Inverse Trigonometric Functions

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LEVEL
High school or university students

OBJECTIVES
To use the calculator to study the different characteristics of the inverse trigonometric functions.

Corresponding eActivity
Invtrig.g1e

OVERVIEW
The study of the domain and range of inverse trigonometric functions has important consequences in pre-calculus and calculus. In this paper, we will study this concept using the graphical and table features of the calculator.

EXPLORATORY ACTIVITIES
[Note] We shall use small letter *x* instead of capital X as shown on the calculator throughout the paper. Using SHIFT SET UP we specify that the angle will be in radian mode.

Activity 1: Determine the domain and range of the following functions:

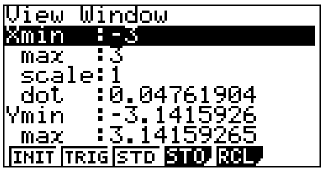
(a) $f(x) = \sin^{-1}(x)$
(b) $g(x) = \cos^{-1}(x)$
(c) $h(x) = \tan^{-1}(x)$

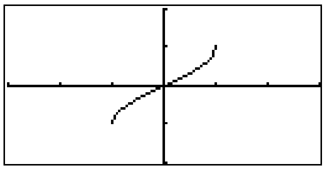
Solution:

(a) $f(x) = \sin^{-1}(x)$

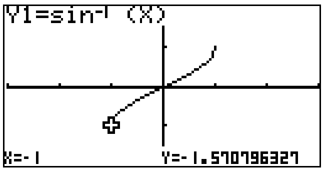
A. Graphical exploration: We open the Graph Editor and graph the function using an appropriate View Window:

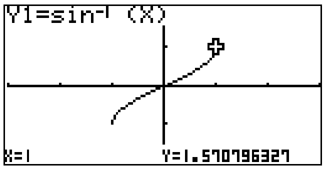
__Inverse Trigonometric Functions__



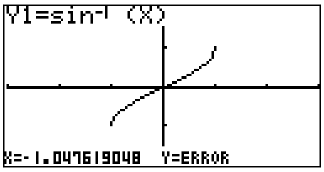


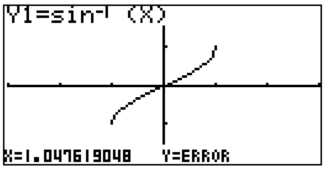
We use [Trace] to explore the domain and range for the function *y* = f(x) as follows:



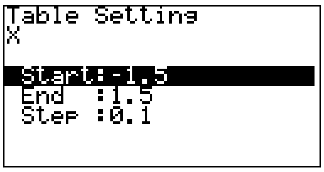


The domain of the function is $x \in [-1,1]$ and the range is approximately the interval [-1.5708,1.5708]. Notice that if we trace along the graph and consider the *x* values outside the interval [-1,1], an error message appears.

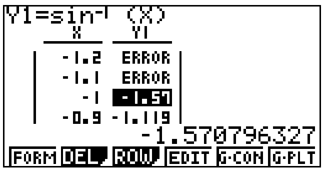


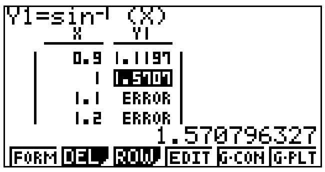


<u>B. Tabular exploration:</u> Enter the function p(x) in the <u>Table Editor</u> and use the following table setting:



We obtain the following partial list of data.





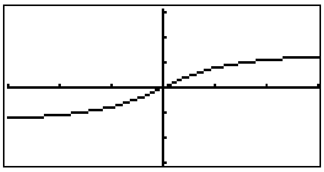
We can verify from the table that the domain of *f*(*x*) is [-1,1] and the range is [-1.5708,1.5708]. Note that if *x* is not in the interval [-1,1], an error message occurs.

(b) To obtain the domain and range of $g(x) = \cos^{-1}(x)$, we follow the same steps given in (a). The domain is [-1,1] and the range is [0,3.142].

(c) Let us explore further the case of $h(x) = \tan^{-1}(x)$.

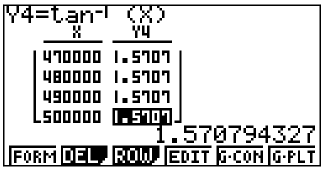
<u>A. Graphical exploration</u>

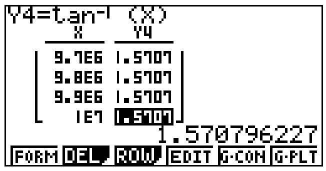
Graphing h, we see from the graph that its domain is $(-\infty,\infty)$ and the graph has horizontal asymptotes.

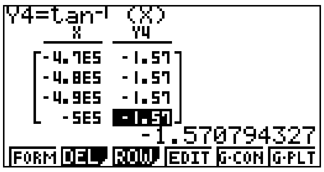


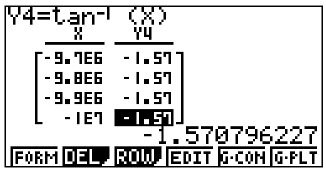
<u>B. Tabular exploration</u>

To determine the asymptote, we investigate the limit of h as x approaches ∞ and as x approaches $-\infty$ using a tabular approach. One can explore the limiting value by choosing suitable viewing windows. For instance, we observe that as we take large values of x, (positive direction), h approaches 1.5708. On the other hand, if we take large values of x(negative direction), h approaches -1.5708.









Thus the range of h is (-1.5708, 1.5708).

Remarks: The results we obtain here regarding the domain and range of the inverse trigonometric functions is consistent with the following property of inverse functions: the domain of f = range of f^{-1} and the domain of f^{-1}= range of f. Consider for example, the fact that our results say that the domain of $\tan^{-1}(x) = (-\infty,\infty)$, which is the range of tan x and the range of $\tan^{-1}(x)$= (-1.5708, 1.5708) which is actually the domain of tan x $=(-\frac{\pi}{2},\frac{\pi}{2})$.

EXERCISES

Exercise 1.

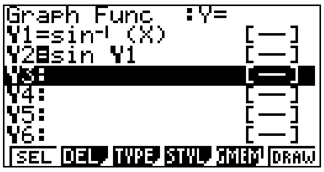
a) For what values of x is $\sin(\sin^{-1}(x))=x$, $\cos(\cos^{-1}(x)) = x$ and $\tan(\tan^{-1}(x))= x$?
b) For what values of x is $\sin^{-1}(\sin (x))= x$, $\cos^{-1}(\cos x) =x$ and $\tan^{-1}(\tan x) = x$?

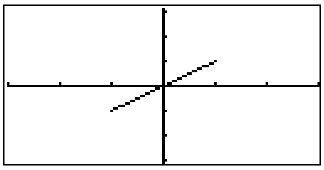
Exercise 2. Let $f(x)= \sin^{-1}(x)$ and $g(x)= \cos^{-1}(x)$. Determine $(f+g)(x)$.

SOLUTIONS

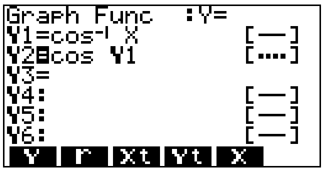
Solution to Exercise 1.

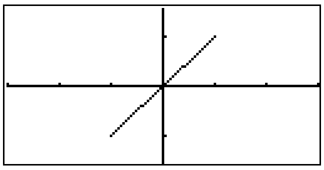
a) We first graph the composite $\sin(\sin^{-1}(x))$ in the graph window. An important observation we can make is that the graph we obtain is that of the identity function $i(x) = x$ given specifically on the interval [-1,1].

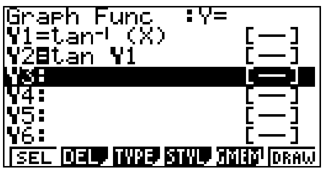


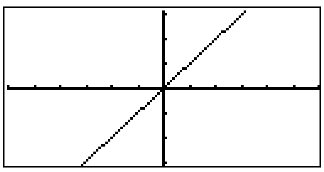


Similarly, when we graph $\cos(\cos^{-1}(x))$ we obtain the identity function $i(x)= x$ on the interval [-1,1] . On the other hand, if we graph $\tan(\tan^{-1}(x))$, we obtain $i(x)= x$ defined for all real numbers.



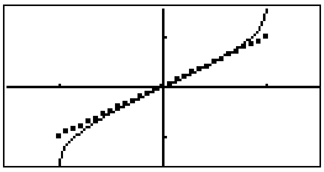
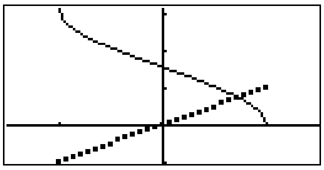
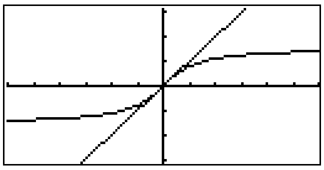




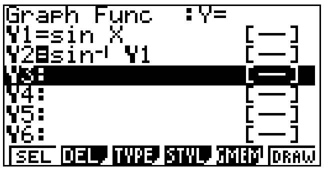


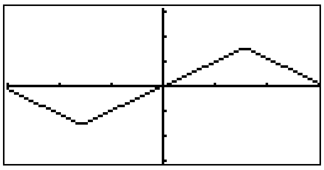
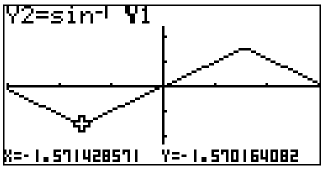
The above graphs show that $\sin(\sin^{-1}(x))=x$ and $\cos(\cos^{-1}(x)) = x$ for all x in [-1,1] and $\tan(\tan^{-1}(x))= x$ for all real numbers.

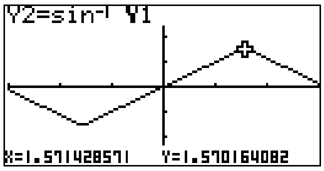
If we graph the functions $\sin^{-1}(x)$ and $\sin(\sin^{-1}(x))$ in the same Graph window, we can observe that $\sin(\sin^{-1}(x)) = x$ is true for all x values in the domain of $\sin^{-1}(x)$. It is also true that cos(cos^{-1}(x)) = x and tan(tan^{-1}(x))= x for all x values in the domain of cos^{-1}(x) and tan^{-1}(x) respectively.

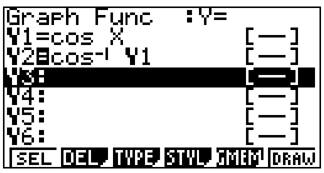
b) This time we graph the composite function sin^{-1}(sin (x)). From the graph we can approximate that sin^{-1}(sin (x)) = x in the interval [-1.571,1.571].

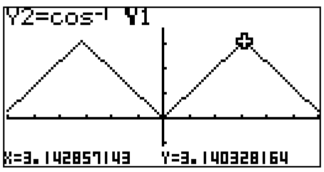


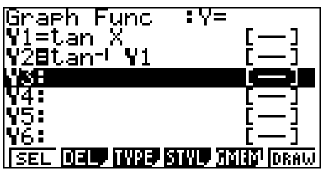



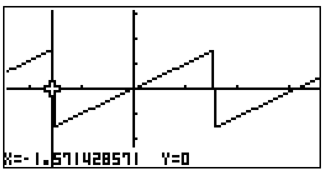


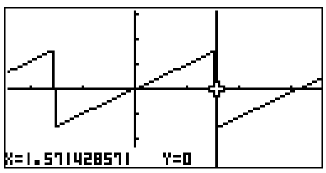
We graph the composite function cos^{-1}(cos (x)) and tan^{-1}(tan (x)). We can approximate that cos^{-1}(cos (x)) = x in the interval [0,3.143]; tan^{-1}(tan (x)) = x in the interval [-1.571,1.571]. In the graph below, we give the vertical asymptotes of tan^{-1}(tan (x)). These are the horizontal tangents of tan^{-1}(x).







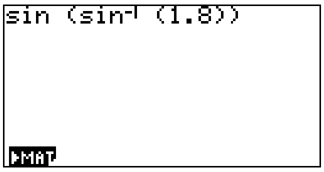


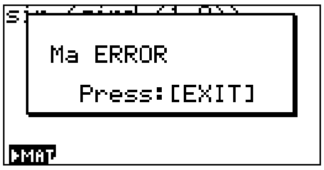


Relating this exercise with the exploratory activity given earlier, it is therefore true that $\sin^{-1}(\sin(x)) = x$, $\cos^{-1}(\cos x) = x$ and $\tan^{-1}(\tan x) = x$ for all values of x in the range of $\sin^{-1}(x)$, $\cos^{-1}(x)$ and $\tan^{-1}(x)$ respectively.

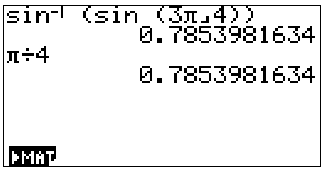
Remarks: The results given in exercise 1 are important properties of inverse trigonometric functions. Here are some selected examples that highlight the consequences of these properties.

1. Evaluate $\sin(\sin^{-1}(1.8))$. Since 1.8 is not in [-1,1], the domain of $\sin^{-1}$, we cannot evaluate this expression. There is no number with a sine of 1.8. Since we cannot determine $\sin^{-1}(1.8)$, we say that $\sin(\sin^{-1}(1.8))$ does not exist. In the Run Editor of the calculator, we get an error message when evaluating $\sin(\sin^{-1}(1.8))$.

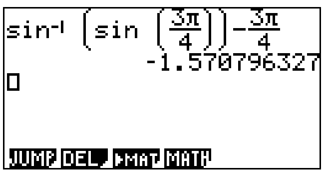




2. Simplify $\sin^{-1}(\sin(\frac{3\pi}{4}))$. Since $\frac{3\pi}{4}$ is not in [-1.571,1.571], the range of the $\sin^{-1}$ function, we cannot apply $\sin^{-1}(\sin(x)) = x$. Thus, we evaluate $y = \sin(\frac{3\pi}{4})$ first, then determine $\sin^{-1}(y)$. In the Run Editor, the value is given directly as follows:



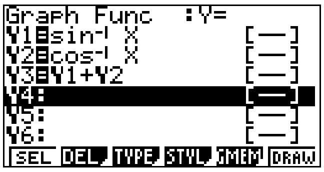
A way to verify for instance, that $\sin^{-1}(\sin(\frac{3\pi}{4})) \neq \frac{3\pi}{4}$ is to calculate $\sin^{-1}(\sin(\frac{3\pi}{4})) - \frac{3\pi}{4}$ in the Run Editor. Note that the answer we obtain is nonzero:

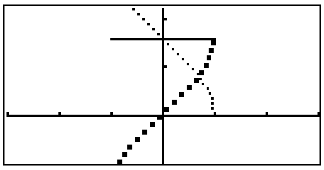
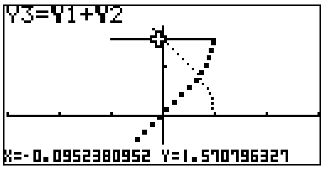


Solution to Exercise 2.

A. <u>Graphical exploration</u>

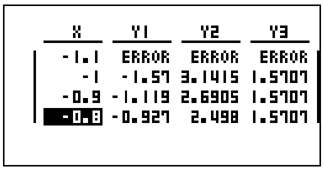
We graph the functions $\sin^{-1}(x)$, $\cos^{-1}(x)$ and $\sin^{-1}(x) + \cos^{-1}(x)$ in the <u>Graph window</u>. We observe that $\sin^{-1}(x) + \cos^{-1}(x) \approx 1.571$

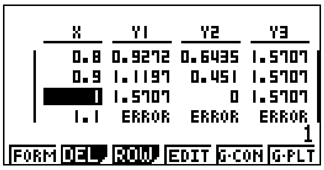


<u>B. Tabular exploration:</u>

This can be verified further in the table window as follows:





Note that it can be verified from the table that $\sin^{-1}(x) + \cos^{-1}(x) \approx 1.571$ is true only if $x \in [-1,1]$. An error message appears if x is not in the interval [-1,1]. This observation can lead us to the analytical solution as follows:

<u>C. Analytical solution:</u>

Let us consider $x \in [-1,1]$. Suppose $y = \sin^{-1}(x) + \cos^{-1}(x)$.
Now, Sin y = Sin $[\sin^{-1}(x) + \cos^{-1}(x)]$
$= \text{Sin}(\sin^{-1}(x))\text{Cos}(\cos^{-1}(x)) + \text{Cos}(\sin^{-1}(x))\text{Sin}(\cos^{-1}(x))$
Since $\sin^{-1}(\sin(x)) = x$, $\cos^{-1}(\cos x) = x$ for $x \in [-1,1]$, we have

$$\text{Sin } y = x^2 + \sqrt{1-x^2}\,\sqrt{1-x^2}$$

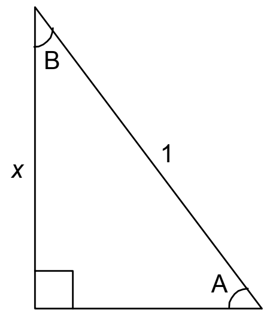
or Sin y = 1.

This implies $y = \dfrac{\pi}{2} \approx 1.571$.

<u>D. Geometric solution:</u>

A geometric interpretation of the problem is as follows:

Consider the right triangle with hypotenuse of length 1. Let A, B be the acute angles of the given triangle.



From the given figure, sin $A = x$ and cos $B = x$. Now, x is in the interval [0,1] so that $\sin^{-1}(x) = A$ and $\cos^{-1}(x) = B$.
Note that, $A + B = 90^0$. It follows that $\sin^{-1}(x) + \cos^{-1}(x) = 90^0$.

REFERENCE

[1] Bittinger et al. *Algebra and Trigonometry*, *Graphs and Models*, 2nd Edition. Addison Wesley Verlag, 2001.