Area between curves

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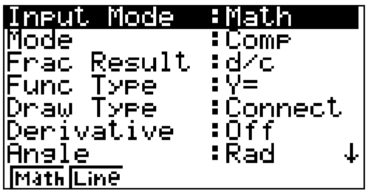
LEVEL
High school or university students who have studied the integration.

OBJECTIVES
To explore how to make use of calculator in finding the area bounded by curves.

Corresponding eActivity
C0401.g1e

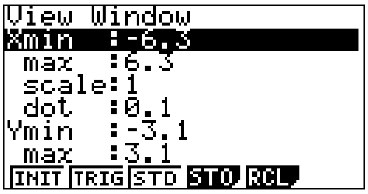
OVERVIEW
Finding the intersections of two curves usually is time consuming traditionally by hand; we will see from this note how calculator can aid us speed up this process and help us find the area bounded by curves.

EXPLORATORY ACTIVITIES
[Note]
(a) We shall use small letter *x* instead of capital X as shown on the calculator throughout the paper.
(b) Unless otherwise specified, we choose MATH mode in the SET UP menu. Go to RUN mode in the Menu. Then press Shift + SET UP and choose Math in the Input Mode, which we show below:

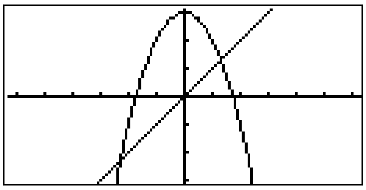


Activity 1: Find the area of the region bounded by the graphs of $y = 3 - x^2$ and $y = x$.
Solution:
Step 1. (Open **Act1. graph)** Sketch the graphs:
We enter the equations above into Y1 and Y2 respectively in the graph editor strip. We

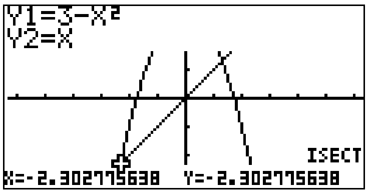
use the default V-Window (or use INIT on the submenu):

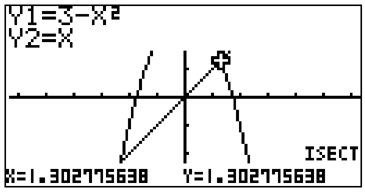


The graph is shown below:


Graph 1

Step 2. We look for the intersections. In this case, we could find the intersections by hand: Set $3-x^2=x$ and we get $x=\dfrac{-1\pm\sqrt{13}}{2}$. In other cases, it will be more difficult to find the intersections by hand. We use F5 or [G-Solv] to find the intersections (ISCT), which we show below:



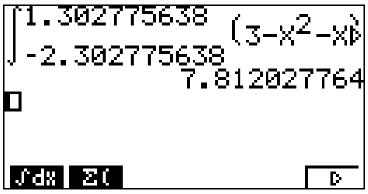


Step 3. Set up the integral.

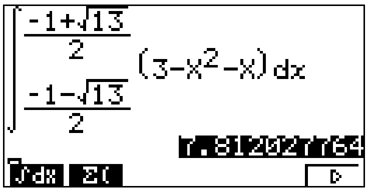
- For those readers who have basic idea of finding the area and setting up the integral, we notice from Graph 1 above that Y1 is greater than Y2 between those two intersections, therefore we will proceed setting up the integral as

$$\int_{\frac{-1-\sqrt{13}}{2}}^{\frac{-1+\sqrt{13}}{2}}\left(3-x^2-x\right)dx\,.$$

- For those readers who just started learning how to find the area under a curve may jump to Step 4 first.
- Of course, if we use the numerical values for the intersections to set up the integral, the area we obtain will be only an approximation to the true value. We use both numerical and symbolic values for the intersections to set up the integral below.
- We open **Act1. numerical**.

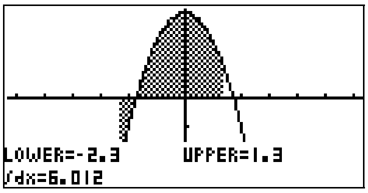


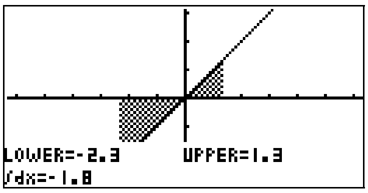
- Set up the integral using symbolic intersections.
- Open **Act1. symbolic**. We obtain the answer to be about 7.812 as shown below:



Step 4. Double check the answer.
We show that we may set up the integrals for both functions separately as shown below:

- Open **Act1. graph** again, and select Y1 (deselect Y2 for the moment), draw the graph for Y1 and press [F5] or [G-Solv] and [F6] [F3]. Use the left arrow ◀ key to select an approximate lower bound, in this case we select LOWER=-2.3. Press [EXE] and the right arrow key ▶ to select an approximate upper bound, in this case we select UPPER=1.3. We show the screen shot of the area under Y1 on the following left. We repeat the same process as described above except we will only select Y2 this time and we show the screen shot of the area under Y2 on the following right. We remark that the LOWER and UPPER bounds shown below will be different when we choose different V-Window, we use 'INIT' in our View Window set up in this case.





- We notice that the area of Y1-Y2 is about what we anticipated (approximately) 7.81.

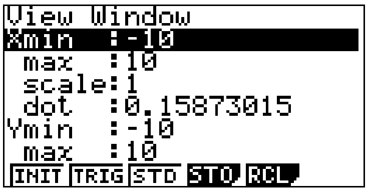
Activity 2: Find the area of the region bounded by the graphs of $f(x) = 3x^3 - 3x^2 - 10x$ and $g(x) = -x^2 + 2x$.

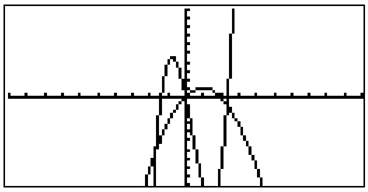
Solution:

Step 1. Sketch the graphs. (Open **Act2. graph**)

- We enter functions *f* and *g* into Y1 and Y2 of graph editor strip respectively.
- By selecting a proper V-Window (shown on the left below), we obtain the graphs of *y=f(x)* and *y=g(x)* below (shown on the right):

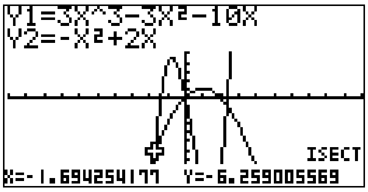
Area between curves

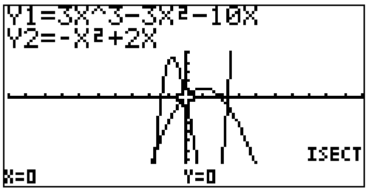


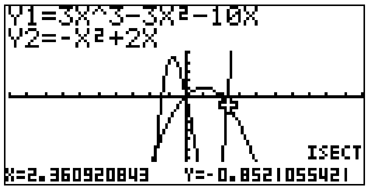


Step 2. Find the intersections.

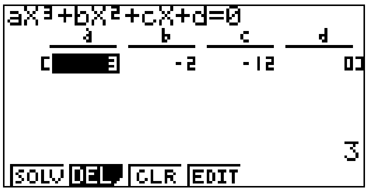
- While the graphs of *y=f(x)* and *y=g(x)* are open as shown above, press [F5] or [G-Solv] and ISCT for intersections. By pressing ▶, we get the following three intersections:



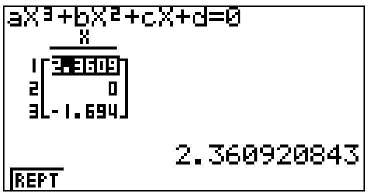




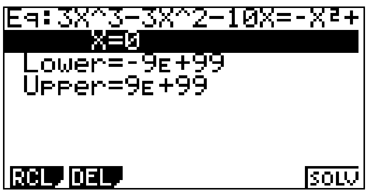
- We can also use the equation solver to find the intersections. First, we write *f(x)=g(x)* as *f(x)-g(x)=0*, or $3x^3 - 2x^2 - 12x = 0$.
- Open **Act2. poly.** We choose degree 3 and enter the coefficients of this degree 3 polynomial equation as shown below:



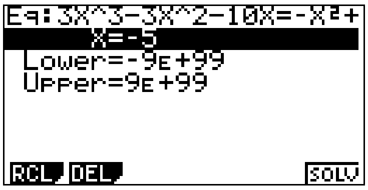
- We find the zeros for $3x^3 - 2x^2 - 12x = 0$ are about at x=2.36092,-1.69425, and 0 respectively, which we show below:

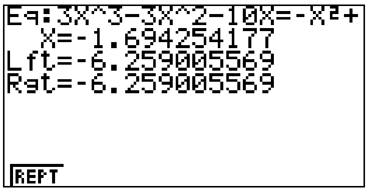


- Another way to find the intersection is to use 'Solver', which we open **Act2. solver.** We first enter the equation of *f(x)=g(x)* as shown below:

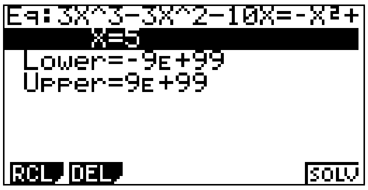


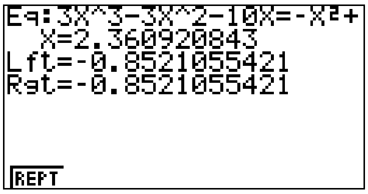
- We next guess a value for X, say X=-5 (shown on the left below), we get the first answer shown on the right below:





- We repeat the process by choosing X=5 shown on the left below and we get the other intersection as shown on the right below:



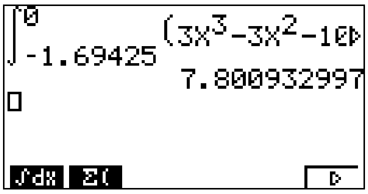


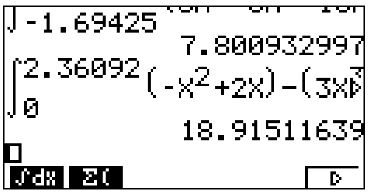
- The last intersection of X=0 is trivial, which we omit the process here.

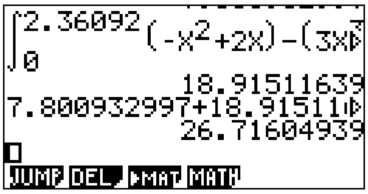
Step 4. Set up the integral. Open **Act2. integral**.

$$\int_{-1.69425}^{0}\left(\left(3x^3-3x^2-10x\right)-\left(-x^2+2x\right)\right)dx+$$
$$\int_{0}^{2.36092}\left(\left(-x^2+2x\right)-\left(3x^3-3x^2-10x\right)\right)dx.$$

- We calculate the above in two separate integrals and we find the answer to be about 26.71605 as shown below (far right):







EXERCISES

Exercise 1. Find the area of the region bounded by $f(x)=x^2+2x+1$ and $g(x)=-x^2+3x+2.$

Exercise 2. Find the area of the region bounded by $f(x)=x^3-4x+3$ and

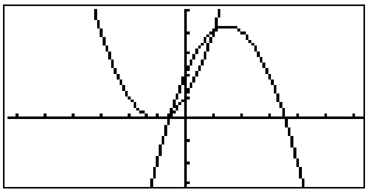
__Area between curves__

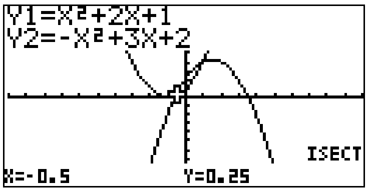
$g(x) = -x^2 + 2x + 3.$

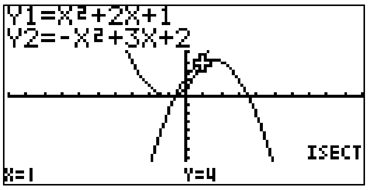
SOLUTIONS

Exercise 1

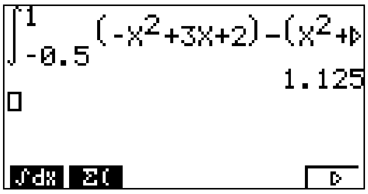
- Open **Ex1 intersection:** The intersections are at (-0.5, 0.25) and (1,4) as shown below:







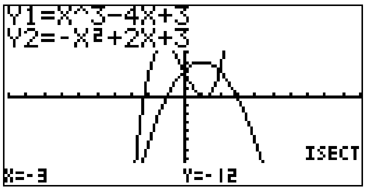
- Open **Ex1 integral:** We set up the integral as shown below:

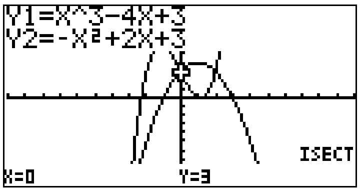


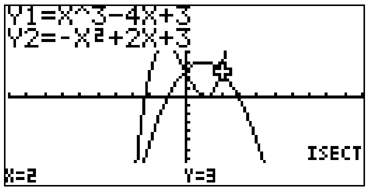
- The area is 1.125.

Exercise 2.

- Open **Ex2 intersection:** The intersections are at (-3, 12), (0,3) and (2,3) as shown below:





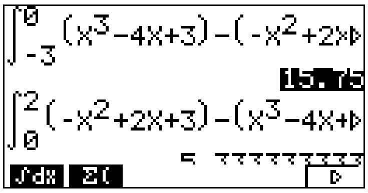


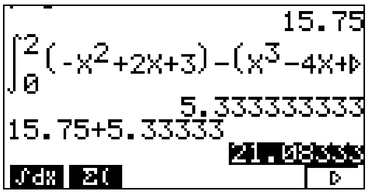
- Open **Ex2 integral:** We set up the integral as shown below.

$$\int_{-3}^{0} \left(\left(x^3 - 4x \quad + 3 \right) - \left(-x^2 + 2x + 3 \right) \right) dx +$$
$$\int_{0}^{2} \left(\left(-x^2 + 2x + 3 \right) - \left(x^3 - 4x \quad + 3 \right) \right) dx.$$

- The screen shots are shown below:

Area between curves





- The area is 21.08333.

REFERENCE

[LEF] Larson, Edwards, Falvo, *Brief Calculus-An Applied Approach*, by Houghton Mifflin Company, 2003, sixth ed., ISBN 0-618-21870-X.