**Unit**

**14**

**INTEGRATION**

# KEY CONCEPTS

**TEACHER’S GUIDE**

**Definition of Integration**

[F(𝑥)] = f(𝑥), then = F(𝑥) + *c*, where *c* is an arbitrary constant.

# Properties of Indefinite Integrals

=

= a

# Integration of power functions

= + c, where n ≠ -1

= + c

# Definite Integrals

If = f(x), then = F(b) – F(a).

# Integration of trigonometric functions

= sin x + c

= – cos x + c

= tan x + c

= sin(ax + b) + c

= – cos(ax + b) + c

= tan(ax + b) + c

# Integration of Exponential functions

= + c

= + c

# Integration of and functions

= ln x + c, where x > 0

= ln (*ax* + *b*) + *c*, where *x* > 0

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# WORKED EXAMPLES

**TEACHER’S GUIDE**

**Level 1**

### Worked Example 1

Find (4*x* – 3) d*x*. Evaluate (4𝑥 – 3) dx.

Solution

(4*x* – 3 *x*2) d*x* = – d*x*

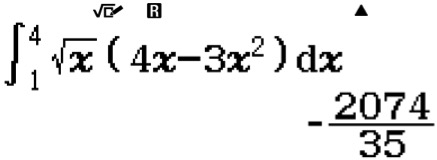
= – + *c*

= – + *c*

(4*x* – 3*x*2) d*x* =

= – – –

= –



**ClassWiz steps**

Press y. Key in (4*x* – 3*x*2) d*x* and press

=. The calculator will display – .

### Worked Example 2

Find d*x*.

Solution

d*x* = – + d*x*

= – + + *c*

= – + + *c*

### Worked Example 3

Find d*x*.

Solution

d*x* = d*x*

= – + *c*

= + *c*

Unit 14 **Integration**

### Worked Example 4

**TEACHER’S GUIDE**

Find tan2 (2*x* – 1) d*x*.

Solution

tan2 (2*x* – 1) d*x* = [sec2 (2*x* – 1) – 1] d*x*

= sec2 (2*x* – 1) + 4 d*x*

= – + 4*x* + *c*

= – 2 tan (2*x* – 1) + 4*x* + *c*

### Worked Example 5

Find (3*x*) d*x*. Evaluate (3*x*) d*x*.

Solution

From double angle formula, cos 2A = 1 – 2sin2A.

With some manipulation, 1 – cos 2A = 2 sin2A.

sin2 A =

(3*x*) d*x* = d*x*

= – cos 6*x* d*x*

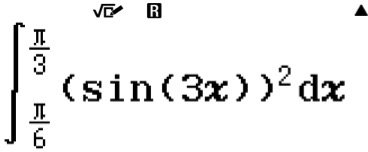
= *x* – + *c*

= – sin 6*x* + *c*

The calculator will display .

Press y. Key in (3*x*) d*x* and press =.

**ClassWiz steps**

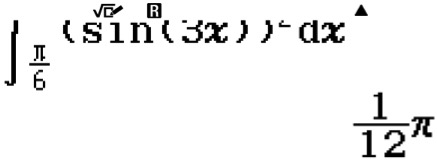


(3*x*) d*x* =

= –

= –

=



**ClassWiz steps**

Do note that for sin2 (3*x*), you have to key it as [sin 3*x*]2. The same applies for other trigonometric function with powers.

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### Worked Example 6

**TEACHER’S GUIDE**

Find + 2 d*x*. Evaluate + 2 d*x.*

Solution

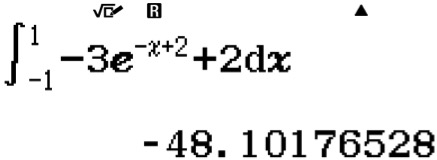
+ 2 d*x* = + 2*x* + *c*

= + 2*x* + *c*

+ 2d*x* =

= –

= –48.1 (3 s.f.)



Press y. Key in + 2 d*x* and press

=. The calculator will display –48.10176528.

**ClassWiz steps**

### Worked Example 7

Find d*x*. Evaluate d*x*.

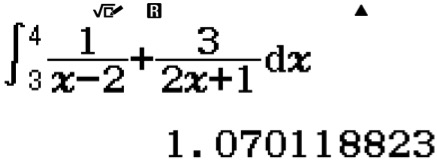
Solution

= ln (*x* – 2) + ln (2*x* + 1) + *c*

d*x* =

= –

= 1.07 (3 s.f.)



**ClassWiz steps**

Classwiz steps: Press y.

Key in d*x* and press =.

The calculator will display 1.070118823.

Unit 14 **Integration**

**TEACHER’S GUIDE**

# Level 2

### Worked Example 8

Find the area of the shaded region for the given intervals, bounded by the curve *y* = + 1 and the

x-axis in the diagram.

*y*

*y* =

–3

2

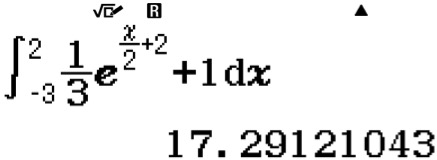
*x*

Solution

+ 1 d *x* =

= – []

= 17.3 units2



**ClassWiz steps**

Press y. Key in + 1 d*x* and press

=. The calculator will display 17.29121043.

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### Worked Example 9

**TEACHER’S GUIDE**

In the diagram, the area of the shaded region is bounded by the curve *y* = *x*2 + 3*x* + 3 and the line *y* = *x* + 11 from point *A* to point *B*.

*y*

*y = x* +11

A

*x*

B

*y = x*2 +3*x* +3

1. Find the coordinates of points A and B.
2. Find the shaded region.

Solution

1. *x*2 + 3*x* + 3 = *x* + 11

*x*2 + 2*x* – 8 = 0 (*x* – 2)(*x* + 4) = 0

*x* = 2 or –4

*y* = 13 or 7

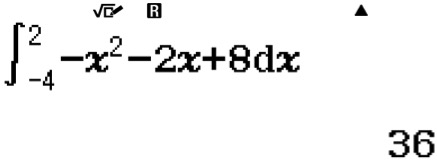
A (–4, 7) and B (2,13)

1. + 11 – (*x*2 + 3*x* + 3) d*x* = – 2*x* + 8 d*x*

=

= –

= 36 units2



Press y. Key in – 2*x* + 8 d*x* and press

=. The calculator will display 36.

**ClassWiz steps**

Unit 14 **Integration**

# CLASSWIZ WORKSHEETS

**CLASSWIZ WORKSHEETS**

**Level 1**

1. Find each of the following

|  |  |
| --- | --- |
| (a) ∫ d*x* | (b) ∫ – 24 d*x* |
| (c) ∫ 5 + 3 d*x* | (d) ∫ –2+ 3*x* – 4 d*x* |
| (e) ∫ – + 2*x* d*x* | (f) ∫ + d*x* |
| (g) ∫ (2*x* – 9)(*x* + 1) d*x* | (h) ∫ (3*x* – 4)2 d*x* |
| (i) ∫ (+ 5) d*x* | (j) ∫ (– 2)(3 – 2) d*x* |
| (k) ∫ + 1 d*x* | (l) ∫ + d*x* |
| (m) ∫ d*x* | (n) ∫ d*x* |

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1. Find each of the following

**CLASSWIZ WORKSHEETS**

|  |  |
| --- | --- |
| (a) ∫ (*x* + 1)2 d*x* | (b) ∫ (3 – 2*x*) 4 d*x* |
| (c) ∫ (1 – 3*x*)–2 d*x* | (d) ∫ d*x* |
| (e) ∫ d*x* | (f) ∫ d*x* |
| (g) ∫ d*x* | (h) ∫ d*x* |
| (i) ∫ d*x* | (j) ∫ d*x* |
| (k) ∫ d*x* | (l) ∫ d*x* |
| (m) ∫ d*x* | (n) ∫ d*x* |

Unit 14 **Integration**

1. Find each of the following

**CLASSWIZ WORKSHEETS**

|  |  |
| --- | --- |
| (a) ∫ sin *x* + 1 d*x* | (b) ∫ 3 cos *x* + 2 d*x* |
| (c) ∫ – 4 sec2 *x* d*x* | (d) ∫ 3 sin *x* – d*x* |
| (e) ∫ – cos (3*x* – 1) d*x* | (f ) ∫ 3 sin (*x* + 5) d*x* |
| (g) ∫ 6 sec2 (2 + 3*x*) d*x* | (h) ∫ 5 – sin *x* d*x* |
| (i) ∫ cos (2*x* – ) + 5 d*x* | (j) ∫ d*x* |
| (k) ∫ d*x* | (l) ∫ tan2 *x* d*x* |
| (m) ∫2 – 3 tan2 (4*x* + 3) d*x* | (n) ∫ 2 sin2 (5*x* – 2) d*x* |
| (o) ∫ 3 – 4 cos2 () d*x* | Teachers to note that trigonometric functions with powers cannot be integrated directly. The use of trigonometric identities is required. |

**Casio ClassWiz Mathematics Workbook**

1. Find each of the following

**CLASSWIZ WORKSHEETS**

|  |  |  |
| --- | --- | --- |
| (a) ∫ + 2 d*x* | | (b) ∫– 3 d*x* |
| (c) | ∫ 2e–4*x* + 1 + 1 d*x* | (d) ∫ d*x* |
| (e) | ∫ – + 3 d*x* | (f) ∫ 3*x* – dx |

1. Find each of the following

|  |  |
| --- | --- |
| (a) ∫ – + 1 d*x* | (b) ∫ 2 – d*x* |
| (c) ∫ d*x* | (d) ∫ – d*x* |
| (e) ∫ + d*x* | (f) ∫ – + d*x* |

Unit 14 **Integration**

# Level 2

**CLASSWIZ WORKSHEETS**

1. Evaluate each of the following. You may use your ClassWiz calculator.

**ClassWiz set-up**

Press y and key in the function and the given intervals.

|  |  |
| --- | --- |
| (a) + 3*x* – 4 d*x* |  |
| (b) + 2*x* d*x* | (c) (+ 5) d*x* |
| (d) d*x* | (e) d*x* |
| (f) d*x* | (g) d*x* |
| (h) (3*x* – 1) d*x* | (i) – sin *x* d*x* |
| (j) cos (2*x* – ) + 5 d*x* | (k) – 4 (– ) d*x* |
| (l) + 1 d*x* | (m) d*x* |
| (n) – d*x* | (0) + d*x* |

**Casio ClassWiz Mathematics Workbook**

1. Find the area of the shaded region for the given intervals, bounded by the curve and the x-axis in the following diagrams.

**CLASSWIZ WORKSHEETS**

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| (a) *y* =  *y*  *y* =  *x*  3  - 5 | | | | (b) *y* = 2 cos (3*x* + 2)  *y* = 2 *cos* (3*x*+2)  *x*  *y*  -  - | | |
| (c) *y* = 3*e*2*x* + 6 | |  |  | (d) *y* =  *y* =  *y*  *x*    3 |  |  |
| *y =* 3*e*2*x* +6  *y*  *x*  -5  -3 | |  |  |
|  | |  |  |
|  | |  |  |
|  |  |  |  |  |  |  |

Unit 14 **Integration**

1. Find the area of the shaded region for the given intervals, bounded by the curve and the x-axis in the following diagrams.

Note that the shaded region is below the x-axis, hence the result will be a negative area. To obtain a positive area, we need to use modulus function on the integrals.

**CLASSWIZ WORKSHEETS**

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| (a) *y* = –  *y*  0 | 10  *y* = | *x* | (b) *y* = – sin2 (4*x* – 3) – 1  *y* = – sin2 (4*x –* 3) – 1  *y*  *x*  2 | | – 1 | |
| (c) *y* = – |  |  | (d) *y* =  *y* =  *y*  *x*  -4  -2 |  |  |  |
|  |  |  |  |  |  |
| *y* = –  *y*  *x*  3 |  |  |  |  |  |
|  |  |  |  |  |  |
|  |  |  |  |  |  |

**Casio ClassWiz Mathematics Workbook**

1. Find the area of the shaded region for the given intervals, bounded by the curve and a line in the following diagrams. [[Note that the shaded region is below the curve and above the line. To obtain a positive area, we need to subtract the integral of the line from the integral of the curve.]]

**CLASSWIZ WORKSHEETS**

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| 1. *y* =   2*y* = 3*x* – 9  *y*  *x*  1  4 |  | 2*y* = 3*x* – 9  *y* = | 1. *y* = 2 sin *πx*   5*y* = 6(1 – *x*)  1    *y*  *x* | *y* = | 2 sin *πx*  5*y* =6(1 – *x* | ) |
| 1. *y* = + 1 *y* = (x + 1)   *y*  *x*  1  -1  *y* = + 1 |  | *y* = (*x* + 1) | 1. *y* =   12*y* = 5(*x* – 1)  12*y* =5(*x* –1)  1  *x*  *y*  *y* =  4 |  | | |

Unit 14 **Integration**

1. Find the area of the shaded region for the given intervals, bounded by the curve and a line in the following diagrams.

Note that the shaded region is below the line and above the curve. To obtain a positive area, we need to subtract the integral of the curve from the integral of the line.

**CLASSWIZ WORKSHEETS**

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
| (a) *y* = (2*x* – 5)2  *y* = 4*x* – 7  *y*  2 | 4 | *y* =(2*x –* 5)2  *y* =4*x –* 7  *x* | 1. *y* = 2 cos (3*x* + π)   *y* = *x* + 1  *y*  *x*  *y* = 2 cos (3*x* + π)  *y* = *x* + 1  1  0 | | | | |
| (c) *y* = –2*e*3*x* + 1  *y* = *x* + 2 – |  | *x*  0 | (d) *y* = –  *y* = 2*x* + 3 |  |  |  |  |
| *y* = *x* +(2 – )  *y* =–2*e*3x +1 | *y* |  |  | *y* | *y* = *2x* +3 |  |
|  |  |  |  | 3 |  |  |
|  |  | *y* = |  |  |  |  |
|  |  |  |  |  |  |  |
| -1 |  |  | –1 | 0 |  | *x* |

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# Level 3

**CLASSWIZ WORKSHEETS**

1. Given that the gradient function of a curve is – sin (3*x* – 2) – 2 and the curve passes through the

point (, ), find the equation of the curve.

1. Given that the gradient function of a curve is 9 – and the tangent to the curve at *x* = 1 is given as

*y* + 5*x* = 19, find the equation of the curve.

1. A curve for which = 4*x* – 5 passes through (0, 9) and (1, ). Find the equation of the curve.
2. A curve for which = 2*x* – 4 has a maximum point at (–1, 6).
   1. Find the equation of the curve.
   2. Find the coordinates of the other turning point.

Unit 14 **Integration**

1. The rate of change of the radius, *r*, of a circle is given by = 5*t*2 – 7 and its initial radius is 6 units. Find

**CLASSWIZ WORKSHEETS**

the equation of *r* in terms of *t*.

6. (a) Find (5*x* ).

(b) Hence, find d*x*.

7. (a) Find [ ln (1 – *x*)].

(b) Hence, find d*x*.

**Casio ClassWiz Mathematics Workbook**

1. The diagram shows part of the curve *y* = (1 – *x*) (2*x* – 3) (*x* + 4) and a line *y* + *x* + 4 = 0.

**CLASSWIZ WORKSHEETS**

*y*

2

*–*4

*x*

*y* + *x* +4 *=* 0

*y =* (1 – *x*)(2*x* – 3)(*x* +4)

* 1. Find the coordinates of the points of intersection.
  2. Hence, find the shaded region.

Unit 14 **Integration**

1. The diagram shows part of the curve *y* = and a line *y* = *x* + 1 + . Find the shaded region.

**CLASSWIZ WORKSHEETS**

*y*

*y* = *x* + 1 +

*y* =

*x*

-1

1. A particle travels in a straight line so that, *t* seconds after passing a fixed point *O*, its velocity, *v* m/s, is

given by *v* = 16*t* – 7.

* 1. Find an expression for the displacement, *s* m, of the particle at any time *t* seconds.
  2. Find the displacement when *t* = 2.

**Casio ClassWiz Mathematics Workbook**

# Level 4

**CLASSWIZ WORKSHEETS**

1. Express as partial fractions. Hence, evaluate d*x* without the use of a

calculator.

1. Express as partial fractions. Hence, evaluate d*x* without the use of a

calculator.

Unit 14 **Integration**

Teachers to note that for these two questions, integration against the y-axis is required.

**CLASSWIZ WORKSHEETS**

1. The diagram shows the curve *y* = . Find the area of the shaded region.

*y*

*y* =

*x*

1. The diagram shows the curve *x* = 5y + 6 – *y*2 and the line 5*y* + 2*x* = 25.

*x*

*y*

*x* = 5*y* + 6 - *y*2

A

B

5*y* + 2*x* = 25

* 1. Find the coordinates of A and of B.
  2. Find the area of the shaded region.

**Casio ClassWiz Mathematics Workbook**

1. A particle *Q* travels in a straight line through a fixed point *O*. The distance, *s* metres, from *O*, is given by

**CLASSWIZ WORKSHEETS**

*s* = *t*3 – 8*t*2 + 5*t* – 9, where *t* is the time, in seconds, measured from the start of the motion of particle *Q*.

* + - 1. Calculate the value(s) of *t* when the particle is instantaneously at rest.
      2. Calculate the time when the acceleration is 2 m/s2.
      3. Find the average speed of *Q* during the first two seconds.
      4. Find the distance travelled in the first 6 seconds.

Unit 14 **Integration**