more frequently, more calculations are needed, so that monthly compounding requires 100.5% of the original amount per month. Students may be surprised to find that more frequent compounding makes very little difference in practice, although it is frequently advertised otherwise.

Module 11
Lists, sequences and series

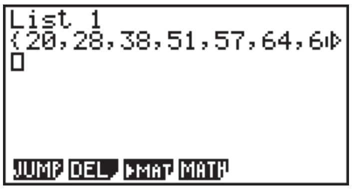
As well as individual numbers, the calculators have the capacity to deal efficiently with ordered sets of numbers called *lists*. There are some situations in mathematics where we need to deal with several numbers at once, so that a list is ideal. An obvious example is a list of data for analysis, as you saw in Module 4.

The calculator contains six files, each of which contains 26 lists for storing data, which are named *List 1*, *List 2, ..., List 26*, as you have already seen earlier. Altogether, the calculator can store 156 lists using the files. Each list can be created, accessed and modified in Statistics mode or in Run-Mat mode. You can also change from one File to another in Statistics mode, using the SET UP menu. You can store material in one file while you work in another.

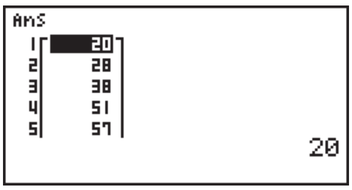


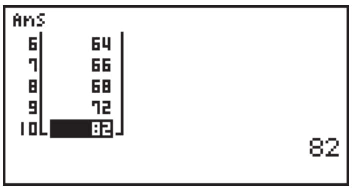


To see some of the list processing capabilities, tap MENU 1 to return to Run-Mat mode, and then [OPTN] [F1] to select the list commands. You can recall any of the lists in the calculator using the LIST ([F1]) command, although it is just as easy to use the keyboard directly with [SHIFT] [1]. The screen below shows that the list is given as an ordered set. (If you get a vertical column of data instead of a horizontal display, you may have the calculator set to Linear mode instead of Math mode, as described in Module 1.)



The list contains the most recent data you have stored into *List 1*. To see the list as a column, first select it with [▲] and then tap [EXE]. You can scroll the data on the screen with the cursor keys [▼] and [▲]. The screens below show the first five elements and the last five elements of the list of years used in the hotel data from Module 4. The number of each element is shown just to the left of the list itself. Use the [EXIT] key to return to the main screen in Run-Mat mode.





As well as recalling lists, a more powerful use of the calculator is to *transform* lists, in order to efficiently produce new lists. There were some examples of this in the data transformations section of Module 4. List transformations can also be dealt with directly in Run-Mat mode. There are some examples following for you to try.

[3] [SHIFT] [1] [1] [EXE]	produces a new list, with 3 times each *List 1* element
[5] [SHIFT] [1] [1] →[SHIFT] [1] [2] [EXE]	copies five times each element of *List 1* into *List 2*

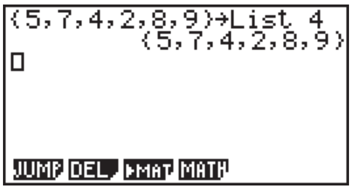
4*List 1*+2 →*List 5* produces a new list, stored into *List 5*, each element of which is two more than four times the corresponding original element of *List 1*. (This new list replaces any previous *List 5*.)

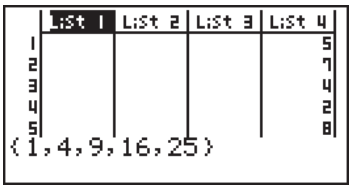
log *List 2*→*List 6* produces a new list, *List 6*, each element of which is the logarithm of the corresponding original element of *List 2*. (This new list replaces any previous *List 6*)

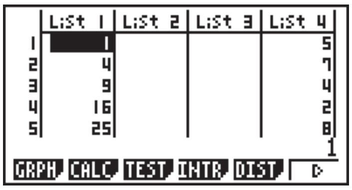
1.8*List 1*+32→*List 1* produces a new list, stored into *List 1*, each element of which is 32 more than 1.8 times the corresponding original element of *List 1*. (This new list *replaces* the original *List 1*). You might want to do this if the elements of *List 1* were Celsius temperatures, and you wanted to change them all to Fahrenheit, using the transformation, F = 1.8 x C + 32, as in Module 4.

In these kinds of ways, data transformations can be made for analysis. After making the changes above, you can check most easily in Statistics mode (MENU 2) that the transformations you made have actually been carried out.

As well as transforming existing lists, you can make a new list in Statistics mode, as you saw in Module 4. It is also possible to enter a sequence directly into a list, using the curly brackets (with SHIFT × and SHIFT ÷) that you saw previously in Module 2. The list completely replaces any existing list in this case. There are two examples below:

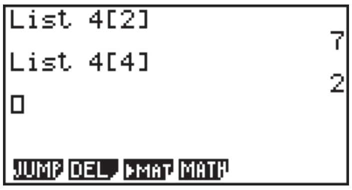


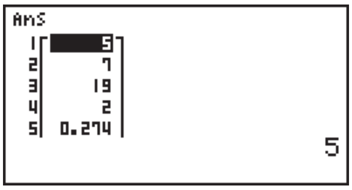




In the left screen, *List 4* (with six list elements) has been created in Run-Mat mode. The next two screens shows how to create a new *List 1* (with 5 list elements) in Statistics mode. First highlight the header at the top of *List 1*, and then type the list elements, using curly brackets. The new list will appear as soon as EXE is pressed.

Individual elements of a list can be retrieved or changed using the square brackets on the calculator. (Use SHIFT + and SHIFT − for these.) For example, the first screen below shows how to retrieve the second and the fourth elements of *List 4*.





In the middle and right screens, the third element of *List 4* has been replaced with the value 19, using the square brackets again. The fifth element has been replaced by a random number between 0 and 1.

Dealing with sequences

As well as dealing with data already entered into the calculator, lists can be used to *generate* data. In this way, *sequences* can be created and their terms added to produce a *series*. A sequence is a list of numbers in a certain order. The *Seq* command, to produce a list consisting of terms of a sequence, is available in the List menu after pressing [OPTN] [F1] and then Seq ([F5]).

The syntax for the *Seq* command contains five parts:

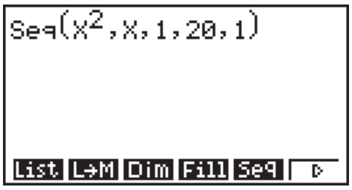
Seq(*formula,variable,first,last,step*)

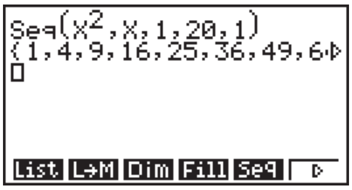
In this statement, *formula* refers to the general term, given as a function of the *variable*. Notice the use of the comma between each pair of elements.

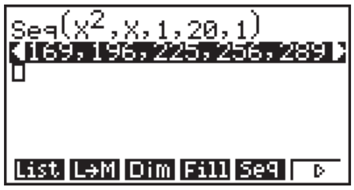
For example, use this command in Run-Mat mode to give the squares of the first twenty integers:

Seq(X^2,X,1,20,1)

The sequence is generated in set form, and can be scrolled right and left using [▶] and [◀] in order to see all the elements. First press [▲] to highlight the list itself.

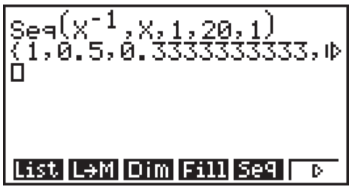


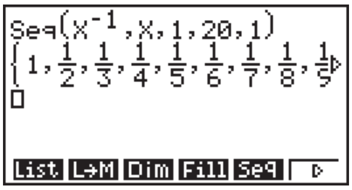


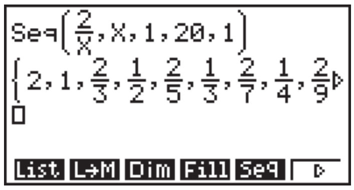


In Math mode, when a sequence comprises more than 20 elements, it is represented in the form of a vertical list instead of a set. You can scroll the sequence on the screen, using [▼] and [▲]:

The Sequence command generates lists elements according to the formula. For example, look carefully at the screens below. In the first screen, the formula for reciprocals, X^{-1}, generates results as decimals. In the middle screen, after tapping the [F↔D] key, the resulting list is given as fractions. In the third screen, the formula itself uses fraction notation, and so the resulting sequence is given as fractions.





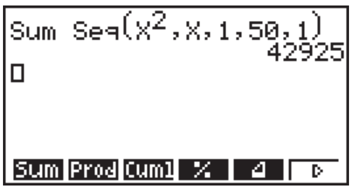


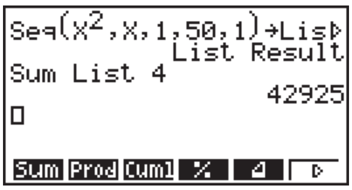
Although it is convenient to use *X* as the variable (because the [X,θ,T] key is on the keyboard, other variables can also be used. When using other variables, the [ALPHA] key must be pressed each time, so it takes a bit longer to construct the command.

A sequence generated with a *Seq* command can be stored directly as a list, using the memory store arrow. An example is the command:

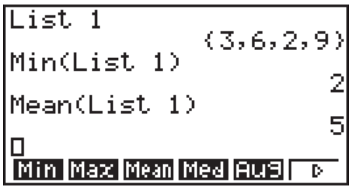
Seq(X^2,X,1,50,1)→ *List 4*[EXE]

The terms of a sequence can be added, using the *Sum* command (which is available in the List menu after pressing the continuation key [F6] twice). Either *Sum Seq*(X^2,X,1,50,1)[EXE] or in the above case, *Sum List 4*[EXE] will suffice.

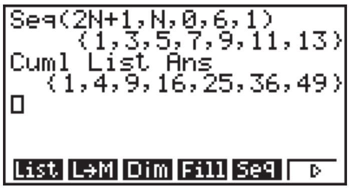




You can analyse lists as data, using other list commands for finding the minimum, maximum, mean and median of a data list. These commands are available in the List menu after pressing the continuation key [F6].) The syntax is illustrated in the following screen.



A useful list command (especially for frequencies and for series) is *Cuml* which stands for *cumulative*. This command has the effect of adding successive terms of a list. To illustrate, the screen below shows the cumulative command, obtained with Cuml ([F3]), on a list that consists of the first six odd numbers. This screen also illustrates the use of the *Ans* variable (obtained with [SHIFT] [(−)]) to access the most recent list.

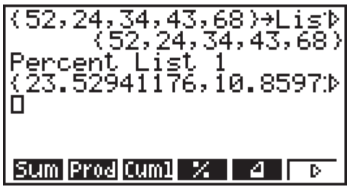


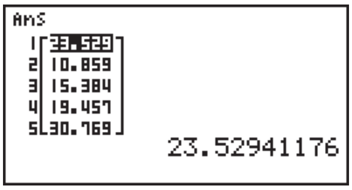
The resulting list contains the cumulative sums, 1, 1+3, 1 + 3 + 5, 1 + 3 + 5 + 7, … . You can see from the screen the surprising result that the results are all square numbers. (A sequence of this sort involving successive sums is called a *series*, described later in this module.)

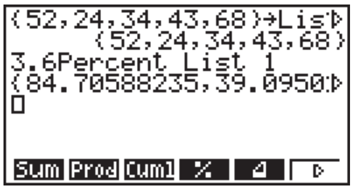
The List percent command % ([F4]) is also very useful, as it converts each of the values in a list into a percentage of the total of the list. For example, consider the following data showing the numbers of students in a school who play a guitar:

Year level	8	9	10	11	12
Frequency	52	24	34	43	68

Store the frequencies in *List 1*. The percent command will convert them to percentages of the total of *List 1*:





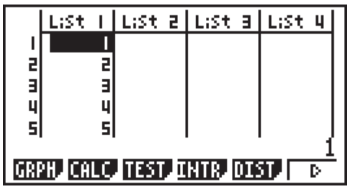


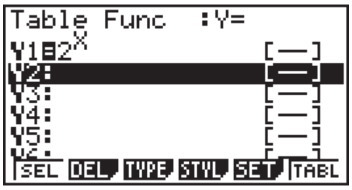
The result shows that almost 31% of the school's guitarists are in Year 12, while about 23.5% are in Year 8. If you wanted to draw a pie chart of these data, notice how the command in the third screen would automatically convert the frequencies in *List 1* into the appropriate number of degrees in the pie chart. It seems that 85^{o} for the Year 8 slice and 39^{o} for the Year 9 slice are appropriate.

Lists and tables

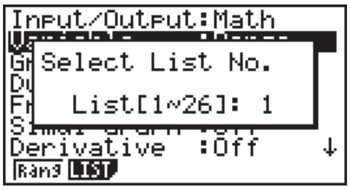
A table provides another method of generating a sequence, when an explicit formula for the sequence is available. Consider, for example, the first fifteen terms of the sequence of powers of 2.

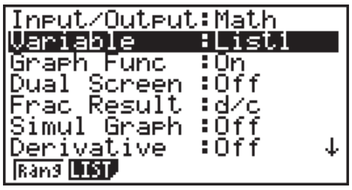
Start by storing the first fifteen counting numbers into a list, *List 1* as shown. Then switch to Table mode and define the sequence with the function $f(x) = 2^x$.

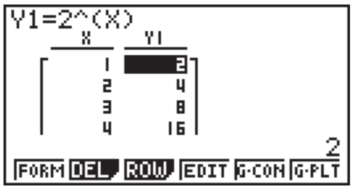




Use the SET UP menu to select *List 1* for the *Variable* instead of *Range*.

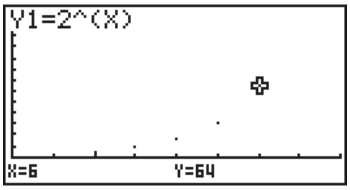




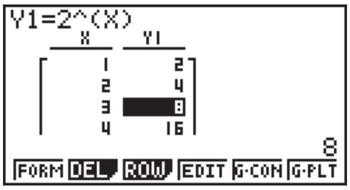


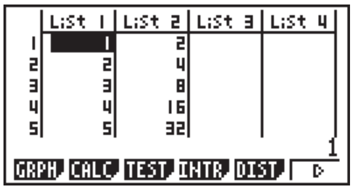
After you return with [EXIT], draw the table. The effect of the *Variable* change is that the function is evaluated for values of X in *List 1* only, as shown above at right. (The *Start*, *End* and *Step* values in the SET menu are ignored, when a finite list is used for the variable.)

The sequence of powers of 2 can be plotted using either G.CON ([F5]) or G.PLT ([F6]).



The sequence can also be copied into a list, using the *List Memory* command. Firstly, make sure the cursor is in the column to be copied. Press LMEM ([OPTN] [F1]) to select the *List Memory* menu and then select *List 2*, finishing with [EXE].

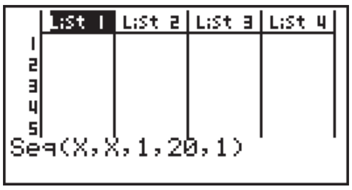


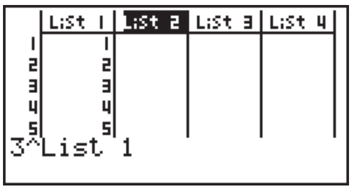


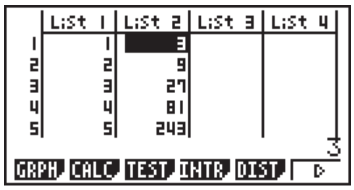
The results are shown above in Statistics mode, with the sequence numbers (1 to 15) in *List 1* and the terms of the sequence itself in *List 2*.

Make sure that you restore the *Variable* setting to *Range* in Table mode, when you have finished.

It is also possible to generate sequences directly in Statistics mode, without the use of tables at all. The screens below show firstly how to generate the sequence of the first twenty counting numbers in List 1 and then how to use this to generate the corresponding powers of three in *List 2*.





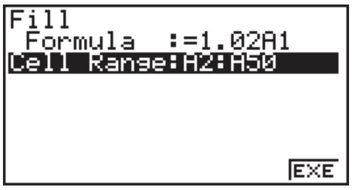


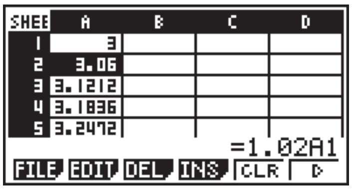
In the first screen, the *List 1* header has been highlighted and a sequence generated using the *Seq* command (obtained with [OPTN] [F1] [F5]). In the second screen, the *List 2* header has been highlighted and a transformation used to generate the powers of three.

Lists and spreadsheets

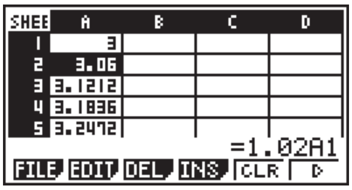
When sequences are defined recursively, it is necessary to use Recur mode or a spreadsheet rather than Table mode to generate and explore a list. Sequences can be generated within the spreadsheet and individual columns of the spreadsheet saved as lists. In addition, you can import existing lists into a spreadsheet for analysis. Enter Spreadsheet mode with MENU 4.

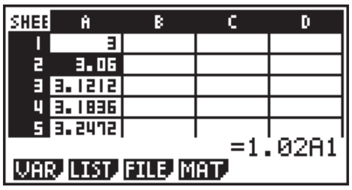
The following spreadsheet shows a sequence of city populations in which each term is 2% more than the previous term, to reflect annual growth rate of 2%. The population begins with 3 million people, represented by 3 in cell A1. The formula used generates a total of 50 terms, including the first one. To enter the formula, start with EDIT (F2) and then FILL (F6 F1).



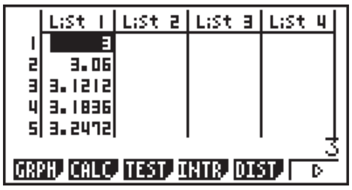


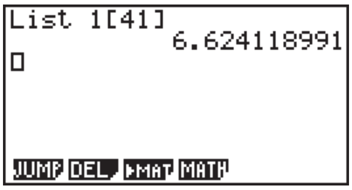
To transfer all 50 terms to a List, make sure the cursor is in column A, then select F6 and STO (F3). To store the column as a list, choose LIST (F2) and complete the Store command as shown.





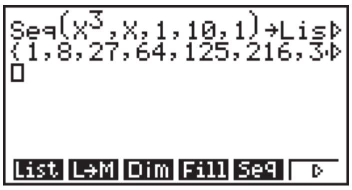
You can check that the list has been saved correctly in either Statistics mode or in Run-Mat mode.

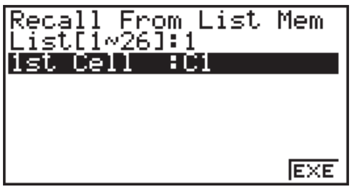


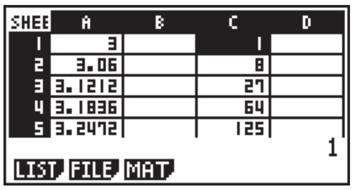


The screen at right above shows the 41st term of the list is about 6.624, suggesting that the population will grow to about 6.6 million after forty years if it continues to grow at 2% per annum.

To transfer a list into a spreadsheet, use RCL rather than STO. For example, the first ten cubes are generated and stored in *List 1* below with a *Seq* command in the first screen.







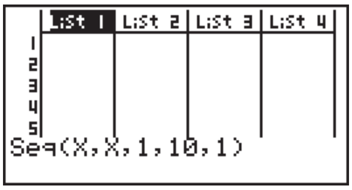
To import this list into the spreadsheet, open the spreadsheet, tap RCL (F6 and then F4). Since a list is involved, select LIST (F1) and complete the details as shown in the middle screen above. When you tap EXE, the ten terms in the list are pasted into the spreadsheet, starting with the designated cell, C1, as shown in the third screen

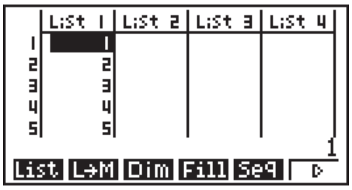
Graphing a list

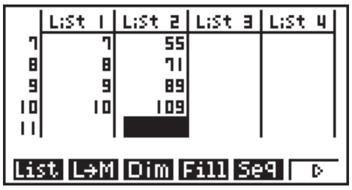
List elements can be graphed in several ways, as you have already seen, in Statistics mode, in Table mode, in Recur mode and also in the Spreadsheet. Graphing a list can often help to clarify its mathematical character, and may even be useful in a search for a pattern to describe an unknown sequence. Consider, for example, a sequence with the first ten terms as follows:

1, 5, 11, 19, 29, 41, 55, 71, 89, 109, …

To examine this sequence graphically, begin by generating a list of the first ten positive integers and storing them into *List 1*. You can do this efficiently in Run-Mat or Statistics mode, using a *Seq* command. Then store the ten terms of the sequence above into List 2.

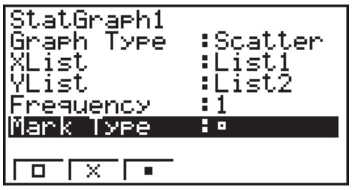


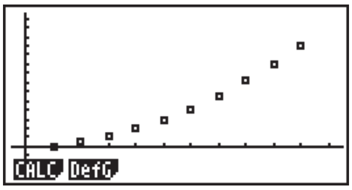




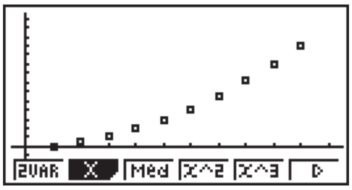
Notice that each of the two lists in this example has the same number (10) elements. Otherwise, the calculator would not be able to graph them.

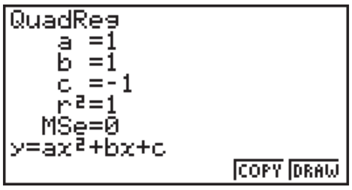
Graph the sequence using a scatter plot. (Refer to Module 4 if you are not sure about how to do this.) The *x*-values are the sequence numbers, which are in *List 1*. The terms of the sequence in List 2 are the *y*-values in this case.

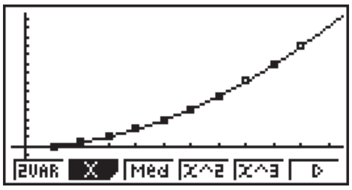




The scatter plot provides a visual image of the sequence. The sequence is curvilinear, clearly not linear, and it may even be quadratic. To test out this hunch, tap CALC (F1) and then X^2 (F4).

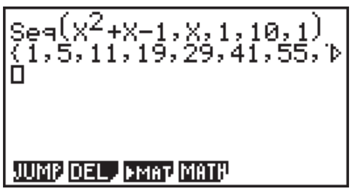




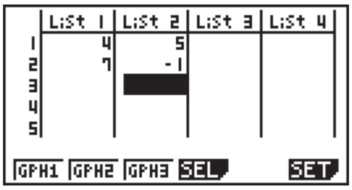


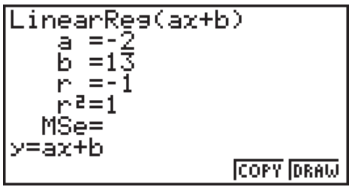
You can use the regression coefficients to find the coefficients of quadratic or linear functions to fit the data points. In this case, the data are readily seen to be quadratic, so the X^2 regression is appropriate. The screens above show that the calculator finds the coefficients $a = 1$, $b = 1$ and $c = -1$, suggesting (correctly) that the relationship is $y = 1x^2 + (1)x + (-1)$ or $y = x^2 + x - 1$.

The *Seq* command below verifies that this function does indeed seem to generate the sequence we started with.



You can use a similar method to quickly determine the equation of the unique line through two points or the unique parabola through three points. The screens below show an example, with the points (4,5) and (7,-1) as the data. The linear regression function shows the equation of the line through the two points, given by $y = 13 - 2x$





. You can check mentally that each of the two points is on this line.

Dealing with series

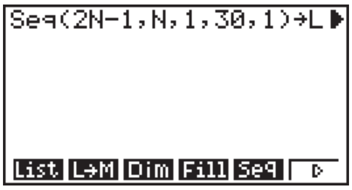
Every sequence has a *series* associated with it. A series is a special kind of sequence: it is the sequence of *partial sums* of the sequence. In the table below, for example, each term of the series is the sum of the previous terms of the sequence. (e.g., 4 = 1 + 3, 9 = 1 + 3 + 5, etc …).

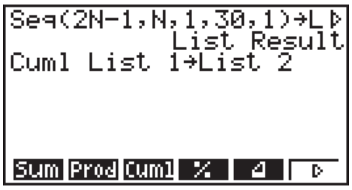
Term	1	2	3	4	5	...
Sequence	1	3	5	7	9	...
Series	1	4	9	16	25	...

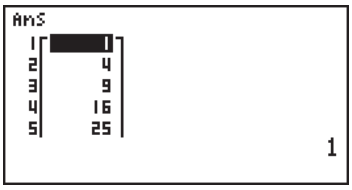
The *Cuml* command is a useful way of producing a series, when you have an explicit definition of the corresponding sequence, since it allows you to accumulate successive terms easily. Consider the example above. The sequence can be defined by the general relationship for the *n*th term, T_n:

$$T_n = 2n - 1,\ n \geq 1$$

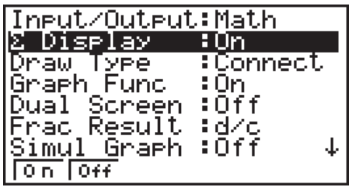
The first 20 terms of the sequence of odd numbers can be generated and stored in *List 1*, as shown below. Then the *Cuml* command can be used to store the series (i.e., the sequence of partial sums) into *List 2*, as shown in the second and third screens. In this case, the series is especially interesting, as successive terms are the squares of the natural numbers.



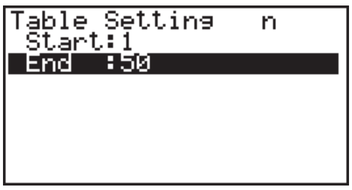


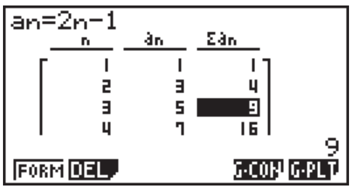


For many sequences, especially those with recursive definitions, the calculator capabilities in Recur Mode are even more useful. Press MENU 8 to enter Recur mode, and select a_n ([F1]) as the type of function, as described in the previous module. To illustrate the calculator procedures, we will continue to use the same sequence and series.

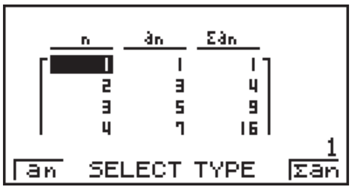


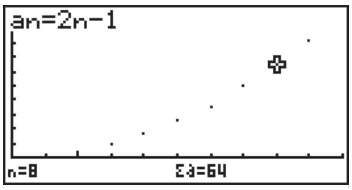
Tap [SHIFT] [MENU] to access the SET UP for this mode, and turn on *Σ Display*, as above, which will display a series associated with a sequence. (Σ is the Greek capital letter equivalent to the English S, and is often used in mathematics to denote a sum.) Odd numbers are one less than even numbers, so define the sequence of odd numbers by the relationship that the nth term is $a_n = 2n - 1$, as before. Set the values for *n* from 1 to 50 and draw a table, as shown in the following screens:





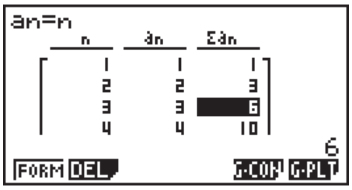
The table has the sequence of the first fifty odd numbers in the middle column, and the associated series in the right column. The table may be scrolled in the usual way. The information can also be plotted. When you tap G.PLT (F6) you need to choose between plotting the sequence or the series.

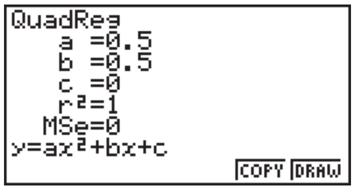




The screen above shows a plot of the series, after a viewing window was suitably defined. The characteristic parabolic shape suggests that the series is a quadratic (as is fairly clear from the first few elements, which are the squares of the positive integers.)

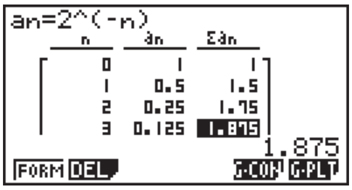
The list memory and statistics capabilities can be used to explore series in Statistics mode after they have been constructed in RECUR mode, as noted earlier in this module. For example, the series below shows the sums of successive counting numbers. Transfer the first and third columns to Statistics mode, using OPTN F1. Once in Statistics mode, a scatter plot suggests that the relationship is quadratic, and the quadratic regression provides the coefficients, as shown above and suggests that the sum of the first n counting numbers is $S_n = 0.5n^2 + 0.5n$. A better way of expressing this formula is $S_n = \frac{n(n+1)}{2}$. Check mentally that this formula seems to give the series values for $n = 1, 2, 3, \ldots$. You may recognise this sequence as the sequence of triangular numbers.

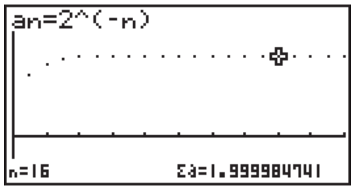




As a final example, consider the series given by successive halvings: $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$

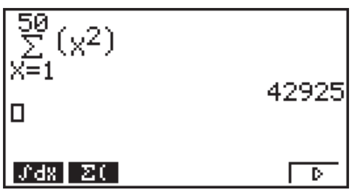
You can think of this as related to the sequence of negative powers of two. The three screens below show how the sequence can be defined, the series tabulated and then plotted. The table and the graph suggest that this series *converges* very quickly to 2.





If you don't want to find *all* the terms of a series, but only need particular values, the Σ command in Run-Mat mode is very useful, and very powerful. Tap MATH (F4) then F6 and then Σ (F2) in Run-Mat mode to see the summation command. The usual mathematical notation is used in Math mode, so the command shown below finds the sum of the squares of the first 50 counting numbers:

$$\sum_{x=1}^{50} x^2 = 1^2 + 2^2 + 3^2 + \ldots + 50^2$$



Exercises

The main purpose of the exercises is to help you to develop your calculator skills.

1 a In Run-Mat mode, insert {2,5,3,4,6,1,12} into *List 1*. Display the list.

b Create *List 2* to consist of five more than four times the elements of *List 1*.

c In Statistics mode, copy *List 2* into *List 3*. Then use a transformation to replace each element of *List 3* with its common logarithm (use the [log] key to get common logarithms, $\log_{10}$).

2 a Use the command Seq (3X+2,X,1,30,1) in Run-Mat mode to generate a list of the first thirty terms of the arithmetic progression: 5, 8, 11, 14, … Use the list to find the 25^{th} term.

b Use the *cumulative* command to generate the arithmetic series corresponding to the arithmetic progression in part a. Find the 20^{th} term of the series.

c Use the command Seq (2x3X-1,X,1,20,1) and the *cumulative* command to generate the first twenty terms of the geometric progression 2, 6, 18, 54, … and its associated geometric series. Find the tenth term of the geometric progression and the eighth term of the series.

3 In Run-Mat mode, generate a sequence of the first ten cubes of counting numbers. Store the sequence in a list. Then use the *Sum* command in the List menu to evaluate $1^3 + 2^3 + ... + 10^3$.

4 a Use a single command to find the sum of the first one hundred counting numbers.

b Edit your command in part a to find the sum of the cubes of the first 100 counting numbers.

c Which is larger: the sum of the cubes of the first hundred counting numbers or the cube of the sum of the first hundred counting numbers?

5 As part of an investigation to determine how many matches were needed to make certain patterns, the following table of data was generated:

Shape number (*S*)	1	2	3	4	5	6	7	8	9	10
Matches (*M*)	2	7	16	29	46	67	92	121	154	191

a The number of matches (*M*) seems to be a quadratic function of the shape number (*S*). Use Statistics mode to determine the precise function. Copy it to the function list before you Draw it.

b In Table mode, use the copied function to make a table to check that the function actually fits all of the data in the table above.

c Plot the sequence from Table mode, using G.PLT ([F6]) to plot discrete points. The viewing window has already been set by the calculator in Statistics mode. Notice that you can trace the plotted points to check that the data have been completely recovered by the quadratic model.

6 The *n*th term of a sequence is given by $T_n = n^2 - 5n$, for $n = 1, 2, 3, ...$ Find the sum of the first 15 terms of this sequence. Then find the sum of the following 15 terms.

7 Use Statistics mode to find the equation of the line joining the points (3,-1) and (-5,-7).

Activities

The main purpose of the activities is to help you to use your calculator to learn mathematics. You may find that some of them are too advanced for you. Ignore activities you don't yet understand.

1 Generate some sequences using Seq (3^{-X},X,0,*k*,1) for some different values of *k*. Study the command carefully and start with $k = 1$, $k = 2$ and $k = 3$ to see what is happening.

Investigate what happens when you add the terms of this sequence with Sum Seq (3^{-X},X,0,*k*,1).

Try some larger values for *k* as well.

2 a There is only one parabola (with formula $y = ax^2 + bx + c$) that goes through the three points, (0,4), (1,6) and (4,0). Use your calculator to find the unique values of *a*, *b* and *c*. Check in Table mode that the function generates the three points. Can you find a parabola passing through *any* three points?

b Use your calculator to find a cubic function passing through (2,21), (0,-1), (-1,-9) and (1,5).

c In similar ways, investigate functions that pass through five points.

3 The exponential function, e^x, can be defined by a remarkable *infinite series*

$$e^x = \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$$

a Use Sum Seq(1/X!,X,0,10,1) to find the sum of the first ten terms for e^1. Then try some larger numbers of terms. (Edit your previous command each time.) You may prefer to use Recur mode instead of Run-Mat mode.

b Of course, a calculator cannot give an infinite number of terms, so you should expect only an approximation to the result. How many terms does it take to get close to *e*= 2.718281828?

c Investigate this series for some different values of *x*, such as $x = 2$ and $x = 3$.

4 Use Statistics mode to help you to find a formula S_n for the sum of the squares of successive counting numbers. Check that your formula works to give the sum of the first ten squares correctly:

$$S_{10} = 1^2 + 2^2 + 3^2 + \ldots + 10^2 = 385$$

5 The harmonic series of first powers,

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \ldots$$

does not *converge* to a value, unlike the other harmonic series. It *diverges*, or continues to grow in size as more terms are taken.

Investigate this series on your calculator.

6 Find a formula for the sum of the cubes of the counting numbers.

What is the relationship between the triangular numbers (1, 3, 6, 10, 15, ...) and such sums?

Notes for teachers

This module illustrates several ways in which lists can be used on the calculators to explore various aspects of sequences and series, to help students understand their meaning, operations and significance. Lists can be used to store the terms of a finite sequence and list commands can be used with the list elements, most notably to find a particular term or evaluate a series. The text of the module is intended to be read by students and will help them to start to see how the calculator can be used efficiently for various kinds of explorations. The Activities are appropriate for students to complete with a partner or in a small group, so that they can discuss their observations and justify their conclusions.

Answers to Exercises

2. (a) 77 (b) 1025 (c) 39366, 6560 3. 3025 4. (a) Sum Seq (X,X,1,100,1) gives 5050
(b) Sum Seq (X,X,1,100,1) gives 25 502 500 (c) cube of the sum is much larger
5. (a) $M = 2S^2 - S + 1$ 6. 640 and 6490 7. $y = 0.75x - 3.25$

Activities

1. This activity allows students to explore the nature of this sequence. Small values of k will allow them to see successive terms as fractions, while the process of adding the terms will help them to see the process of rapid convergence. Work of this kind offers a good introduction to the idea of an infinite series.

2. This activity extends the idea of fitting curves to points, introduced in the text as fitting a line to two points, using graphing capabilities in Statistics mode. Three points determine a parabola and four a cubic, subject to some restrictions. [Answers: (a) $y = -x^2 + 3x + 4$ (b) $y = 2x^3 - x^2 + 5x - 1$]

3. While calculators cannot deal directly with infinite series, activities of this kind permit students to investigate a number of terms, in order to study ideas of convergence and divergence at first hand. There are several ways of doing this, using both Run-Mat mode and Recur mode, and you should encourage pairs of students to use more than one approach to deepen their understanding. This particular series converges very quickly, and students should find the result for e without difficulty. [Answers: the sum of the first ten terms is 2.718281801, while only 12 terms are needed to obtain 2.718281828.]

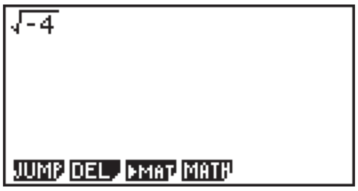
4. In this activity, students will need to generate a sequence of squares of integers and then use a suitable method to generate the associated series. This can be done in either Recur or Run-Mat mode. They will then need to use Statistics mode to look for a formula for the series, similar to the example in the text for summing integers. They may need to be careful with the accuracy of the calculations, depending on the numbers of terms generated, and interpret small values as zero. [Answer: the expression $n^3/3 + n^2/2 + n/6$ or $n(2n + 1)(n + 1)/6$ gives the sum of the first n squares.]

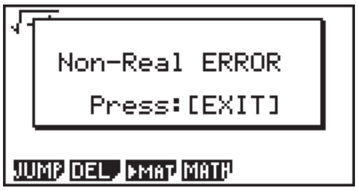
5. This activity is included in part for students to appreciate that some sequences do not converge, even though intuition might suggest that they should (with each successive term adding little to the total). Encourage students to explore a range of values, and to notice that the sum is always larger for more values, in contrast to some of the convergent sequences studied. Use Σ in Run-Mat mode for an efficient, although slow, method. (Of course a formal proof of the result is well beyond secondary school.)

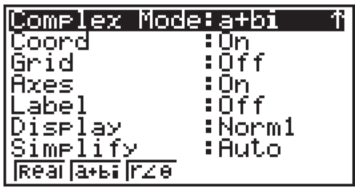
6. This activity is similar in character to Activity 4, although less direction is offered to students, and they are required to make use of their calculator to develop hypotheses about relationships, for later careful study or proof. [Answer: $n^4/4 + n^3/2 + n^2/4$ or $n^2(n + 1)^2/4$ gives the sum of the first n cubes; this is the square of the nth triangular number]

Module 12
Complex numbers

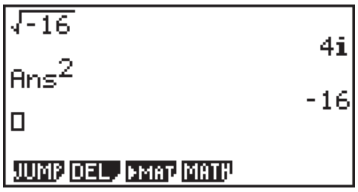
You have already seen that sometimes the calculator produces results that are *complex numbers* The most likely place to see this is when you find all the solutions to quadratic or cubic equations. There are three solutions to each cubic equation and two to each quadratic, but sometimes the solutions are not real numbers. Another possible place for complex numbers to occur is when you find square roots, since square roots of negative numbers are complex numbers. If you try to find the square root of a negative number, you may get an error, as shown below, since the calculator defaults to real numbers when not in complex mode.







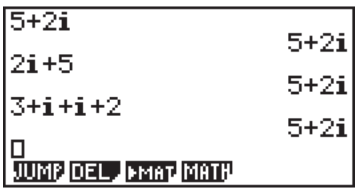
The screen above shows how to change the SET UP in Run-Mat mode to allow complex results to be displayed. Set *Complex Mode* to a + b**i** for most purposes. The complex number ***i*** is a square root of -1, defined by the relationship: $i^2 = -1$. Then, complex numbers are displayed when needed.

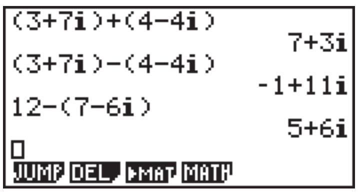


Square roots are related to quadratic equations. The screen shows that $\sqrt{-16} = 4i$; this is one of the solutions to the equation $x^2 + 16 = 0$, the other solution being $-4i$. (Notice that the calculator only gives one of the square roots of any number, including real numbers.)

Complex numbers in general consist of two parts, a *real part* and an *imaginary part.* These are historical names, and the words do not have their usual meanings. In particular, the imaginary part is not a figment of the imagination – it is just as 'real' as the real part. Complex numbers have been studied for hundreds of years, and first arose in the context of solving equations, especially cubic equations. Nowadays, they are used for all sorts of purposes, including the study of electric circuits, where they are especially suitable for representing *impedance* and *current.*

The complex number ***i*** may be entered into the calculator from the keyboard, using SHIFT 0 (even in Real mode). The screen below shows that the calculator expresses complex numbers by giving the real part followed by the imaginary part. This is sometimes referred to as the *algebraic* form of the number. (Complex numbers are sometimes expressed in other forms, as you will see shortly.)



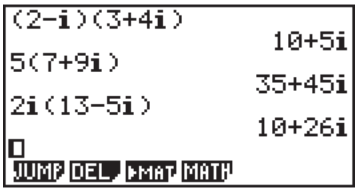


Complex numbers are added and subtracted by adding and subtracting their respective real and imaginary parts, as the screen at right above shows. This is a bit like adding 'like terms' in algebra. In most cases, it is as easy to do this in your head as to use the calculator. Multiplication and division are a little harder to do in your head, however, since they usually require you to use the relationship $\boldsymbol{i}^2 = -1$, in order to simplify a result. So a calculator may save you work in such cases.

For example, to evaluate the product of $2 - \boldsymbol{i}$ and $3 + 4\boldsymbol{i}$, you can think of them as binomials, as shown below:

$$\begin{aligned}(2 - \boldsymbol{i})(3 + 4\boldsymbol{i}) &= 2\text{x}3 + 2\text{x}4\boldsymbol{i} - \boldsymbol{i}\text{x}3 - \boldsymbol{i}\text{x}4\boldsymbol{i} \\ &= 6 + 8\boldsymbol{i} - 3\boldsymbol{i} - 4\boldsymbol{i}^2 \\ &= 6 + 5\boldsymbol{i} + 4 \\ &= 10 + 5\boldsymbol{i}\end{aligned}$$

As you can see below, the calculator handles simplification work like this automatically:

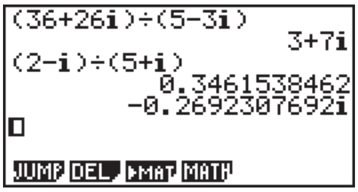


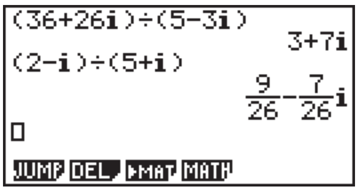
Division of two complex numbers is even more complicated, since you need to make sure that the denominator does not have any complex parts to end up with your answer in the appropriate form. Look carefully at the example shown below, dividing $36 + 26\boldsymbol{i}$ by $5 - 3\boldsymbol{i}$:

$$\begin{aligned}\frac{36 + 26\boldsymbol{i}}{5 - 3\boldsymbol{i}} &= \frac{36 + 26\boldsymbol{i}}{5 - 3\boldsymbol{i}} \text{ x } \frac{5 + 3\boldsymbol{i}}{5 + 3\boldsymbol{i}} \\ &= \frac{(36 + 26\boldsymbol{i})(5 + 3\boldsymbol{i})}{(5 - 3\boldsymbol{i})(5 + 3\boldsymbol{i})} \\ &= \frac{102 + 238\boldsymbol{i}}{34} \\ &= 3 + 7\boldsymbol{i}\end{aligned}$$

Notice the effect in the denominator of multiplying the complex number $5 - 3\boldsymbol{i}$ by its *conjugate*, $5 + 3\boldsymbol{i}$, is to give a real number, $5^2 - (3\boldsymbol{i})^2 = 34$. The conjugate of a complex number has the same real part as the original number, while the sign of the imaginary part is reversed.

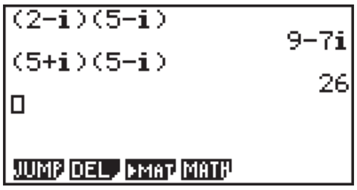
The calculator handles the division process efficiently, without showing all the intermediate steps. However, it will not automatically give fractional answers when they might be expected or preferred, as you can see in the first screen below for $(2 - \boldsymbol{i}) \div (5 + \boldsymbol{i})$.





However, the [F↔D] key allows you to represent a (rational) result in fractions instead of decimals. The result of tapping the [F↔D] key after the result is given is shown in the screen at right above. In fact, as the next screen shows, if the division had been entered as a fraction, using the [a b/c] key, a fractional result would have been obtained immediately:

The source of the last result on the calculator is not immediately clear. If you want to see what is happening, it may be helpful to use the calculator to do *some* of the arithmetic, and to then do the rest yourself. Firstly, note the two results below, multiplying each of the numerator and the denominator of the quotient (2 – ***i***)÷(5 + ***i***) by the complex number (5 – ***i***).

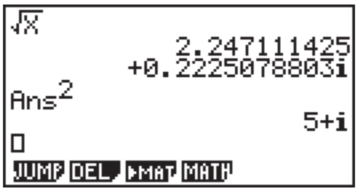


These results from the calculator can be used to help you to write down the result that was produced by the calculator using the [÷] key.

$$\frac{2-i}{5+i}=\frac{(2-i)(5-i)}{(5+i)(5-i)}=\frac{9-7i}{26}=\frac{9}{26}-\frac{7i}{26}$$

The choice of multiplying by (5 – ***i***) was made in the example above to ensure that the numerator did not include a complex number. Note that (5 – ***i***) and (5 + ***i***) are conjugates of each other.

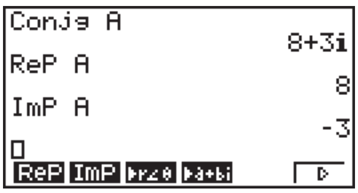
Complex numbers can be stored in the alphabetic memories and recalled in the same way as real numbers. The next screens show a few examples.



Most of the 'scientific' functions will *not* work on the calculator in complex mode, however, usually giving an Ma Error (i.e., a mathematical error, suggesting that the operation concerned is not mathematically meaningful).

In Run-Mat mode, bring the complex menu (CPLX) to the screen with [OPTN] [F3]. This allows for some operations with complex numbers to be performed easily. It also allows you to access ***i*** with a single key: [F1](***i***) instead of [SHIFT] [0], which requires two key steps. If you are using complex numbers a lot, this may be preferable.

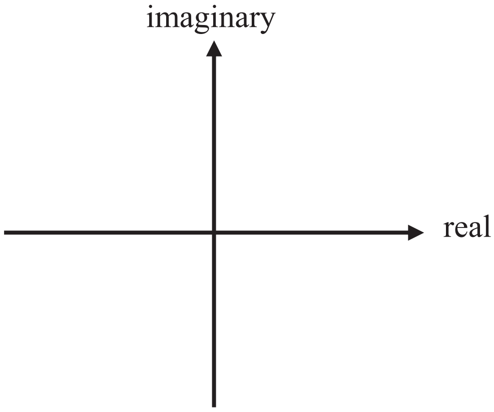
The functions in the complex menu can be used with either variables or complex numbers. For example, to get the conjugate of a complex number, use Conj ([F4]). The real part (ReP) and the imaginary part (ImP) of a complex number are available in the other complex menu, after [F6].



In each case, the command comes first, followed by the number or variable, as shown above for the stored value A = 8 – 3***i***.

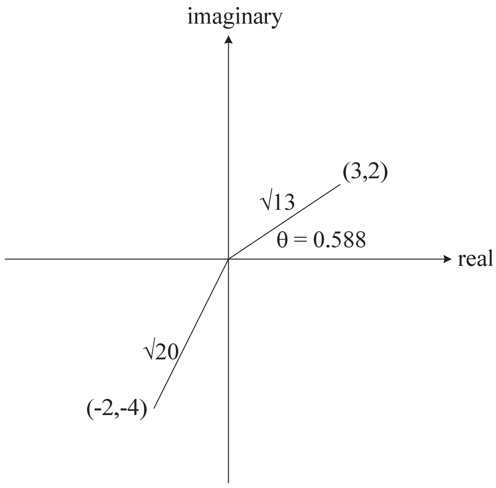
Graphing complex numbers

Real numbers can be graphed on a number line. Since complex numbers have two parts, a real part and an imaginary part, they can be graphed on a plane. The plane formed by the two axes shown below is called the *complex plane.* The *x*-axis is called the *real axis* and the *y*-axis is called the *imaginary axis.* Each complex number can be represented as a point on the complex plane using the real and imaginary parts as coordinates. Diagrams representing complex numbers in this way are often called *Argand diagrams,* named after one of the originators of the idea around two hundred years ago.



Sometimes complex numbers are represented in coordinate form, to help visualise their location on the complex plane. For example, the complex number $3 + 2\boldsymbol{i}$ can be represented as (3,2) and $-2 - 4\boldsymbol{i}$ can be represented as (-2,-4). This is sometimes also called Cartesian form.

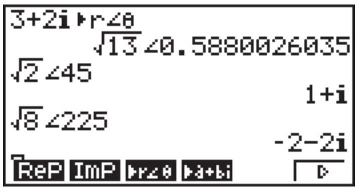
These two complex numbers are both graphed below. The points are also often represented by line segments joining the point to the origin. This leads to another way of representing complex numbers, using polar coordinates. Each number on the complex plane can be represented in terms of its distance from the origin and the angle the line segment makes with the positive real axis. The angle can be measured in either radians or degrees.



The length of the line is called the *modulus* or the *amplitude,* and is generally represented with the absolute value symbol | |. Sometimes, as on these calculators, it is called the *absolute value* of the complex number. The angle is generally called the *argument* of the complex number, often abbreviated to *arg.*

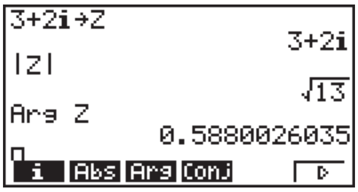
For example, in the diagram above, if the complex number is $z = 3 + 2\boldsymbol{i}$, we can represent z in coordinate form as (3,2). Also, $|z| = \sqrt{13} = 3.606$ and $\arg z = 0.588$ (in radians). So we could represent z in *polar form* as [3.606,0.588]. (Remember that polar coordinates are shown with square brackets so they don't get confused with rectangular coordinates.)

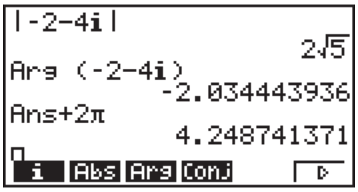
The calculator will allow you to move between coordinate form and polar form. Start in the Complex menu by tapping F6. Then F3 and F4 allow for the two representations to be used. The screen below shows how to represent a number in polar form.



To represent a number in coordinate form, use the angle symbol (in yellow on the keyboard, accessible with SHIFT F↔D). For the second two commands in the screen above, the angles were first changed to degrees in SET UP.

The calculator contains functions for both the absolute value Abs (F2)and the argument Arg (F3) of complex numbers. Set the angle measures to radians before you start. Look carefully at the examples in these two screens:

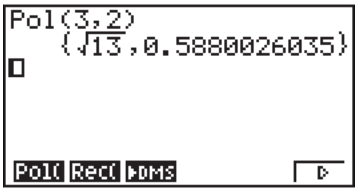


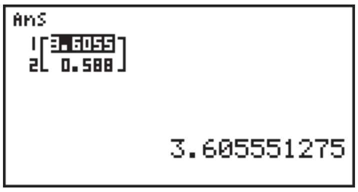


The values seem correct for the first example on the left, but the argument of the second complex number -2 – 4$\boldsymbol{i}$ is given as a negative number. Complex numbers can have many arguments, since you can turn more than one full turn (in either direction) before reaching them.

Rather than try to represent an infinite number of arguments, each differing by 2π, the calculator just displays one argument in the interval from -π to π, with negative values meaning turns in a clockwise direction. This is sometimes called the *principal argument* of the complex number. In this case, if a positive argument is sought, you need to add a full turn in the anticlockwise direction by adding 2π to the value given. So the complex number -2 – 4$\boldsymbol{i}$ shown on the Argand diagram above can be represented in polar form as [2√5,4.249] or [2√5,-2.034] or, indeed, an infinite number of other ways.

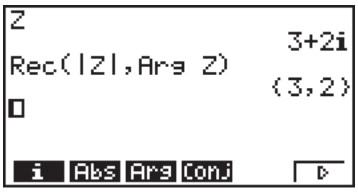
There is another way of moving between rectangular and polar forms of complex numbers on the calculator, using the polar-rectangular conversions in the Angle menu. Tap OPTN F6 F5 to enter Angle menu and then F6 to get the conversion menu in Run-Mat mode.





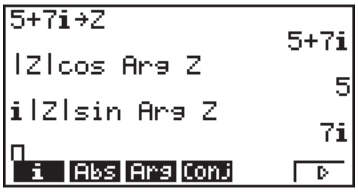
The first result gives a list. To obtain the second screen, use ▲ to highlight the list and then EXE.

In the screens below 3 + 2$\boldsymbol{i}$ has been saved into memory z, as above.

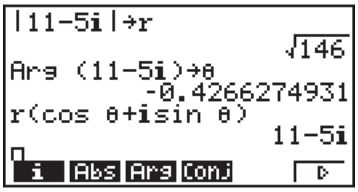


There is a connection between the rectangular form and the polar form of a complex number, arising from the trigonometry of the situation.

If a complex number is represented as $[r,\theta]$ then its coordinates are $(r \sin \theta, r \cos \theta)$. The number can also be represented as $r(\cos\theta + \boldsymbol{i}\sin\theta)$. This is sometimes abbreviated to r cis θ, where the symbol 'cis' refers to 'cos + ***i*** sin'. You can also see these connections using the calculator, as shown below:



After the complex number $z = 5 + 7\boldsymbol{i}$ is defined, the calculator's complex and trigonometric functions can be used to recover the coordinates. You can use the variables r and θ to store the modulus and the argument of a complex number if you wish, since the calculator has reserved memories for these. (See the pink writing above the x^2 and ^ keys.) The screen below shows how to do this, for the complex number $11 - 5\boldsymbol{i}$:



The final command in the screen shows how the algebraic form of the number can be recovered from the values for r and θ, using the cis property.

Apart from representing them on the complex plane, there is a good reason for representing numbers in polar form: it makes it easier to multiply pairs of complex numbers, because of an unexpected relationship. If two complex numbers are represented in polar form as $[r,\alpha]$ and $[s,\beta]$, then their product can be found by multiplying the moduli and adding the arguments:

$$[r,\alpha] \times [s,\beta] = [rs,\alpha + \beta]$$

If you multiply a complex number by itself a few times (that is, raise it to a power), this result leads to the truly remarkable *de Moivre's Theorem*:

$$[r,\theta]^n = [r^n,n\theta]$$

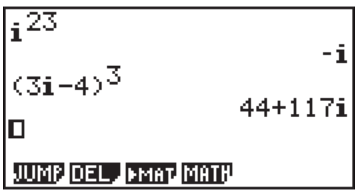
or

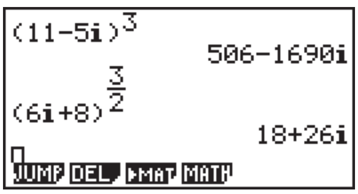
$$(r(\cos\theta + \boldsymbol{i}\sin\theta))^n = r^n(\cos n\theta + \boldsymbol{i}\sin n\theta)$$

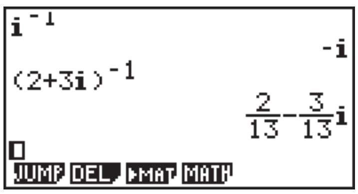
This result was first used by Abraham de Moivre in the first half of the eighteenth century. It is very useful for finding powers and roots of complex numbers, as shown in the next section.

Powers and roots

Powers of complex numbers can be obtained in the same way that powers of real numbers are obtained, using either the ^ key or the x^2 key (for squares).



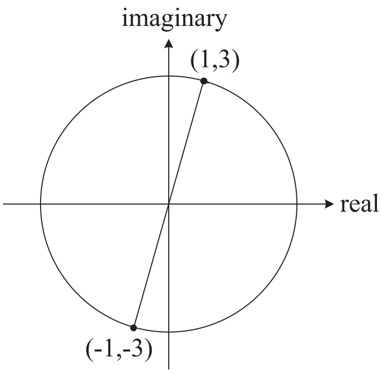




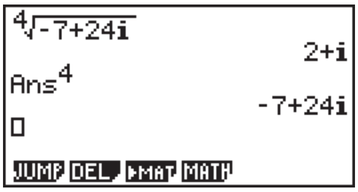
Reciprocals can also be obtained in the usual ways, although you may need to use the F↔D key to obtain 'friendly' results, as shown in the right screen above.

The calculators will find a square root of a complex number successfully, using the square root command. The other square root is the opposite of the one found. The screen below shows these things for finding the two square roots of -8 + 6***i***, which are 1 + 3***i*** and -1 – 3***i***. The final command on the screen shows that, indeed, (-1 – 3***i***)^2 = -8 + 6***i***, as required.

Note that the two square roots are opposite each other on an Argand diagram, and the same distance from the origin. (So you can think of them as two points on a circle with centre at the origin.)

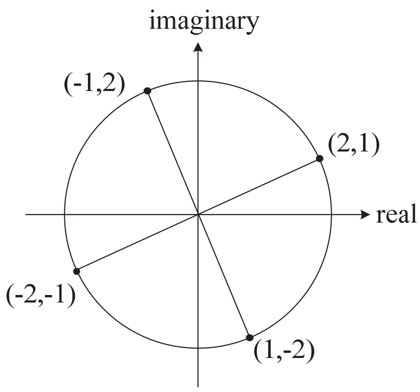


You can use the root command with [SHIFT] [^] to get roots of numbers, but the calculator only gives one result. The screen below shows (and verifies) that one of the 4th roots of -7 + 24***i*** is 2 + ***i***.

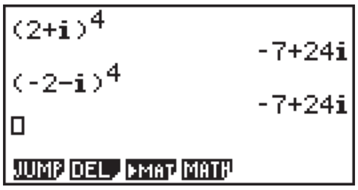


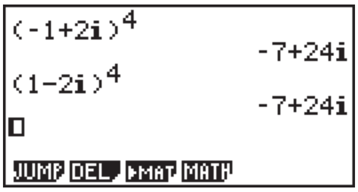
In the same way that a number has two square roots, it has three cube roots, four 4th roots, and so on. Remarkably, when plotted on an Argand diagram, the *n*th roots of a complex number are all equally spaced on a circle whose centre is the origin. If joined, the *n* roots form a regular *n*-sided polygon.

So, in the case of the four 4th roots of -7 + 24***i***, once one of them is found, the rest can be written down by modifying the coordinates to find a number in each quadrant. The number opposite 2 + ***i*** on the circle is -2 – ***i***, while the other two roots are given by -1 + 2***i*** and its opposite 1 – 2***i*** . It is easier to see the relationships among these from the coordinates, (2,1), (-2,-1), (1,-2) and (-1,2). These are plotted on the Argand diagram below, making it is easy to see that they form a square.



The calculator can be used to verify directly that all three additional roots are in fact 4th roots, as shown in the screens below:





Square roots and 4th roots are special cases however. To find other roots of numbers, you will need to use de Moivre's Theorem with fractional exponents:

$$Z^{\frac{1}{n}} = [r,\theta]^{\frac{1}{n}} = r^{\frac{1}{n}}(\cos\theta + i\sin\theta)^{\frac{1}{n}}$$

$$= r^{\frac{1}{n}}(\cos\frac{\theta+2k\pi}{n} + i\sin\frac{\theta+2k\pi}{n})$$

where k takes values 0, 1, 2 for cube roots, and 0, 1, 2, 3 for fourth roots, and so on.

In the case of cube roots of a number represented in polar form by $[r,\theta]$, the first root ($k = 0$) is

$$[r,\theta]^{1/3} = r^{1/3}(\cos\frac{\theta}{3} + \boldsymbol{i}\sin\frac{\theta}{3})$$

After the first root is found, the other two values are found by increasing the angles from $\frac{\theta}{3}$ to $\frac{\theta}{3} + \frac{2\pi}{3}$ and then to $\frac{\theta}{3} + \frac{4\pi}{3}$, corresponding to $k = 1$ and $k = 2$.

When represented on an Argand diagram, the three cube roots of a complex number are equally spaced around a circle.

Exponential form

As well as coordinate form and polar form, a complex number z can be represented also in exponential form, relying on the remarkable relationship:

$$e^{i\theta} = \cos\theta + \boldsymbol{i}\sin\theta$$

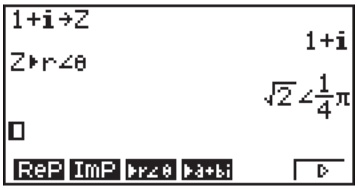
So, if $z = r(\cos\theta + \boldsymbol{i}\sin\theta)$, then $z = re^{i\theta}$.

The great mathematician Karl Freidrich Gauss used this representation to derive an extraordinary relationship, which shows a surprising connection between the five most famous numbers in mathematics: e, $\boldsymbol{i}$, π, 1 and 0:

$$e^{i\pi} + 1 = 0$$

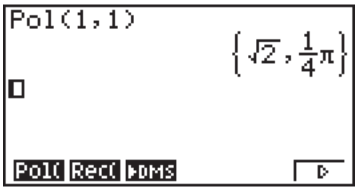
This remarkable result can be demonstrated on the calculator directly:

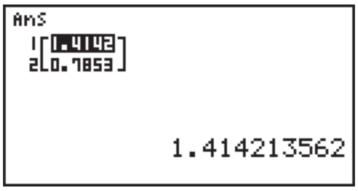
It is possible to move freely between the three different representations of complex numbers, using the calculator. To illustrate, consider the complex number $z = 1 + \boldsymbol{i}$. To represent z in polar form, use the transformation command in the Complex menu, as shown below.



Alternatively, you can use the separate absolute value and argument commands to get the same result, as shown at right above.

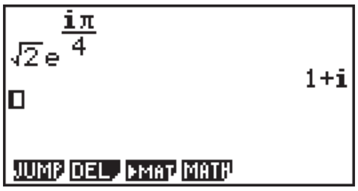
The screen below shows that the Angle menu (via OPTN) can also be used for the same purpose, even though it does not refer explicitly to complex numbers.



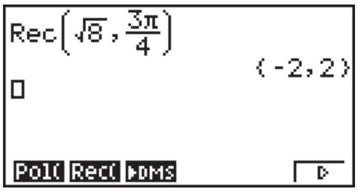


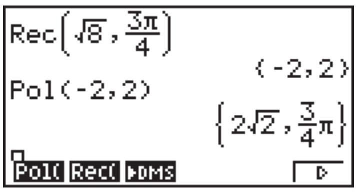
You may recognise the results here: 1.414213562… = $\sqrt{2}$ and 0.7853981634… = $\pi/4$. The calculator gives only rational approximations to these irrational numbers.

Finally, the exponential form of $z = re^{i\theta}$ uses these values for r and θ from the polar form. The calculator screen below verifies that this is a correct representation of z.



Similarly, any complex number represented in polar form, such as $\left[\sqrt{8}, \frac{3\pi}{4}\right]$ can be represented in coordinate form through the use of the Angle menu or can be represented in exponential form on the calculator directly. The screens below show the relationships involved.





Exercises

The main purpose of the exercises is to help you to develop your calculator skills.

1 Give both square roots of -49.

2 Evaluate $(7 - 5\boldsymbol{i})^2$.
Check by finding the square root of your answer.

3 Evaluate $(52 + 77\boldsymbol{i}) \div (5 - 8\boldsymbol{i})$.
Check by multiplying your answer by $(5 - 8\boldsymbol{i})$.

4 Find both square roots of $-7 + 24\boldsymbol{i}$.
Check by squaring.

5 Express $\frac{4 + \boldsymbol{i}}{5 + 4\boldsymbol{i}}$ as a complex number, without using decimals.

6 When the complex number $-2 - 7\boldsymbol{i}$ is plotted on an Argand diagram, how far is the point from the pole?

7 Which complex numbers in Cartesian form are given in polar form by $[4, 50^{\circ}]$ and $[5, \frac{\pi}{4}^{R}]$?

8 Give two different pairs of polar coordinates for representing $8 + 3\boldsymbol{i}$ on an Argand diagram.

9 Evaluate $(6 - \boldsymbol{i})^5$.
Store the result in memory A and then find |A|.

10 Two complex numbers in polar form are $[2, 40^{\circ}]$ and $[5, 35^{\circ}]$.
Give the polar form of their product.

11 A complex number z has polar form $[\frac{5}{2}, \frac{\pi}{4}]$.
Give the polar form of z^4.
In what quadrant of an Argand diagram does z^7 lie?

12 Evaluate $e^{3 + \boldsymbol{i}}$ and $e^{3 - i}$.

Activities

The main purpose of the activities is to help you to use your calculator to learn mathematics. You may find that some of them are too advanced for you. Ignore activities you don't yet understand.

1 How does the product of a complex number and its conjugate compare with the absolute value (or modulus) of the number?

Try some examples to find out; then look for an explanation of your result.

2 Is the conjugate of the sum of two complex numbers the same as the sum of the conjugates of the two numbers?

Try some examples to find out; then look for an explanation of your result.

3 Find the three cube roots of -8. Check by finding the solutions of $z^3 + 8 = 0$.

You will notice that two of the roots are conjugate pairs of each other. Are the complex roots of an equation always arranged in pairs, or is this particular equation a special case? Try some other examples to explore this question.

4 Explain why multiplying a number by ***i*** has the effect of rotating its position on an Argand diagram 90^o anti-clockwise. What would you need to multiply by in order to rotate through different angles and directions?

5 A complex number can be represented in two forms: polar or Cartesian. For example, in polar form, the number [√5, 30^o] can be regarded as a stretch of √5 from the origin with an anticlockwise rotation of 30^o. The same number represented in Cartesian form is 2 + 1i, which indicates that it can be plotted at the point (2,1) on an Argand diagram.

Consider two complex numbers, say [4,50^o] and [3,20^o] in both polar and Cartesian forms. Combine these two numbers using addition, subtraction, multiplication or division and notice the geometric effects of the operations.

Which of the two forms of representation are more helpful to understand the effects of the operations?

Try with some other pairs of complex numbers.

6 In solving a quadratic equation of the form $ax^2 + bx + c = 0$, you find the values of the roots of the function $f(x) = ax^2 + bx + c$. That is, when you solve the equation, you find where the graph of the function crosses the x-axis.

You can use the quadratic formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ to solve the equation.

(a) Solve the equation $x^2 – 4x + 5 = 0$, using the quadratic formula. Then graph the associated function, $f(x) = x^2 – 4x + 5$.

(b) What do you notice about the solutions and the graph? Discuss this with another student.

(c) Explain why the complex solutions are conjugates of each other.

Notes for teachers

This module illustrates several ways in which the calculator can be used to explore various aspects of complex numbers, to help students understand their meaning, operations and significance. The text of the module is intended to be read by students and will help them to start to see how the calculator can be used efficiently for various kinds of explorations. The Activities are appropriate for students to complete with a partner or in a small group, so that they can discuss their observations and justify their conclusions.

Answers to Exercises

1. $7\boldsymbol{i}$, $-7\boldsymbol{i}$ 2. $24 - 70\boldsymbol{i}$ 3. $-4 + 9\boldsymbol{i}$ 4. $3 + 4\boldsymbol{i}$, $3 - 4\boldsymbol{i}$ 5. $\frac{24}{41} - \frac{11}{41}i$ 6. $\sqrt{53} \approx 7.280$ 7. $2.571 + 3.064\boldsymbol{i}$ and $\frac{5\sqrt{2}}{2} + \frac{5\sqrt{2}}{2}i$ 8. $[\sqrt{73}, 0.359^R]$ and $[\sqrt{73}, (0.359+2\pi)^R] = [\sqrt{73}, 6.642^R]$ 9. $5646 - 6121\boldsymbol{i}$, 8327.302 10. $[39.0625,\pi]$, z^7 in 4th quadrant 11. $10.852 + 16.901\boldsymbol{i}$, $10.852 - 16.901\boldsymbol{i}$

Activities

1. This activity allows students to see for themselves the results of multiplying a complex number by its conjugate. Encourage them to work in pairs and to choose several examples to look for the generalisation that the product is the square of the modulus or absolute value. They should be able to explain this result by multiplying the numbers as binomials.

2. Working in pairs, students should be able to generate some examples to see that the property holds and to explain why it does.

3. Students may have already noticed that complex solutions to quadratic equations are conjugates of each other (as explored also in Activity 6(c)), but may not have explored cubic equations to the same extent. Encourage them to choose some examples and to use Equation mode to find solutions (but check in SET UP that the calculator is in complex mode). Although they will not be able to justify their conclusions theoretically, they should have little difficulty seeing patterns in complex solutions of higher order polynomial equations. [Answers: three cube roots are -2, $1 + \sqrt{3}\boldsymbol{i}$, $1 - \sqrt{3}\boldsymbol{i}$]

4. Ask students to explore some examples and to look for a pattern to verify that the result described actually happens. They should plot the numbers on an Argand diagram to study them more carefully. They might notice that lines on the Argand plane representing $a + b\boldsymbol{i}$ and $-b + a\boldsymbol{i}$ have slopes b/a and –a/b respectively, so are at right angles. If they represent complex numbers in polar form, $\boldsymbol{i} = [1,90^o]$ will make it easier to see the result. Encourage them to investigate the effects of multiplying by multiples of $\boldsymbol{i}$.

5. Students should explore the effects of adding, subtracting, multiplying and dividing by firstly choosing some pairs of numbers and using the calculator to get the results. If students work in pairs, with one person using only Cartesian form and the other using only polar form, they should be able to see that addition and subtraction are easily handled mentally in Cartesian form, while multiplication and division are easily handled mentally in polar form. This activity will help them to appreciate and understand better De Moivre's Theorem. [Answer: note $[4,50^o] \times [3,20^o] = [12,70^o]$ is easily done mentally, but is much more difficult when numbers are in Cartesian form.]

6. In this activity, students should appreciate the importance of the discriminant $b^2 - 4ac$ in determining complex roots. The ± sign makes clear that the two roots are conjugates if the discriminant is negative. For the example given, both roots are complex conjugates as the discriminant is negative. The graph of the function does not intersect the x-axis. [Answers: roots are $2 + \boldsymbol{i}$ and $2 - \boldsymbol{i}$. The real part of the solutions corresponds to the line of symmetry of the graph.]

Module 13
Matrices

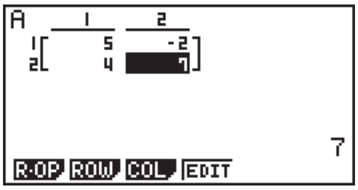
A *matrix* is a rectangular array of numbers, used in mathematics and its applications to store and manipulate information. The dimensions of a matrix record how many rows and columns it has. In fact, you have probably already seen matrices on the calculator. In Module 9, matrices of coefficients for systems of linear equations were used. To represent a system of two linear equations in two variables, a 2 x 3 matrix was needed. In Module 4, you used lists to store statistical data. You can think of a list as a matrix with one column, so that a set of 20 temperature readings can be stored in a 20 x 1 matrix. Statisticians often refer to their 'data matrices', in which the raw data are collected.

On the calculator, matrices are defined (that is, their dimensions are set) and the matrix entries are entered and edited in Run-Mat mode.

To define a new matrix, use the MAT (F3) key to highlight its name and then enter each of the two dimensions, followed by EXE after each. It is important to use the order: rows, then columns.

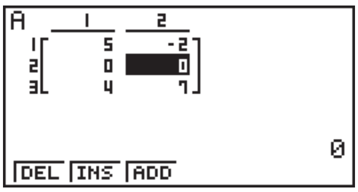
After the dimensions are set, the calculator will show a matrix with zeroes in each cell. You can overwrite any of the zeroes to enter numerical data in the matrix, followed by EXE each time. You can also move around a matrix with the cursor keys to change any entries.

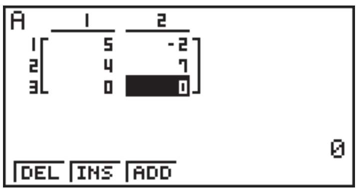
The screen below shows a 2 x 2 matrix, with name **A**. (In mathematics, matrix names are usually written in bold type, as they are in this module.)



Press EXIT to return to the matrix list from a matrix. When in the matrix list, highlight a matrix and press EXE to bring it to the screen for editing.

When a matrix is being edited, you can delete, insert or add a row or a column using ROW (F2) or COL (F3). Adding and inserting are a little different. To see the difference, notice that the screens below show the effect of inserting a new row into **A** (on the left) and adding a new row to **A** (on the right). The added row (or column) is on the 'outside' of the matrix.

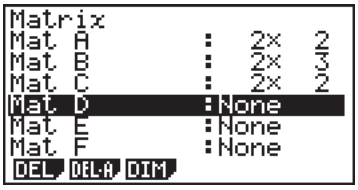




In each case, **A** has become a 3 x 2 matrix. Return to the matrix list with EXIT to check this. Then return to the matrix and use the ROW command to delete the new row of zeroes, and restore **A** to its previous state as a 2 x 2 matrix. Return to the matrix list to define and enter **B** and **C** as well:

$$\mathbf{A} = \begin{bmatrix} 5 & -2 \\ 4 & 7 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & -2 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} 3 & 11 \\ 0 & 2 \end{bmatrix}$$

The matrix list will show you which matrices have been defined, as well as their dimensions:



You can delete a matrix by defining its dimensions again (which will make all of the entries equal to zero). You can also use the function keys shown above to delete matrices. Notice especially that all matrices can be deleted at once with DEL.A (F2).

Matrices can also be defined directly in Run-Mat mode, using the square brackets [ ] on the calculator keyboard. Press SHIFT + and SHIFT − to obtain these. Each row of a matrix is entered as a set of values separated by commas and surrounded by square brackets. Rows are written directly after each other and the entire set of rows is enclosed in square brackets.

So, the matrices

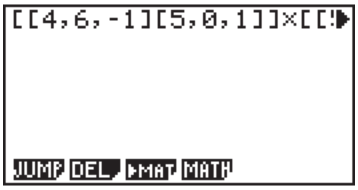
$$\begin{bmatrix} 4 & 6 & -1 \\ 5 & 0 & 11 \end{bmatrix} \text{ and } \begin{bmatrix} 5 \\ 8 \\ -4 \end{bmatrix}$$

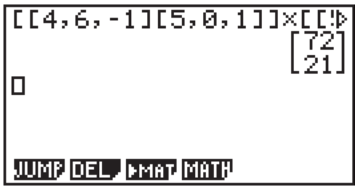
are represented on the calculator as

[[4,6,-1][5,0,11]] and [[5][8][-4]]

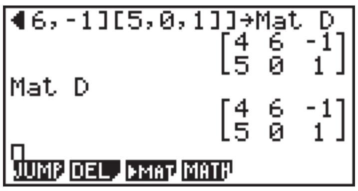
respectively. Notice that there are no commas between rows; the commas are used only to separate elements of a row.

If you merely wish to do some matrix arithmetic, it is sometimes quicker to use this method rather than defining and storing matrices in the matrix list. For example, the screen below shows the product of the two matrices above to get the resulting 2 x 1 column matrix.





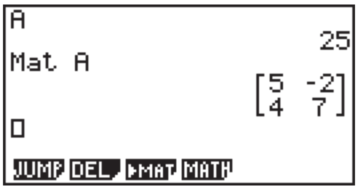
You can also define a matrix by name in Run-Mat mode. The next screens show how the above 2 × 3 matrix can be stored as a named matrix from RUN mode, in much the same way that variable memories are defined. The matrix command Mat (SHIFT 2) is on the keyboard.



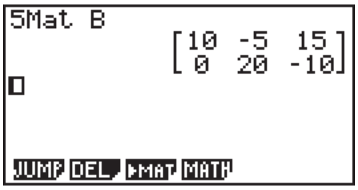
Be careful that you do not accidentally redefine an existing matrix by doing this, however, as the calculator will *replace* an existing matrix with one defined in this way. Notice above that matrix **D** has been recalled to the screen using the Mat command to check that it has been defined correctly.

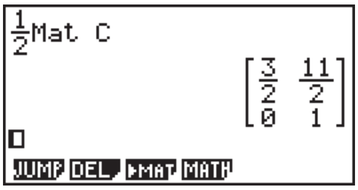
Matrix arithmetic

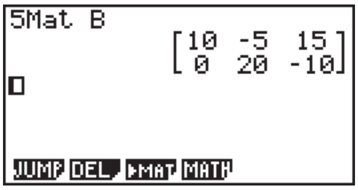
Matrix operations can be carried out in Run-Mat mode. On the calculator, A refers to the variable memory *A* while Mat A refers to matrix **A**. The variable *A* and the matrix **A** do not necessarily have anything to do with each other. The matrix symbol is on the keyboard with Mat (SHIFT 2).



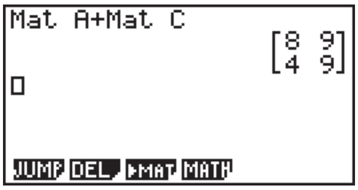
Scalar multiples of matrices and powers of (square) matrices are produced with the commands shown below. Notice that $\mathbf{A}^3$ (shown at right below) does not mean that each element of **A** is cubed; rather, it means the matrix product **A** x **A** x **A**.

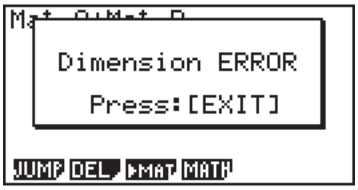






Matrices with the same dimensions can be added and subtracted. The terms in the same position for each matrix are added and subtracted to get the result. So you will get a *dimension error* if you try to add or subtract matrices with different dimensions, such as **A** + **B**:

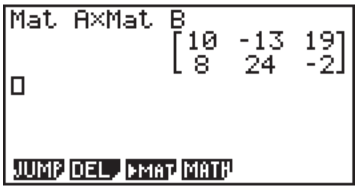


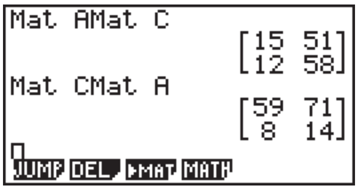


Similarly, matrix multiplication is only defined when the number of columns of the first matrix is the same as the number of rows of the second matrix. The first element in the product **AB** comes from multiplying the first row of **A** by the first column of **B**, and adding the results:

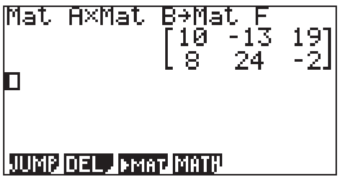
$$5 \times 2 \;+\; -2 \times 0 = 10$$

The number of elements in the row (i.e., the number of columns of **A**) must match the number of elements in the column (i.e., the number of rows of **B**), or the matrices cannot be multiplied in the order **AB**. This explains why there is a dimension error for **BA**, but not for **AB** in this case:

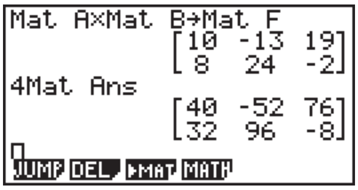




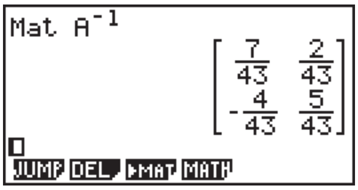
Use the calculator as above to check that **AC** does not give the same result as **CA**, showing that matrix multiplication is not commutative. Notice that it is unnecessary to use a multiplication sign to indicate matrix multiplication, as for other kinds of multiplication on the calculator, although it is still a good idea to do so for clarity. As with variables, you can store the result of a matrix operation directly into a new matrix, using the → key, even if the matrix has not yet been defined. The next screen shows this for defining **F** = **AB**.



You can always retrieve the most recent matrix answer with the command. For example, after the screen above, notice the effect of the next command:



Matrix division is not defined directly. Instead, *inverses* of matrices are used. The inverse of a square matrix, if it has one, can be obtained with the usual inverse key, x^{-1} (SHIFT)) The screens below show that the elements of the inverse are given as fractions when the original matrix has only integers for each term:

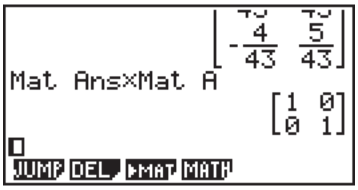


If decimal values are preferred, the F↔D key can be used, as shown above at the right.

In general, the inverse of a 2 x 2 matrix represented as $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is given by $\begin{bmatrix} \frac{d}{D} & -\frac{b}{D} \\ -\frac{c}{D} & \frac{a}{D} \end{bmatrix}$

(*D* refers to the *determinant* of the matrix, given by *ad* – *bc*.)

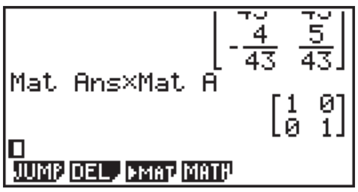
You can check that the calculator result for the inverse of matrix **A** by multiplying it by the original matrix. Both $\mathbf{AA}^{-1}$ and $\mathbf{A}^{-1}\mathbf{A}$ should give the 2 x 2 identity matrix, $\mathbf{I}_2$, with ones in the diagonal and zeroes elsewhere. (**I** stands for the identity matrix, which has ones in the main diagonal and zeros everywhere else.) Note the automatic use of *Mat Ans* below to continue the calculation verifying this relationship. In the screen below, after calculating $\mathbf{A}^{-1}$, the multiplication key was pressed first:



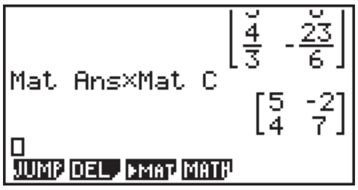
Scalar numbers can be divided directly, but they can also be divided by multiplying by an inverse. For example, to divide 2 by 7, i.e., to find 2 ÷ 7, you can evaluate 2×7^{-1}. You can think of the result of dividing 2 by 7 as a number which, when multiplied by 7, will give 2:

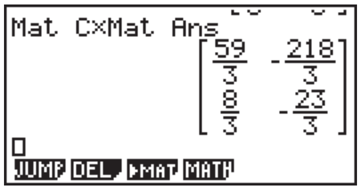
$$2 \div 7 = 2 \times 7^{-1} = 2 \times \frac{1}{7} = \frac{2}{7}$$

Unlike numbers, direct division is *not* defined for matrices, however. But the same idea involving inverses is used. To find **A** ÷ **C**, you need to find $\mathbf{A} \times \mathbf{C}^{-1}$. You can do this directly on the calculator with a single command:



The result is the matrix that, when multiplied by **C**, will give matrix **A** again. The calculator can verify this property for you:



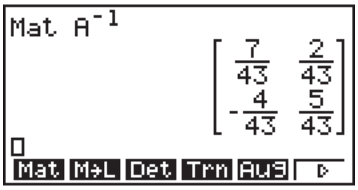


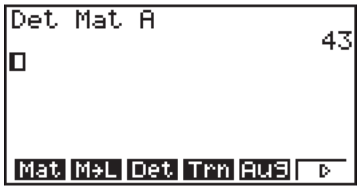
Again, notice that the multiplications must be performed in the correct order, since multiplication is not commutative. In this case, you must *post*-multiply $\mathbf{AC}^{-1}$ by **C**: $\mathbf{AC}^{-1}\mathbf{C} = \mathbf{AI} = \mathbf{A}$. The other order, *pre*-multiplying $\mathbf{AC}^{-1}$ by **C** to get $\mathbf{CAC}^{-1}$, does *not* result in **A**, as shown at right above.

The Matrix menu

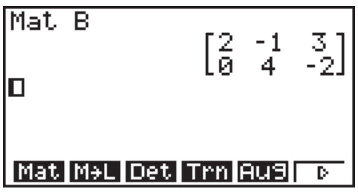
Various other useful matrix commands are available in the Matrix menu, accessed via the OPTN key with MAT (OPTN F2). Notice in this menu that the F1 key allows you to access the *Mat* command with a single keystroke (rather than the SHIFT 2 we have used so far.) If you working a lot with matrices, it may be convenient to keep the Matrix menu on the screen for this convenient reason.

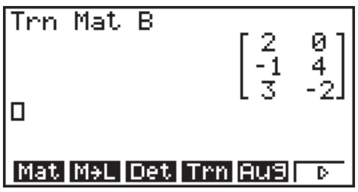
The determinant of a square matrix can be found with Det (F3) In the case of **A**, the inverse suggests that the determinant is 43, as indeed it is, shown in the screen below. (Notice that each element of the inverse is a fraction with 43 in the denominator.) As shown on the previous page, the determinant is used to construct the inverse of a 2 x 2 matrix; a matrix with zero determinant does not have an inverse. (It's harder to see the connections with larger matrices.)

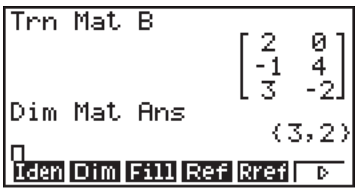




The *transpose* of a matrix is a new matrix with the rows and columns switched. The command Trn (F4) can be used to find a transpose. In the middle screen below, notice that for the transpose of **B**, sometimes represented by **B'**, each row is the same as one of the previous columns. Where **B** is a 2 x 3 matrix, **B'** is a 3 x 2 matrix. The third screen below shows this with the dimension command, Dim (F2) which is in the second screen of the Matrix menu.







Transformation matrices

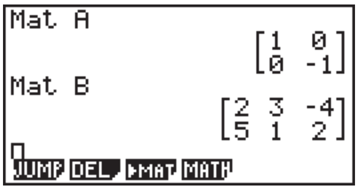
Matrices are especially useful to describe transformations in the plane. For example, consider the 2 x 2 matrix below used to find the images of the three points (2,5), (3,-1) and (-4,-2):

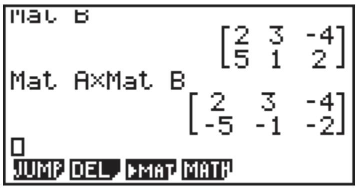
$$\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}2\\5\end{bmatrix}=\begin{bmatrix}2\\-5\end{bmatrix} \qquad \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}3\\-1\end{bmatrix}=\begin{bmatrix}3\\1\end{bmatrix} \qquad \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}-4\\-2\end{bmatrix}=\begin{bmatrix}-4\\2\end{bmatrix}$$

These examples demonstrate a consistent pattern. In each case, the *x*-value of the image is unchanged, while the *y*-value is reversed in sign. These examples show that this matrix has the effect of reflecting points about the *x*-axis.

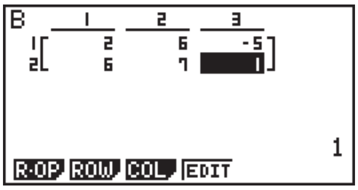
In this case, the numbers are easy enough to do the matrix multiplication in your head, but other situations may involve more difficult numbers, and hence require a calculator.

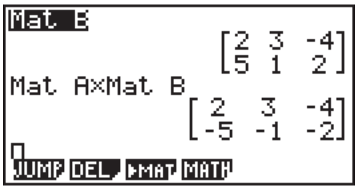
One way to do this would be to define the transformation matrix as a 2 x 2 matrix **A** in the calculator, and then to define a column vector (a 2 x 1 matrix) for each of the three points, as above. An easier way is to combine the three column vectors into a single 2 x 3 matrix, **B**, with one column for each of the three points. Then, the matrix multiplication **AB** gives all three results simultaneously. The screens below show **A**, **B** and **AB** respectively:

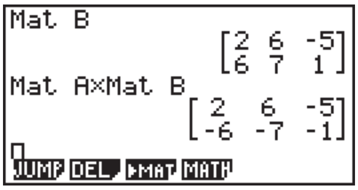




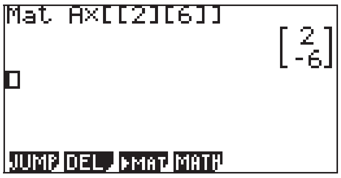
If you have a succession of transformations of this kind to make, the calculator editing features will be very useful. For example, to effect the same reflection transformation of the three points, (2,6), (6,7) and (-5,1), firstly use the MAT ([F3]) key to edit **B.** Then tap [EXIT] (twice) to return to the results screen. Use the [▲] key to highlight the original command used to display Matrix **B**, and then tap the [EXE] key. The calculator will then repeat the earlier sequence of commands, including the transformation command, with the new matrix **B**, as shown in the right screen above.

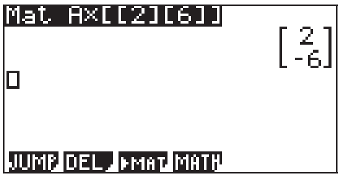


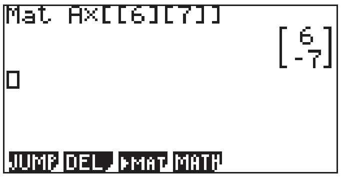




Another way to use the calculator to determine the images of several points under a transformation defined by **A** is shown below, using [[2][6]] on the calculator to define a column vector as the first point to be transformed. (Remember that the square brackets needed here are available with [SHIFT] [+] and [SHIFT] [−].)







To enter the next point, edit the command, starting with [▲] [▲] [◀] to enter the next point and then tap [EXE] to transform the point. The advantage of this way of finding images under transformations is that you can continue to do so until you have enough information about the transformation to understand its effect.

Matrices and equations

A major use of matrices is to represent and then to solve systems of linear equations. You have already seen examples of this in the module on equations, where the calculator's inbuilt equation solver can be used to solve a linear system, once the matrix of coefficients is provided. Precisely the same task can be performed using matrices. Consider the following system of linear equations:

$$2x + 6y = 10$$

$$3x + 7y = 11$$

This can be represented as a matrix equation:

$$\mathbf{BX} = \mathbf{C}$$

where $\mathbf{B} = \begin{bmatrix} 2 & 6 \\ 3 & 7 \end{bmatrix}$, $\mathbf{X} = \begin{bmatrix} x \\ y \end{bmatrix}$ and.$\mathbf{C} = \begin{bmatrix} 10 \\ 11 \end{bmatrix}$

Then the equation can be solved by pre-multiplying the equation by the inverse of the matrix of coefficients, $\mathbf{B}^{-1}$:

$$\mathbf{B}^{-1}\mathbf{BX} = \mathbf{B}^{-1}\mathbf{C}$$

$$\mathbf{X} = \mathbf{B}^{-1}\mathbf{C}$$

After **B** and **C** are entered into the calculator, the result can be obtained directly:

The solution $x = -1$ and $y = 2$ can be read from the result matrix **X** above. You can check this solution by substitution into the original equations.

With this calculator, it is unnecessary to use matrices for this purpose, however, as a system of up to six linear equations in six unknowns can be solved more easily in EQUA mode, described in Module 9. You only need to use matrix methods to solve linear equations when there are more than six equations or when there is not a unique solution.

To solve a system of linear equations, you can also use an elimination method, which involves *row operations*(This is the method normally used, when calculators or computers are not available.) The method relies on the fact that a solution to a system of equations is not changed if either:

(i) an equation is multiplied by a constant; or
(ii) a multiple of one equation is added to another equation.

When equations are represented by their coefficients in matrices, the permissible row operations are equivalent to these:

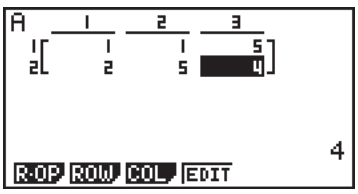
(i) a row is multiplied by a constant; or
(ii) a multiple of one row is added to another row.

These elementary row operations can be performed in Run-Mat mode. To illustrate, consider the system of two linear equations:

$$x + y = 5$$

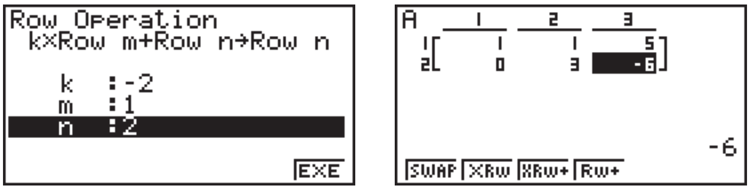
$$2x + 5y = 4$$

Enter the 2 x 3 matrix of *all* the coefficients into the calculator (i.e., including the numbers on the right of the equals sign as another column of the matrix.). Then tap R.OP (F1) to get the menu of elementary row operations.

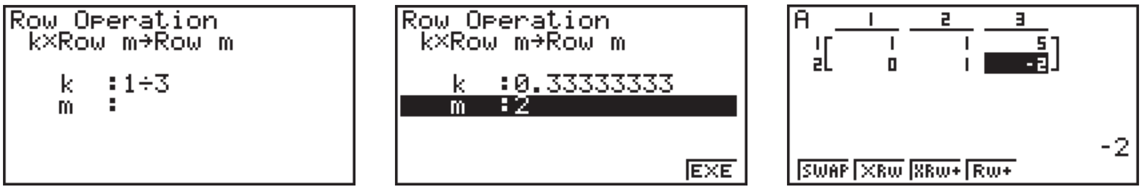


The aim is to transform the matrix using row operations so that there is only one variable in each corresponding row of equations and the coefficient of the variable is 1. In this case, begin by subtracting 2 times *Row 1* from *Row 2* to change the first coefficient to 0.

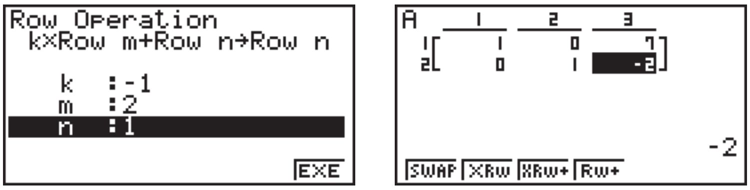
Use XRw+ (F3) On the screens below, enter k = -2, m = 1 and n = 2 to have this effect. (Tap EXE after each value is entered.) Check the results: *Row 1* is unchanged, and each element of *Row 2* has been changed by subtracting twice the corresponding element of *Row 1* from it.



The next step is to divide *Row 2* by 3, to change the second coefficient to 1. This is the same as multiplying by one third (i.e., k = 1/3). XRw (F2) is used:



Finally (just for completeness, since you could easily solve the system from here in your head), add -1 times *Row 2* to *Row 1* to change the second coefficient to zero:



The resulting matrix allows you to read the solutions of $x = 7$ and $y = -2$ directly in the final column. The final matrix shows that an equivalent system of equations to the original system is:

$$x = 7$$
$$y = -2$$

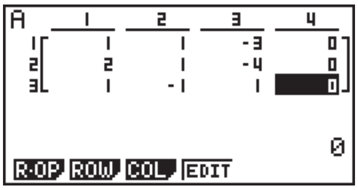
This sort of procedure is a certainly too tedious to use when there is a much faster alternative available in Equa mode, described in Module 9. However, sometimes there is no alternative. For example, consider the system below:

$$\begin{aligned} x + y - 3z &= 0 \\ 2x + y - 4z &= 0 \\ x - y + z &= 0 \end{aligned}$$

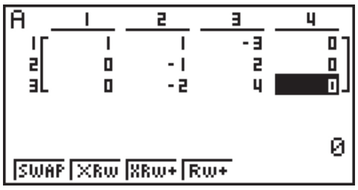
An obvious solution is $x = 0$, $y = 0$ and $z = 0$, but are there other solutions? In this case, the matrix of coefficients does not have an inverse (so the matrix is called *non-singular*), which indicates that at least one of the rows is linearly dependent on the other rows.

For the same reason, the calculator will give a Ma ERROR if you try to solve the system in Equa mode. So, it may seem at first that the calculator is not much help.

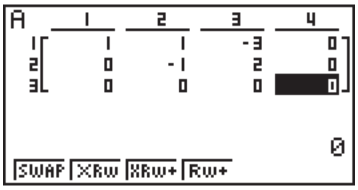
However, if you use the elementary row operations on the system, a complete solution can still be found. The screens below shows some steps along the way, starting with the 3 x 4 matrix of coefficients:



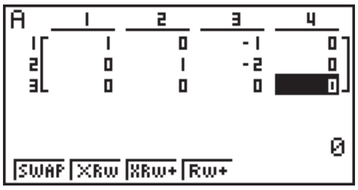
After *Row 1* is used to remove the first coefficient from *Row 2* and *Row 3*, with two successive row operations, the following matrix is left:



Notice that *Row 3* is twice *Row 2*, indicating the linear dependency. After subtracting twice *Row 2* from *Row 3* (and thus creating a row of zeroes) and reversing the signs in *Row 2*, we get:



Finally, subtract *Row 2* from *Row 1* to give the result below:



The first two lines of the matrix show that the system is equivalent to the system

$$x - z = 0$$
$$y - 2z = 0$$

The solution to this system is readily seen to be $x = z$ and $y = 2z$. Whenever z has a value, you can work out the corresponding values of x and y. That is, there are an infinite number of solutions to the system of equations. Rather than describe the solutions in terms of one of the variables, it may be better to describe them in terms of another parameter, t say.

So the solution to the original system can be given as $x = t$, $y = 2t$ and $z = t$. One of the solutions is (0,0,0), when $t = 0$. Another is (3,6,3) when $t = 3$.

Exercises

The main purpose of the exercises is to help you to develop your calculator skills.

1 Enter these two matrices into your calculator:

$$\mathbf{A} = \begin{bmatrix} 2 & 11 & 3 \\ -1 & 0 & 5 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 4 & 7 \\ 2 & 3 \end{bmatrix}.$$

Use these two matrices in the questions that follow.

2 Find 5**A** and $\mathbf{B}^2$.

3 Which of **AB** and **BA** is not defined? Evaluate the product that *is* defined.

4 Find the determinant of **B**.

5 Find $\mathbf{B}^{-1}$. Check that $\mathbf{B}^{-1}\mathbf{B} = \mathbf{B}\mathbf{B}^{-1} = \mathbf{I}_2$, the 2 x 2 identity matrix.

6 Edit **B** so that

$$\mathbf{B} = \begin{bmatrix} 4 & 9 \\ 2 & 3 \end{bmatrix}$$

and edit **A** so that

$$\mathbf{A} = \begin{bmatrix} 2 & 11 & 3 \\ -1 & 0 & 5 \\ 4 & -6 & 0 \end{bmatrix}.$$

7 Find **A'**, the transpose of **A** and then find **A'A**. Check that **A'A** is a symmetric 3 x 3 matrix

8 Use the transformation matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

to find the images of the four points (0,3), (3,6), (5,-2) and (7,7).

Describe the geometric effect of this transformation.

9 Express the following system of equations in matrix form, **BX** = **C**:

$$x + y + z = 5$$
$$2x - y - z = 8$$
$$x - 5y - 3z = 3$$

Solve the system with $\mathbf{X} = \mathbf{B}^{-1}\mathbf{C}$.

10 Use elementary row operations on the calculator to solve the linear system:

$$4a - b = 10$$
$$3a + 5b = 19.$$

Activities

The main purpose of the activities is to help you to use your calculator to learn mathematics. You may find that some of them are too advanced for you. Ignore activities you don't yet understand.

1 You have already seen that matrix multiplication is not commutative. That is, in general, **AB** ≠ **BA**.

Notice that matrix multiplication can only be defined in pairs: the symbol **ABC** means **A** x **B** x **C**; but this can mean either **AB** x **C** or **A** x **BC**.

Define some matrices **A**, **B** and **C** and check on the calculator whether or not (**AB**)**C** = **A**(**BC**). That is, check whether or not matrix multiplication is *associative*.

2 Transformation matrices transform points in the plane. Start with the rectangle defined by (1,0), (4,0), (4,2) and (1,2) to determine the geometric effect of multiplying by the following matrices.

$$\mathbf{A} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix} \qquad \mathbf{D} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Try some other transformation matrices and starting with some other shapes of your own choosing.

How do the areas of the images compare with the areas of the original shapes?

3 The determinant of a transformation matrix provides information about the areas of shapes and their images. Use the answers from Activity 2 to help you understand this relationship.

4 Investigate the effects of using a transformation matrix which is the product of two other transformation matrices.

For example, consider again the matrices in Activity 2. What is the effect of the transformation matrix **AB**? **CA**? **BA**? $\mathbf{A}^2$? $\mathbf{D}^2$?

Use your calculator to check your predictions.

5 Determine whether these systems of equations have unique solutions, infinite solutions or no solutions.

$x + y + z = 9$	$x - 2y + 3z = 5$	$2x + y - 2z = 11$
$3x - y - z = 8$	$2x - 4y + 6z = 10$	$x + 2y - 3z = 17$
$2x - 2y - 2z = 1$	$x + 3y + 3z = 7$	$-3x - y + 5z = 36$

Find the solutions for those systems that have solutions.

6 Examine carefully the rows of this matrix to explain why it has a zero determinant.

$$\begin{bmatrix} 1 & 2 & 4 \\ -1 & 5 & 3 \\ 1 & 9 & 11 \end{bmatrix}$$

Invent some other 3 x 3 matrices with zero determinant.

Invent a 4 x 4 matrix with zero determinant.

Check your inventions with your calculator.

Notes for teachers

This module illustrates several ways in which the calculator can be used to explore various aspects of matrices, to help students understand what they are and how to use them efficiently. The text of the module is intended to be read by students and will help them to start to see how the calculator can be used efficiently for various kinds of explorations. The Activities are appropriate for students to complete with a partner or in a small group, so that they can discuss their observations and justify their conclusions.

Answers to Exercises

2. $\begin{bmatrix} 10 & 55 & 15 \\ -5 & 0 & 25 \end{bmatrix}$, $\begin{bmatrix} 30 & 49 \\ 14 & 23 \end{bmatrix}$ 3. **AB** is undefined; **BA** = $\begin{bmatrix} 1 & 44 & 47 \\ 1 & 22 & 21 \end{bmatrix}$ 4. -2 5. $\begin{bmatrix} -3/2 & 7/2 \\ 1 & -2 \end{bmatrix}$

7. **A'A** = $\begin{bmatrix} 21 & -2 & 1 \\ -2 & 157 & 33 \\ 1 & 33 & 34 \end{bmatrix}$ 8. $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 0 & 3 & 5 & 7 \\ 3 & 6 & -2 & 7 \end{bmatrix} = \begin{bmatrix} 0 & 3 & 5 & 7 \\ 6 & 12 & -4 & 14 \end{bmatrix}$. Vertical stretch, factor 2.

9. $x = 13/3$, $y = -1/3$, $z = 1$ 10. $a = 3$, $b = 2$.

Activities

1. The intention of this activity is to encourage students to use some examples to see for themselves that matrix multiplication is not commutative, although it is associative. It is much easier to do this on a calculator than by hand and students can edit matrices and perform calculations again efficiently to generate a range of examples.

2. Encourage students to plot the original rectangle on grid paper and then to plot its image under various transformations, perhaps in a different colour. To use the calculator efficiently, it is best to define the rectangle with a 2 × 4 matrix stored in the calculator. Students working in pairs can try the transformations with different starting objects and compare notes. The area concept (related to determinants) is addressed also in Activity 3. [Answers: **A** is a reflection about the *y*-axis; **B** is a vertical stretch with factor 3; **C** is a horizontal stretch with factor 5 and a vertical stretch with factor 2; **D** is a reflection about the *x*-axis.]

3. Encourage students to study their examples from Activity 2 and to generate some further examples to compare the areas of shapes before and after transformations have been applied. The determinant of the transformation matrices reflects the extent to which an area is affected by the transformation.

4. In this activity, students can use the calculator to explore combinations of transformations, and should find that matrix products of the kinds suggested will describe compositions of transformations. Students should notice that, as the transformations are used for pre-multiplication, that the second transformation is effected first.

5. Students might choose to use EQUA mode here as well as row operations. [Answers: the first system has no solutions; the second system has an infinite number of solutions; the third system has the unique solution, $x = 0$, $y = 1$, $z = -5$.]

6. Students should experiment with row operations here to see that the second row added to two times the first row gives the third row. This linear dependency means that the matrix has rank 2 and has no inverse. Similar dependencies will allow students to create other matrices with zero determinant.

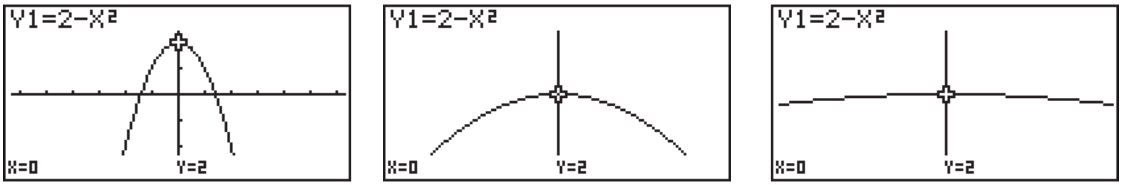
Module 14
Calculus

The Casio fx-9860GII and fx-CG 20 graphics calculators are both numerical machines, and so can only give approximations to the calculus, which is primarily concerned with continuous mathematics. However, some important aspects of calculus can still be explored effectively with the calculators, and good approximations to many key concepts are available.

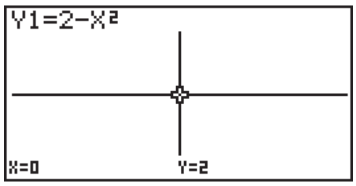
Local linearity

If you draw the graph of a continuous elementary function, and zoom in on the graph far enough, the graph will become linear. This is the critical idea of 'local linearity'. The slope of the graph is the derivative of the function at the point in question. It is unnecessary to think of the derivative as related to the slope of tangents or secants or chords. You can think of the slope of the curve itself, taken over a small enough interval.

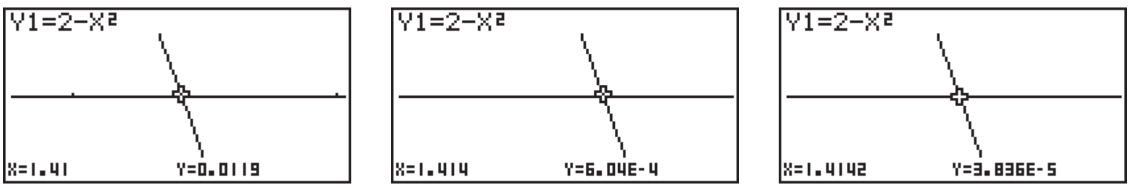
To see an example of this, draw the graph of $y = 2 - x^2$ on the Initial screen. Trace to the point (0,2) and zoom in repeatedly, as the next screens show. (It is a good idea to use the FACT (F2) menu to set the zoom factors to a higher than usual number, such as 10, in each direction before you start.)



The curve appears to 'flatten out' at this point; if you continue to zoom in, the curvature will completely disappear and the graph will become a horizontal line, as below. These screens strongly suggest that the derivative (the slope of the curve) is zero at the turning point (0,2) of the graph.



Repeated zooming in elsewhere on the graph produces a line, too, a slightly jagged one because of the screen resolution. The three screens below show this near the positive root of the function:

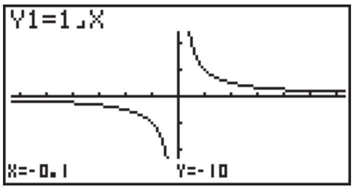


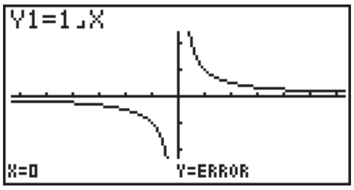
You can think of the derivative at a point on the graph as the gradient of the curve at that point.

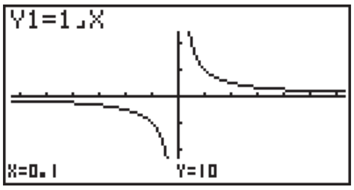
Continuity and discontinuity

Many of the functions used in school mathematics are *continuous*, and so do not present particular difficulties for use in mathematics. (In general, a function is continuous if you can draw its graph on paper without lifting your pencil off the page.) However, some functions are *discontinuous*, and so need to be treated with special care. The calculators have various ways of identifying discontinuities.

An example of a discontinuous function is the reciprocal function, $f(x) = 1/x$, which has a *jump discontinuity* at the point $x = 0$. The function is not defined at this point. The screens below show this function graphed on the Initial screen.

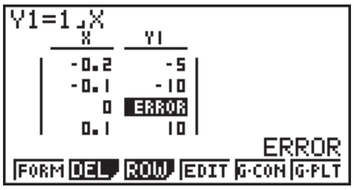




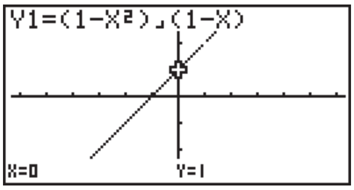


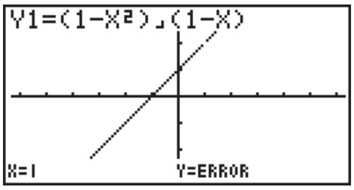
Notice that, although *Draw Type* (in SET UP) has been set to *Connect*, rather than *Plot*, the calculator does *not* connect the points over the point of discontinuity. If you trace the curve from left to right, notice that the cursor 'jumps' over the points where $x = 0$ and does not plot a point. Instead, it indicates that there is an error.

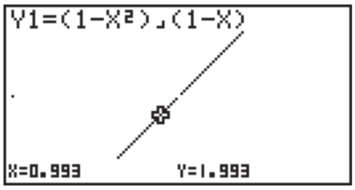
Similarly, if you tabulate this function near the point of discontinuity, the calculator will identify the discontinuity (attempted division by zero) with an error message:



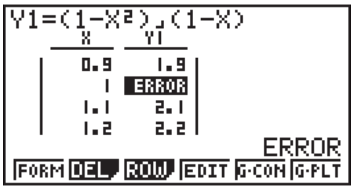
Another kind of discontinuity is illustrated with the function $f(x) = \frac{1-x^2}{1-x}$. This function has a *removable discontinuity* at $x = 1$. It is called 'removable' as if the function were redefined by defining $f(1) = 2$ it would then be continuous. The graphs below all show an apparent 'hole' at $x = 1$. The error is identified at the hole at (1,2) when tracing. The first two graphs are drawn on the Initial screen; the third shows the result of zooming in twice with factors of 10 on each axis.

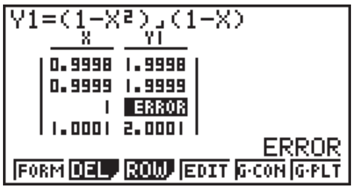






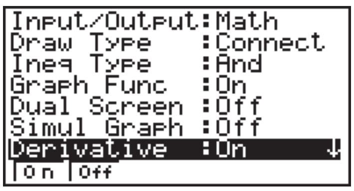
Once again, tables of values also identify the discontinuity at $x = 1$, when $1 - x = 0$, although the function is defined and continuous for other values of x, as the next screens suggest.



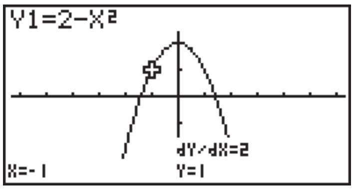


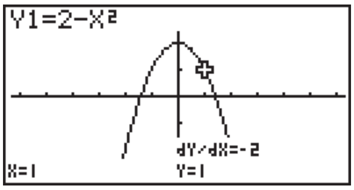
Numerical derivatives

The derivative of a function at a point can be found from a graph by first turning on the *Derivative* command in the Graph mode SET UP menu, as shown below:

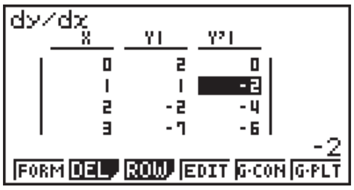


Tracing a graph will now give you an approximation to the derivative of the function as well as the coordinates of each point on the graph:





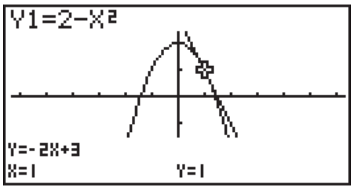
To see many values of the derivative at once, press MENU 7 to enter Table mode. When the derivative trace is turned on, the numerical derivatives of functions are provided in a table, together with the function values. The screen below shows that there seems to be a clear connection between the values in the third column (the derivatives) and the first: suggesting that $y'(x)$ = dy/dx = $-2x$ in this particular case. Scrolling down the table will give more values consistent with this relationship.

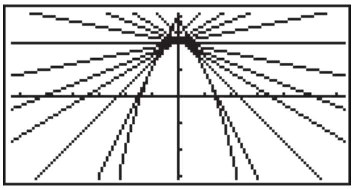


X	Y1	Y'1
0	2	0
1	1	-2
2	-2	-4
3	-7	-6

-2

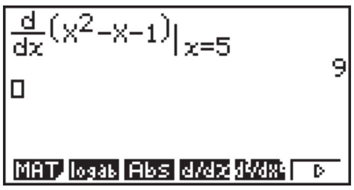
You can get a more visual display of derivatives for a graph by using the *Sketch* menu. After drawing a graph, press Sketch (F4) and then Tang (F2) to draw a tangent at any point. Then trace to the point you want and tap EXE. For example, the first screen below shows the tangent $y = 2x + 3$ to $y = 2 - x^2$ at $x = 1$, while the second screen shows many other tangents for this function as well.



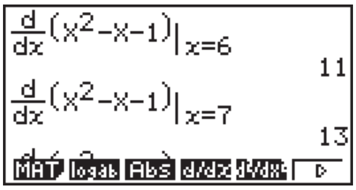


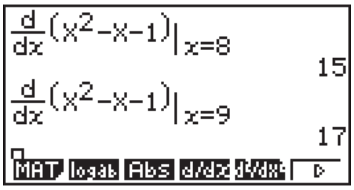
It seems clear from these successive tangents that the gradient of the curve (and so the derivative of the function) changes from positive to negative as you go from left to right across the screen, and is zero at the turning point at the top. Note that, since the tangents are sketched, they will be erased if you change the screen in any other way (such as zooming, or adding another function and then drawing again.) You will also erase tangents if you shift to a different mode and then return. You can erase a sketch with the clear screen command CLS (F1) in the Sketch menu.

In Run-Mat mode, a numerical derivative command d/dx can be found in the MATH menu, accessed with MATH (F4). Select d/dx (F4) and then complete the expression on the screen to show the function (of X) concerned and the X-value at which you want the derivative, after using the ▶ key. Tap EXE to evaluate the derivative, shown below for $f(x) = x^2 - x - 1$ when $x = 5$.



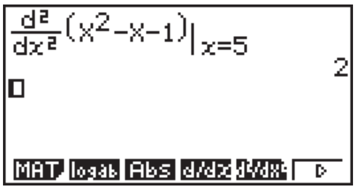
To evaluate the derivative of this function at several points in succession, as above, press EXE after each result, edit the value with ▲ ▲ ◀ ◀ and press EXE again. It appears in this case that the value of the derivative increases by 2 for every increase of 1 in the variable *x*.

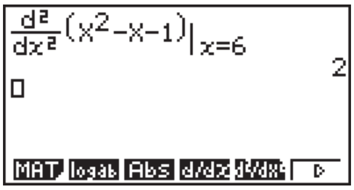


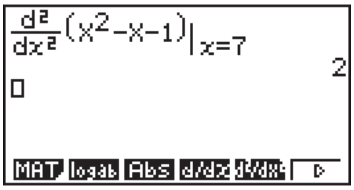


As well as using the Math menu, note that you can also access calculus functions, including the derivative function, in an OPTN menu, obtained with CALC (F4). (If the calculator is already in Run-Mat mode, however, it is quicker to use the MATH menu.)

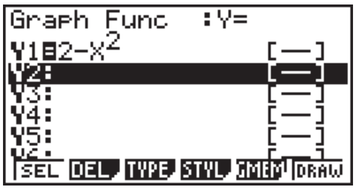
The Math menu contains a second derivative command as well, which works in the same way as the first derivative. For $f(x) = x^2 - x - 1$, the second derivative is constant, consistent with the first derivative increasingly steadily. In the screens below, the command was edited each time, starting with ▲ ▲ ◀, to produce successive results.

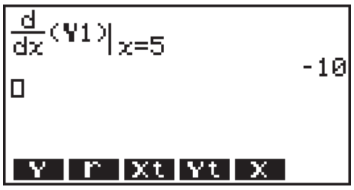


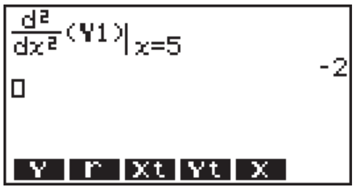




The screens below shows that you can find derivatives of a function already defined in the function list, while in Run-Mat mode. To enter the ***Y*** symbol for ***Y1***, you first need to choose VARS (VARS) then GRPH (F4) to put the menu shown above on the screen. The ***Y*** comes from F1 (***Y***).





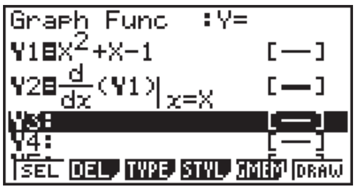


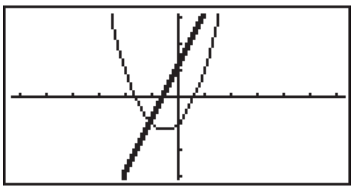
The calculator provides *approximations* to these derivatives, rather than exact results. However, as you can see from these few examples, the approximations are very good ones, and results which are exactly integers will usually (but not always) be found as integers.

When you have finished exploring derivatives, it is a good idea to turn the Derivative trace *off* in SET UP, as it will be quite distracting otherwise, and it takes the calculator longer to do some tasks.

Dealing with derivative functions

You can use the calculators to graph or tabulate many values of derivatives at once, in order to study derivative functions. Enter the two functions shown below into the functions list. The second function, ***Y2***, is the derivative of the first function, ***Y1*** at *all* points, not just one particular point.

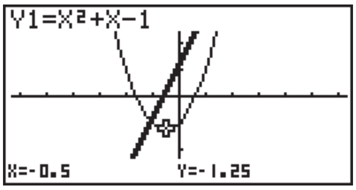


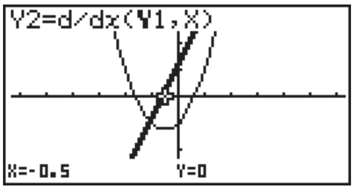


To enter the derivative command for ***Y2***, tap OPTN and then CALC (F2) to access the CALC menu, followed by d/dx (F1). Enter the symbol for ***Y*** using the menu at the bottom of the screen. Instead of a particular value for *x*, use the X,θ,T key to indicate all values of the variable. After the function is defined, use the STYL (F4) menu to draw Y2, the derivative function, with a bold style to distinguish it from the original function, as in the screen at right above.

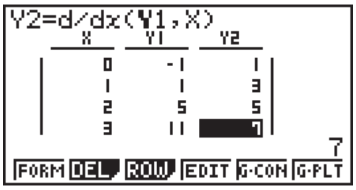
The derivative function requires the calculator to find the numerical derivative of function ***Y1*** at each pixel point across the screen, and to then plot the point, so it takes a bit more time to plot than the original function. Use this time to predict in your mind what the graph should look like, by thinking about the sign and the value of the gradient of the function at different points.

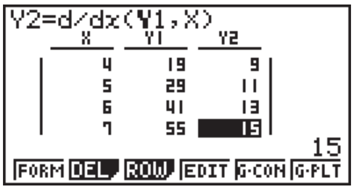
Drawing a pair of graphs like this is illuminating in many ways, allowing you to compare features of the graph of the function and its derivative. For example, notice that the turning point on the graph of the function is where the derivative graph crosses the *x*-axis:



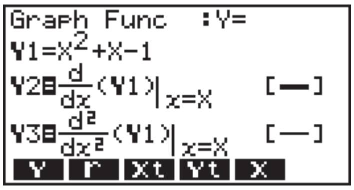


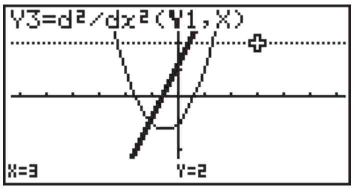
Now press MENU 7 and examine the table. Set the table step equal to 1. From the table, it is fairly easy to see in this case that the derivative function has the rule $y = 2x + 1$, especially if you get a few more values by scrolling down a bit further.

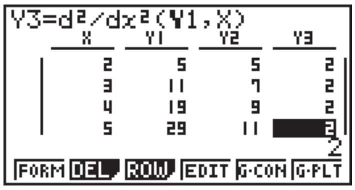




To explore the relationship between *x* and the derivative function *Y2* when the connection is less obvious, you can use the list memory and statistics facilities described in Module 11 to analyse the data from the table. The same ideas apply to the second derivative function, which can be defined symbolically, graphed and tabulated in the same way as the first derivative function. Because of the many computations involved, the calculator takes longer to graph the second derivative function than the first derivative function, but you will still find it useful to see all three together as shown below. In this case, we drew the second derivative function with a dotted line, to distinguish it from the other two). You can make good use of the time taken to draw the graphs by imagining what the graphs of the derivative functions *should* look like, before they appear on the screen.







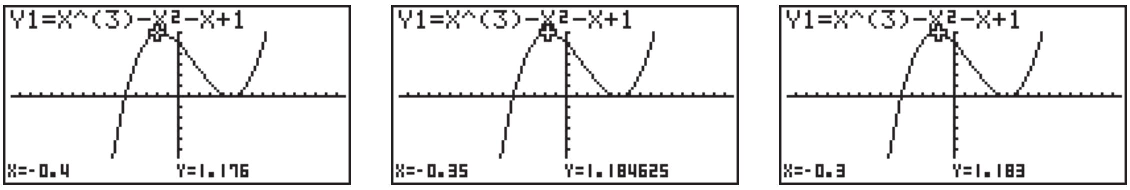
In this case, since the original function is quadratic, its first derivative is a linear function and its second derivative is a constant function. Both the graph and the table show these characteristics quite well. Notice, for example, that all the values for the second derivative in the table are the same, so you can easily see that its rule is $y = 2$.

Relative maxima and minima

The quadratic function used as an example in the previous section had a *relative minimum (*at the bottom of the parabola) which was also a *global* minimum – the lowest value obtained by the function for any real value of *x*. Other functions are different from this, and we are often interested in *local* minima or *local* maxima that may *not* be global. On a graph, these are readily seen as places where the curve loops (up or down respectively).

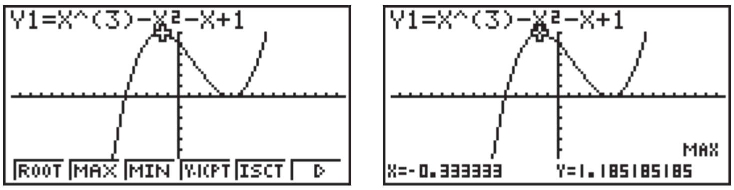
You can get an approximate idea of a local maximum or minimum by tracing, as the graphs of

$y = x^3 - x^2 - x + 1$ below shows. The graphs are drawn on the Initial screen, after zooming in once with Zoom Factor of 2 and setting the tick marks at every 0.2 for each axis. For these graphs, $-3.15 \le x \le 3.15$ and $-1.55 \le y \le 1.55$. There seems to be a relative maximum near $x = -0.4$ and a relative minimum near $x = 1$. The screen is not sharp enough to see whether the x-axis is below, through or above the local minimum point.

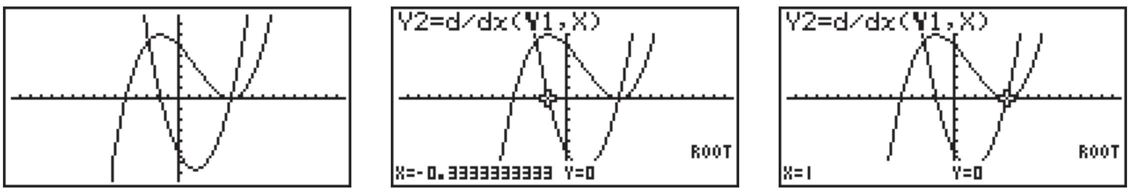


Pay particular attention to the y-values as you trace. It seems from these screens that there is a local maximum to the right of $x = -0.4$ and to the left of $x = -0.3$. As the graph would continue going up to the right of the screen, however, it is clear that this is not a *global* maximum of the function.

You could get a better idea of the local maximum, and the associated x-value by zooming in near the loop, but a quicker way is to use a *graph solve* command. Tap G.Solve ([F5]) to select the menu and then Max ([F2]) to find the maximum. In this case, since only one function is defined, the calculator will proceed to look for any relative maximum points on the graph. Although the value given is only an approximation, it looks as if there is a relative maximum at $x = -1/3$, which is between $x = -0.4$ and $x = -0.3$, as expected, and a relative minimum at $x = 1$.



Another way to find relative maxima and minima from a graph is to use the derivative function. The relative extrema will occur where the (first) derivative is zero – that is, where the graph of the first derivative crosses the x-axis. In this case, since the function itself is cubic, the first derivative is quadratic, and there are two solutions.



These two solutions can be found using the *ROOT* command in the *graph solve* menu. You will need to select the derivative function (with ▲) and use the ▶ key to get the second root after the first one is found. *Be patient!* The calculator has to do a great deal of computation to find these solutions, which may take a few seconds to appear. It seems that the relative maximum is at $x = -1/3$ and the relative minimum is at $x = 1$, which is *on* the x-axis.

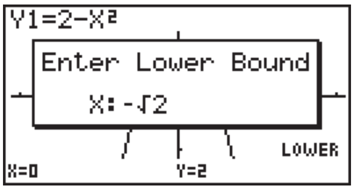
As well as finding relative extrema using graphs, you can get good numerical approximations to them in Run-Mat mode. Tap CALC ([OPTN] [F4]) and the continuation key, [F6], to see the *FMax* and *FMin* commands. The syntax in each case is: *FMax*(*function,left,right,precision*). In this statement, the *function* rule must use x as the variable; *left* and *right* are the two endpoints of an interval on which you want to find the maximum or minimum and *precision* is an integer from 1 to 9, to choose how many decimal places of accuracy you want the result to have. A higher number is more precise, but will take longer to get. You can omit a number for the precision if you wish, in which case the calculator will choose a value of 4 for you. If a function is already defined in the function list of Graph or Table mode, you can use the *Y* variable from the [VARS] menu instead of a rule.

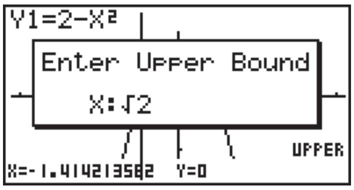
Numerical integration

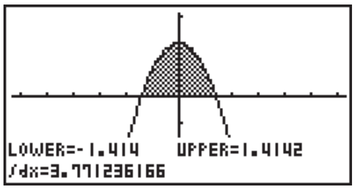
As for differentiation, the calculators can give you good numerical approximations to definite integrals, in either Graph or Run-Mat modes.

Once a graph of a function is drawn, you can find the definite integral of the function as the area under the graph between two values, using the *graph solve* menu. For example, consider the function $f(x) = 2 - x^2$, drawn on the Initial screen below. To find the area under the curve and above the x-axis, tap G.Solv (F5) and select ∫dx(F6 F3). (On the fx-CG 20, there is a larger suite of four integration commands, but we will deal with only the first command here, which is available on each calculator.)

After selecting a function (if necessary, when more than one is defined in the function list), the calculator needs an input for the lower limit and the upper limits for the area. You can trace to a suitable value for the lower and upper bounds, and tap EXE after each, but be aware that, if your choices are made with the cursor, their accuracy depends on the screen resolution. It is usually better to enter bounds in turn directly from the keyboard and tap EXE after each:





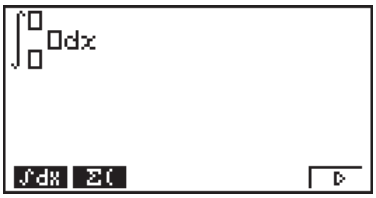


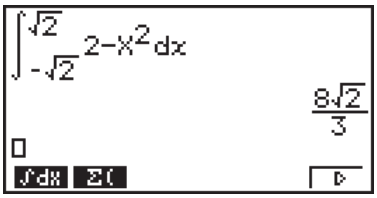
After the lower bound is chosen, the calculator draws a vertical line, and asks for the upper bound. In this case, the choices of $x = \pm\sqrt{2}$ allow us to find the area under the curve and above the x-axis. After the upper bound is chosen, the calculator will shade the area selected and print its area on the screen. In this case, the area is calculated to be about 3.771 units, as shown at right above.

In this particular case, if you trace instead of entering values, the closest you can get to the two x-intercepts are $x = -1.4$ and $x = 1.4$, different from the *exact* values of $x = -\sqrt{2}$ and $x = \sqrt{2}$.

Alternatively, numerical integration can be performed in RUN mode, using the integration command in either the MATH menu or the CALC menu. This method relies on your knowing the limits of integration, and not needing to check the graph for any reason.

When set in Math mode, the calculator displays the integral in conventional notation, as the first screen below shows. Enter the information in the following order: expression ▶ lower limit ▶ upper limit EXE.



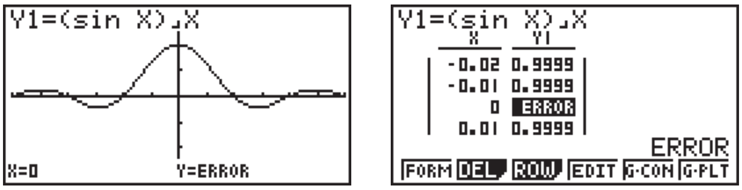


This method of numerical integration can be quicker than the graph-based method, but relies on your knowledge of the situation; in many cases, it is wise to draw a graph first, in order to understand what the integral is measuring.

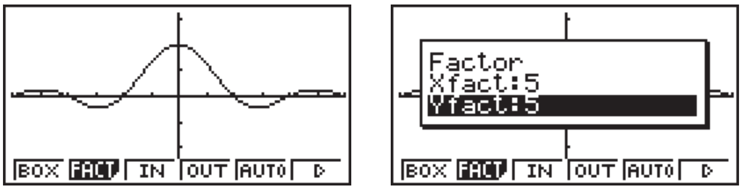
Notice that in this case the calculator has given an *exact* result for the integral. In other cases a numerical approximation will be provided by the calculator and the theoretical methods of the calculus would be needed to find an exact result.

Limits and asymptotic behaviour

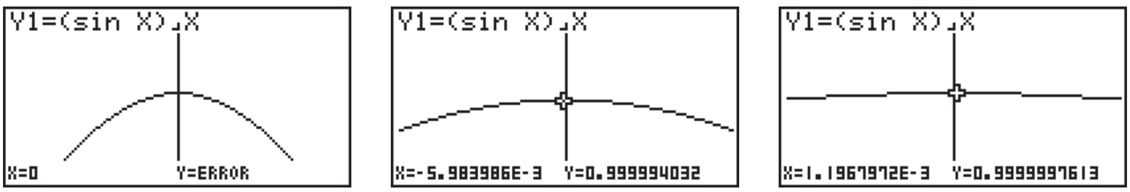
You can use graphical approaches to limits with careful use of the zooming facilities of the calculator. These provide a good foundation for later, more analytic, treatments. For example, to evaluate the limit of $\frac{\sin x}{x}$ as $x \to 0$, first make sure that the calculator is set to radian mode. Then draw the graph of $y = (\sin x)/x$ on the Trig screen. Trace to position the cursor near the point (0,1). Notice that the calculator indicates that the function is undefined at $x = 0$ in both Graph and Table modes.



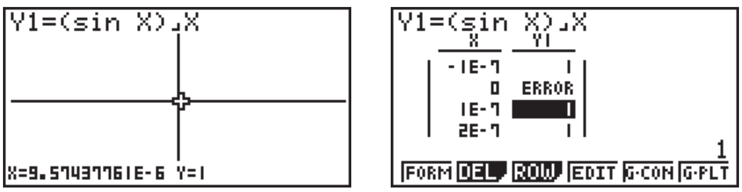
In Graph mode, open the zoom menu and use FACT (F2) to change the zoom factor from the normal setting of 2 to a larger value to speed up the process of examining the limit. The screen below shows the zoom factors set to be 5 in each direction:



Tap EXIT to return to the zoom menu and *Zoom In* (F3) repeatedly. Here are some screens obtained with these zoom factors:

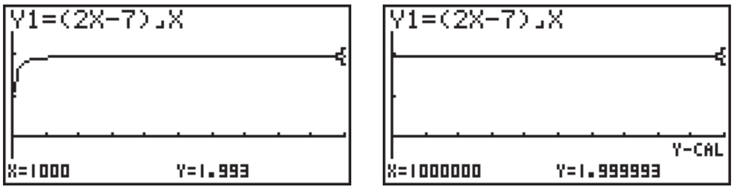


If the resulting curve is traced, the eventual value of the function is clearly 1. The calculator graph *does* eventually show a limiting value of 1 for the function near $x = 0$, even though the function is undefined at that point. Similarly, in Table mode, it is also clear from the right screen below that the limit of $\frac{\sin x}{x}$ as $x \to 0$ seems to be 1:



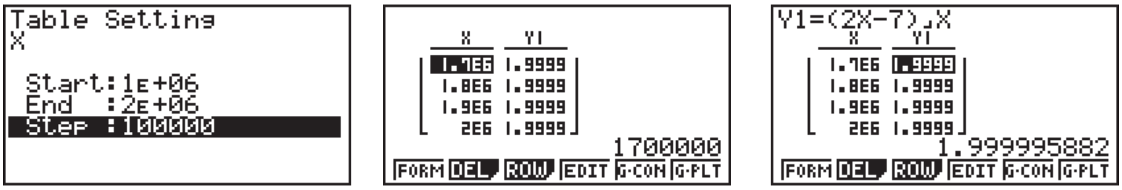
Another way of examining limits graphically is to change the viewing window manually. This is especially helpful for examining limits to infinity; often, a very large value for the independent variable will help you see what the limit appears to be, and it is quicker to change it manually than to zoom in just the horizontal direction.

For example, to evaluate the limit of $\frac{2x-7}{x}$ as $x \to \infty$, first draw the graph of $y = (2x-7)/x$ and then change the viewing window to show large values of x, such as *XMin* = 0 and *XMax* = 1000. Change the *X Scale* value as well, so that the x-axis is not too chunky. Experiment to get suitable values for the y-axis. The screen on the left below has $-1 \leq y \leq 3$, and shows that the limit appears to be 2.

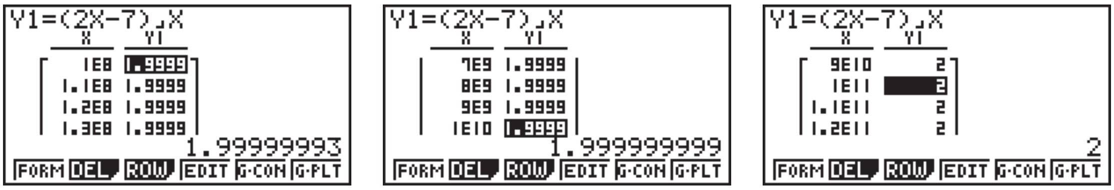


To verify this apparent limit, leave the y-values the same, and change the x-values to go from 0 to 1 000 000, with *X Scale* = 100 000. The screen on the right suggests that the limit as x tends to infinity is 2.

As well as graphical approaches, a table of values can also help you to see the likely limits of a function at a point or at infinity. For example, switch to table mode, and examine values of the function for very large values of x. Use the SET command to tabulate the function from x = 1 000 000 to x = 2 000 000, in steps of 100 000:



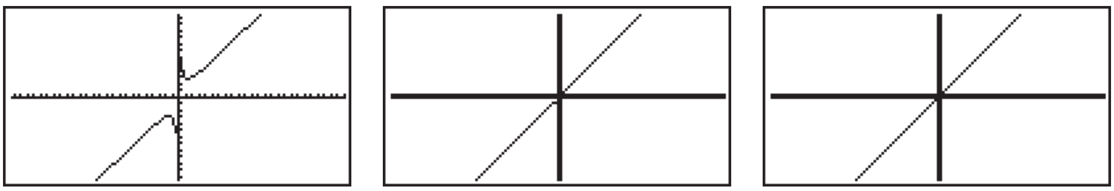
Notice that the calculator uses scientific notation for very large numbers. You can get a sense of the limiting process by using ⏶ and ⏷ to scroll tables. The table extract above shows that, for x = 1 700 000, the value of the function is very close to 2. Further down the same table, for x = 2 000 000, the value is even closer to 2. Even larger values of x can be chosen, by editing the values in the x column directly. The next three screens show very large x values of 100 million, 10 billion and 100 billion respectively. The resulting value of the expression is closer and closer to 2.



In such ways, the calculator suggests that the value of the expression is 2, within the limits of accuracy of the machine. Of course, analytic methods are needed to *prove* that the limit is exactly 2. The calculator merely provides very strong evidence that it is likely to be 2.

The same kinds of ideas are useful for examining asymptotic behaviour of functions graphically.

For example, to examine the limiting behaviour of $f(x) = \frac{x^2+2}{x}$, first draw the graph of $y=(x^2 + 2)/x$ on the Initial screen. Only two small bits of the graph appear in the window. However, if you zoom out repeatedly, as shown below, the limiting curve appears to be the identity line $y = x$ after two, four and six zooms, with standard zoom factors of 2 in each direction.



Once again, some analysis of the function is necessary to *prove* this result, but the calculator helps you to 'see' it visually rather than mentally.

Exercises

The main purpose of the exercises is to help you to develop your calculator skills.

1 a Find the derivative of $f(x) = 3 - x^3$ at x = -1.2, 2.3 and 1.7, without using a graph. (Use ⏶ after each result to save typing.)

b Check your answers to part a by tracing a graph, with the *Derivative* turned on.

c Check your results in part a again by using a table.

2 Draw a graph of $y = \cos x$ on the default Trig viewing window. Trace to a point on the graph and then zoom in several times to see the local linearity. Use Zoom ORIG to return to the original viewing window, and try another point in the same way.

3 The function $f(x) = \dfrac{x^2 + x + 1}{x^2 - x - 6}$ has two points of discontinuity on the interval $-5 \le x \le 5$. Use a table to identify these two points.

4 Draw a graph of $y = |3 - x| - 1$ on the Initial screen with the Derivative trace turned on. Trace to the vertex of the graph and zoom in several times. Explain what you see.

5 Graph the function $f(x) = x^3 - 3x + 1$ and its derivative function on the same screen. What are the values of x at the turning points of $f(x)$? How did you find out?

6 Find the minimum value of $f(t) = \dfrac{4 + t^2}{2t}$ for $0 < t \le 5$.

7 a Draw a graph of $f(x) = 2x + 1$. Then use a *graph solve* command to evaluate. $\int_0^1 (2x+1)dx$

Check your answer by finding the area of the trapezium.

b Use the integral command in the Math menu in RUN mode to evaluate $\int_0^1 \sqrt{1-x^2}\,dx$

and explain why it is approximately equal to $\dfrac{\pi}{4}$.

c Evaluate $\int_0^1 x^2 e^x dx$ without drawing a graph. (The answer should be $e - 2$.)

8 Find the limit, as $x \to 0$ of $\dfrac{1 - \cos x}{x}$.

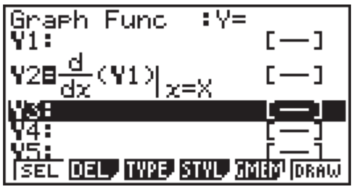
9 Describe the asymptotic behaviour of the function $f(x) = \dfrac{5 + x}{2x}$.

10 Evaluate the limit of $\dfrac{2x + 7}{1 - 3x}$ as $x \to \infty$.

Activities

The main purpose of the activities is to help you to use your calculator to learn mathematics. You may find that some of them are too advanced for you. Ignore activities you don't yet understand.

1 Make a derivative function transformer by defining the second function in the function list to be the derivative of the first one.



Use your transformer to investigate various kinds of functions.

For example, what happens if the first function is linear? quadratic? cubic?

2 In Table mode, with $f(x) = x^2 - x$, tabulate both f and its derivative for $x = 0$ to 5 in steps of 1. Study the table carefully and then guess what $f'(7)$ will be.

Check your guess with the calculator.

Predict what will happen if you study a related function, such as $g(x) = x^2 - x - 2$ in the same way. Use the calculator to check your predictions.

3 Examine the asymptotic behaviour of some rational functions like

$$f(x) = \frac{x^2 + 3}{x - 2} \quad \text{and} \quad g(x) = \frac{x^3 + 2x + 5}{x + 1}.$$

Use both graphs and tables to study the functions.

Explain your results.

4 Draw the graph of $y = \frac{x - 1}{x^2 - x - 2}$ and explain any discontinuities you can see.

5 Draw a graph of the function $f(x) = 3x^2 - 18x + 24$ between $x = 0$ and $x = 6$. Find the area between the graph of the function and the x-axis for the following intervals:

(i) $0 \leq x \leq 2$ (ii) $2 \leq x \leq 4$ (iii) $4 \leq x \leq 6$ (iv) $0 \leq x \leq 6$

What do you notice about these areas?

Investigate different functions and find where the area under the x-axis is greater than the area above the x-axis for a chosen interval.

6 An object starting from rest moves so that its velocity v metres per second after t seconds is given by $v = t^2$, or $f(t) = t^2$.

Use a suitable graph to calculate the distance travelled by the object in each of the first few seconds of motion (i.e., between $t = 0$ and $t = 1$, and then between $t = 1$ and $t = 2$, and so on.) Use your results to describe the motion of the object.

Investigate the distance travelled in the first few seconds of motion for other velocity functions such as $f(t) = 10 - t^2$ and $f(t) = (t - 6)^2$.

Notes for teachers

This module illustrates several ways in which the calculator can be used to explore various aspects of calculus, to help students understand some key introductory concepts. These include the idea of a derivative as a rate of change, the concept of a derivative function, continuity, limits, asymptotes and integration as area under a curve. The text of the module is intended to be read by students and will help them to start to see how the calculator can be used efficiently for various kinds of explorations of relevance to the calculus. The Activities are appropriate for students to complete with a partner or in a small group, so that they can discuss their observations and justify their conclusions.

Answers to Exercises

1. -1.32, -15.87, -8.67 3. x = -2, 3 4. The graph does not change shape, even after repeated zooming. The derivative is -1 to the left of the vertex and 1 to the right of the vertex, and so is not defined at the vertex itself. Despite this, note that the calculator gives a value of 0. 5. x = -1, 1 are found by finding the roots of the derivative function. 6. y = 2 7. (a) Area is 2 (b) Area is approximately 0.78543981634, as it is a quarter of a circle (c) 0.7182818285 8. 0
9. Horizontal line, y = ½ 10. 2/3

Activities

1. Activities using a derivative function transformer are very powerful for understanding how derivatives describe change in functions, and we suggest that students work on them together. They are also useful for whole class discussions. Students should readily see that derivatives of quadratic functions are linear, of cubic functions are quadratic, and so on. In addition, students should see that roots of derivative functions identify turning points of functions.

2. Explorations of this kind will help students appreciate the idea of a derivative function and see the linear nature of the derivative of a quadratic function. Choosing functions differing only by a constant will also allow them to see that the derivative functions may be the same for different functions, an important idea for later study of differential equations.

3. This activity exploits the potential of the calculator to help students understand asymptotic behavior of rational functions. As they zoom out, adjusting the axes as they do so in some cases, they will see that a ratio of a quadratic function and a linear function is linear in nature eventually, for example. Again, encourage students to work in pairs to explore functions in this way. They may need help in setting suitable zoom factors, to zoom out efficiently.

4. Students should not have difficulty seeing the jump discontinuities in this case (at x = 2 and -1). If necessary, suggest that they think about factors of the expressions to understand their source. Encourage them to generate further examples for themselves.

5. The key idea of this activity is that integrals find a *signed* area, so that care is needed to determine whether or not a graph is above the x-axis before interpreting an integral as an area, and hence the importance of drawing a graph. Suggest that students explore the absolute value function, $f(x) = |3x^2 - 18x + 24|$ to find positive areas. [Answers: (i) 20 (ii) 4 (iii) 20 (iv) 36 Note that the fourth value is not the sum of the first three values.]

6. For students studying rectilinear motion, activities of this kind are helpful. They will need to rewrite $v = t^2$ as $y = x^2$ to suit the calculator. By exploring how the integrals change each second, and understanding that they represent distances travelled, they should be able to appreciate how the object is accelerating as it travels further in each second. Explorations with other functions will show different patterns of course, including deceleration.

Module C1
Functions

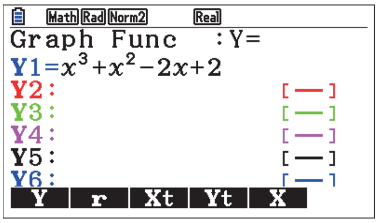
Note: This module is for the CASIO fx-CG 20 only. If you have the CASIO fx-9860GII calculator, please refer instead to Module 2.

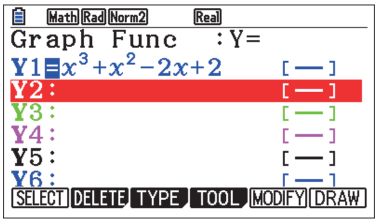
Functions are very important in mathematics, and are used to represent relationships in many ways and for many purposes. Graphics calculators have many capabilities helpful for understanding, representing and using functions. In this module, we focus especially on representing functions in three ways: symbolically, graphically and numerically, making use of the Graph and Table modes.

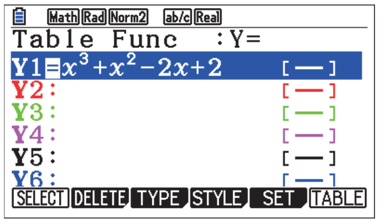
Symbols

Tap MENU 5 to access the symbolic menu for graphing and MENU 7 to access the same symbolic menu for tabulating. Use the ▲ and ▼ keys to select one of the 20 locations (Y1 to Y20) in which to store a function. The rule (or the symbolic formula) for a function can then be entered using the [X,θ,T] key for the variable, *X*, followed by the [EXE] key.

If there is already a formula in the location you choose, you will replace it with any new formula you type. You can also delete any highlighted function by first tapping either DEL ([F2]) or the [DEL] key. The screen below shows the function $f(x) = x^3 + x^2 - 2x + 2$ stored in location Y1. Functions like this are also written as rules like $y = x^3 + x^2 - 2x + 2$. The calculator uses the same conventions as algebra: $2x$ means $2 \times x$, but it is not necessary to tap the multiplication key.

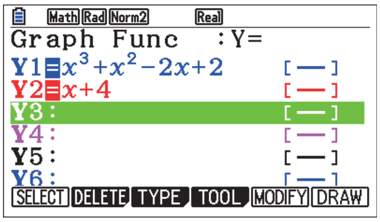






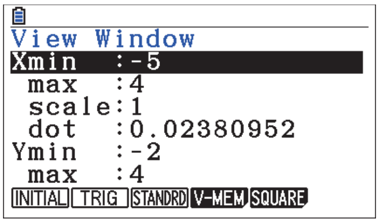
The calculator requires functions to be written as a function of *x* with *y* on the left of the equals sign. So a linear function such as $3x - y = 5$ must first be rearranged to $y = 3x - 5$. Tapping MENU 5 and MENU 7 allows you to jump from the symbolic menu in graph mode to the same menu in Table mode.

To deal with more than one function at a time, enter the two formulas in different places. The screen below shows two functions, $f(x) = x^3 + x^2 - 2x + 2$ and $g(x) = x + 4$. We will use these two functions to learn how to use the calculator.



Graphs

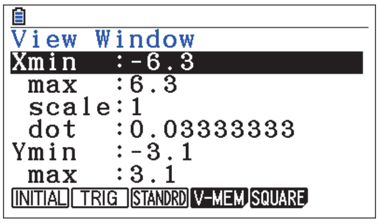
The graphics screen shows only a part of the coordinate plane, which extends infinitely both horizontally (left and right, like the horizon) and vertically (up and down). So you need to tell the calculator which part of the coordinate plane to use. This is called the *viewing window*. Like the calculator screen, the viewing window has the shape of a rectangle. Tap V-Window ([SHIFT] [F3]) to see what the current viewing window is. The settings will be whatever they last were, since the calculator keeps them the same until they are changed.

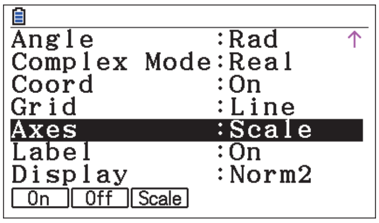


The minimum and maximum horizontal values (called ***Xmin*** and ***Xmax***) and the minimum and maximum vertical values (called ***Ymin*** and ***Ymax***) are enough to define the viewing window. The calculator will automatically draw the horizontal and vertical axes if you choose a window that includes them (although, as you will see later, you can prevent axes from being drawn if you wish). The settings above show Xmin = -5, Xmax = 4, Ymin = -2 and Ymax = 4. That is, they define a viewing window for which $-5 \leq x \leq 4$ and $-2 \leq y \leq 4$.

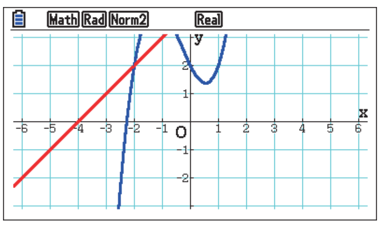
Use the ▲ and ▼ keys to see the current settings on your screen. The ***scale*** values for *x* and *y* tell you what the tick marks on each axis represent. For the settings above, both scales have been set to 1. You can change any of the screen settings by typing a new value over a highlighted one, and tapping EXE to register your choice. To leave a setting unchanged, use the ▲ and ▼ keys to pass over it. Tapping the EXIT key or tapping the EXE key when no change has been made will result in returning from the view window screen to the function list.

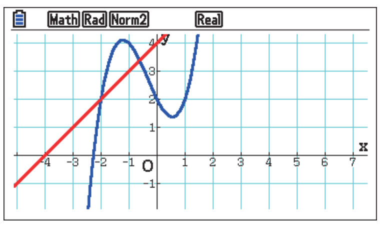
For now, tap INITIAL (F1) to automatically choose the initial settings shown at left below. These have the advantage that every pixel across and down the screen will represent 0.1 units. Pixels are the little black rectangles that go to make up the screen – there are 127 pixels across and 63 pixels down on this calculator, so the screen is about twice as wide as it is high. With these settings, the origin (where the two axes cross) will be in the centre of the screen. Another advantage of this INITIAL screen is that the scales on the two axes are the same: units on the horizontal axis and the vertical axis are the same, so that the graph does not distort the shape of the function.





Tap SET UP and set the Grid to *Line* (to draw grid lines on the screen) and Axes to *Scale* (to number the tick marks on the axes) as shown above. To draw graphs of the stored functions in whatever the current viewing window happens to be, tap DRAW (F6). The left screen below shows the graphs of the above two functions, $f(x) = x^3 + x^2 - 2x + 2$ and $g(x) = x + 4$ drawn on the INITIAL window. Check that the tick marks seem to be about right.

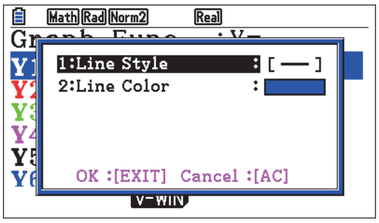


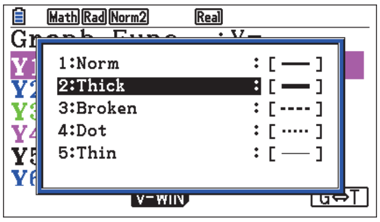


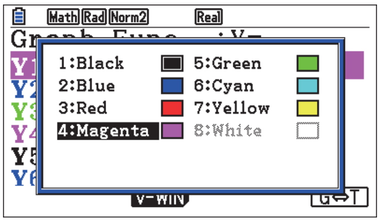
Immediately after graphs are drawn, you can tap the cursor keys ◀ ▶ ▲ ▼ to move the centre of the viewing window a little in any of the four directions. This is quite handy if the viewing window chosen is not quite convenient. In the case above, in which the top loop of the cubic graph is not showing, check the effect of tapping ▲ and then ▶ to get the graph screen on the right.

To change the viewing window, first display it with V-Window ([SHIFT] [F3]). Default settings are available with INITIAL ([F1]) and TRIG ([F2]) and STANDRD ([F3]) or you can change any of the settings manually by typing over the present value and then tapping [EXE]. The INITIAL settings are often useful to start with. The TRIG settings are especially useful for trigonometric functions. The STANDRD (Standard) settings give $-10 \leq x \leq 10$ and $-10 \leq y \leq 10$, which is usually a poor choice since the screen shape is not square. After making choices, tap [EXIT] to return to the function list.

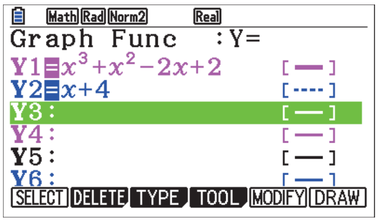
When more than one graph is drawn, as for the examples here, it is a good idea to use a different line *color* or *style* for each graph to make it easy to tell them apart. On the calculator, eight colors and five line styles are available for graphs. If a graph is showing, tap [EXIT] to display the formulas. Select a formula and tap FORMAT ([SHIFT] [5]) and [EXE] to display the choices as shown below. Use [▲] and [▼] to move the cursor, [EXE] to select and [EXIT] when finished.

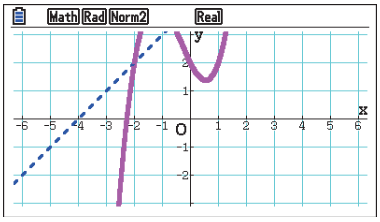






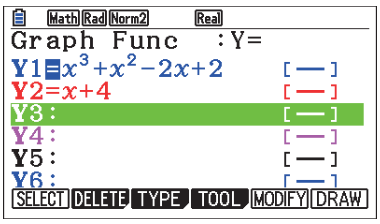
The screens above show the process of changing the first graph to have a thick style in magenta color, as below. The function list shows both colors and line styles (in square brackets).

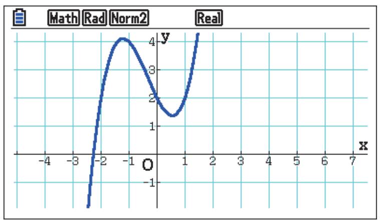




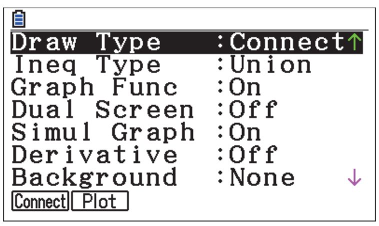
Switch between graphs and symbols by tapping [F6], which activates the G-T command. (The 'T' stands for 'text'.) When the graphs are showing, you can also return to the function list with the [EXIT] key. When the function list is showing, you can return to the graph screen with DRAW ([F6]), but this will cause the graphs to be drawn again. The G-T command ([SHIFT] [F6]) is much quicker.

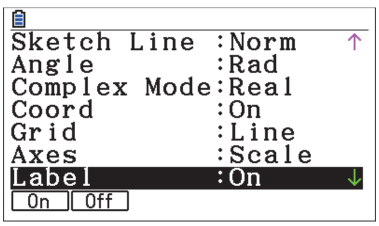
To temporarily turn off a graph, highlight a formula and then tap SEL ([F1]). Notice that the shading on the equals sign is removed to show that the function will not be graphed. The following screens show how only the first function (*Y1*) is graphed. Tap SEL ([F1]) again to turn a graph back on.





You can adjust various settings for graphs using the SET UP menu by first tapping [SHIFT] [MENU] while in Graph mode. Make sure that you are still in graph mode to see the screens below, which show some of the most useful possibilities:



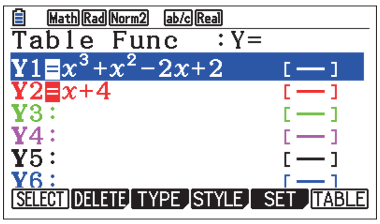


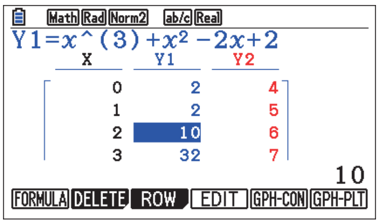
Draw Type	Graphs can be drawn by connecting successive points or by just plotting points. *Connect* is usually the better choice; sometimes, however, it will lead to graphs being joined improperly.
Ineq Type	When several inequalities are graphed (as described in the next module), you can choose which areas are shaded (by selecting *Union* or *Intersection*)
Graph Func	When this is turned on, the formula for a graph will be on the screen. This is usually helpful, especially if you want to know which graph you are tracing. It can be removed by turning *Graph Func* off.
Dual Screen	When this is turned on, the screen is divided in half, with either a pair of graphs or a graph and a table showing.
Simul Graph	When is turned on, all graphs are drawn simultaneously. Most people prefer to turn it off, so that they can see each graph drawn separately.
Derivative	Controls whether or not the numerical derivative of a function is displayed when tracing or tabulating. (See Module 15 for more information about numerical derivatives.) It is best to leave this turned off for now.
Background	For most purposes, set the background for graphs to *None.* A background (such as a picture) is used sometimes, as explained in the Picture Plot module.
Coord	This is usually left on, so that coordinates are showing on the screen. Turn this off if you don't want them displayed.
Grid	Places a grid or background lines on the screen, corresponding to tick marks. This is most likely to be useful when the scales on the two axes are the same and also when the scales are neither very big nor very small.
Axes	Controls whether or not the x- and y- axes are displayed. If *Scale* is chosen, numbers are shown on axes.
Label	Controls whether or not axis labels (i.e. x and y) are shown

As for other calculator settings, any changes you make will be retained by the calculator even after it is switched off. You will find it interesting to explore the use of these settings.

Tables

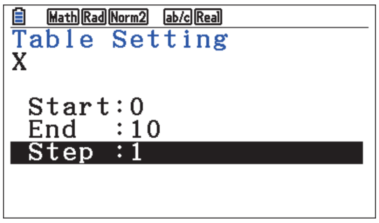
A table of function values is always a helpful way to represent a function and is available from MENU 7. Notice that the function list shown at left below is identical to that in graph mode. Tap TABL (F6) to show a table of values for the two functions defined above.

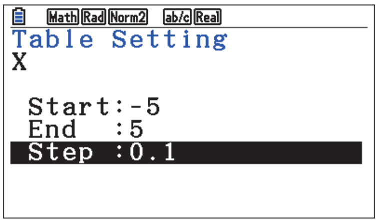


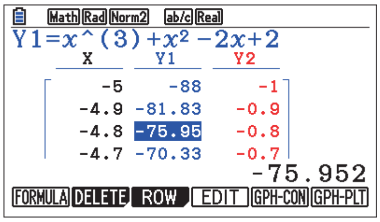


You can scroll tables vertically and horizontally using any of the four cursor keys. The function is shown in symbols at the top of the screen, and highlighted y-values are shown large at the bottom of the screen. You can scroll quickly by keeping your finger on a cursor key. Notice what happens if you scroll ◀ or ▶ repeatedly.

To set or change the values that are tabulated, tap EXIT and then SET (F5). Change the first (Start) and last (End) values for X, and the increment (Step) manually. Tap EXE after each change and then tap EXIT to return. The two tables above were obtained with the setting screen shown at left below:



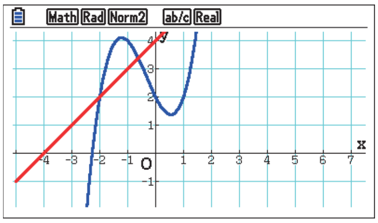


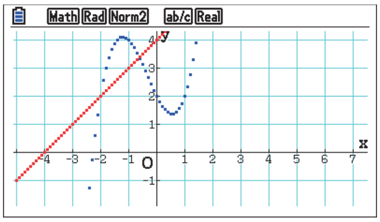


With the table setting screen changed to the middle screen above, the table below contains values of *X* from -5 to 5, going up in steps of 0.1. Notice that *Y1*(-4.8) = -75.952 is shown in full at the bottom of the screen, although the value in the table is approximated to save space.

Be careful not to make the *Step* too small, or the range from *Start* to *End* too large, or the calculator may run out of memory to store all the table values. To return to the formula screen from a table, tap FORM (F1) or EXIT.

You can draw a graph of table values in the current viewing window. Choose either G.Con (F5) for a continuous graph or G.PLT (F6) for a discrete plot of table values. The next two screens show what happens with each of these two options, using the same viewing window as previously:





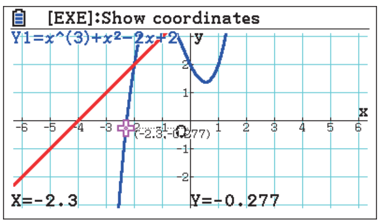
The continuous graph is the same as the one that we drew in graph mode above. Notice that the graph on the right only plots individual points that are represented in the table (in this case, values of *x* going from -5 to 5 in steps of 0.1).

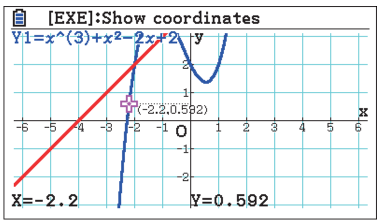
After you have drawn a graph, tap G-T (SHIFT F6) a few times to see how to toggle between a graph and the table. Tap EXIT or FORM (F1)from the table to return to the function list.

Tracing and zooming graphs

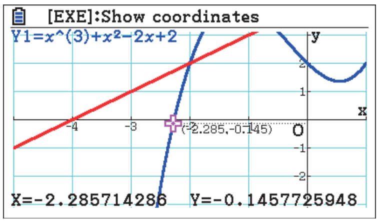
Once you have drawn some graphs, you will usually want to explore them to find out more about the functions concerned. The two main ways of doing this involve tracing and zooming. First, return to the graph screen with MENU 5 and DRAW (F6). Use the INITIAL screen for now.

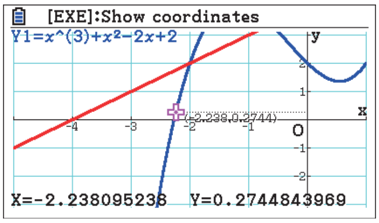
Tracing a graph is rather like running your finger along it, and seeing the coordinates of each point that has been plotted. Tap Trace (F1) to activate the graph trace. Use ◀ or ▶ to trace, and ▼ or ▲ to shift between graphs. From these two screens, you can tell that the function $f(x) = x^3 + x^2 - 2x + 2$ has a value of zero for *x* close to -2.3. To trace quickly, press your finger down on ◀ or ▶. (Keep your eye on the coordinates if you do this, or you may miss the cursor flashing past.)





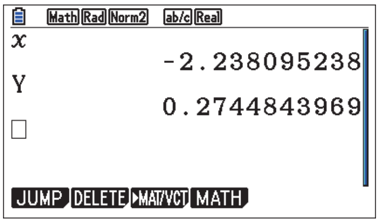
Notice that the x-values for the above traces go up in steps of 0.1 as you trace from left to right across the screen. This is a major advantage of the INITIAL screen (which takes advantage of the number of pixels on the graph screen, 127×63). Another advantage is that the scales on the two axes are the same. If you change the viewing window manually (say to $-5 \leq x \leq 1$), a less friendly step for x will be used by the calculator and the scales on the x and y axes will no longer be the same, as shown on the next two screens:





On the calculator, the coordinates of points on the screen are represented by (X,Y). It is important to understand how the calculator uses its memories for the two variables X and Y. Whenever you trace, the values of X and Y are changed automatically in the memories labeled X and Y, and you can access the most recent values in the Run-Matrix mode. To see how to do this, first Tap MENU 1 to return to Run-Matrix mode.

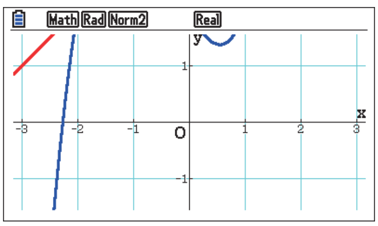
When you recall the contents of the X and Y memories, you will see that they contain the values from the most recent trace. The Y value is recalled with ALPHA −; the X value can be recalled either with ALPHA + or merely with the X,θ,T key. Using the X,θ,T is easier, since it requires only one key.

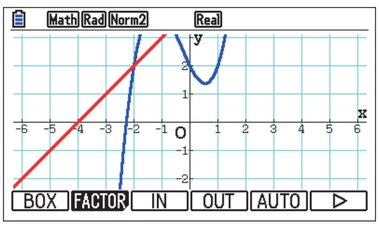


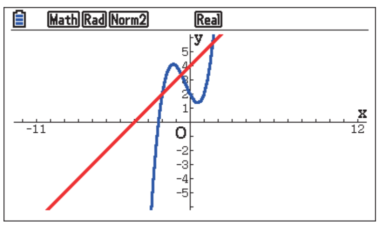
Return to graph mode with MENU 5 and draw the graphs again in the INITIAL window. Tap Zoom (F2) to access the *zoom* menu. Zooming allows you to change the viewing window very quickly, rather than having to change each of the *Xmin*, *Xmax*, *Ymin* and *Ymax* values separately.

Tap IN (F3) to zoom in; move the cursor to your preferred position for the middle of the screen and then tap EXE to conclude the zoom. 'Zooming in' is a bit like using a magnifying glass, to look more closely at some parts of graphs. (But notice that the 'thickness' of the graph and the axes don't change, so it's not exactly like a magnifying glass!). Similarly, tap Zoom (F2) and then OUT (F4) to zoom out. 'Zooming out' is a bit like looking at graphs from further away, so that you can see more of them. It is sometimes helpful to zoom out to see the overall shapes of graphs.

The next three screens show what happens after zooming in (the left screen) or zooming out (the right screen) from the middle of the INITIAL viewing window (the middle screen). Notice that the tick marks have not changed in the three cases (and continue to be at every one unit on each axis) although the grid is not shown for the third screen (as it would look too crowded with that scale).



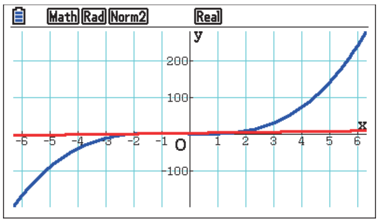


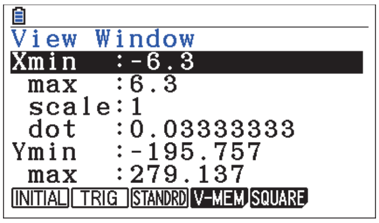


It's easy to get a bit lost after several zooms. If this happens, you can return to the Zoom menu with

F2, tap the continuation key F6 and then tap ORIGINAL (F1) to zoom back to the original viewing window. To undo a zoom, you can zoom back to the immediately previous screen by tapping the continuation key F6 and then PREVIOUS (F5). Another useful zoom is SQUARE (F2) which 'squares up' the axes, by giving the same scale to each.

The automatic scaling zoom, AUTO (F5) has the effect of adjusting the *y*-axis so that all the values associated with those on the *x*-axis are shown on the graph. This is sometimes useful to give an indication of the key features of a graph, but it can also lead to very distorted graphs. For example, the graph below shows the effect of using the automatic scaling zoom after drawing the sample graphs on an INIT window. Notice that there is a *y*-value plotted for each *x*-value on the screen.





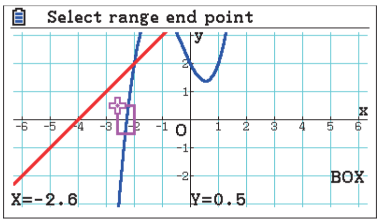
The viewing window that is produced by this zoom is also shown; this helps to make clear how distorted the graphs are in this case. The thickening on the *y*-axis is caused by putting a tick mark at every 1 unit (between -195.757 and 279.137), while the thickening on (part of) the *x*-axis shows the graph of $y = x + 4$.

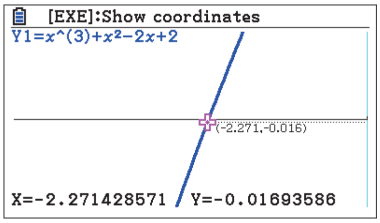
The factors for zooming in or out can be changed in the FACT (F2) menu after selecting Zoom (F2). The values selected are the values used to multiply or divide the values on the axes. The default value is for a factor of 2 on each axis. You can zoom very quickly and still keep some of the friendly aspects of the INITIAL screen by changing both zoom factors to 10. Tap EXE after each change. The two zoom factors do not have to be the same, and sometimes it is useful to use different factors on each axis. For example, to zoom in only one direction (vertically or horizontally), change the zoom factor for the *other* direction to 1.

When the graph screen is showing, the quickest way to zoom (using just one key) is to use the + key to zoom in and the − key to zoom out, using the existing zoom factors.

The BOX (F1) zoom allows you to define a new screen by making a rectangular 'box'. First use the cursor keys to move to one corner of the box and then tap EXE. Then move the cursor to the *diagonally opposite* corner of the box and tap EXE again. (Notice you make a box on the screen as you move the keys.) This is a good way of quickly focusing on the parts of a graph of most interest.

For example, in the screen below on the left, a box zoom is being used to highlight the intercept of the cubic graph near the origin. The small rectangle outlined becomes the whole screen after EXE is tapped the second time.

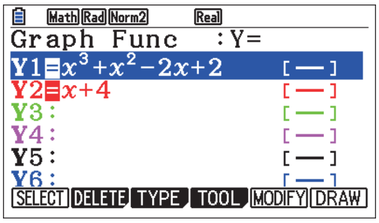


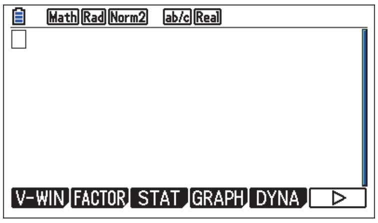


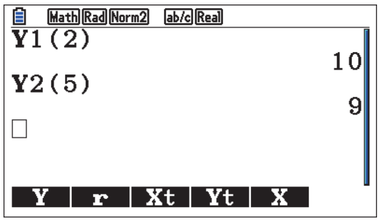
Usually, both tracing and zooming will be needed to answer important questions about functions from their graphs. For example, to find a good approximation to the intercept in the box, you will need to trace after the zoom has been completed. It seems from the graph at right that there is an *x*-intercept close to -2.27. Notice that tracing after a box zoom has been completed will usually *not* result in convenient steps in the *x*-direction. This is illustrated in the screen at the right above.

Evaluating functions

Sometimes you may want to find the value of a function for just one or two values of the variable (x), rather than for large set of values, in the form of a graph or a table. To do this, after the functions have been defined in Graph or Table mode, return to Run-Matrix mode with MENU 1. In the screen at left below, the present functions are shown.





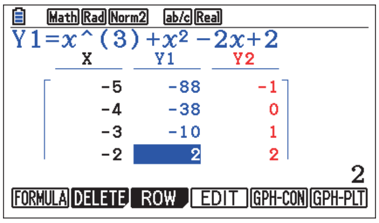


Consider the function $f(x) = x^3 + x^2 - 2x + 2$, which is represented in the calculator at the moment as *Y1*. In mathematics, we represent the value of the function when $x = 2$ as $f(2)$. The calculator uses a similar notation: the value of the function when $X = 2$ is represented as *Y1*(2).

To show this on the screen, and to obtain the numerical value of *Y1*(2), the **Y** symbol is needed. This is a different **Y** symbol from that for the Y memory: it is a function and not just a variable. Tap the VARS key (to access calculator variables) and then GRAPH (F4) to access the familiar menu you have already seen when entering functions. The **Y** symbol is showing at F1. As the last screen above shows, this variable allows you to enter *Y1*(2) and obtain its value of 10 with the EXE key. Similarly *Y2*(5) = 9.

Tracing and zooming tables

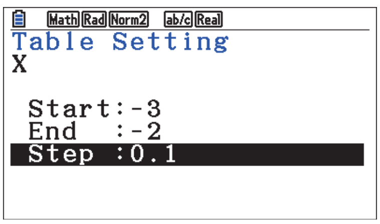
Just as you can trace and zoom a graph to find out more about the functions concerned, you can do something similar with a table. Return to Table mode with MENU 7. To trace a table, simply move the cursor keys ⊖ and ⊖. This is like running your finger up and down the values in the table to read them carefully.

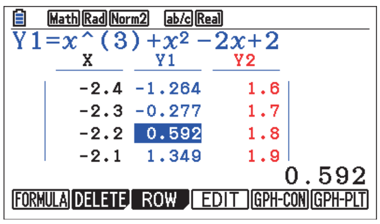


The screen shows that when $x = -3$, the value of the function is negative (*Y1(-3)* = -10) while for $x = -2$ the value is positive (*Y1*(-2) = 2). This suggests that there is a value of x between -3 and -2 for which $f(x) = x^3 + x^2 - 2x + 2$ has the value zero.

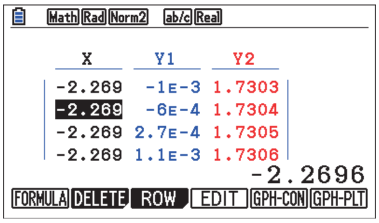
You can get a good approximation to this value by zooming in. To do this, tap FORM (F1) and then SET (F5) to make a table of values that goes up with a smaller step. A good choice is a step of 0.1 rather than 1, with x-values between -3 and -2.

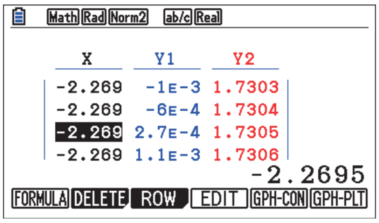
Now return to the table (by tapping EXIT and then TABL (F6)). Trace the table of *Y1* values to see that the appropriate value of x now seems to be between -2.3 and -2.2, as shown below.





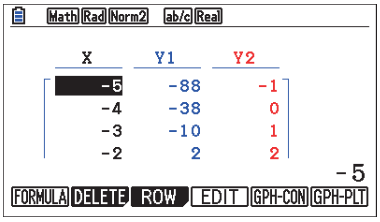
This process can be continued, and, if you decrease the step by a factor of 10 each time, each successive zoom will give another decimal place of accuracy. After using a step of 0.0001 below, we can see that the intercept is closer to x = -2.2695 than to x = -2.2696, since the corresponding y-value is closer to zero.

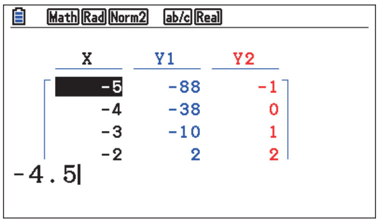


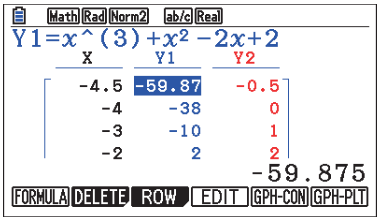


You can zoom out in a table in the same kind of way as zooming in, by making the interval and the steps bigger instead of smaller each time.

Another way of using tables to study the values of functions efficiently involves changing the X values directly in the table. To see how this works, start with any table as at left below. To find the value of the functions when x = -4.5, say, simply enter -4.5 somewhere in the X column of the table and tap EXE, as shown below. Notice that the table values are automatically updated. Change the x-value again, and watch the results.

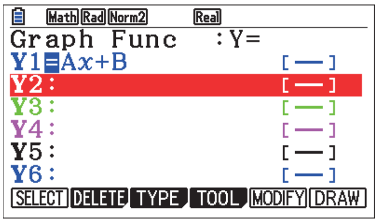


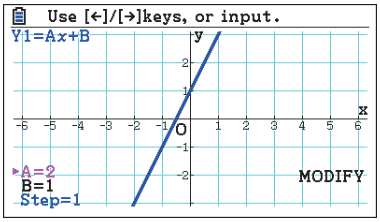




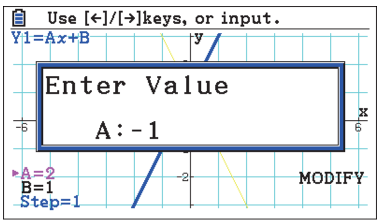
Modifying functions

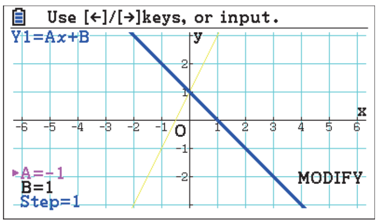
As Module 3 shows, sometimes it is helpful to consider several graphs in succession. One way of doing this efficiently is to graph a function with parameters and use the MODIFY commands to vary the parameters. For example, linear functions can be represented as $y = Ax + B$, with a parameter A for the slope and B for the y-intercept. Enter this function into the list, using the ALPHA key for the parameters. Use the INITIAL View Window and tap MODIFY (F5) instead of DRAW (F6) to draw the graph.

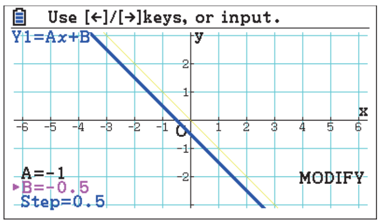




The graph shown depends on the values of A and B, shown in the bottom left of the second screen above. You can change the highlighted value by entering a new value, followed by EXE, or by using ◀ and ▶ to change the value by the value of *Step* (also shown above). Use ▼ and ▲ to highlight a different parameter or *Step*. Change *Step* (after highlighting it) by entering a new value. There are examples of changing parameter values below to graph $y = -x + 1$ and $y = -x - 0.5$.

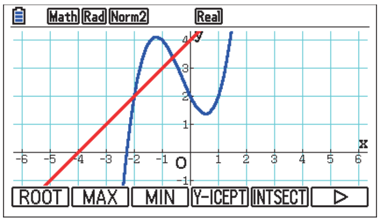


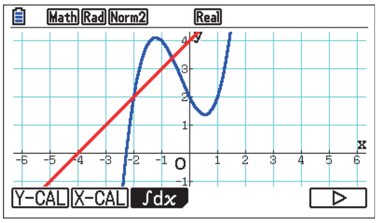


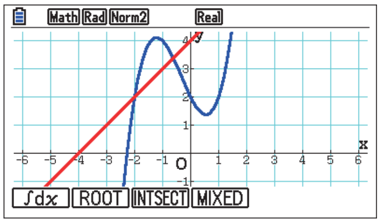


Interpreting graphs automatically

Many useful things about functions can be found out by tracing and zooming graphs and tables. However, sometimes these processes can be a little tedious, and it is more efficient to use the calculator's powerful automatic capabilities. To see how these work, first use MENU 5 to draw the graphs again on the INITIAL screen. Tap ▲ to move the screen down a little. Then Tap G-Solv (F5) to bring the *graph solve* menu to the screen. The screens below show the complete menu; note that the third screen menus appear only after tapping ∫dx (F3) in the middle screen.





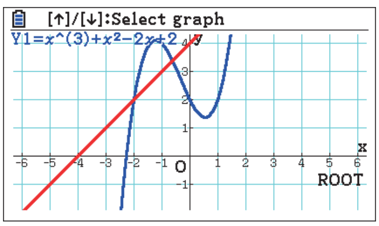


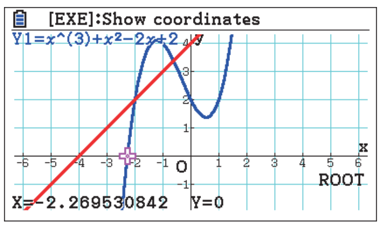
The immediate menu commands in this menu will automatically locate good numerical approximations to various values associated with graphs:

ROOT	An x-intercept, a point at which a graph crosses the x-axis gives a *root* of a function, a value of x for which the function has a value of zero.
MAX, MIN	The points at which a function has a *relative* or *local* maximum or minimum value. On a graph, these points are at the top or bottom of a curve. Local maxima and minima do not necessarily correspond with *global* maxima or minima, which are concerned with the entire set of values for the function.
Y-ICEPT	The point where a graph crosses the y-axis (i.e. for which $x = 0$).
INTSECT	A point of intersection of a *pair* of graphs.
Y-CAL	Calculates the y-value associated with a particular x-value for a function.
X-CAL	Calculates the x-values associated with a particular y-value for a function.
∫dx (Menu)	Various commands related to definite integrals, described in Module 15 and also briefly below:
∫dx	A definite integral of a function, found as the area between a graph and the x-axis between two particular x-values. The area above the x-axis is regarded as positive, and that below the x-axis as negative.
ROOT	A definite integral of a function between two roots.
INTSECT	A definite integral as an area between intersections of a pair of graphs.
MIXED	A definite integral between selected roots, intersections and other points.

The calculator will only provide ***approximations*** *to these important values. To find* ***exact*** *answers to questions of these kinds, you will need to use some mathematical analysis.*

We will use the menu to find a root of $f(x) = x^3 + x^2 - 2x + 2$. Tap ROOT (F1) as shown in the screen at left below, a small square cursor is located on the appropriate graph at the bottom, but the calculator needs to be informed which graph is the one to be used. (In this case, there is a choice, since two functions have been graphed.)



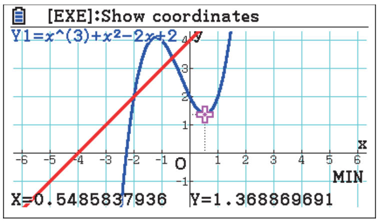


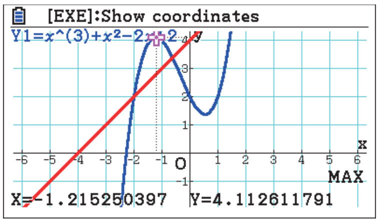
Move between graphs as in tracing with ▼ and ▲. Since *Graph Func* is turned on, the screen displays the formula for the graph which is currently selected and the graph selected is flashing.

Make your choice by tapping EXE. After a few seconds, the calculator will find the first root of the cubic function, from left to right across the screen: $x \approx -2.269530842$. The y-value given allows you to see how good the approximation is. Notice that the value is consistent with the value we obtained earlier by tracing and zooming both graphs and tables. Only rarely would so many decimal places be needed in practice, so $x \approx -2.27$ or even $x \approx -2.270$ may be sufficiently accurate approximations to two and three decimal places respectively.

It is clear from the graph that there are no further roots of this function for these values of x. (It is clear, too, if you know about the shapes of graphs of cubic functions, that there will be no roots for other values of x either.) Had the graph shown that there *were* other roots, however, the calculator would have found the succeeding ones after the ▶ key was tapped.

The other commands in the *graph solve* menu work similarly. The next two screens show the *MIN* and *MAX* commands for the cubic function. On the left screen below, notice that the minimum shown with the cursor is not the minimum value of the function overall, but merely the *relative* or *local minimum*near the region where the curve turns. That is, it is not a *global* minimum value. It is clear from the graph that the function has smaller values than this, negative values for example.





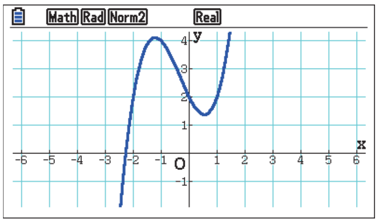
Notice above that the calculator finds a relative maximum point, at $x \approx -1.22$, where $y \approx 4.11$. The calculator would do so even when the y-value is not showing on the screen, because it searches systematically the x-values from left to right across the screen, regardless of whether the associated y-values are visible.

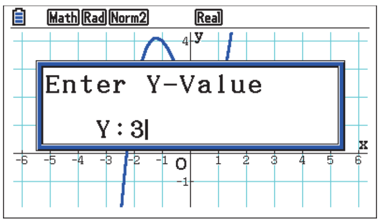
The *INTSECT*command needs you to identify two functions (by tapping EXE for each one in turn) if there are more than two graphed. In this case, since there are only two graphs on the screen, this step is omitted. The calculator finds the points of intersection from left to right across the screen.

Tap ▶ to move to the next point of intersection to the right or ◀ to move to the next point to the left. Although the points on the screen are given to many decimal places, they are still not exact. In this case, mathematical analysis shows the *exact* three points of intersection are at

$$(-2,2), \left(\frac{1-\sqrt{5}}{2}, \frac{9-\sqrt{5}}{2}\right) \text{ and } \left(\frac{1+\sqrt{5}}{2}, \frac{9+\sqrt{5}}{2}\right).$$

You can tell from the graph at the left below that there are three values for which the function $f(x) = x^3 + x^2 - 2x + 2$ has the value 3. (Imagine a horizontal line at $y = 3$ to see this.)





To find these three values, use the *X-CAL* command in the *graph solve* menu. As only one graph is now showing, it is unnecessary to select which graph to use. When you start *X-CAL*, you will need to insert the y-value (3), followed by EXE, as shown in the graph at right above. Three approximate solutions are produced ($x \approx -1.80$, $x \approx -0.45$, $x \approx 1.25$) by tapping the ▶ key after each result.

Exercises

The main purpose of the exercises is to help you to develop your calculator skills.

1 Draw a graph of $y = x^2 - x - 3$ on the INITIAL screen. The bottom of the graph does not quite fit on the screen. Use the cursor keys (◀ ▶ ▲ ▼) to move the origin so that the graph does fit. Change the style of the graph from thin to thick.
- a Trace the graph to find y when x =-0.7.
- b Trace to find the two values of x for which $y = 1.59$.
- c Change to Run-Matrix mode. Tap [X,θ,T] [EXE] and [ALPHA] [−] [EXE] to see traced values on the screen.

2 Draw a graph of $y = x^2 + 7$ on the INITIAL screen. Explain what you see.

3 Draw a graph of $f(x) = 15 \times 1.19^x$ on the interval $-1 \le x \le 10$ and $-10 \le y \le 100$. Put tick marks at every unit on the x-axis and at every ten units on the y-axis.
- a Trace to find the approximate value of x for which $y = 40$.
- b Use *X-CAL* in the *graph solve* menu to get a better approximation.

4 Make a table of values for the functions $f(x) = 4 - x^2$ and $g(x) = x^3 + x - 1$, for values of x from 1 to 1.5, going up in steps of 0.01.
- a For which value of x do the two functions have approximately the same value?
- b Use G-CON ([F5]) to draw graphs of the two functions on the INITIAL screen.
- c Use a box zoom to draw a box around the point of intersection of the two graphs near $x = 1.3$. Then trace the graphs to find the coordinates of the point of intersection to two decimal places.
- d Use *INTSECT* in the *graph solve* menu (in Graph mode) to get a better approximation.
- e Turn off the graph of the parabola. Zoom back to the original screen and use *graph solve* to find the value of x for which $x^3 + x - 1 = 0$.

5
- a Construct a table of values for $f(x) = 2.3^x$ for $0 \le x \le 10$ with *Step* = 1. In the viewing window, set *XMin* to 0 and *XMax* to 10.
- b Use G-PLOT ([F6]) to plot the table values. Use Zoom AUTO ([F5]) to automatically choose y-values to suit the x-values. Check the View Window to see the scales chosen.
- c Trace the plotted points. Between which two values of x is $2.3^x = 100$?
- d Zoom on the table twice by factors of 10 to get a better approximation to the value of x for which $2.3^x = 100$

6.
- a Draw graphs of $x - y = 3$ and $4x - y = 2$.
- b Draw a graph of $x - y = 4$ in bold style and of $8x + 10y = 5$ in thin style. Trace the graphs to find their point of intersection.

7 Draw graphs of $y = x - 3$ and $y = 2 - 4x$ on a standard screen. (Use STANDRD ([F3]) in the viewing window.) Redraw the graphs on the INITIAL screen.

Notice the differences in scales and graphs.

8
- a Enter the function $y = Ax^2 + Bx + C$ and draw the graph using the MODIFY command.
- b Highlight the parameters in turn using ▼ and ▲ and set $A = 1$, $B = -1$ and $C = -1$.
- c Set *Step* to 1
- d Use ◀ and ▶ to change the value of C. Describe what happens to the graph.
- e Use ◀ and ▶ to change the value of A. Describe what happens to the graph.

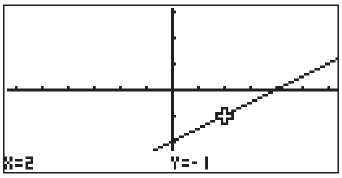
Activities

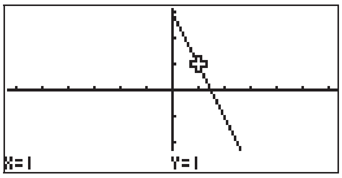
The main purpose of the activities is to help you to use your calculator to learn mathematics. You may find that some of them are too advanced for you. Ignore activities you don't yet understand.

1 a Draw graphs of $y = x + 2$, $y = 2x + 1$, $y = x - 1$, $y = 3 - x$ and $y = 1 - 0.5x$ all together on the INITIAL screen. Which graphs are parallel to each other? Which graphs are perpendicular to each other?
b Change the viewing window so that *Xmin* = -8 and *Xmax* = 8. What effect does this have on parallel and perpendicular pairs of graphs?
c Change to the STANDRD screen. What effect does this have on parallel and perpendicular graphs?
d Change back to the INITIAL screen, and find some more functions whose graphs are lines that are either parallel or perpendicular to each other.

2 It's a good idea to imagine what a graph will look like before you draw it. Try to do this in each part of this activity.
a Find a pair of quadratic functions whose graphs cross each other exactly twice.
b Find a pair of quadratic functions whose graphs do not intersect.
c Find a quadratic function and a linear function whose graphs intersect at exactly one point.

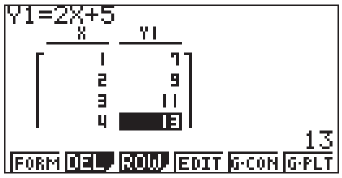
3 a Which linear functions have been graphed here on the INITIAL screen?





b Invent some questions like this to give to a partner. Make sure that they can see both intercepts on the screen and turn off the *Graph Func*. (Before you give them the calculator to look at, it might be advisable to define a function near the bottom of the list (such as Y_{20}), and then return with F2 to the top of the list, just in case they are tempted to peek!)

4 In Table mode, draw a table of values of a linear function for integer values of *x* from 1 to 10. Use the cursor in the table to scroll down the values of the function. The screen shows this for $y = 2x + 5$.



a Use the pattern of values of the function (in this case, 7, 9, 11, 13, …) to predict the next values of the function as you scroll down. Predict the remaining six values, i.e., those for $x = 5, 6, 7, 8, 9$ & 10.
b Now try a different linear function, like $y = 2x + 6$.
c Then try a different linear function, like $y = -3x + 6$.
d Try some others for yourself, until you can see how to make the predictions efficiently.

5 Draw a graph of $y = x + \frac{1}{x}$ on the default screen.
Describe and explain what happens if you zoom in and out several times.

6 Draw graphs of some absolute value functions, such as $y = |x|$, $y = |x - 2|$ and $y = |x - 2| - 4$. Predict what the graphs of $y = |x + 4|$ and $y = |x + 1| + 5$ will look like. Graph them to check your predictions. Compare the graphs of function pairs such as $y = 2 - x^2 - 3x$ and $y = |2 - x^2 - 3x|$. Make up some more examples.

Notes for teachers

This module is important for new users of the fx-CG 20 calculator, as the major advantage of graphics calculators over scientific calculators is the ability to represent, manipulate and explore functions on a graphics screen. The calculator provides many opportunities for learning about the behaviour of functions, and it is important that students develop sufficient expertise with the calculator to be able to realise these. Once basic calculator operations are mastered, there are many ways in which the calculator can be used to enhance learning about functions, some of which are reflected in the activities below. The text of the module is intended to be read by students and will help them to start to see how the calculator can be used efficiently for various kinds of calculations.

Answers to Exercises

1. (a) -1.81 (b) -1.7, 2.7 2. No graph is visible because of the screen settings. Tap ⏶ a few times. 3. (a) $x \approx 5.635$ (b) $x \approx 5.638$ 4. (a) $x = 1.28$ (c) (1.28,2.37) (d) (1.278163073,2.366299159) (e) $x = 0.682$ 5. (b) Note $1 \le y \le 4142$ (c) between $x = 5$ and $x = 6$ (d) $x = 5.53$ 6. (a) Note that functions need to be written with y on the left of the equals sign, such as $y = x - 3$. (b) (5/2,-3/2) 7. Graphs appear perpendicular (incorrectly) in STANDRD screen. 8. (d) the parabola moves vertically as C changes (e) the parabola expands and contracts as A changes.

Activities

1. This is an essential activity for students to explore both the relationships between slopes and linear graphs and also the effects of choosing different scales on each axis. [Answers: Graphs 1 and 3 are parallel and each perpendicular to graph 4; graphs 2 and 5 are perpendicular. Parallel lines remain parallel in STD, but perpendicular lines are no longer perpendicular.]

2.This activity is intended to encourage students to explore graphs of linear and quadratic functions, and to see how changing coefficients changes graphs. It will also provide students with lots of experience of graphing functions on the calculator. Encourage students to work in pairs and to compare their solutions with each other.

3. An activity of this kind can be used as a classroom activity or as an activity between two students. The purpose is to help them to understand the significance of both slopes and intercepts of linear functions. Tracing a graph will help students see how the slope determines the rate of change and the intercepts. Leaving a grid on the screen may also help students' thinking.

4. This activity focuses on tabulated values to help students see the idea of a rate of change, or slope, of a linear function. Working in pairs will encourage students to verbalise and understand the rate of change; for example, for $y = 2x + 5$, y increases by 2 when x increases by 1, which is the essential idea of a slope of 2.

5. Activities of this sort help students to appreciate that the appearance of a graph depends critically on the scales chosen. In this case, zooming in might show no graph at all, including a point of discontinuity at $x = 0$, while zooming out shows a graph that is close to the identity line $y = x$. Activities of this sort also help students with the idea of asymptotic behaviour, important in later studies of the calculus.

6. An absolute value command is available on the calculator via OPTN F5 and then F1. This activity illustrates how the calculator can be used to understand particular kinds of transformations by exploring several examples of them. It is helpful for students to undertake these explorations with a partner, in order to discuss their observations.

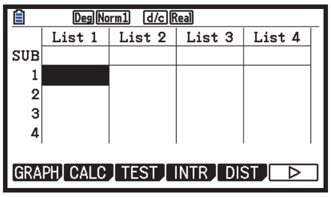
Module C2
Data analysis

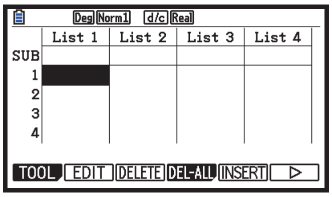
Note: This module is for the CASIO fx-CG 20 only. If you have the CASIO fx-9860GII calculator, please refer instead to Module 4.

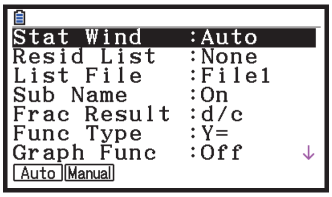
For data analysis purposes, a graphics calculator like the CASIO fx-CG 20 differs from a scientific calculator in two important ways. Firstly, substantial data sets can be stored, and so can be checked, edited and transformed. Secondly, graphical analyses are possible, as well as numerical analyses; the fx-CG 20 has color capabilities that are especially helpful for graphing statistical data. In this module, you will see how to use the calculator to help you with data analysis.

Entering and editing data

Press MENU 2 to access Statistics mode. Data can be entered into any of 26 variables, called *List 1* to *List 26*. Use the cursor keys to see if there are any data already stored in your calculator. Press the continuation key, F6, to get the middle screen below. Note that F3 and F4 are used to DELete data. DEL (F3) deletes a single data point, while DEL.A (F4) deletes a whole column (list) of data. Use the DEL·A command, several times if necessary, to clear any data from your calculator before you start this module. Also, access the SET UP menu check that the *Stat Wind*(*ow*) option is set to *Auto*, and *Graph Func* turned off, as below.





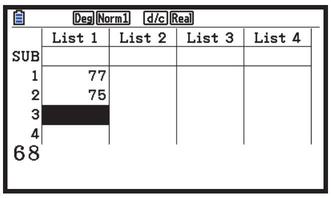


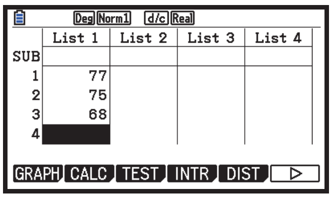
To explore some of the calculator's capabilities, consider the following data set, showing March rainfall (measured in millimetres) for a country town, collected annually for the past 30 years:

77, 75, 68, 81, 110, 90, 88, 42, 68, 88, 95, 62, 72, 120, 79, 80, 90, 81, 88, 77, 101, 91, 84, 85, 63, 62, 84, 82, 87, 76

This is *not* in fact a very large set of data. It is only an example and the same ideas used here will apply for a set of up to 999 data points. It is hard to get a reliable impression of large sets of data without some form of analysis.

Enter Statistics mode with MENU 2 to enter the data into *List 1* of the calculator. Place the cursor in *List 1* and enter the data points one at a time, pressing EXE after each. As you type each data point, it is shown in normal size at the bottom left of the screen, and can be edited in the usual ways before pressing EXE. Notice that the cursor moves downward each time you press EXE. The screen after the first two entries in Statistics mode is shown below:

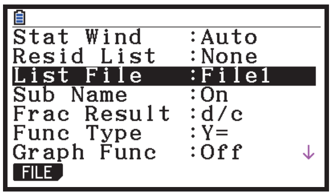




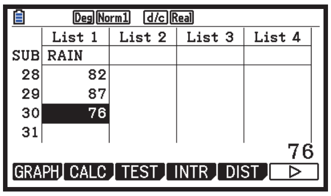
Use the cursor to move around to correct any typing errors by retyping the correct value. Notice that the entry highlighted in the list is reproduced in normal size at the bottom right of the screen, as the second screen above shows.

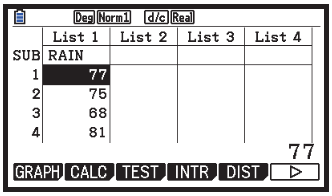
There are 26 data lists available, with each one permitted to have up to 999 elements. In addition, the calculator will allow you to enter and deal with even more data if you wish, using *data files*. A data file is a separate collection of 26 lists, *List 1* to *List 26*. The calculator has six data files, called *File 1* to *File 6*. (There are *overall* limitations on calculator memory of course, which may affect how much data you can enter, depending on what other material is stored in your calculator.)

To change from the present data file to another one, first enter SET UP to see the file which is being used at present, reported in *List File*, as shown below. Select your preferred file starting with FILE (F1) Choosing a new file does not delete any previous data you have entered in another file. Once data are stored in the calculator, in either lists or files, they will stay there until you delete or replace them, even when the calculator is turned off. Changing files allows you to store data in the calculator for later use, while still using the six lists for everyday use.



When entering data, it is easy to make a key pressing error, so you should be alert to ways of checking that the data have been correctly entered. One way is just to scroll up and down *List 1* and check against the original data. As you scroll, look out for entries that are obviously incorrect (such as those with only one digit or with more than three digits). An incorrect entry can be corrected by retyping it. The screen below shows the data after they were entered in Statistics mode. This gives a check that the correct number of data points have been entered as the 30th data point is 76, matching the original data; with so many data points, it is easy to miss one or enter one twice.

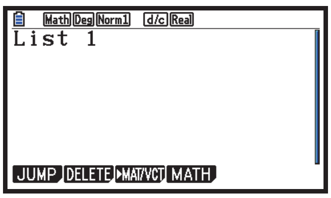


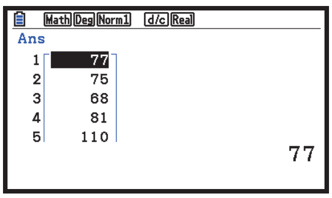


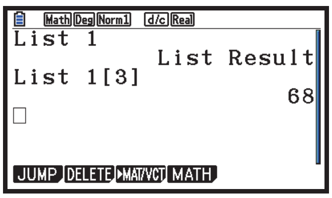
Errors and omissions can be corrected using the EDIT, DEL(ete) and INS(ert) commands shown above, which are available after tapping the continuation key (F6) in the original screen. The DEL.A menu can be used to delete an entire list, if necessary. You can scroll in both directions using (▲) and (▼) or can use the TOOL ((F1)) menu to jump to the TOP ((F3)) or BTM ((F4)) of the list.

You can give a short name to each list, to help you remember what the data represent. Move the cursor to the SUB row and then use (ALPHA) and the letter keys to do this. In this case, the data have been labelled RAIN. You can turn off the *Sub Name* feature if you wish in the SET UP menu.

Once entered into the list, data can be recalled in Run-Mat mode also. The *List* command is available on the keyboard as (SHIFT) (1). Recalling a large list puts a scrollable version on the screen, as shown below. Tap (EXIT) to escape the list. Particular list elements can be recalled by number using the square brackets on the keyboard ((SHIFT) (+) and (SHIFT) (−)), as shown below on the right.

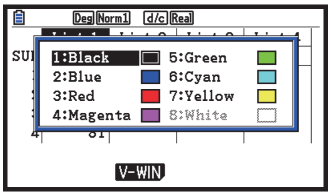


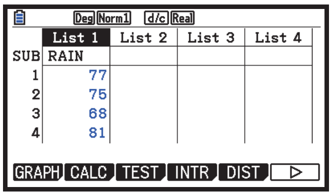


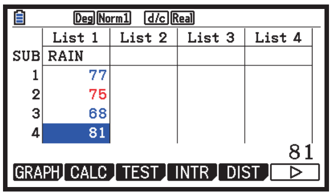


Coloring data

It is possible to use different colors to represent data, by coloring the data in a list; these colors will be used later when graphical displays of data are generated. To color all the data in a list, first move the cursor to the list heading at the top of a column and select FORMAT (SHIFT 5), as shown in the screen at left below. Choose a color for the data, either by using the cursor and EXE or simply by tapping the associated number. In the middle screen below, 2 was used to color the data blue.







Notice that neither the name of the list (*List 1*) nor the name of the variable (RAIN) are colored by this process, and that the color is used to highlight data by the cursor. You can color individual data points if you wish, by highlighting them with the cursor and then using the FORMAT command, as shown in the third screen above.

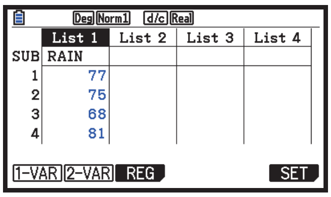
It is unwise to color the data until they are all entered, as later data entries will automatically be colored black. In our case, we have colored all of List 1 blue again before proceeding.

Summarising data numerically

Once data are entered, your calculator allows you to summarise them in various ways, to help you to interpret them. Two important ways are to calculate some statistics and to sort the data.

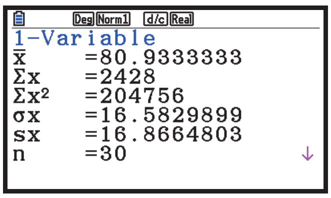
Statistics are numbers that can be used to help describe samples of data. Some statistics such as the *mean* and the *median* are associated with *central tendency*, since they summarise where the data are on average, in some sense. Other statistics are associated with *spread*; these include the *standard deviation*, the *quartiles*, and maximum and minimum data values. Other statistics such as the *mode* and the sum of the scores Σx can also be used in various ways to understand a set of data better.

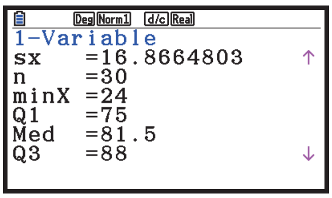
Calculations of statistics are controlled from the CALC menu in the main Statistics screen. Tap CALC (F2) to activate the menu and then SET (F6) to set the calculations to those you want.

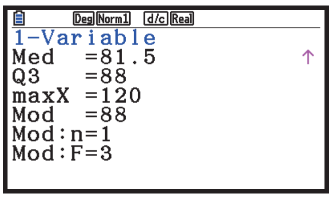


Calculations are available for one variable at a time (univariate, 1VAR) or two variables at a time (bivariate, 2VAR) as shown above. You need to set the calculator according to your data.

In this case, the choice of *List 1* above matches the data you stored in *List 1*. Notice that the 1Var Frequency is set to 1, as each data point represents a single year's rainfall. Press EXIT from the SET screen and then 1VAR (F1)to display the univariate statistics for the rainfall data. There are many statistics provided, as shown below: you will need to scroll with ▼ and ▲ to see them all.

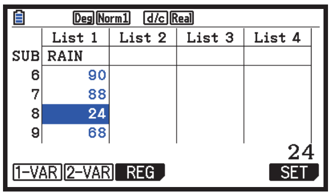


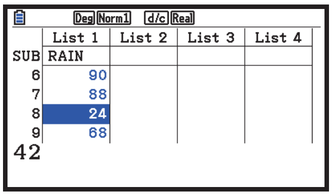


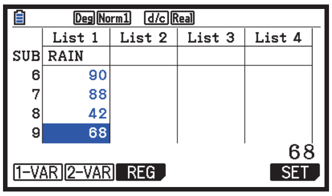


As well as helping to interpret the data, these calculations provide a few checks on your data entry. For example, the value for $n = 30$ is correct, the maximum value of 120 is correct, but the minimum value of 24 is *not* correct. (It is fairly easy in this case to scan the original data to find the largest and smallest values to check these.) The mode of 88 is correct. There is only one mode and it has a frequency of 3. There is at least one key press error here, as the minimum value is incorrect.

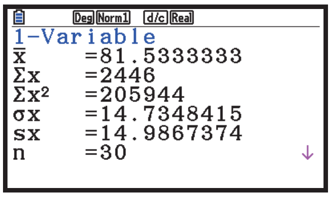
A very common error in typing is *transposition* – entering two characters in the wrong order. In this case, a scroll down to check the data shows that the 8th entry has indeed been transposed, as shown below. The error is easily fixed by typing over it with 42 EXE:

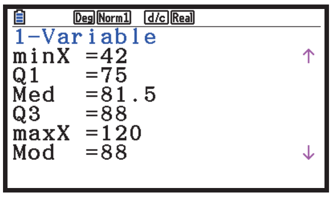




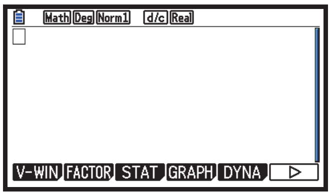


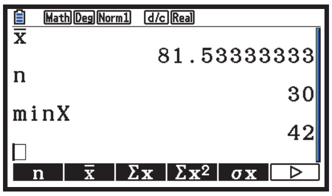
Now, if you return and calculate the univariate statistics again, they will be correct, as shown below. Check that yours match these, in case you have made any data entry errors.



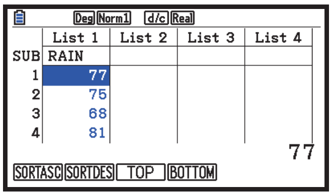


Conveniently, the more important statistics can be retrieved in the variables menu in Run-Mat mode *after* data have been analysed. Press VARS and STAT (F3). You will recognise the symbols in X (F1) shown at the right below. Use the continuation key F6 to find what you want.





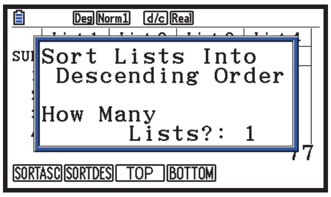
If the data are not in order of size when you enter them (as for the present example), it is a good idea to *sort* them from smallest to largest (*ascending* order) or from largest to smallest (*descending* order). A useful first analysis of the data, sorting may also reveal typing errors. Return to the main screen with EXIT and press the continuation key F6 and then TOOL (F1) to get the next screen. There are two sort commands, SRT.A (F1) to sort into ascending order and SRT.D (F2) to sort into descending order.

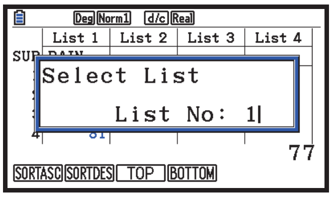


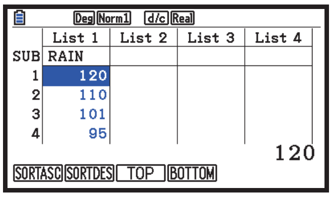
Be careful here! Once the data have been sorted, you cannot unsort them.

So it is usually not a good idea to *start* an analysis by sorting the data. Since the original order of data cannot be restored after sorting, it would also be unwise to conduct a sort if you later wanted to look at the annual trends in rainfall from year to year, for example. (It may be wise to make a copy of your data in an empty list, if one is available, before sorting. See the data transformations section later in this module for how to do this.)

In this case, we will sort the data from largest to smallest (i.e. into descending order) with F2. The calculator asks how many lists are to be sorted together. With only one list, tap 1 EXE.



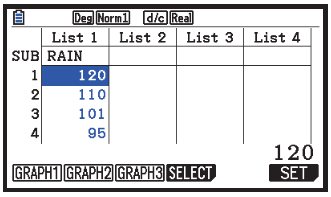


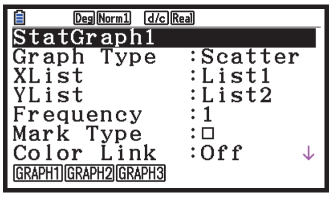


The list concerned is *List 1*, so tap 1 EXE in response to the next question. The data are then immediately sorted, as the third screen above shows. Scrolling this list will give you a better feel for the rainfall data than the original list. You can see, for example, that there are three rainfall readings of 88 mm, as suggested above with a single Mode of 88, for which the frequency is 3.

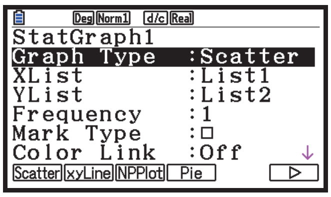
Displaying data graphically

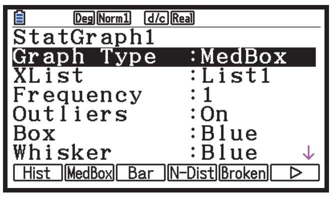
A good reason for using a graphics calculator to handle data is to deal with large amounts of data, for which a quick visual summary is hard to produce or interpret. A common way to represent larger amounts of univariate data is with a box plot or a histogram, each of which is accessible on this calculator. Graphical displays are controlled from the GRPH menu in the main statistics screen. Press GRPH (F1) to activate this menu and then SET (F6) your choices, as for the CALC menu in the previous section.

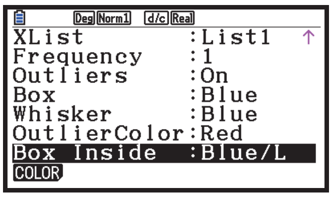




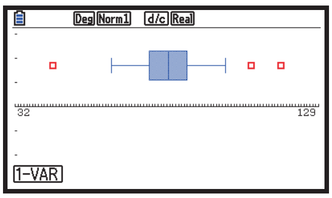
There are three graphs available at any time, labelled GPH1, GPH2 and GPH3. For each graph, move the cursor to see the settings and their choices. The previous screen shows that GPH1 is set to draw a scatter plot of *List 1* and *List 2* data, which is not useful here. Scroll down to highlight *Graph Type* and note the choices at the bottom.





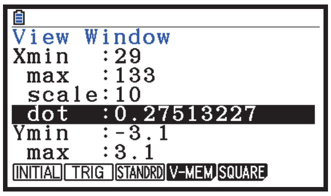


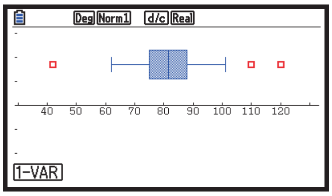
Use the continuation key F6 several times to see more choices. We will choose MedBox (F2) to choose a box plot for the rainfall data, showing medians and quartiles. A box plot can also be directed to display *outliers*, unusually large or small data points, less than the maximum and greater than the minimum scores; these may need closer inspection and explanation. (They may represent unusual data or they may simply be typing errors at some stage.) Set your screen to match the choices shown above on the right, including turning the outliers *on* and displaying them in a different color. When color settings are needed use COLOR (F1) and make a choice, as earlier with the FORMAT command. Then press EXIT to return and GPH1 (F1) to get the graph below.

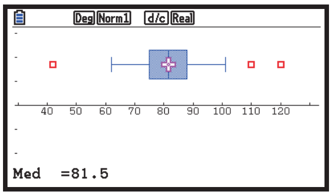


The box plot makes it clear that the rainfall is generally quite similar from year to year, with occasional very high and low years, as the 'box' is quite small in relation to the 'whiskers'.

Since the *Stat Wind*(*ow*) was set to *Auto* at the start of this chapter, the calculator chooses settings for the *x*-axis to suit the data. This is useful, as it will also show any data entry errors not yet detected. However, the choice of *scale* for the *x*-axis is not done automatically, so it's a good idea to adjust this for yourself. The result of changing the *x*-scale from 1 to 10 is shown below.

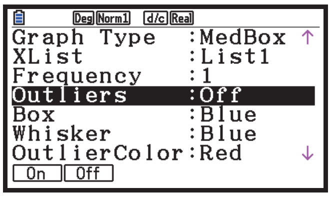


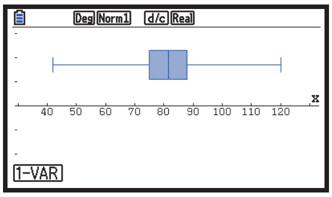




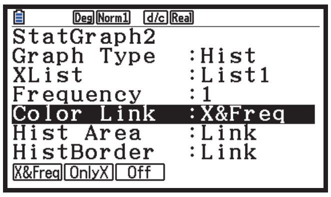
The changed scale makes the graph easier to read. The graph can also be traced with SHIFT F1 to show the five numbers used to construct it (The minimum and maximum scores, the median and the first and third quartiles) as well as the outliers; e.g., the median of 81.5 is shown above at the right.

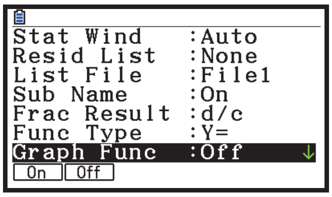
The original definition selected outliers to be displayed. Had you not chosen this option, a different box plot would result as shown below, where outliers are included in the 'whiskers'. This box plot gives a different impression of the data, since it doesn't recognise that three of the extreme points are outliers, suggesting that March rainfall is more variable than it actually seems to be.





As well as a box plot of these data, a histogram may give a useful visual summary. A histogram has data values on the horizontal axis and frequencies on the vertical axis. It is a convenient form of summary to give a visual impression of the 'big picture' of a set of data. Return to the GRPH menu and tap SET (F6). Choose the second graph with GPH2 (F2) to be a histogram as shown below. The *Color Link* settings will have the effect of graphing the data using the same color as the data, set earlier.

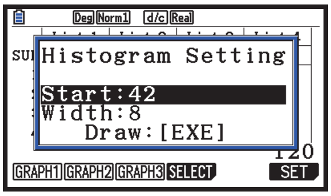


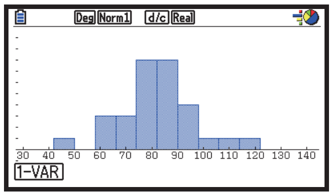


The advantage of allowing the calculator to automatically choose ranges for the histogram is that it will graph all the data, even those entered in error, allowing you to detect any data entry errors. But it is better to make some decisions yourself, in order to control what happens.

It is usually best to leave the *Stat Wind*(*ow*) on Auto, as the calculator will allow you to make changes. Use SHIFT MENU to access the SET UP menu, and adjust the *Stat Wind*(*ow*) if necessary. It is also a good idea to turn the *Graph Func* to *off*, so that writing on the screen does not obscure the picture. Press EXIT to finish.

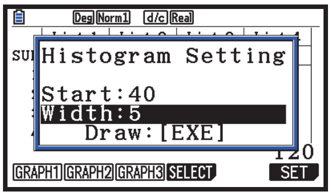
When you graph the histogram using F2 (GPH2), you will be presented with a choice of where to start the histogram and how wide to make each interval (or bin) for the data. The automatic calculator choice will often be inappropriate, as below. (It starts at the minimum point and chooses an inconvenient bin width to suit the data, which is quite hard to interpret when graphed.)

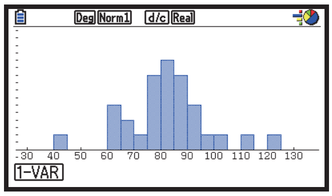




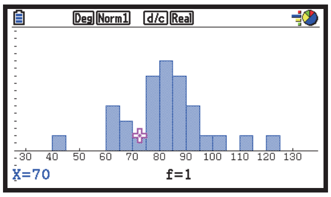
The histogram is shown in blue, the same color as the data, because the *Color Link* is turned on. The symbol in the top right corner of the graph screen indicates this.

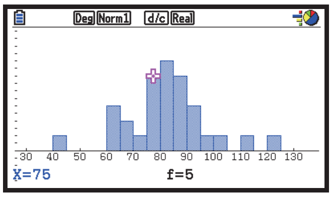
Better choices here for the histogram intervals might be to start at 40 mm and group the data into intervals of 5 mm (the *Width* on the calculator). Redraw the histogram, making these changes. Make sure you press [EXE] after each of the *Start* and *Width* entries. The result is shown below.

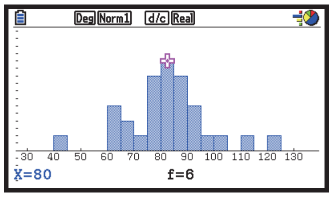




The histogram shows a strong clustering of annual rainfalls around 75 mm – 95 mm, with some rather wet years, a few dry years and a bad drought year. You can trace this histogram with TRACE ([SHIFT] [F1]) as shown below. Use ▶ and ◀ to trace. For each interval, the calculator shows the *left* endpoint as *X* and the interval frequency as *f* at the bottom of the screen.







These screens show that there is only one measurement in the interval $70 \le x < 75$, five in the interval $75 \le x < 80$ and six in the interval $80 \le x < 85$, all consistent with the original data. Notice in particular that the rainfall of 75 mm is placed in the middle of these three intervals and the 80 mm rainfall is in the third one.

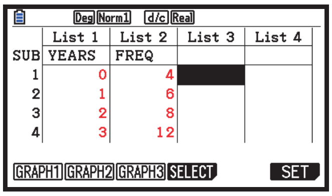
You can easily produce a different histogram by starting at a different point or (more importantly) changing the width of the intervals. There are few hard and fast rules on how to deal with data of these kinds, and the calculator will help you to explore the data in a number of ways in order to reach plausible conclusions. Because the shapes of histograms depend on the choices you make, you should always be cautious in interpreting histograms, both your own and those made by others.

Frequency data

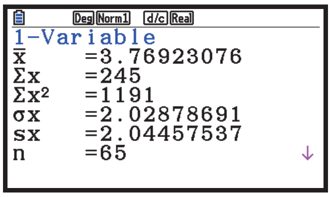
Some data have already been sorted into frequency distributions, so that it is convenient to analyse them taking the frequencies into account. E.g., the following data show the number of years (rounded to the nearest whole number) for which the 65 houses in a certain street had been occupied by the same people.

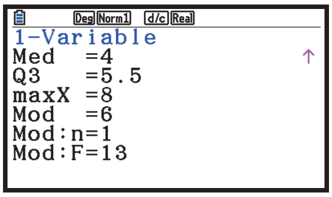
Number of years	0	1	2	3	4	5	6	7	8
Frequency	4	6	8	12	10	9	13	0	3

Although it would be possible to enter these data into a large list in the calculator (with four 0's, six 1's, eight 2's, and so on), there is an easier way. Enter the number of years into *List 1* and the frequencies into *List 2,* as shown below.

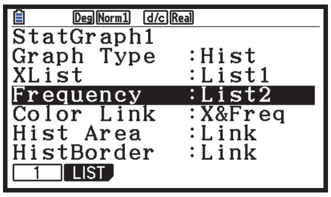


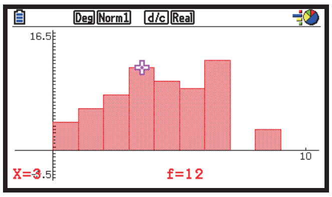
When setting the (univariate) calculations, identify *List 2* as the frequencies (*1VAR Freq*).





The *n* value shows that the calculator recognises that there are 65 data points in the data set, not just 8. Residents in the street have been in their houses about 3.8 years, on average. The median length of stay is 4 years, and the most common (the mode, with 13 households) is 6 years. Similarly, when setting graphical displays, identify *List 2* as the frequency associated with *List 1* data, as shown below, to create a histogram of the data. Set the *Width* = 1 so that each column represents one year.



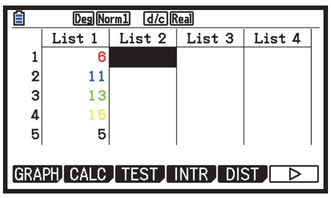


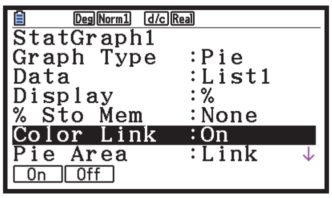
Categorical data

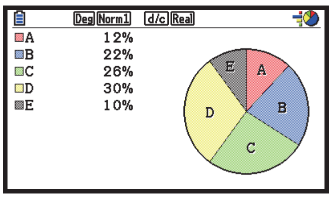
Some statistical data are arranged into distinct categories, and we are interested in comparing the categories in some way. The calculator is helpful for offering graphical representations for this purpose, especially pie charts and bar graphs. For example, consider the following table, showing the numbers of sweets of various colours in a box.

Colour	Red	Blue	Green	Yellow	Black
Number	6	11	13	15	5

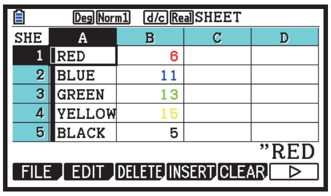
Enter the numerical data into List 1 as shown below. Select a Pie graph and choose to represent the *Display* as a percentage (which allows for good comparisons with other pie graphs), as shown below. The data have been colored (using SHIFT 5) to match the colors of the sweets, and the Color Link turned on accordingly. The resulting Pie graph is shown below. Notice that the graph can be traced to show which sector of the graph corresponds with which category. The categories are described as A, B, C, D, etc. according to the order in which the data were listed.

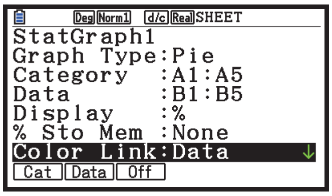


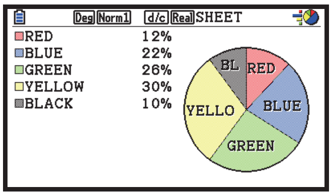




While this is a helpful graph, as better version is available if the data are stored in Spreadsheet mode, as the category names can be included as well. Use MENU 4 to access a spreadsheet. The screens below show the equivalents of those shown above for Statistics mode.

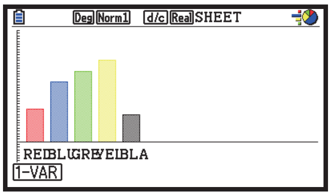


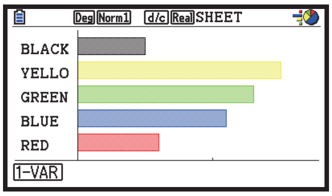


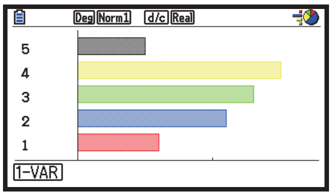


Notice that the data are referenced in cells, rather than lists, with A1:A5 referring to the column containing A1 down to A5. Notice importantly that the category names are included; to enter these, it is necessary to precede the text with inverted commas (SHIFT EXP). It is a good idea to use the Alpha-Shift Lock command (SHIFT ALPHA) to make entering names more efficient; shorter names are better than longer names. The GRAPH menu is available after tapping the continuation key, F6 .

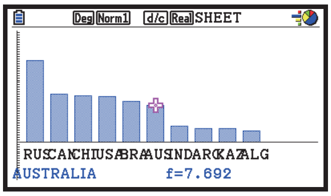
Data of this kind can also be represented with a bar chart, in both Statistics and Spreadsheet modes. The screens below show the choice of horizontal or vertical and the effects of choosing to use the Spreadsheet mode to include category labels (showing in the first two screens).

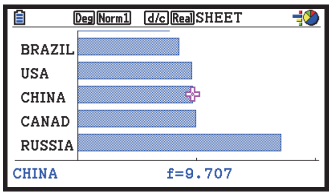


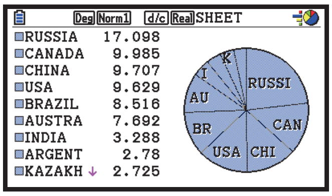




Sometimes, categorical data include numerical information, not frequencies. A bar graph is a suitable way of representing such data graphically, but care is taken to choose a suitable graph and label it suitably. The example below concerns the areas of the ten largest countries, according to *Wikipedia*. Data have been entered into the spreadsheet with full country names and areas (in millions of square kilometres). Some possible graphs from the spreadsheet are shown below:







The first bar graph is perhaps the best of these three representations, as all ten country populations are seen at once, although country names are abbreviated. The second bar graph requires that the ⏶ and ⏷ keys are needed as soon as the graph appears to choose which countries are shown, as there is insufficient screen space for ten categories. The pie chart shows many country names better (on the left of the screen) but not all fit on the screen and some are not shown at all on the graph.

Bivariate data

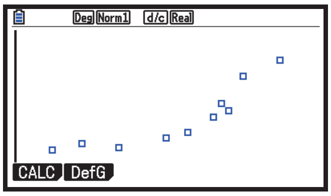
To explore some of the bivariate data analysis capabilities of this calculator, enter the following data set into the *List 1* and *List 2* columns, pressing EXE after entering each value.

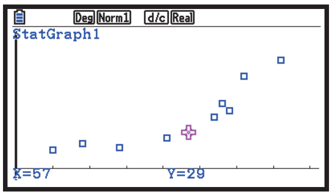
X (year after 1900)	20	28	38	51	57	64	66	68	72	82
Y (number of hotels)	15	20	17	25	29	42	53	47	75	88

The data refer to the number of hotels (Y) in a small growing city over some years (X) in the twentieth century. The years have been abbreviated to their last two digits, so in 1968 ($X = 68$), there were 47 hotels. Before graphing the data, go to GRAPH mode with MENU 5 and either delete with DEL (F2) or turn off with SEL (F1) any functions listed.

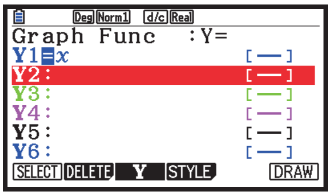
In Statistics mode tap GRPH (F1) to access the graph menu. Tap SET (F6) to define the first graph (GPH1) as shown below, a scatter plot with *List 1* on the horizontal axis and *List 2* on the vertical axis. The frequency of each pair is 1. Press EXIT when finished.

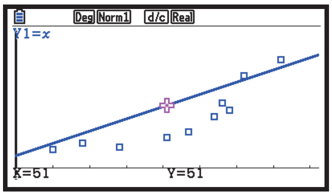
As for univariate data, it's best to use the *Auto* Stat Window (in SET UP) at first, to make sure all data are plotted. Choose GPH1 (F1) to draw the graph you have just defined, as below. Notice that the scatter plot can be traced, using SHIFT F1. This is a good way of checking your data entry.



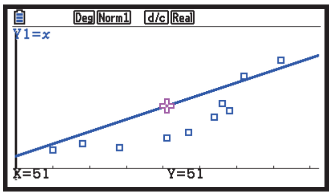


One way of exploring the relationship between *X* and *Y* here is to look for functions that fit the data approximately. In order to do this manually, tap EXIT and then F1 to redraw the graph. Then select DefG (F2). The calculator is now in GRAPH mode. Delete or turn off with SEL (F1) any existing graphs. Enter a guess for a function that might fit the hotels data. For example, a first rough guess might be that the number of hotels is approximately the same as the year, $Y = X$.



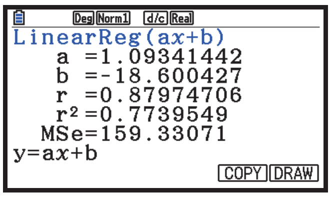


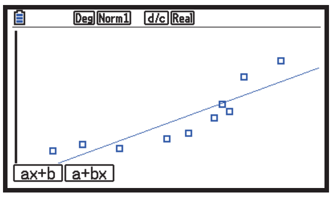
Use DRAW (F6) to draw the graph on top of the scatter plot. You can trace either the data or the graph to compare them and move between one and the other with ▲ and ▼. In the screens here, (*Graph Func* has been turned *on* in SET UP.) The guess $Y = X$ does not seem a very good one, as most points are below the line. A better guess may be for a slightly steeper line, dropped a little. The graph below shows such a guess with $Y = 1.2X – 20$.



To calculate and draw a regression line called *a line of best fit* through the data points, tap EXIT, redraw the scatter plot and then tap CALC (F1). Choose one of the immediate menu items at the bottom of the screen, or one of the others shown by pressing the continuation key (F6). Various kinds of relationships between the two variables, *List 1* and *List 2*, can be represented.

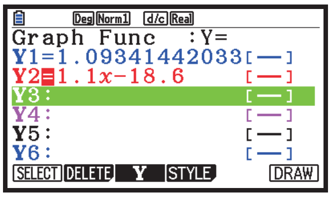
After the regression coefficients are displayed, tap DRAW (F6) to draw the line on the scatter plot. (Use COPY(F5) to copy the regression function to the Graph and Table function list for later analysis. It is not essential to do this, but it is a good idea to allow for options later such as those described later in this section.) The screens below show a regression line with X (F2), and then *ax*+*b*, to obtain a linear regression function through the data in the form $y = ax + b$.

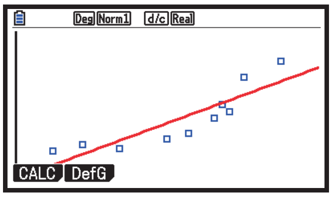




The regression coefficients *a* and *b* show that the best fitting line is given by $y = 1.0934x – 18.600$, close to the earlier guess, but the visual evidence still suggests the line is a poor fit to these data.

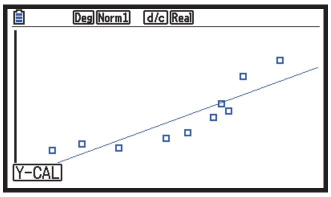
In this case, especially since the original data are all whole numbers, the calculator probably provides more precision than necessary with the regression coefficients. A more suitable line for you to use may be $y = 1.09x - 18.60$ or even $y = 1.1x - 18.6$. To enter this function, first tap EXIT and select DefG (F2), as shown below in *Y2*, having temporarily turned off *Y1* with SEL (F1).

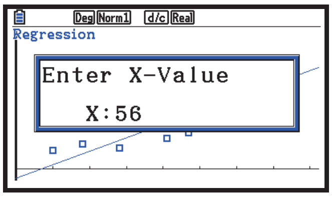


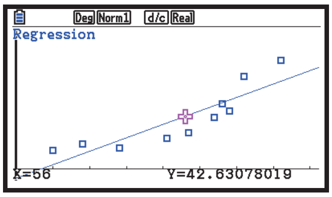


The line is a *model* for the data; it is an attempt to provide a simplified description of the relationship between the two variables. Of course, a linear model like this one is only one kind of model; there are many other possibilities to model relationships of different kinds. The *correlation coefficient* in this case $r \approx 0.88$, gives a measure of how well the data can be modelled by this linear function. The possible values for r are $-1 \le r \le 1$, with the positive value indicating that the larger values of y tend to be associated with the larger values of x (i.e., the slope of the line is positive). The square of the correlation coefficient, $r^2 \approx 0.77$ is also useful to indicate how well the model accounts for the data. In this case, it can be interpreted as 77% of the variation in hotel numbers can be accounted for by the year in the linear model.

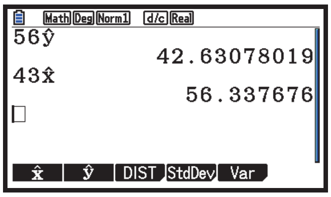
The regression line just found by the calculator can be used to make predictions. One way is to use the G-Solve (F5) command when the line is showing, as shown below. The predicted value is 42.6.

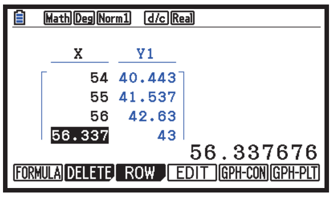




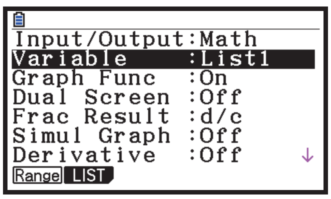


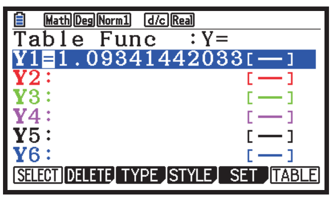
You can also use the STAT menu under OPTN, but you will first need to return to Run-Mat mode with MENU 1. The two menu choices in STAT are shown below. To predict the y-value associated with $x = 56$, enter 56 followed by $\hat{y}$ (F2) EXE. To find the x-value associated with a y-value of 43, enter 43 followed by $\hat{x}$ (F1) EXE. You can check manually that these are the values found by substituting in the linear regression function above. Be careful when making predictions that go beyond the data you started with (this is called *extrapolation.* You can also explore the model in Table mode (provided you copied the function previously), as shown on the right below.

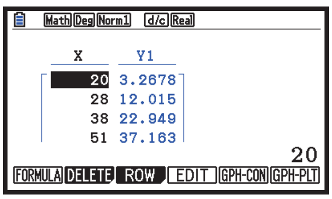




There is a third, different, way of using the table functions of the calculator, especially useful in this situation. The screen below shows that the SET UP for Table mode includes *Variable*, which can be set to *Range* or *List*. Change the setting to *List* and select *List 1* as the list, as shown below.



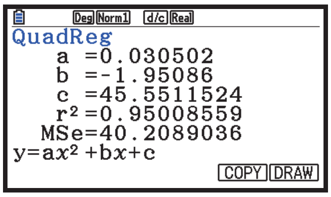


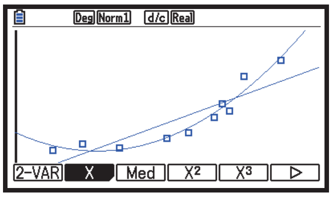


Now, when the table is constructed with TABL (F6), the calculator uses only the *List 1* values for the independent variable, *x*, as shown below. This method allows you to check the predictions from the linear model more efficiently. The predictions do not match the data very well in this case. Make sure you return the *Variable* setting to *Range*, the usual preference in Table mode, after you have finished with your statistical analyses.

Return to Statistics mode with MENU 2 and draw the regression line again. As noted earlier, the line doesn't seem to fit the data particularly well, since there are several points that are not close to the line, and the general shape of the scatter plot does not closely resemble a line anyway. That is, the linear model is not a particularly good model for these data. In fact, it appears that the relationship between the two variables is *curvilinear* (i.e., a shape that isn't a line).

You can make more than one choice, to look for a good fit to the data. In this case, the data suggest that a quadratic model obtained with X^2 (F4), may be a better choice than a line. The screens below confirm that this seems to be the case, with both a line and a quadratic curve fitted to the data:





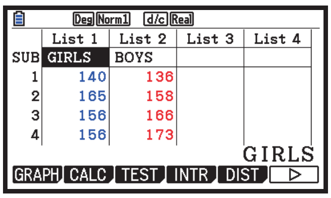
The quadratic model $y = 0.03x^2 - 1.95x + 45.5$ can also be explored in the same way as the linear model in Graph, Table and Run-Matrix modes. In general, you should be wary of choosing models that are very sophisticated, unless there are good historical or other reasons for doing so, especially with very small data sets of this kind.

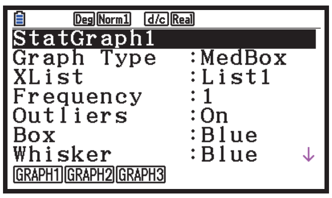
Comparing groups

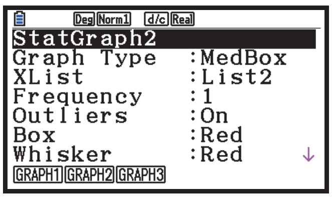
Sometimes, data are gathered on several groups, but there is no pairing of the data, as there was with the hotels data. The intention of the data collection is often to make comparisons between different groups. For example, the data following show the heights of the boys and girls separately in a class of fourteen-year olds.

Girls	Boys
140, 165, 156, 156, 126, 139, 145, 131, 145, 147, 135, 157, 172, 162, 160, 148	136, 158,166, 173, 128, 132, 158, 157, 182, 162, 169, 148, 145, 129, 151, 181, 150

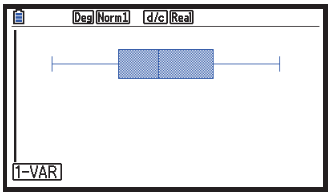
Enter the data into *List 1* and *List 2* for girls and boys respectively. A graphical comparison is often the most effective; the following screens show how to set up a box plot for each group separately.

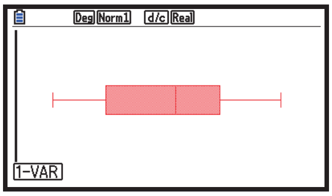




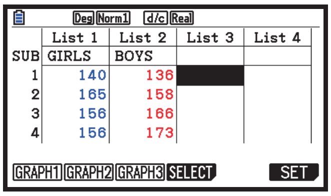


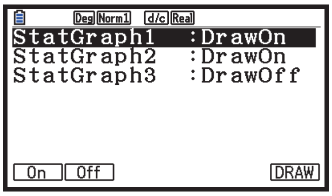
When the box plots are drawn (using, as usual, the *Auto Stat Window*), they both seem to be similar, with the box plot for boys (on the right) a bit lower on the screen than that for girls. These do not provide a useful basis for comparison, partly as each box plot occupies the entire screen width.

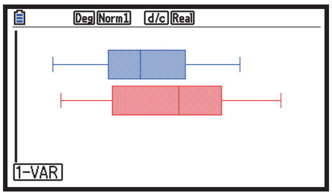




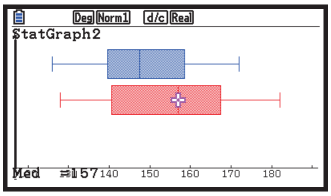
A better alternative is to draw both graphs on the screen at once. To do this, enter the GRPH menu and choose SELECT (F4)to select graphs to be drawn, as shown below.

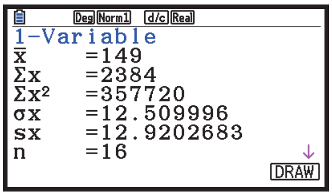


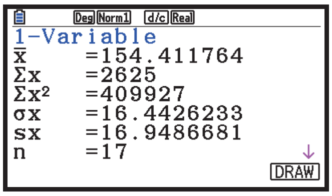




Use the cursor and F1 keys to turn *both* GPH1 and GPH2 on, as shown on the screen above in the middle. Then, when you tap DRAW (F6), both graphs are drawn together.







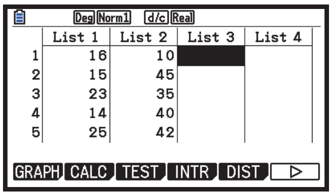
Notice that the graph for boys is still below the graph for the girls, as originally. It is now clear that there are substantial differences between the two groups. You can trace the graphs to see the differences in the five numbers used to create them.

If you redraw the graph and choose the one variable statistics with 1VAR (F1), the calculator will require you to select which of the two groups you want to obtain statistics for. Use the cursor keys to select and press EXE to enter your choice. Return to the box plots with DRAW (F6).

The previous screens show that the mean height for girls is about 5.4 cm less than that for boys, but the boys' heights are more varied, reflected in the higher standard deviation. These observations are consistent with the box plots.

Data transformations

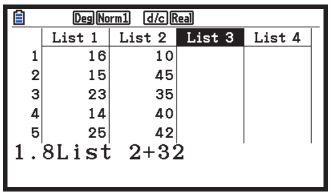
As well as entering data manually, as we have done so far in this module, lists of data can be *transformed* to make new variables. To illustrate the procedure, consider the small data set below, consisting of five data points in each of *List 1* and *List 2*. (In this case, the list names have been turned off in SET UP, with *Sub Name*.)

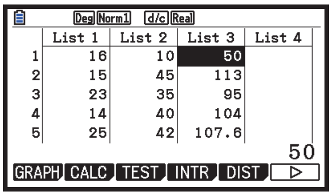


Suppose that the data in *List 2* were all temperatures in degrees Celsius, and you wanted them in degrees Fahrenheit instead. The relationship between these two temperature scales is:

$$\textit{Fahrenheit} = 1.8 \times \textit{Celsius} + 32$$

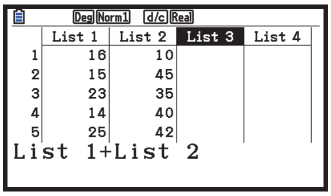
To transform the data in *List 2* in this way, we will make a new variable, *List 3*. First move the cursor to the heading at the top of the *List 3* column, highlighting the name, *List 3*, as shown below.

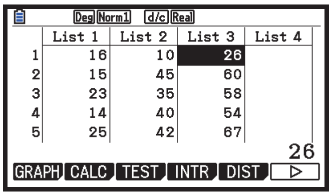




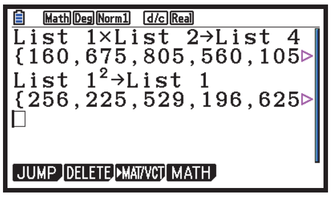
Then enter the transformation, 1.8 List 2 + 32, as shown above. For the List command, you will need to tap SHIFT 1. When the transformation is entered, press EXE and it will be carried out immediately. The screen on the right shows the results. Notice that the entire set of *List 2* data is transformed in the same way. Check some of these for yourself to see that the data points in *List 3* are the Fahrenheit temperatures associated with Celsius temperatures shown in *List 2*.

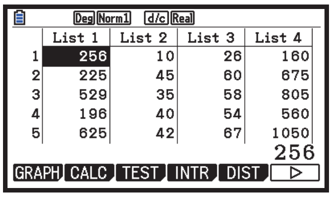
Transformations can involve several variables, too. For example, suppose you wanted to create a new variable that was the sum of the first two variables above. Using the transformation List 1 + List 2 in the *List 3* header will have this effect, as shown below.





Data transformations are usually best carried out in Statistics mode, as you can see the contents of the lists on the screen, and ensure that new variables created do not overwrite existing variables unless intended. They can also be conducted in Run-Mat mode, using the → key, as shown below.





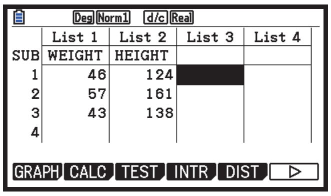
Return to Statistics mode and check the final results carefully, to understand the importance of the order in which these transformations were carried out.

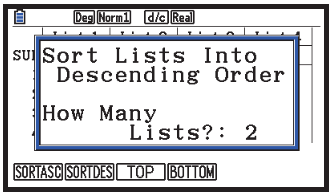
Sorting bivariate data

In the previous section, the data were sorted to help make sense of them and as a kind of check on data entry. You should always take care when sorting data, but be especially careful when sorting bivariate data, represented by *pairs* of lists. Consider people's weights and heights shown below. Each row shows one person. It is important that the weights (*List 1*) and heights (*List 2*) for each person are kept together when sorting, or the data will become nonsensical.

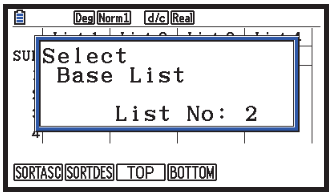
Name	Weight	Height
Ong	46	124
Kym	57	161
Anil	43	138

To sort the data (person's pairs of measurements) in descending order of height, first enter the data into *List 1* and *List 2*, tap F6 and TOOL (F1) and then SORTDES (F2). Enter 2 EXE to indicate that you have two lists altogether, as shown below.



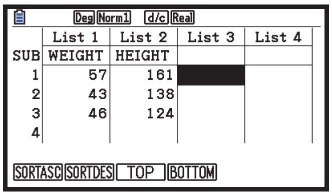


In this case, the *Base List* is height, since we want the data sorted by height. As heights are stored in *List 2*, enter [2] [EXE] to indicate this choice:

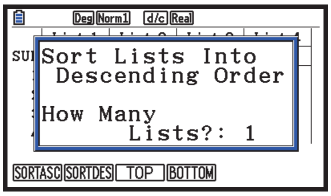


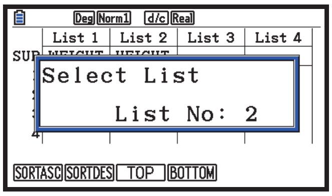
The calculator will then ask you for the other list to be sorted. In this case, it is *List 1*, so enter a [1] and then tap [EXE] to complete the process, as shown below:

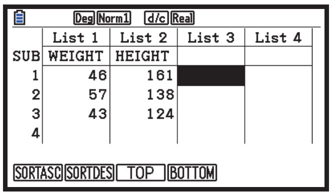
The result of the sorting is shown at the right below. Notice that the heights and weights are still paired correctly with each other, and the data have been sorted into descending order of the heights in *List 2*.



The next screens show what happens if you sort the second list into descending order, *without* informing the calculator that there are two lists to be considered at once.







The connections between height and weight have now been ***permanently*** lost, since the weights have retained their original order, while the heights have been sorted.

Exercises

The main purpose of the exercises is to help you to develop your calculator skills.

1 a A record was kept in a golf tournament of the approximate distance (in metres) from a certain hole of the first shots of players. Here are the data:

Distance (m)	1	2	3	4	5	6	7	8	9	10
Frequency	7	10	18	16	14	12	12	39	13	11

Enter these data into *List 1* and *List 2* of your calculator. Name the lists DIST and FREQ respectively. Color the data red. Use CALC to set the 1Variable statistics using the frequencies appropriately. Find the mean distance of the golf shots from the hole.

b Draw a histogram of the golf data, using the *Color Link* to match the color of the data. Trace the histogram to check that the data have been correctly entered. Which observation appears to be an outlier? Explain why you think it may be an outlier.

c In fact, there were only nine golf shots that finished eight metres from the hole. Change the data accordingly, and recalculate the mean distance.

d Now draw a box plot of the corrected golf data. Trace to find the median distance.

e Use a transformation to change the distances of the golf shots from metres into feet. (1 metre is about 3.28 feet). Store the transformed data in *List 3*. Draw a box plot of the transformed data, and find the mean distance of the golf shots in feet.

2 a The data in the table show the height of a stone thrown into the air from the roof of a house at various times after it was thrown until it hit the ground. The heights and times were estimated from a video replay of the incident.

Time (seconds)	0	0.2	0.4	0.7	1	1.2	1.6	1.9	2
Height (metres)	15	17	20	22	24	23	18	8	1

Enter these data into your calculator. How high was the house?

b Draw a scatter plot of the stone data. What sort of relationship (linear or curvilinear) appears to be involved?

c Change the plotting symbol on your scatter plot to a small cross.

d Find the regression coefficients for the line that best fits these data. Draw the line on the scatter plot. Does it seem to provide a good fit?

e Return to Run-Mat mode, and use the STAT menu to predict where the stone was after 1.5 seconds and the time at which the stone will return to its original height of 15 m. Then use the VARS menu to retrieve the correlation coefficient.

f Return to Statistics mode. Draw a quadratic curve through the data. Find the equation of the curve and the correlation coefficient. Which fits the data better – the line or the curve?

3 Children in a school were asked in a survey to nominate their favourite lunch food for a class picnic. Sandwiches were preferred by 53 children, 62 preferred fruit, 105 preferred rice, 44 preferred salad and 34 preferred pies.

a Use a spreadsheet to represent these data with a pie chart using colors of your choice; then use the pie chart to find the percentage of children preferring fruit.

b Represent the data with a suitable bar graph.

Activities

The main purpose of the activities is to help you to use your calculator to learn mathematics. You may find that some of them are too advanced for you. Ignore activities you don't yet understand.

1 Choose a random sample of ten students in your class by selecting ten names from a complete list in a hat. Find out how many hours each student watches TV each week, and then use the calculator to find the mean, median and mode of these data. Represent the data using a box plot and histogram.

If one student changes their mind about the number of hours of TV watching each week, what difference will this make to the results?

2. Roll a standard six-sided die 50 times and make a frequency table of the results. Summarise the results in various ways, including finding the mean, mode and median and drawing suitable graphs such as a box plot or histogram. Compare your findings with your partner. Repeat these processes.

3. A group of students had their mathematics and science test results as follows:

Mathematics	20	28	38	51	57	64	66	68	72	82
Science	15	20	17	25	29	42	33	47	75	88

Plot these data and find an appropriate equation to predict the science result of a student in the class with a test mark of 65 for mathematics, but who was absent for the science test. Use box plots to help make some statements about the differences and similarities of the marks.

Try to obtain some marks on a pair of different tests in your own class, and use methods like these to compare them.

4. Two groups of patients were randomly selected and given medication to lower their blood pressure. One group was given a drug for this purpose, while the other group was given a placebo (a pretend drug); patients were unaware whether their drug was real or a placebo. The following results were recorded.

Drug

Before	100	106	105	103	100	70
After	100	90	90	95	75	80

Placebo

Before	110	110	105	95	85	82
After	100	105	95	100	90	90

Plot these data and use statistics to comment on the effectiveness of the drug.

5 The table shows the mean heights of 12 sets of parents, all of whom have at least one son, and the height of their eldest son at age 21.

Parent mean height (cm)	168	179	175	179	161	183	170	172	174	186	175	163
Son mean height (cm)	170	186	172	186	162	191	181	195	179	199	176	171

Plot the data and determine the best equation to predict the heights of sons from their parents. use your equation to make some predictions. Determine some heights for parents and their sons or daughters in your class and make some suitable comparisons.

6 Find out the population in your country over the past five decades. Find the best-fitting equation for these data over time and use this to predict the population in 2030.

What reservations might you have about predictions of this kind?

Notes for teachers

This module illustrates several ways in which the fx-CG 20 calculator can be used to explore various aspects of statistics, to help students understand the elements of univariate and bivariate data analysis and to take advantage of the calculator's color capabilities. The text of the module is intended to be read by students and will help them to start to see how the calculator can be used efficiently to summarise and represent data in various ways, and become a tool for analysing their own data. The Activities are appropriate for students to complete with a partner or in a small group, so that they can discuss their observations and justify their conclusions.

Answers to Exercises

1. (a) 5.99 m (b) 8^{th}, entered as 39 instead of 9 (c) 5.49 m (d) 5.5 m (e) 15.01 ft 2. (a) 15 m (b) curvilinear (d) $y = -5.49x + 21.93$, not a good fit (e) 13.7 m, 1.26 seconds, $r \approx -0.53$ (f) $y = -15.49x^2 + 26.20x + 13.13$, $r \approx 0.97$ 3. (a) 20.8%

Activities

1. Activities of this kind are important, as they allow students to deal with real data of local interest to them. As the data are stored, and can be edited, they should be able to see for themselves the effects of making changes to the data, noting particularly the fragility of the mode and the possible effects of outliers on the mean.

2. Activities involving random data lend themselves to statistical analysis of the kinds suggested, and it is a good idea for students to work together, to see the variations between students and between trials. As well as comparing results, you might encourage students to amalgamate their results, to produce a more stable outcome.

3. The information provided is intended to provide a context for students to examine bivariate data, although a more substantial sample would normally be expected for work of this kind. This activity is intended to help the students refine their data analysis skills and to use regression results for a practical purpose. Many contexts will provide good data for this purpose, including the suggestion to obtain school test data. While this is frequently confidential, other bivariate data would serve a similar purpose and you should encourage students, or pairs of students, to obtain some real bivariate data for analysis in the manner suggested.

4. This activity informally introduces an important use of statistics in practical and research settings, which often involve small samples. More formal consideration of issues of this kind, and the comparisons involved, require statistical hypothesis testing, addressed briefly in a later module.

5. An activity of this kind will involve students refining their bivariate data analysis skills, making use of suitable plots and regression equations. It may be relatively easy for a class of students to obtain data of this kind from students in the class (together with their parents), so that real data are being used, rather than the data provided.

6. Students will usually be able to locate some suitable bivariate data (year, population) from government bodies (such as a bureau of statistics) or from Internet sources, although advice for your particular country may be needed. Over short periods, natural population growth is often linear, although over longer periods, different models are more appropriate. The most likely models to suit population natural population growth are exponential, which will require students to have some experience with exponential functions. Students should appreciate that population growth is rarely entirely natural, but is influenced by government policies (e.g. for limiting or encouraging family growth, warfare, or for migration), as well as improvements in health practices and reductions in child mortality, so that care is needed not to extrapolate models carelessly.

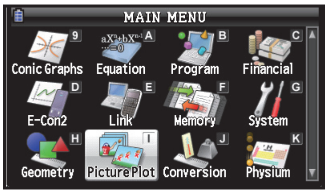
Module C3
Modelling with graphs

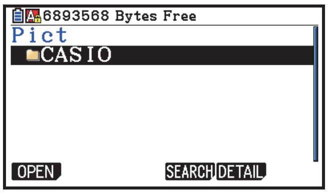
Note: This module is for the CASIO fx-CG 20 only. It requires the use of the *Picture Plot* software, not presently available on the fx-9860GII series.

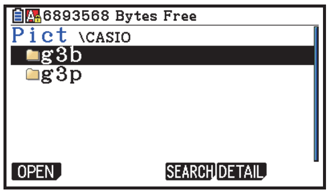
The *Picture Plot* software allows you to understand the mathematical basis of everyday settings, represented in photographs stored on the calculator. The processes of describing the world using mathematics and using the mathematical descriptions to answer everyday questions are both important parts of mathematical modelling. In this module, we will see how the calculator permits and supports work of that kind. It is assumed that you are already familiar with the use of the fx-CG 20 for data analysis, as described in Module C2.

Resources

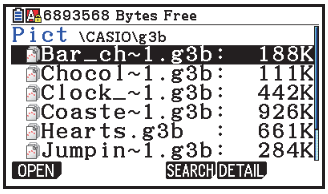
The *Picture Plot* module is available as Add-in software on the CASIO website at http://edu.casio.com and you should check there for software updates, extra photographs and suggested activities. *Picture Plot* is available on the fx-CG 20, as shown below. When you tap the icon, the picture storage area (*Pict*) will be shown. When you open the CASIO file (with OPEN (F1)) or just with EXE) you will see two further files, each of which contains many images.

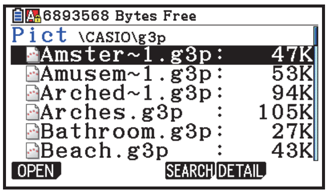


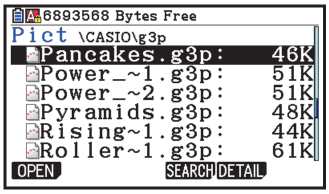




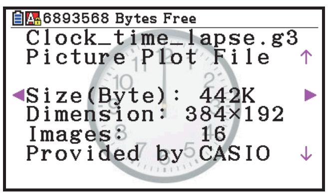
The available images are of two kinds: picture files (labelled g3p) and animation files (comprising a sequence of still picture files, labelled g3b). Opening these will reveal a list of the contents. The full title of each file often is too large to fit on the screen, so is abbreviated; e.g., *Chocolate_flow* is abbreviated to *Chocol~1*. You can scroll lists in either direction using cursor keys ▼ and ▲ or go directly to an item in a list using the alphabetic characters on the keyboard. E.g., tapping 4 takes you directly to picture files starting with P, as shown in the third screen below.

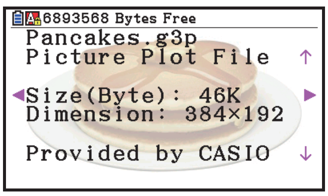






To obtain more information about a file before opening it, use the DETAIL (F5) command. This will provide information about the number of images in the set for an animation file, as well as a faint version of the images, as shown by the *Clock_time_lapse* example on the left, indicating that 16 images are involved. In the case of a picture file, the detail shows a faint version of the picture, along with other information as shown in the *Pancakes* file on the right.

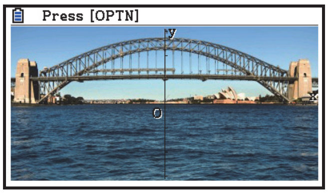




You can EXIT from the details and check other details, but it is unnecessary to OPEN (F1) any files until you are sure that they are the ones you wish to use.

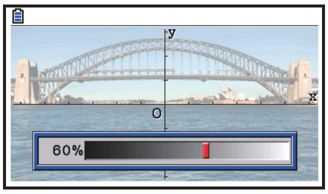
Setting up a model

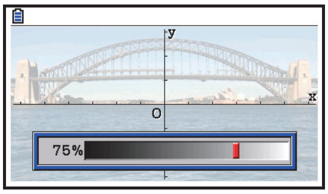
To illustrate how *Picture Plot* can be used, we will explore the picture file called *Harbour_bridge.g3p*. After you select it in the list and tap OPEN (F1), a picture of Australia's famous Sydney Harbour Bridge is displayed on the screen. This bridge connects north and south Sydney across the harbour, and can be well described with graphs.



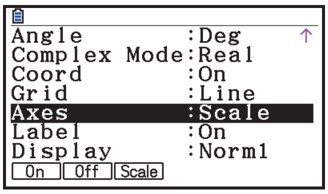
To describe the bridge with graphs, it is necessary to have a suitable set of axes. When the picture is first displayed, a pair of axes is shown on the screen, depending on the View Window. For this symmetrical image, the Initial view window (F1) is a good choice.

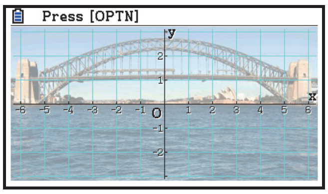
In this case, the axes are a little difficult to see, as the color of the picture is quite intense. So, it's a good idea to reduce the intensity a little to allow the mathematics to be more visible. To do this, tap the OPTN key then F6 F6 and then select FADEI/O (F3). Use the cursor to choose less intense colors. The examples below show a compromise between making the mathematical work clear and still retaining the image. In this module, we will use the 60% version throughout.



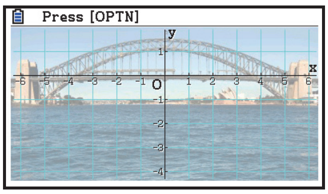


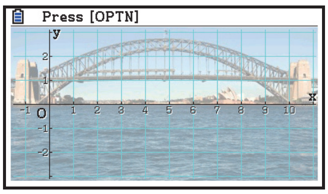
It is a good idea when modelling to make sure that a numerical scale is showing on the axes. A grid is also useful, to allow you to see how mathematical functions can be constructed to represent the picture. (These are matters of personal preference, however.) To do this, use SET UP to configure the *Grid* and the *Axes* as shown below. The resulting screen that we will use is shown on the right.





You can move the origin of the coordinate system both vertically and horizontally using the cursor keys or by choosing appropriate settings in the View window. Several other choices are possible, such as making the horizontal base of the bridge the *x*-axis, or choosing the origin to be at the left end of the bridge, as illustrated below.





The choice of axes is arbitrary, although of course different results will be obtained with different axes. We will use the original choice in the centre of the screen in this module, although it is perfectly adequate to make a different decision.

Refining an initial model

We will begin by trying to model the lower curve of the bridge, which seems to cross the x-axis near $x = \pm 5$ and the y-axis a little above $y = 2$. It's a good idea to think about the relationships involved, using the picture to help you. The curve looks like an inverted parabola, symmetrical about the y-axis, so we might expect it to be represented by a function of the form

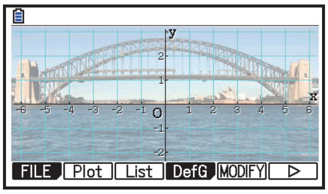
$$y = a - bx^2.$$

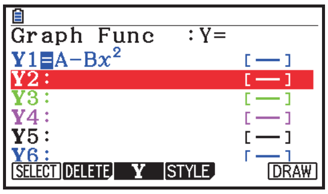
When $x = 0$, the value of y is a little more than 2, so we might expect a to be a little more than 2, say about 2.3.

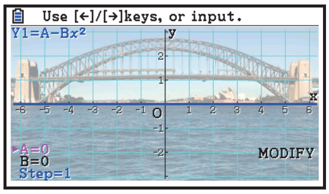
When $y = 0$ (i.e., the intercepts on the x-axis), $bx^2 = a$. Since $x \approx 5$ and $a \approx 2.3$, it seems as if we might expect b to be about $2.3/25 \approx 0.1$.

Thinking like this gives a good chance to predict a reasonable first approximation to a suitable function. We will then use the Modify feature to adjust our initial guess.

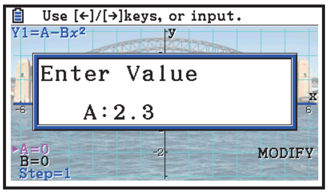
To define an initial guess for the curve, tap OPTN and then DefG (F4), as shown below. Define the function as $y = A - Bx^2$ and tap EXE. Tap DRAW (F6) and then MODIFY (F5) to adjust the function to fit the curve better. (The function shown graphed in blue in the third screen below has $A = 0$ and $B = 0$, so the function is $y = 0$ (the x-axis), although you will see a different result if the variables already had a different value in your calculator, and will get a correspondingly different initial graph.)

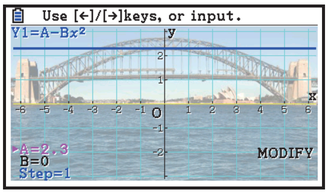




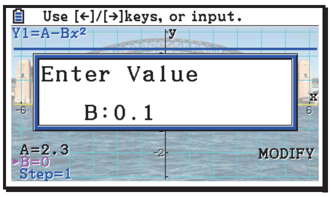


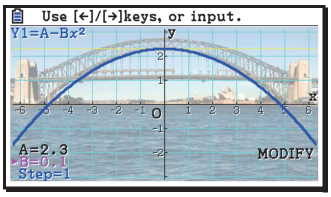
A good initial guess is $A = 2.3$ and $B = 0.1$. Since the value for A is already marked with a small arrow on the left of the screen, you can enter 2.3 and EXE to give A the value 2.3. The result shown below indicates that the top of the curve (presently a line) is about right.



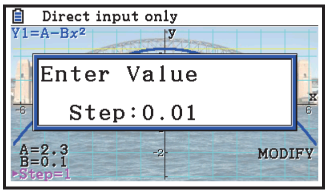


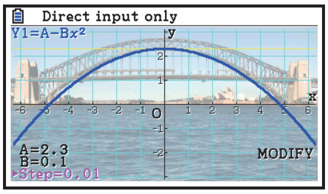
Then move the cursor down to mark variable B. Enter 0.1, followed by EXE, to give B the value 0.1. The graph now shows an inverted parabola, as expected, that seems quite close to the lower curve of the bridge.



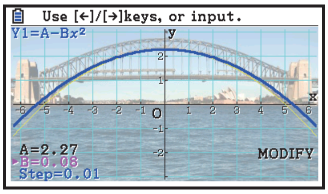


Although this graph provides a reasonable model for the lower curve of the bridge, it can still be improved. To do this, move the cursor down to mark the *Step* command and then give it a suitable value to allow for fine adjustments of the model to the bridge. In this case, a good choice is 0.01, which will allow us to easily change the second decimal place of the values for A and B.





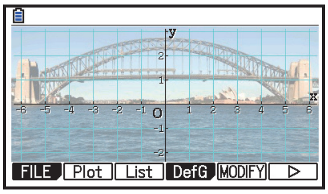
Then move the cursor back to *A* and *B* and use ▶ and ◀ to adjust the values until a good fit of the curve to the bridge is evident. Each tap of the cursor moves the corresponding value 0.01. The screen below suggests that $y = 2.27 - 0.08x^2$ is a good model for the lower curve of the bridge.

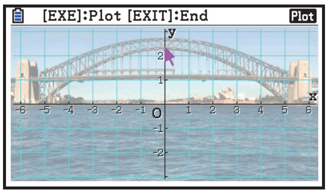


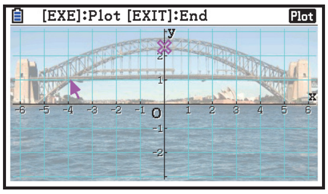
Statistical curve-fitting

A different approach to modelling the curve of the bridge is to use statistics, and in particular regression functions of the kind studied in Module C2 Data analysis. To see how to do this, start by removing the previous model. Tap EXIT to stop using the Modify capability and then OPTN and DefG (F4) to return to the function list. Finally, tap SEL (F1) to deselect the function, which will remove it from the screen. Return to the picture with EXIT.

To use statistics, we need some data. Data points are generated by using the Plot (F2) command to plot some points on the curve.

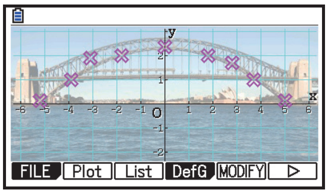






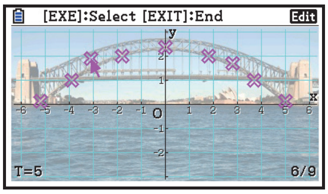
In the screens above, a cursor has been obtained after Plot has been selected. Move the cursor to points on the curve in turn and tap the EXE key to mark a point with a cross. The right screen above shows a point being marked, at the apex of the parabola. Select a range of points across the whole curve, in order to find a good-fitting model.

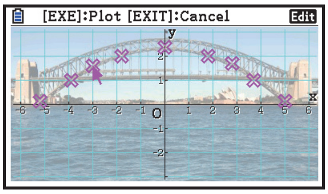
You may find that moving the cursor is quite slow; you can hold your finger down to move it more quickly. Another good way of moving it quickly and efficiently is to firstly use the keyboard numbers 1, 2, 3, … , 9, which will move the cursor to corresponding parts of the screen, and then use the cursor for smaller movements. Tap EXIT when you are satisfied that you have enough points. As points are labelled in order of placement, it is generally a good idea to plot points from left to right.

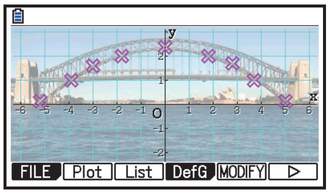


The screen above shows one possible set of points on the curve, but you might choose others.

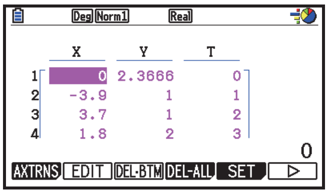
It is possible of course that some of the points don't lie on the curve properly and you want to adjust them to improve their position. In the screen above, for example, the third point from the left has been poorly placed, and will lead to inaccurate modelling. To adjust points before an analysis is undertaken, tap F6 and then EDIT (F3). You can then move among the points with ▶ and ◀ until you reach the point to be changed. In the case below, it is the sixth point (labelled T = 5) that needs to be changed. Tap EXE to select the point and then move it to a more appropriate position and tap EXE to plot it. After all points are correct, tap EXIT to finish editing.



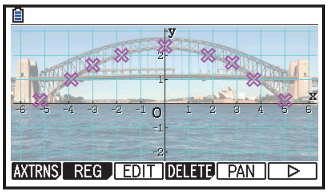


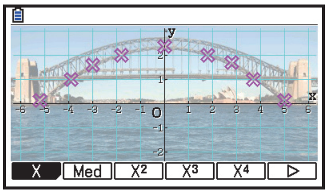


You can see the details of the points plotted by tapping LIST (F3) and even adjust the values in the table by highlighting them with the cursor and then entering a preferred value with the keyboard.

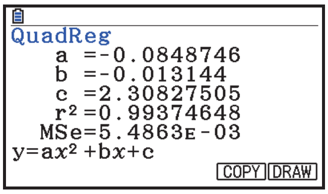


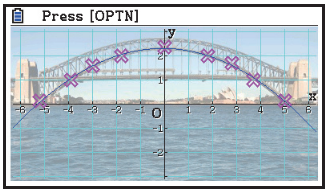
When you are satisfied with your selection of points, you can obtain a suitable regression function to fit the points by first tapping OPTN and F6, followed by REG (F2). You will see the familiar range of regression functions that you have used in Statistics mode, as shown below.





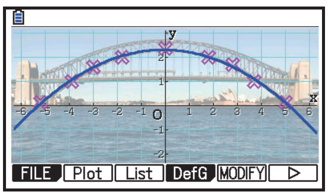
A good choice to fit what looks like a parabolic curve is the quadratic regression X^2 (F3). The result below is similar to the results from Statistics mode. Copy the function and then draw it to see that it fits the points (and hence the curve of the bridge) very well.

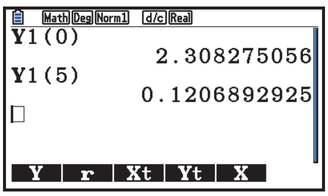


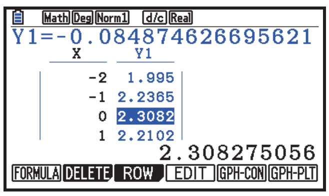


The error term is very low (MSe ≈ 0.0055), suggesting that the fit is very good. So a suitable model is $y = -0.08x^2 - 0.01x + 2.31$. If we regarded the linear term (with x) as too small to be important, a model of $y = -0.08x^2 + 2.31$ is very similar to that obtained in the previous section by manipulating the graph directly.

Since the model was copied after being generated, you can return to the function list with OPTN and DefG (F4) and DRAW it on the screen, to show again how well it fits the bridge. You could also use it to make predictions in Run-Matrix mode or in Table mode, as shown below.

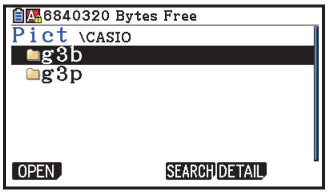


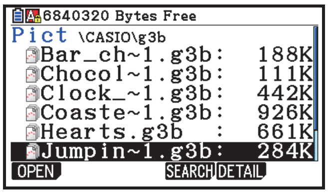


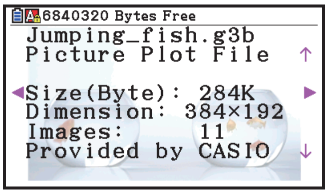


Modelling movement

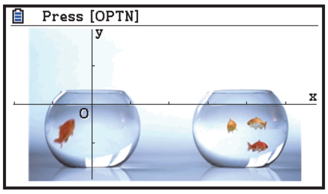
As well as representing and understanding real objects, you can use Picture Plot to analyse real situations that involve movement, using the animation files. We will use the file *Jumping_fish.g3b* shown below. Use OPTN FILE (F1) and then OPEN (F1) to select the animation files in the g3b folder. After opening the folder (using OPEN (F1) or EXE), highlight the file name and use DETAIL (F5) to see that 11 images are involved in this animation.



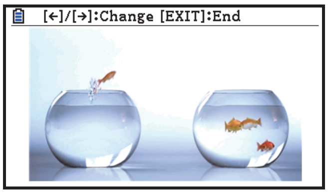


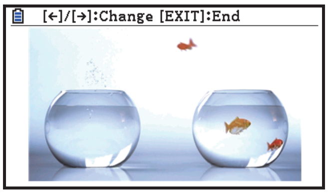


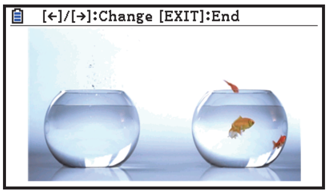
Tap EXIT and then open the file to see the opening screen of the animation, a pair of goldfish bowls as shown below.



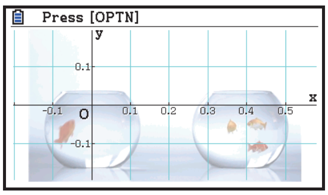
To see the animation, tap the OPTN key, followed by F6 F6 and then PLAY (F2). You can advance the images one at a time in with MANUAL (F2), but it is best to watch the animation first using AUTO (F1). The animation of a goldfish jumping from bowl to bowl will play through three times. Some screens are shown below, from Manual mode.

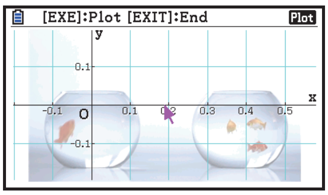






We can use *Picture Plot* to study this amazing motion, in order to understand it better and to see its mathematical basis. As we did earlier in this module, the images can be lightened by fading, and a grid or a scale (or both) added, to increase the clarity. Before you tap the OPTN key, you can move the origin using the cursor keys, although we have left them in their original position in this module, for which the horizontal axis is drawn at the same level as the top of the water in each bowl, and the origin is in the middle of the first bowl. The image below has been faded to 60%.

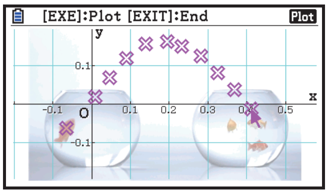




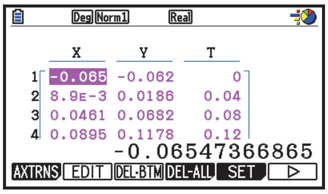
To analyse the motion of the jumping fish, we will create some data during the jump and then use statistical analysis to explore it. Start with OPTN PLOT (F2) to access a screen to plot points, shown at the right above. Each point shows a different position of the jumping fish. You can position the cursor with the normal cursor keys as well as using the keyboard keys; tap EXE to plot a point.

In plotting points for this example, care is needed to choose the same part of the fish each time. For this example, we have chosen to mark a point in the middle of the fish body each time: if you choose different points at different times (such as choosing the head or tail), the resulting model would be less likely to represent the motion well.

Each time you plot a point, the next screen will be automatically shown. That is, unlike the case for still pictures earlier in this module, you cannot add unlimited points to describe the motion. So there is at most one point plotted in every screen, in this case a maximum of eleven points altogether. The more points you plot, the better defined will be the motion of the fish, of course, The screen below shows the maximum number of points plotted (eleven for this animation). As noted on the top of the screen, tap [EXIT] when you have finished plotting.



As previously, the data created in this way can be edited if necessary, using [OPTN] [F5] and EDIT ([F3]), and can be displayed as a list (using [OPTN] LIST ([F3]).

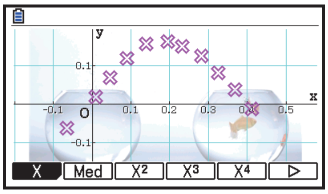


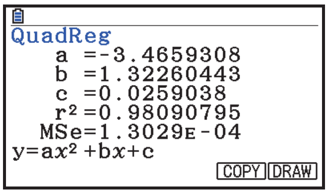
Notice that the times have been added automatically in the third column of the table: there is a point plotted every 0.04 time units, so that the entire jump lasts only 0.4 time units.

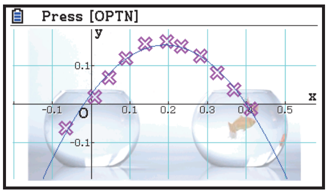
There are three ways to model this motion mathematically, using (i) the relationship between the height (y) and the horizontal distance (x), (ii) The relationship between height (y) and time (T), or (iii) the relationship between distance (x) and time T. We will explore all three of these.

Height and distance relationship

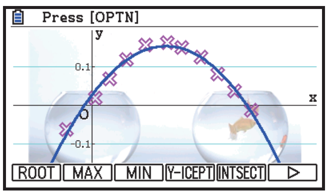
The plotted points shown above appear to show a parabolic relationship. To study this more carefully, tap [OPTN] [F6] and then REG ([F2]) to see some possible regression models. The parabolic choice is X^2 ([F3]), as shown below. COPY ([F5]) it to the function list and then DRAW ([F6]) it on the data to see that the resulting graph seems to model the data quite well.

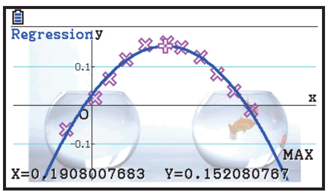






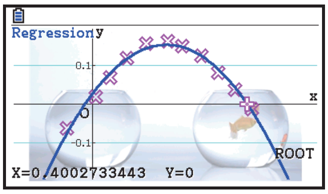
This model $y = -3.47x^2 + 1.32x + 0.03$ can be used to answer some questions about the fish's jump. For example, the maximum height of the fish can be modelled using G.SOLVE ([SHIFT] [F5]):

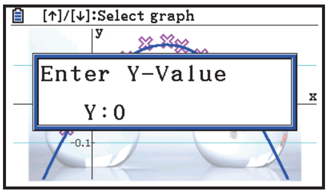


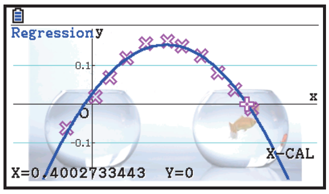


It seems from the model that the fish reached a height of about 0.15 units about 0.19 units to the right of leaving the water.

Similarly, the horizontal length of the jump can be modelled by finding the roots of the function. Again G.SOLVE is useful for this purpose, using the ROOT command, although there are other methods such as using X-CAL as shown below. According to the model, the fish has jumped about 0.4 units in horizontal distance.



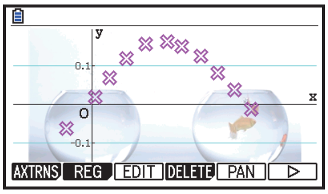


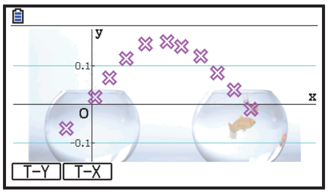


We don't know the units of time or distance involved here, but we can estimate from earlier screens the approximate size of the fish itself to understand better how high and far it has jumped.

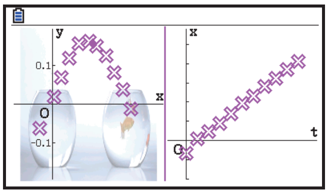
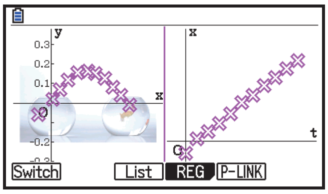
Relationships with time

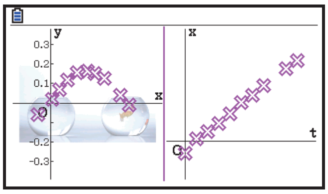
As well as relationships involving vertical and horizontal distances, we might choose to model the fish's jump by describing how the height and distance change over time. The easier of these two cases involves comparing horizontal distance with time. To do this, tap OPTN then F6 and then AXTRNS (F1), as shown below.





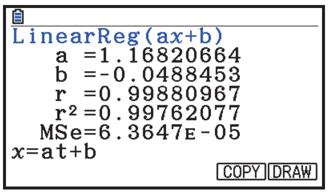
Two possible *Axis Transformations* are available, as shown in the second screen above, allowing for the t-y or the t-x relationship to be explored. Start by selecting T-X (F2) to see how the horizontal values (x) change over time (t).

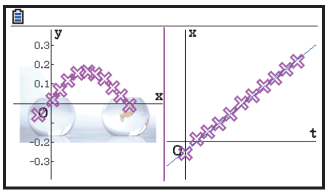





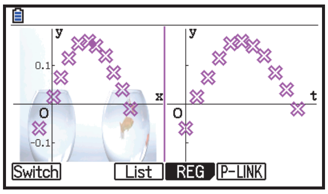
Notice in the resulting graphs that the scales on the axes have been changed, so that both the x-y and x-t graphs can be seen at the same time. Tap OPTN to see commands for studying these two graphs, shown in the middle screen above. You can trace the x-t graph using SHIFT F1 . A more insightful trace is available through P-LINK (F5) which flashes pairs of corresponding points on each graph as you trace. In the right screen above, obtained using P-Link, two matching points are 'off'.

The graph of x versus t looks very strongly linear, and the modelling using REG (F4) with a linear function reinforces this observation, as shown below. The points are very close to the line given by $x = 1.17t - 0.05$. In fact, the variations from the line are likely to be a result of small errors in selecting the points originally. It seems that the distance travelled horizontally by the fish is directly related to the time in the air.

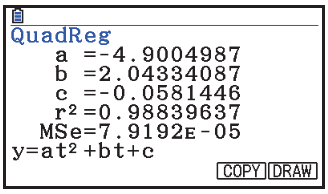


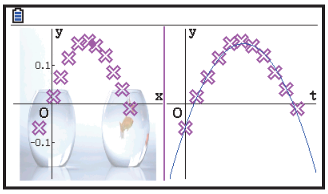


To study the y-t relationship, use the EXIT key to return to AXTRNS and select T-Y (F1). This time, the graph tells a quite different story, as shown below.



You can explore this relationship in the same way as for the x-t relationship, including the P-LINK and regression capabilities. The height (y) of the fish increases to a maximum half-way between the two bowls and then reduces to zero again, when the fish enters the second bowl. Clearly a parabolic model is suggested by the data here, and the regression command allows that to be obtained:





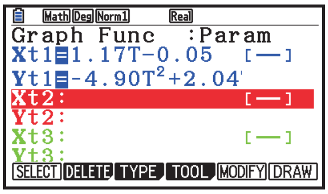
The model suggested by the data is $y = -4.90t^2 + 2.04t - 0.06$. Although the measurements are not given by the calculator, this model suggests that the height is related to an object under the influence of gravity on Earth … and that time is measured in seconds and the other units in metres.

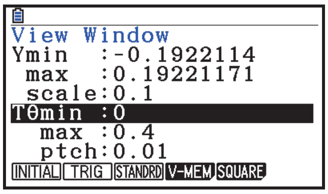
This relationship helps to understand the fish's jump, which involves accelerating out of the water and then falling under the influence of gravity, giving rise to the characteristic parabolic shape for the height versus time graph. The horizontal distance versus time graph is linear, because the longer the fish is in the air, the further it travels horizontally, and there is no acceleration or change of velocity involved, as there are no horizontal forces involved.

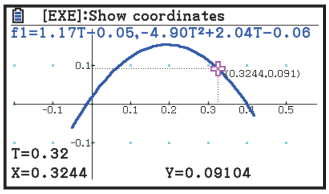
In these kinds of ways, studying and analysing graphs of phenomena help us to understand exactly what is happening.

Modelling in Parametric mode

To see the fish jumping situation from another perspective, you can use the functions obtained to describe a parametric function, with time as the parameter, as shown below. Use your own relationships or the ones shown above. (In parametric mode, the X,θ,T key inserts a T instead of an x in the functions).







The View window can remain unchanged from before, but you need to select suitable values for T that match the values in the animation. In this case, the jump extends over about 0.4 time units, so $0 \le t \le 0.4$ seems appropriate, with a small *pitch* of 0.01 to allow a reasonable number of points to be plotted. The resulting graph shows the y-x relationship as the original picture shows, since it is modelling how each of x and y vary over the values of t.

In summary, it seems that the amazing jump of the fish can be represented very well with these mathematical functions, and so can be thoroughly understood, which is usually one of the purposes of mathematical modelling.

Exercises

The main purpose of the exercises is to help you to develop your calculator skills.

1 One of the picture files available on the calculator is called *Satellite_dish.g3p*. Find and open this file in Picture Plot. How much memory does the file require?

2 One of the animation files available on the calculator is called *Wind_turbine.g3b*. Find and open this file in Picture Plot. How many separate images does the file comprise?

3. Use the *Harbour_bridge.g3p* file for the following exercises, as in the text of the module.

(i) Guess a suitable model for the horizontal roadway of the bridge.
(ii) Check your model visually by using DefG (OPTN F4).
(iii) Adjust your model, by changing the style of the graph to be thin and obtaining the best fit you can visually.
(iv) Model the roadway by plotting some points and choosing a suitable regression function.
(v) Compare your two models for the roadway over the bridge.

4. Use the *Harbour_bridge.g3p* file for the following exercises, as in the text of the module.

(i) Delete any existing plotted points.
(ii) Find a suitable model for the *top* arch of the bridge, starting with $Y1 = Ax^2 + B$. Start by guessing suitable values for A and B.
(iii) Use the MODIFY function to change the values and improve the fit.
(iv) Plot some points for the top of the bridge and use a suitable regression function to find a suitable quadratic model.
(v) Compare your model with that found in Exercise (iii).
(vi) What reservations should be expressed about the suitability of these two models?

5. Use the *Harbour_bridge.g3p* file for the following exercises, as in the text of the module.

(i) Instead of the origin being in the centre of the bridge, as in the module, choose the origin to be on the water line at the left end of the bridge, as suggested at the bottom of page 2.
(ii) Find a suitable model for the lower arch, using $Y1 = Ax^2 + Bx + C$ and MODIFYcommands.
(iii) Obtain a suitable model by plotting some points and then choosing a regression function.
(iv) Compare your two models from parts (ii) and (iii).
(v) Why are these models different from those derived in the module?

6. Use the *Jumping_fish.g3b* animation file for the following exercises.

(i) Use *View window* to set the origin half way between the two bowls so that $-0.4 \leq x \leq 0.4$ and $-0.2 \leq y \leq 0.2$
(ii) PLAY the movie to see approximately how high the fish jumps.
(iii) Use *DefG* to test models of the form $y = A - Bx^2$ to model the flight of the fish.
(iv) Plot some points and use a quadratic regression function to model the fish jump.
(v) Compare your models from parts (iii) and (iv).
(vi) Use AXTRNS to find the *x-t* relationship and the *y-t* relationship
(vii) Compare your results in part (vi) with those in the module

7 Use the relationships from part (vi) of the previous exercise to draw a suitable graph in parametric mode, using the same view window. It should show a similar parabolic path for the fish jump.

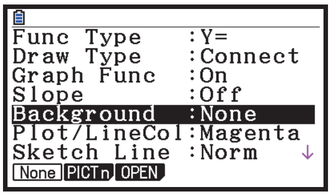
Activities

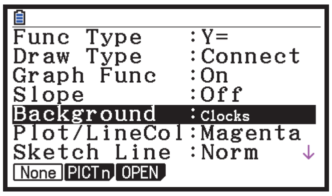
The main purpose of the activities is to help you to use your calculator to learn mathematics. You may find that some of them are too advanced for you. Ignore activities you don't yet understand.

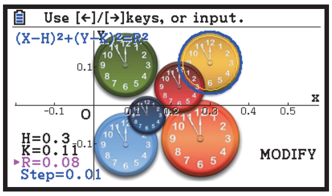
1 In the *Harbour bridge* modelling, distances are defined by the chosen scales on the calculator, not by distances in real life. However, you can explore actual distances if you have some real measurements. Officially, the Sydney Harbour Bridge provides clearance for shipping of 49 m. use this information to determine other measurements (such as the height of the top of the bridge above the water or the distance between the left and right pylons. Check your modelling with actual data available online.

2 The *Jumping_fish.3gb* animation files do not contain measurements, but it seems likely that the scales used are in metres and the time is measured in seconds. Assuming that is the case, use the files to determine approximately how long the fish is, and work out how high and how far it actually jumps.

3 Files in *Picture Plot* can be used in SET UP as background screens in Graph mode and Conic Graphs mode. For example, the screen below shows that the *Background* setting in Conic Graphs mode has been set to Clocks by using OPEN (F3), and then choosing *Clocks.g3p*.







Once a file is set as a background, all graphs are drawn on the background. In this case, you can use MODIFY commands in the usual way to construct circles to match a clock, as the third screen shows. Try to match each clock in turn with a suitable circle. Remember to return the Background to *None* when finished.

Explore some other background screens in this way, in both *Conic Graphs* and *Graph* mode.

4 In Picture Plot, study the *Roller_coaster.g3p* image, which shows various roller coasters. You can fit curves to these shapes by choosing a number of points and then using the regression capabilities. Notice that four points are sufficient for a perfect fit to a cubic function, but insufficient for a quartic function, which needs at least five points. Notice too that although a function can be fit perfectly to some points, it will not necessarily match the roller coaster well unless the points are carefully chosen. Discuss your models with some colleagues, to find the best model you can with the least number of points used.

5 Study the animation file *Pendulum.g3b*, showing a pendulum for a clock swinging backwards and forwards.

What sort of motion seems to be involved in the horizontal and the vertical directions?

The periodic nature of the pendulum suggests that some kind of trigonometric functions might be involved. Make sure that you set the angle to radians before starting modelling the relationships.

6 There are many other files available in Picture Plot, and many other animation files. Choose some files of your interest to see how mathematics can be used to represent the real world well, provided you choose suitable models carefully. Share your observations with others.

Notes for teachers

This module illustrates several ways in which the fx-CG 20 calculator can be used for modelling everyday situations with graphs, and using the graphs to understand the situations better. The text of the module is intended to be read by students and will help them to start to see how the calculator can be used efficiently to summarise and represent data in various ways, and become a tool for analysing their own data. The Activities are appropriate for students to complete with a partner, so that they can discuss their observations and justify their conclusions.

Answers to Exercises

1. Use OPTN and then FILE (F1) to find the file, which requires 71K 2. File contains 20 images 3. (i) $y = 1$ looks close (ii) Model seems close, although default graph is a little thick. (iii) A better fit is $y = 1.1$, more easily seen with a thin line style. (iv) & (v) results will depend on points chosen, although the two models ought to be close to each other. 4. (i) Use OPTN F6 and DELETE (F4) (ii) Good guesses are A = -0.1 and B = 2.8 (iii) $y = -0.06x^2 + 2.7$ is a good fit (iv) & (v) results will depend on points chosen, but the two models ought to be close. (vi) the models assume the shape is quadratic, although at the extremes the shape is clearly not quadratic. 5. (i) Set the View Window to have $-1 \le x \le 11.6$ and $-3.1 \le y \le 3.1$ so that the arch passes through the origin. (ii) $y = -0.09x^2 + 0.95x - 0.26$ appears to be a good fit. (iii) After plotting seven points, we got $y = -0.09x^2 + 0.90x - 0.05$ (iv) Models are not identical, but are close. (v) the change of axis changes the meaning of x in the models. 6. (ii) use OPTN F6 F6 and then PLAY (F2). The fish seems to jump to about $y = 0.15$, although it is difficult to tell as axes are removed while movie plays. (iii) $y = 0.15 - 3.5x^2$ seems to model the flight well. (iv) $y = -3.3x^2 + 0.02x + 0.16$ seems close; results rely a little on chosen points. (v) Models are close, but are not identical, partly because of points chosen. (vi) $x = 1.21t - 0.24$ and $y = -5.03t^2 + 2.11t - 0.06$ (vii) Relationships are similar to those in the module. However, the x-t relationship is changed because the origin has been translated horizontally. The y-t relationship is very similar to that found previously.

Activities

1. Data can be consulted online for this famous bridge, and students can be encouraged to do this as an extension and a check on their mathematical modelling.

2. This activity is intended to encourage students to look more carefully at the measurements involved in the animation. It seems in this case that the fish is around 6 cm in length, and so jumps about 15 cm high and 40 cm in horizontal distance.

3. Students may need some help setting background screens and you may prefer to choose other examples for this purpose, especially if they have not studied conic graphs yet. Some of the pictures lend themselves to polar coordinate graphs (in Graph mode).

4. Students may need help understanding that a fit can be 'perfect' (i.e. $r^2 = 1$) and yet the curve does not match the roller coaster well. Encourage them to choose points over the full length of the roller coaster shown, especially at the end points to minimise this problem. In general, a cubic model is preferred to a quartic model, as it involves fewer parameters.

5. The sinusoidal regression will match the horizontal pendulum movement very well, even though the calculator screen will seem quite cluttered. More sophisticated students can see the idea of simple harmonic motion here.

6. There are many interesting choices that students can make, depending on their mathematical sophistication. Encourage them to undertake some modelling in depth to take best advantage of the opportunities provided by the images.

CASIO COMPUTER CO.,LTD.
Tokyo, Japan

SA1409-001001A