



Derivative at a point

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LEVEL

High school or university students who have studied the pre-Calculus subjects such as polynomial and trigonometric functions.

OBJECTIVES

To explore the use of calculator in understanding the slope of a tangent line at one point or derivative at one point and its applications.

Corresponding eActivity

C0101.g1e

OVERVIEW

Many students know how to compute the derivative of a function at one point by memorizing a proper formula but lack the understanding of the concept. We will use CASIO 9860 model to explore this concept from various angles.

(i) We will study the concept of derivative at one point numerically and also with an aid of animations.

(ii) We will use the concept of the derivative at a point to approximate numbers numerically.

EXPLORATORY ACTIVITIES

[Note]

(a) We shall use small letter x instead of capital X as shown on the calculator throughout the paper.

(b) Unless otherwise specified, we choose MATH mode in the SET UP menu. Go to RUN mode in the Menu. Then press **SHIFT** SET UP and choose **Math** in the **Input Mode**, which we show below:

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Input Mode	:Math
Mode	:COMP
Frac Result	:d/c
Func Type	:Y=
Draw Type	:Connect
Derivative	:Off
Angle	:Rad ↓
MathLine	

The derivative of a function f at a number a in its domain is the limit

$$\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h},$$

we explore the above limit from the right ($h \rightarrow 0^+$) and from the left ($h \rightarrow 0^-$).

Activity 1: For $f(x) = -x^2 - x + 1$, and $a = -1.5$,

a) find $\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$ with h decreasing from $h=0.01$ to 0.001 with step size -0.001 .

b) find $\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$ with h increasing from $h=-0.01$ to -0.001 with step size 0.01 .

Solution:

a)

Step 1. (Open [limit from the right for Activity 1](#)) Notice that we have defined Y1 to be $-x^2 - x + 1$ and Y2 to be $\frac{Y1(-1.5+h) - Y1(-1.5)}{h}$. We use the following table value before we explore the table for Y2.

Table Settings	
X	
Start:	0.01
End :	1E-03
Step :	-1E-03

Step 2. With Y2 being selected, we show the table for Y2 as follows:

Y2=(Y1(-1.5+X)-Y1(-1.5))	
X	Y
4E-3	1.996
3E-3	1.997
2E-3	1.998
1E-3	1.999
1.999	
FORM DEL ROW EDIT G-COM G-PLT	

We notice Y2 is close to 2.

b)

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Step 1. (Open [limit from the left for Activity 1](#)) Notice that we have defined Y1 to be $-x^2 - x + 1$ and Y2 to be $\frac{Y1(-1.5+h) - Y1(-1.5)}{h}$. We use the following table value before we explore the table for Y2.

Table Settings	
X	
Start:	-0.01
End :	-1E-03
Step :	1E-03

Step 2. With Y2 being selected, we show the table for Y2 as follows:

Y2=(Y1(-1.5+X)-Y1(-1.5))/X	
X	Y2
-4E-3	2.004
-3E-3	2.003
-2E-3	2.002
-1E-3	2.001

2.001

FORM DEL ROW EDIT G-COM G-PLT

We notice Y2 is close to 2.

Step 3. We check the answer numerically by opening '[check \(RUN editor\)](#)'. The answer is shown below:

$\frac{d}{dx}(-x^2 - x + 1) _{x=-1.5}$	
	2

MAT MODE ABS DRG CLR

We recall the derivative of a function f at a number a in its domain is the limit

$$\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h},$$

the derivative can be illustrated graphically by drawing a family of secant lines through the point $(a, f(a))$ with slopes

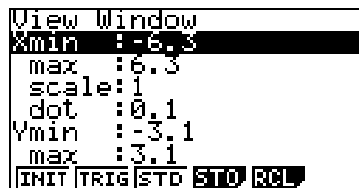
$$\frac{f(a+h) - f(a)}{h}$$

for a variety of values of h that are taken closer and closer to 0. For each chosen value of h , the equation of the line that joins the points $(a, f(a))$ and $(a+h, f(a+h))$ is

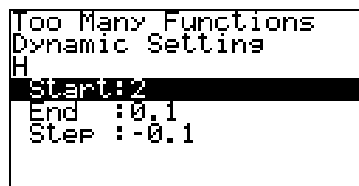
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$$y = f(a) + \frac{f(a+h) - f(a)}{h}(x-a) \quad \text{Equation (1)}$$

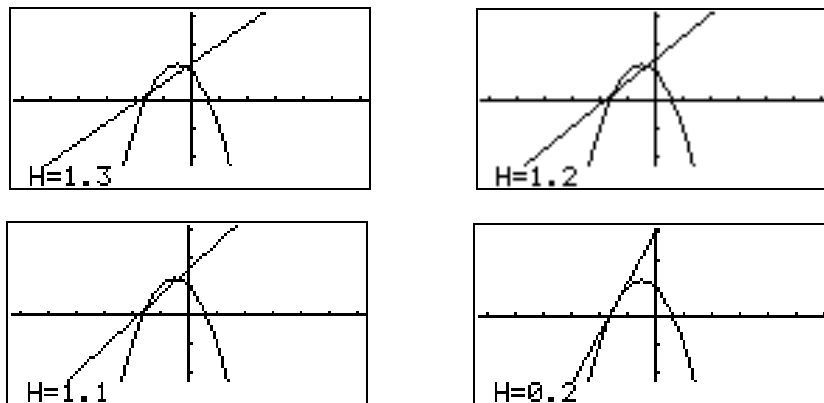
Activity 2: Explore the secant lines with h decreases from 2 to 0.1 and step size -0.1.
 Solution: Open 'animation' under Activity 2. We see that $Y1=f(x)$ and $Y2$ is the tangent line equation with various h 's. We proceed as follows:
 Step 1. We set the View window to be 'Standard':



Step 2. We set the variable H (in Y2) to be as follows:



Step 3. We execute the animation and 'conjecture' that the secant lines will get toward the tangent line when H gets smaller, which we show in the following screen shots below:



Therefore, we conjecture that the

$$\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

is the slope of the tangent line for the function f at $x=a$.

We use the tangent line at $(a, f(a))$ as an approximation to the curve $y=f(x)$ when x is

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near a . An equation of this tangent line is

$$y = f(a) + f'(a)(x - a) \text{ Equation 2.}$$

And the approximation $f(x) = f(a) + f'(a)(x - a)$ is called the **linear approximation** or **tangent line approximation** of f at a .

Activity 3: Use a linear approximation to estimate the values $2^{0.2}$ and $2^{0.4}$.

Solution: Consider Equation 2 above, we see

$$2^{0.2} = 2^0 + f'(0)(0.2 - 0) \text{ and } 2^{0.4} = 2^0 + f'(0)(0.4 - 0).$$

We open linear approx. of Activity 3, and we see the approximations of $2^{0.2}$ and $2^{0.4}$ to be around 1.138629 and 1.2772589, which we show the screen shots respectively below.

Calculator screen showing the derivative of 2^x at $x=0$ and the linear approximation for $2^{0.2}$.

```

d/dx (2^X) | x=0 +M
0.6931471806
2^0 + M x (0.2 - 0)
1.138629436
  
```

Calculator screen showing the derivative of 2^x at $x=0$ and the linear approximations for $2^{0.2}$ and $2^{0.4}$.

```

0.6931471806
2^0 + M x (0.2 - 0)
1.138629436
2^0 + M x (0.4 - 0)
1.277258872
  
```

We note that the approximation of $2^{0.2}$ should be better than that of $2^{0.4}$ because 0.2 is closer to 0 than 0.4 is. We show the true values of $2^{0.2}$ and $2^{0.4}$ below. (Open true values)

Calculator screen showing the true values of $2^{0.2}$ and $2^{0.4}$.

```

2^0.2
1.148698355
2^0.4
1.319507911
  
```

EXERCISES

Exercise 1. For $f(x) = \sin x$, and $a = 0$,

- (i) find $\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$ with h decreasing from $h=0.01$ to 0.001 with step size -0.001 .
- (ii) find $\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$ with h increasing from $h=-0.01$ to -0.001 with step size 0.01 .

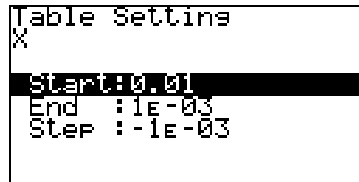
Exercise 2. Use a linear approximation to estimate the values $\sqrt{0.99}$, $\sqrt{1.01}$ and $\sqrt{1.05}$.

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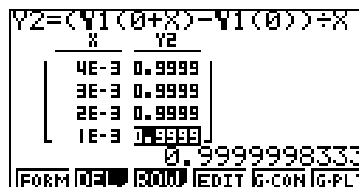
SOLUTIONS

Exercise 1.

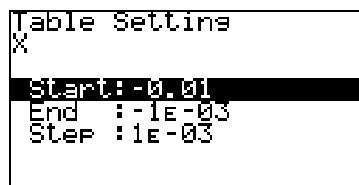
(i) Open Ex 1. (i), we set the values for h as follows:



And we see the table values for $\frac{f(a+h)-f(a)}{h}$ to be close to 1 as shown below:



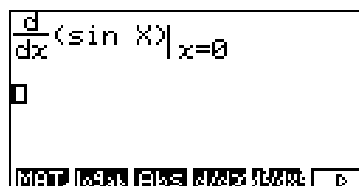
(ii) We open Ex. 1 (ii) and set the values of h to be as follows:



And we see the table values for $\frac{f(a+h)-f(a)}{h}$ is close to 1 as shown below:



We conjecture that the $\lim_{h \rightarrow 0} \frac{f(0+h)-f(0)}{h}$ is to be 1. We open 'Ex1-check' to verify our answer:



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Exercise 2. We use $f(x) = f(a) + f'(a)(x-a)$ and $f(x) = \sqrt{x}$ with $a = 1$. We open Ex2. We show the approximations for $\sqrt{0.99}$ and $\sqrt{1.01}$ respectively as follows:

Calculator screen showing the approximation of $\sqrt{0.99}$ using the derivative of \sqrt{x} at $x=1$. The screen displays the formula $\sqrt{1 + \frac{d}{dx}(\sqrt{x})|_{x=1} \times (0.99-1)}$ and the result 0.995. The bottom of the screen shows the calculator mode as **MAT** and the number of digits as **103%**.

Calculator screen showing the approximation of $\sqrt{1.01}$ using the derivative of \sqrt{x} at $x=1$. The screen displays the formula $\sqrt{1 + \frac{d}{dx}(\sqrt{x})|_{x=1} \times (1.01-1)}$ and the result 1.005. The bottom of the screen shows the calculator mode as **MAT** and the number of digits as **103%**.

We show the approximation for $\sqrt{1.05}$ as follows:

Calculator screen showing the approximation of $\sqrt{1.05}$ using the derivative of \sqrt{x} at $x=1$. The screen displays the formula $1 + \frac{d}{dx}(\sqrt{x})|_{x=1} \times (1.01-1)$ with result 1.005, and the formula $1 + \frac{d}{dx}(\sqrt{x})|_{x=1} \times (1.05-1)$ with result 1.025. The bottom of the screen shows the calculator mode as **MAT** and the number of digits as **103%**.

We open Ex 2.-check to check the true values for these three numbers, which we show below:

Calculator screen showing the true values of $\sqrt{0.99}$, $\sqrt{1.01}$, and $\sqrt{1.05}$. The screen displays the results: $\sqrt{0.99} = 0.9949874371$, $\sqrt{1.01} = 1.004987562$, and $\sqrt{1.05} = 1.024695077$. The bottom of the screen shows the calculator mode as **MAT** and the number of digits as **103%**.

REFERENCE

[S] James Stewart, *Calculus-Concepts and Contexts*, Brooks/Cole, Thompson Learning, 2001, ISBN 0-534-37718-1.

[YL] W.-C. Yang and J. Lewin, *Exploring Mathematics with Scientific Notebook*, Springer Verlag, 1998. ISBN 981-3083-88-3.