Derivative at a point

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LEVEL
High school or university students who have studied the pre-Calculus subjects such as polynomial and trigonometric functions.

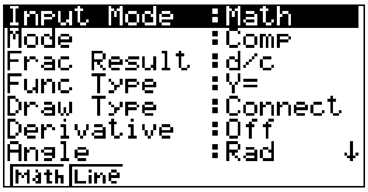
OBJECTIVES
To explore the use of calculator in understanding the slope of a tangent line at one point or derivative at one point and its applications.

Corresponding eActivity
C0101.g1e

OVERVIEW
Many students know how to compute the derivative of a function at one point by memorizing a proper formula but lack the understanding of the concept. We will use CASIO 9860 model to explore this concept from various angles.
(i) We will study the concept of derivative at one point numerically and also with an aid of animations.
(ii) We will use the concept of the derivative at a point to approximate numbers numerically.

EXPLORATORY ACTIVITIES
[Note]
(a) We shall use small letter *x* instead of capital X as shown on the calculator throughout the paper.
(b) Unless otherwise specified, we choose MATH mode in the SET UP menu. Go to RUN mode in the Menu. Then press [SHIFT] SET UP and choose **Math** in the **Input Mode**, which we show below:

Derivative at a point



The derivative of a function *f* at a number *a* in its domain is the limit

$$\lim_{h \to 0} \frac{f(a+h)-f(a)}{h},$$

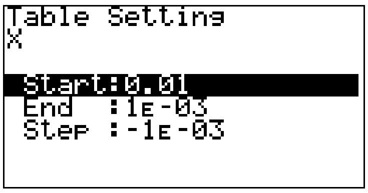
we explore the above limit from the right ($h \to 0^+$) and from the left ($h \to 0^-$).

Activity 1: For $f(x) = -x^2 - x + 1$, and $a = -1.5$,

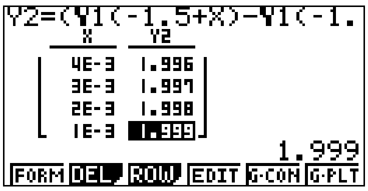
a) find $\lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$ with *h* decreasing from h=0.01 to 0.001 with step size -0.001.

b) find $\lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$ with *h* increasing from h=-0.01 to -0.001 with step size 0.01.

Solution:
a)
Step 1. (Open limit from the right for Activity 1) Notice that we have defined Y1 to be $-x^2 - x + 1$ and Y2 to be $\frac{Y1(-1.5+h)-Y1(-1.5)}{h}$. We use the following table value before we explore the table for Y2.



Step 2. With Y2 being selected, we show the table for Y2 as follows:



We notice Y2 is close to 2.

b)

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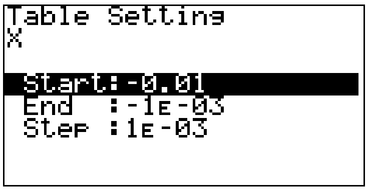
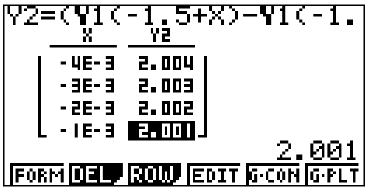
Step 1. (Open limit from the left for Activity 1) Notice that we have defined Y1 to be $-x^2 - x + 1$ and Y2 to be $\dfrac{Y1(-1.5+h) - Y1(-1.5)}{h}$. We use the following table value before we explore the table for Y2.

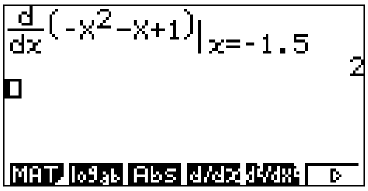


Step 2. With Y2 being selected, we show the table for Y2 as follows:



We notice Y2 is close to 2.

Step 3. We check the answer numerically by opening 'check (RUN editor)'. The answer is shown below:



We recall the derivative of a function *f* at a number *a* in its domain is the limit

$$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h},$$

the derivative can be illustrated graphically by drawing a family of secant lines through the point (*a,f(a)*) with slopes

$$\frac{f(a+h) - f(a)}{h}$$

for a variety of values of h that are taken closer and closer to 0. For each chosen value of *h*, the equation of the line that joins the points (*a,f(a)*) and (*a*+*h*,*f(a*+*h*)) is

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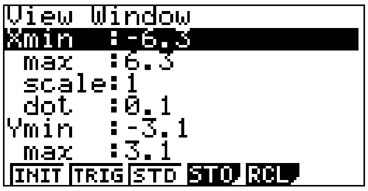
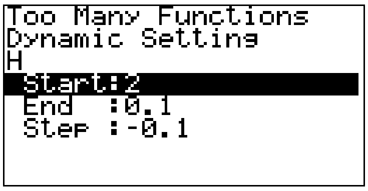
$$y = f(a) + \frac{f(a+h) - f(a)}{h}(x-a) \quad \text{Equation (1)}$$

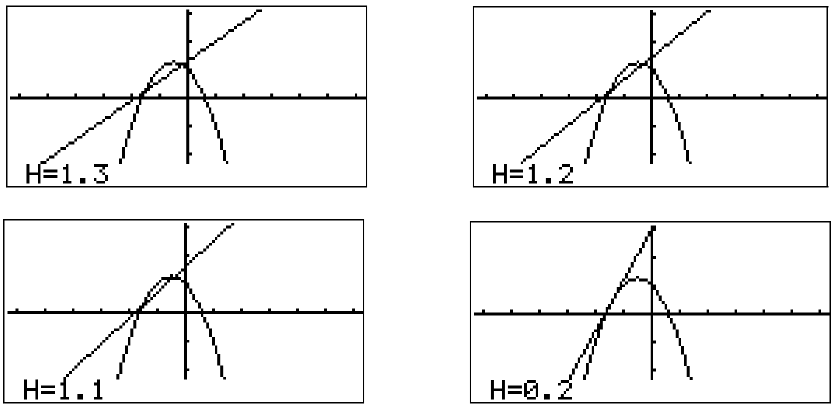
Activity 2: Explore the secant lines with h decreases from 2 to 0.1 and step size -0.1.
Solution: Open '**animation**' under Activity 2. We see that Y1=f(x) and Y2 is the tangent line equation with various *h*'s. We proceed as follows:
Step 1. We set the View window to be 'Standard':



Step 2. We set the variable H (in Y2) to be as follows:



Step 3. We execute the animation and 'conjecture' that the secant lines will get toward the tangent line when H gets smaller, which we show in the following screen shots below:



Therefore, we conjecture that the

$$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

is the slope of the tangent line for the function *f* at *x=a*.

We use the tangent line at (*a, f(a)*) as an approximation to the curve *y=f(x)* when x is near a. An equation of this tangent line is

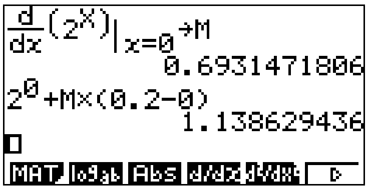
$$y = f(a) + f'(a)(x - a) \text{ Equation 2.}$$

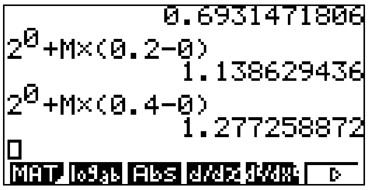
And the approximation $f(x) = f(a) + f'(a)(x - a)$ is called the **linear approximation** or **tangent line approximation** of f at a.

Activity 3: Use a linear approximation to estimate the values $2^{0.2}$ and $2^{0.4}$.
Solution: Consider Equation 2 above, we see

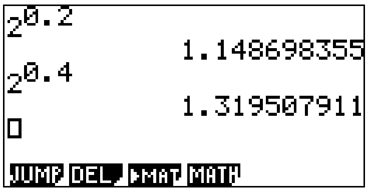
$$2^{0.2} = 2^0 + f'(0)(0.2 - 0) \text{ and } 2^{0.4} = 2^0 + f'(0)(0.4 - 0).$$

We open linear approx. of Activity 3, and we see the approximations of $2^{0.2}$ and $2^{0.4}$ to be around 1.138629 and 1.2772589, which we show the screen shots respectively below.





We note that the approximation of $2^{0.2}$ should be better than that of $2^{0.4}$ because 0.2 is closer to 0 than 0.4 is. We show the true values of $2^{0.2}$ and $2^{0.4}$ below. (Open true values)



EXERCISES

Exercise 1. For $f(x) = \sin x$, and $a = 0$,

(i) find $\lim_{h\to 0} \dfrac{f(a+h) - f(a)}{h}$ with *h* decreasing from *h*=0.01 to 0.001 with step size -0.001.

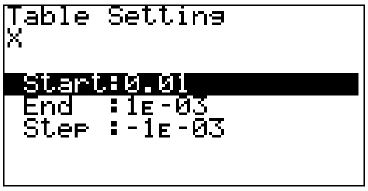
(ii) find $\lim_{h\to 0} \dfrac{f(a+h) - f(a)}{h}$ with *h* increasing from *h*=-0.01 to -0.001 with step size 0.01.

Exercise 2. Use a linear approximation to estimate the values $\sqrt{0.99}$, $\sqrt{1.01}$ and $\sqrt{1.05}$.

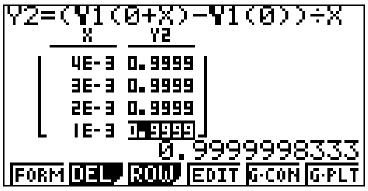
SOLUTIONS

Exercise 1.

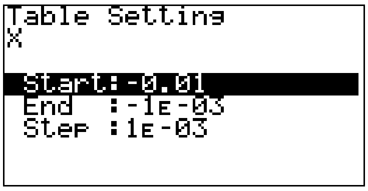
(i) Open Ex 1. (i), we set the values for h as follows:



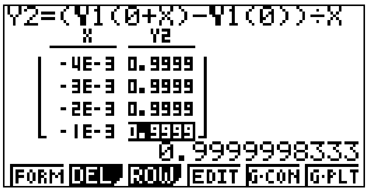
And we see the table values for $\dfrac{f(a+h)-f(a)}{h}$ to be close to 1 as shown below:



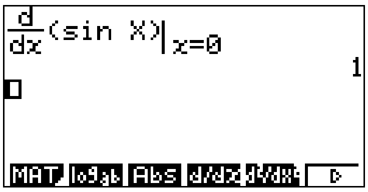
(ii) We open Ex. 1 (ii) and set the values of h to be as follows:



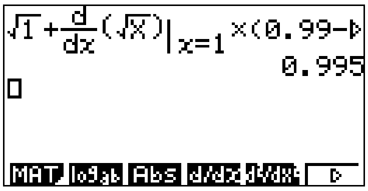
And we see the table values for $\dfrac{f(a+h)-f(a)}{h}$ is close to 1 as shown below:

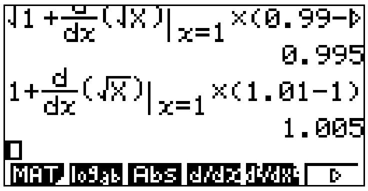


We conjecture that the $\lim_{h\to 0}\dfrac{f(0+h)-f(0)}{h}$ is to be 1. We open 'Ex1-check' to verify our answer:

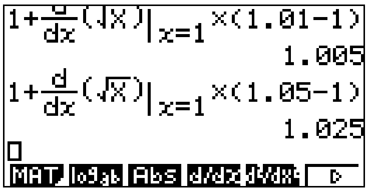


Exercise 2. We use $f(x) = f(a) + f'(a)(x-a)$ and $f(x) = \sqrt{x}$ with $a = 1$. We open Ex2. We show the approximations for $\sqrt{0.99}$ and $\sqrt{1.01}$ respectively as follows:

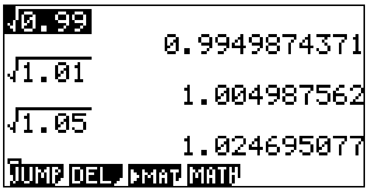




We show the approximation for $\sqrt{1.05}$ as follows:



We open Ex 2.-check to check the true values for these three numbers, which we show below:



REFERENCE

[S] James Stewart, *Calculus-Concepts and Contexts*, Brooks/Cole, Thompson Learning, 2001, ISBN 0-534-37718-1.
[YL] W.-C. Yang and J. Lewin, *Exploring Mathematics with Scientific* Notebook, Springer Verlag, 1998. ISBN 981-3083-88-3.

