**Unit**

**13**

**DIFFERENTIATION**

# KEY CONCEPTS

**TEACHER’S GUIDE**

**Power rule**

If *y* = 𝑥*n* where *n* is a rational number, then = *n*𝑥*n* – 1.

# Constant multiple rule

If f(𝑥) is a function and *k* is a constant, then [*k f*(𝑥)] = *k* ∙ [f(𝑥)]

# Sum and difference rule

1. (*u* + *v*) = +
2. (*u* – *v*) = –

# Chain rule

= ∙

# Product rule

 (*uv*) = *u* – *v* , if both *u* and *v* are functions of 𝑥.

# Quotient rule

() =

# Increasing and decreasing functions

1. If > 0 for all 𝑥in (*a*,*b*), it means that *y* = f(𝑥) is increasing in (*a*,*b*).
2. If < 0 for all 𝑥in (*a*,*b*), it means that *y* = f(𝑥) is decreasing in (*a*,*b*).

# Connected Rates of change

= ∙

Chain Rule

# Nature of stationary points

1. There are 3 types of stationary points: maximum point, minimum point and point of inflexion.
2. Given a curve *y* = *f*(𝑥) and a stationary point *P* at 𝑥= *a*,
	1. If changes from negative to positive as 𝑥increases through *a*, then point *P* is a minimum point.
	2. If changes from positive to negative as 𝑥increases through *a*, then point *P* is a maximum point.
	3. If does not change sign as 𝑥increases through *a*, then point *P* is a point of inflexion.
3. We may use either the first derivative test or second derivative test to find the nature of the stationary point.

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1. Definition of second derivative test

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* 1. If = 0 and < 0 at 𝑥= *a*, then point *P* is a maximum point.
	2. If = 0 and > 0 at 𝑥= *a*, then point *P* is a minimum point.

# Trigonometric functions

1. (sin 𝑥) = cos 𝑥
2. (cos 𝑥) = – sin 𝑥
3. (tan 𝑥) = sec2 𝑥
4. (sin (*a*𝑥+ *b*)) = *a* cos (*a*𝑥+ *b*)
5. (cos (*a*𝑥+ *b*)) = – *a* sin (*a*𝑥+ *b*)
6. (tan (*a*𝑥+ *b*)) = *a* sec2 (*a*𝑥+ *b*)
7. (sin*n* 𝑥) = *n* sin*n* – 1 𝑥cos 𝑥
8. (cos*n* 𝑥) = – *n* cos*n* – 1 𝑥sin 𝑥
9. (tan*n* 𝑥) = *n* tan*n* – 1 𝑥sec2 𝑥

# Exponential functions

1. (*e*𝑥) = *e*𝑥
2. (*eu*) = *eu*
3. (*ea*𝑥+ *b*) = *aea*𝑥+ *b*
4. (*e*f(𝑥)) = f’(𝑥)*e*f(𝑥)

# Logarithmic functions

1. (ln 𝑥) =

2. (ln *u*) =

3. [ln (*a*𝑥+ *b*)] =

4. [ln f(𝑥)] =

Unit 13 **Differentiation**

# WORKED EXAMPLES

**TEACHER’S GUIDE**

**Level 1**

### Worked Example 1

Differentiate the following with respect to 𝑥. Hence, find the gradient of the curve at the given *x*-value.

(a) – 5 + 3 , 𝑥= 4

(b) , 𝑥= 16

 (c) (+ 1)(5 – 2𝑥), 𝑥= 1

Solution

Press z5[^5a2$$

+ 3[^3a2$$$4

= and the calculator will display –91.

 **ClassWiz steps**

Press qy to get Y.

(a)

 (– 5 + 3) = – +

When 𝑥= 4, gradient = – +

= – 91

(b) ()

|  |  |
| --- | --- |
|  | **ClassWiz steps** |
|  |
|  | Press qy to get Y. |
|  | Press a3[d + |
|  | 5[p1R2s[$$$16 |
|   | = and the calculator will display 9.31640625. |
|  |  |

= ( + – )

= + +

When 𝑥= 16, gradient = + +

 = 9

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(c) [(+ 1)(5 – 2𝑥)]

**TEACHER’S GUIDE**

|  |  |
| --- | --- |
|  |  **ClassWiz steps**  |
|  |
|  | Press qy to get Y. Press |
|  | (s[$ + 1) |
|  | (5p2[)$1 = and |
|  | the calculator will display –2.5. |
|   |  |
|  |  |
|  |  |

= (5– 2𝑥+ 5 – 2𝑥)

= (5 – 2 + 5 – 2𝑥)

= – 3 – 2

When 𝑥= 1, gradient = – 3 – 2

= – 2.5

### Worked Example 2

Using the Chain Rule, differentiate – . Hence, find the gradient of the tangent at 𝑥= 1.

Solution

 [–6(2 – 5𝑥)–3] = 18(2 – 5𝑥)–4(–5) = –90(2 – 5𝑥)–4

 **ClassWiz steps**

Press qy to get Y. Press z6a(2p5[) D$$1 = and the

When 𝑥 = 1, gradient of tangent = – 90(2 – 5(1))–4 = –

calculator will display – .

### Worked Example 3

Using the Product Rule, differentiate (𝑥2 + 1).

Solution

[(𝑥2 + 1)] = (𝑥2 + 1)[ (–1)] + [](2𝑥)

### Worked Example 4

= – +

Using the Quotient Rule, differentiate

Solution

 [] =

=

=

=

=

 Unit 13 **Differentiation**

### Worked Example 5

**TEACHER’S GUIDE**

Differentiate the following with respect to 𝑥. Hence, find the gradient of the curve at the given *x*-value.

1. 2 sin 𝑥, 𝑥=

(b) 5 tan (8𝑥– 9), 𝑥= 2

(c) 3 cos2 (2𝑥 – ), 𝑥 =

Solution

(a) (2 sin 𝑥) = 2 cos 𝑥

 **ClassWiz steps**

Press qy to get Y. Press

2j[)${a6

= and the calculator will display 1.732050808.

When 𝑥 = , gradient = 2 cos

=

(b) [5 tan (8𝑥– 9)] = 5[sec2 (8𝑥– 9)](8)

 **ClassWiz steps**

Press qy to get Y. Press

5l8[p9)$2

= and the calculator will display 70.37686346.

= 40 sec2 (8𝑥– 9)

When 𝑥= 2, gradient = 40 sec2 [8(2) – 9]

=

= 70.4 (3 s.f.)

(c) [3 cos2 (2𝑥 – )] = 6 [cos (2𝑥 – )] [–sin(2𝑥 – )] [2]

= –12 cos (2𝑥 – ) sin(2𝑥 – )

 When 𝑥= , gradient = –12 cos [2 () – ] sin[2 () – ]

 = 3

 **ClassWiz steps**

Press qy to get Y. Press 3(k2[p{a3$)) d${a12 = and the calculator will display 5.196152423.

### Worked Example 6

Differentiate 3 ln (2𝑥2 – 3) with respect to 𝑥. Find the gradient when 𝑥= 3.

Solution

 [3 ln (2𝑥2 – 3)] =

=

When 𝑥= 3, gradient =

 **ClassWiz steps**

Press qy to get Y. Press

3h2[dp3)$3

= and the calculator will display 2.4.

=

=

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### Worked Example 7

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Differentiate 3*e*–3𝑥 + 5 with respect to 𝑥. Find the gradient when 𝑥= 2.

Solution

 (3*e*–3𝑥 + 5) = –9*e*–3𝑥 + 5

When 𝑥= 2, gradient = –9*e*–3(2) + 5 = –

 **ClassWiz steps**

Press qy to get Y. Press 3Hz3[ + 5$$2 = and the

calculator will display –3.310914971.

# Level 2

### Worked Example 8

Differentiate the following with respect to 𝑥.

(a)

(b) (2 – 3𝑥) ln (𝑥– 7)

Solution

(a) =

 =

(b) (2 – 3𝑥) ln (𝑥– 7) = (2 – 3𝑥)()– 3 ln (𝑥– 7)

= – 3 ln (𝑥– 7)

Unit 13 **Differentiation**

# CLASSWIZ WORKSHEETS

**CLASSWIZ WORKSHEETS**

**Level 1**

1. Differentiate the following with respect to 𝑥.

|  |  |
| --- | --- |
| (a) 3𝑥2 | (b) – 5𝑥3 + 3𝑥+ 2 |
| (c)  | (d) 3𝑥+  |
| (e) 6𝑥2 + – 5 | (f ) + 5𝑥2 |
|  (g) 7 – 2 | (h)  |
| (i)  | (j) (𝑥+ 3)(5𝑥– 6) |
| (k) 𝑥(– 1) | (l) (𝑥– 3)2 |

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1. Using the Chain Rule, differentiate the following with respect to 𝑥.

**CLASSWIZ WORKSHEETS**

|  |  |
| --- | --- |
| (a) (𝑥+ 1)5 | (b) 3(𝑥– 7)4 |
| (c) – 5(2𝑥+ 5)6 | (d)  |
| (e)  | (f )  |

1. Using product rule, differentiate the following with respect to 𝑥.

|  |  |
| --- | --- |
| (a) 𝑥(3𝑥– 1)3 | (b) (2𝑥+ 3)(4 – 3𝑥)4 |
| (c) 4𝑥3 | (d) (3+ 1) |

Unit 13 **Differentiation**

1. Using quotient rule, differentiate the following with respect to 𝑥.

**CLASSWIZ WORKSHEETS**

|  |  |
| --- | --- |
| (a)  | (b)  |
|  (c)  | (d)  |

1. Differentiate the following with respect to 𝑥.

|  |  |  |
| --- | --- | --- |
| (a) 3 + 4 sin 𝑥 | (b) 7𝑥2 – 2 cos 𝑥 |  |
| (c) 5 tan 𝑥+ 1 | (d) 5 tan (𝑥+ 1) |
|  |
| (e) 𝑥2 tan 𝑥 | (f ) (𝑥+ 1) sin 𝑥 |
| (g)  | (h) (1 – 2 cos 𝑥)3 |
| (i)  | (j) tan 3𝑥 |
| (k) 5 sin (𝑥) | (l) 4 cos (𝑥– ) + 5𝑥 |
| (m) 3 cos2 𝑥 | (n) 4 tan5 𝑥+ 5𝑥 |
| (o) 3 sin3 3𝑥 | Teachers to note: Close confusers |

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1. Differentiate the following with respect to 𝑥.

**CLASSWIZ WORKSHEETS**

|  |  |
| --- | --- |
| (a) *e*𝑥+ 1 | (b) 5*e*𝑥 |
| (c) 2*e*–2𝑥+ *e*3𝑥+ 1 | (d)  |
| (e) 4 | (f )  |

1. Differentiate the following with respect to 𝑥.

(e) 2 ln

Teachers to note: Close confusers

(d) 7 ln (2– 5)2

(c) 3 ln (5𝑥2 – 4)

(b) ln (𝑥+ 3)

(a) ln 𝑥+ 3

Unit 13 **Differentiation**

# Level 2

**CLASSWIZ WORKSHEETS**

1. Differentiate the following with respect to 𝑥.

|  |  |
| --- | --- |
| (a) cos 𝑥 ln 𝑥 | (b) ln (2𝑥+ 1) ∙ *e*2𝑥 |
| (c) ln (sin 𝑥) | (d) *e*tan 𝑥 |
| (e) *e*4𝑥∙ ln (3𝑥– 1) | (f ) 　  |
| (g)  | (h)  |

1. Find the equation of the tangent to the curve *y* = at 𝑥= 1.
2. Find the equation of the normal to the curve *y* = tan (3𝑥 – ) at 𝑥 = .

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1. Find the equation of the normal to the curve *y* = (2𝑥– 3)*e*3𝑥at 𝑥= –1.

**CLASSWIZ WORKSHEETS**

1. Find the equation of the normal to the curve *y* = , for 𝑥≠ 5, at the point where the curve crosses

the *x*-axis.

1. Find the equation of the tangent to the curve *y* = 2𝑥2 – 3𝑥+ 4, which is parallel to *y* – 𝑥+ 2 = 0.
2. Find the equation of the tangent to the curve *y* = 2𝑥ln 𝑥, which is perpendicular to 2*y* + 𝑥– 3 = 0.

Unit 13 **Differentiation**

1. Find the range of values of 𝑥for which *y* = 3𝑥3 – 4𝑥2 + 5 is increasing.

**CLASSWIZ WORKSHEETS**

1. Find the range of values of 𝑥for which *y* = is increasing.
2. Find the range of values of 𝑥for which *y* = is decreasing.
3. Find the range of values of 𝑥for which *y* = is decreasing.

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1. Find the range of values of 𝑥for which *y* = ln is increasing.

**CLASSWIZ WORKSHEETS**

1. Find the range of values of 𝑥for which *y* = 𝑥*e*2𝑥 is decreasing.
2. The length of a rubber band, *s* cm, at time *t* seconds is given by *s* = – *t*2 – *t* + 3. Find the value(s) of *t*

when

* 1. the length is increasing at a rate of 2 cm/s.
	2. the length is decreasing at a rate of 1 cm/s.
1. Given that *y* = ln ( + 1), find . Given that when 𝑥= 3, the rate of increase of 𝑥is 1.2 units per second,

find the rate of increase of *y*.

Unit 13 **Differentiation**

1. Two variables 𝑥and *y* are related by *y* = ln (*e*–𝑥+ 3) for 𝑥> 0. Given that 𝑥is increasing at a rate of 2 units per second when 𝑥= –1, find the rate of change of *y*.

**CLASSWIZ WORKSHEETS**

1. Two variables 𝑥and *y* are related by *y* = sin 4𝑥– 3 cos 2𝑥where 𝑥is in radians. Given that 𝑥is decreasing at a rate of 4 units per second when 𝑥= 0.1, find the rate of change of *y*.
2. Given that *y* = 𝑥+ sin 𝑥where 𝑥is in radians, find the rate of change of *y* when 𝑥= and if 𝑥increases

at a rate of 0.4 radians per second.

1. Given that *y* = and 𝑥changes at a constant rate of radians per second, find the rate of

change of *y* when 𝑥= .

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1. Determine the nature of the stationary point(s) for the curve *y* = 𝑥(𝑥– 3)3.

**CLASSWIZ WORKSHEETS**

1. Determine the nature of the stationary point(s) for the curve *y* = (𝑥2 – 4)3 + 5.

Unit 13 **Differentiation**

1. Determine the nature of the stationary point(s) for the curve *y* = .

**CLASSWIZ WORKSHEETS**

1. Determine the nature of the stationary point(s) for the curve *y* = *e*𝑥(3 – 𝑥)

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1. Determine the nature of the stationary point(s) for the curve *y* = 3𝑥ln 3𝑥.

**CLASSWIZ WORKSHEETS**

Unit 13 **Differentiation**

# Level 3

**CLASSWIZ WORKSHEETS**

1. A curve has the equation *y* = 3𝑥– 2 tan2 4𝑥.
	1. Find *.*
	2. Find the gradient at 𝑥= .

* 1. Hence, find the equation of the normal.

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1. A curve has the equation *y* = *e*2𝑥– 1cos𝑥.

**CLASSWIZ WORKSHEETS**

𝑥

*y*

tangent

normal

*y* = *e*2𝑥-1*cos* 𝑥

0 π

2

* 1. Find *.*
	2. Find the equation of the tangent and normal at 𝑥 = .

* 1. Find the area enclosed by the tangent, normal and *y*-axis.
1. A curve has the equation *y* = ln

Unit 13 **Differentiation**

tangent

*y* = *ln*

normal

2

**CLASSWIZ WORKSHEETS**

* 1. Find the coordinates of the turning point.
	2. Find the equation of the tangent and normal at 𝑥= 2.
	3. Find the area enclosed by the tangent, normal and *y*-axis.

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1. A curve has the equation *y* = .

**CLASSWIZ WORKSHEETS**

* 1. Find *.*
	2. Find the equation of the normal at 𝑥= – 1.
	3. The normal cuts the curve at another point *A*. Find the coordinates of point *A*.
1. A function is defined by *y* = , where 𝑥≠ ±. Show that the curve is always decreasing.
2. A function is defined by *y* = *e*𝑥3 + 3𝑥2 + 3𝑥+ 4. Show that the curve is always increasing.

Unit 13 **Differentiation**

1. Show that *y* = *e–*𝑥tan 𝑥is always increasing.

**CLASSWIZ WORKSHEETS**

1. A piece of wire of length 48 m is used to make the edges of a rectangular tank as shown in the diagram.

2𝑥+ 1

𝑥

*h*

The dimensions of the box are 2𝑥+ 1 m by 𝑥m by *h* m.

* 1. Express h in terms of 𝑥.
	2. Show that the volume of the tank, *V* m3, is given by *V* = – 6𝑥3 + 19𝑥2 + 11𝑥.
	3. Find the value of 𝑥for which the volume is a maximum.

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1. A piece of wire, 70 cm long, is bent to form the shape as shown in the diagram.

𝑥

2𝑥

*r*

**CLASSWIZ WORKSHEETS**

This shape consists of a semi-circular arc, radius r cm and two sides of a right-angled triangle, 𝑥cm and 2𝑥cm respectively.

* 1. By finding the perimeter of the shape, express 𝑥in terms of *r*.
	2. Show that the area of the shape, *A* cm2, is given by *A* = + .
	3. Hence determine the value of *r* for which *A* is either a maximum or minimum.
	4. Determine whether this value of *r* makes *A* a maximum or minimum.
1. A curve *y* = – for 𝑥> 0, has a stationary value at *K*.

(a) Find .

 (b) Find the coordinates of *K*.

 (c) Determine the nature of the stationary value at *K*.

Unit 13 **Differentiation**

# Level 4

**CLASSWIZ WORKSHEETS**

1. A ladder *AB* of length 4 m is leaning against a vertical wall. The foot of the ladder is sliding away from the wall, *AG*, at a constant rate of 0.3 m/s.

*A*

0.3 m/s

*B G*

Find the rate at which the top of the ladder is sliding down the wall at the instance when the vertical height of the top of the ladder from the ground is 2.4 m.

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1. In the diagram, water flows out of the cone at a rate of 10 cm3/min. The height and base radius of the cone is given as 30 cm and 10 cm respectively. The radius and height of the water level is defined by *r* and *h* respectively.

**CLASSWIZ WORKSHEETS**

 10

*r*

30

*h*

* 1. Express the volume of water, *V*, in terms of *h*.
	2. Find the rate of decrease of the water level when *h* = 12 cm.
	3. Given that the area of the circular exposed water surface is *A* cm2, express *A* in terms of *h*.
	4. Find the rate of decrease of the water surface area when *h* = 12 cm.

Unit 13 **Differentiation**

1. A motorcycle and a car, travelling in opposite directions are approaching each other as shown in the diagram. The vehicles are at a perpendicular distance of 40 m apart.

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Let 𝑥m be the horizontal distance and *y* m be the shortest distance between the motorcycle and the car.

40 m

*y* m

𝑥m

Given that *y* is decreasing at a rate of 30 m/s at the instance when *y* = 120 m, find the rate, in m/s, at which 𝑥is decreasing.

1. Air is let out of a spherical balloon at a constant rate of 90 cm3/s.
	1. Find the rate of change of radius when the radius is 2 cm.
	2. Find the rate that the surface area of the spherical balloon is changing at the same instance.