Approaching Logarithmic Equations

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LEVEL
High schools after students have studied logarithmic functions.

OBJECTIVES
To discuss solving logarithmic equations with technology such as graphics calculator.

CORRESPONDING eActivity
LOGEQUA.g1e

OVERVIEW
Traditionally equations involving logarithm are solve algebraically which are quite challenging and may contain pitfalls such as the problem discuss in the first activity. In another activity we discuss solving a problem on logarithmic functions with different bases which is algebraically difficult to solve. The graphics calculator is used in visualizing the problems and for checking results obtained.

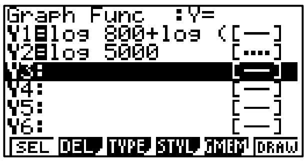
EXPLORATORY ACTIVITIES
[Note]
We shall use small letter *x* instead of capital X as shown on the calculator throughout the paper.

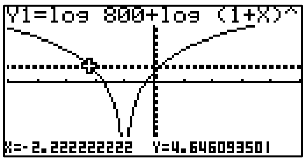
Activity 1: Find x if we are given $\log(800(1+x)^{20}) = \log 5000$.

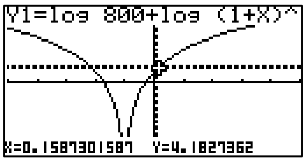
Solution:
There are few approaches we can employ here. We will solve the problem through graphs of the logarithmic functions where we can explore and visualize the solution in the process.

Now open the eActivity LOGEQUA.g1e. First we would define the left side and right side logarithmic functions separately, and graph them on the same set of axes.

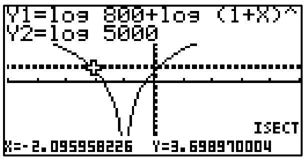
(a) Open the Graph Editor strip "**Act-1A**". Here we assign $\log(800(1+x)^{20})$ as Y1 and the constant $\log 5000$ as Y2. Note that we can enter $\log 800 + \log(1+x)^{20}$ for $\log(800(1+x)^{20})$. Graph both Y1 and Y2 and trace them.

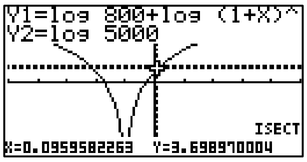






The solutions to this problem are the intersection points of the two graphs. We can obtain more accurate depictions of the two points with [G-Solve].

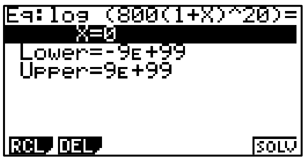


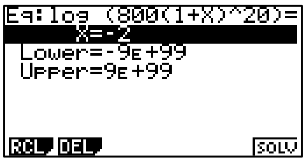


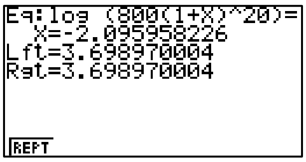
Using [G-Solve] we found that the solutions to the equation are $x \approx -2.096$ and 0.096.

(b) It is usually tedious to check the solutions but with the aid of technology the check and balance is easy to perform and most importantly we can use approaches different from the approach used in solution.

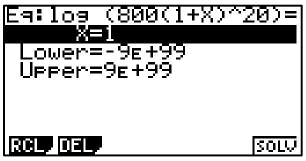
Scroll down to open the Solver strip "**Act-1B**" and enter the equation given. To use the solver we must enter an initial value for the parameters, which in this case is just x. From (a) we understand that one of the solutions is ≈-2.1, therefore we enter x=-2 as an initial value for the first run.

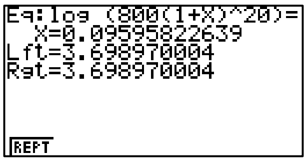






Having check the first solution repeat with the initial value of x=1.





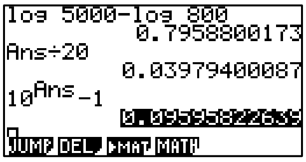
Both runs confirm numerically that the solutions are indeed $x \approx -2.096$ and 0.096.

Discussion

To enhance the discussion we look at a probable algebraic approach which solves the equation using a few fundamental properties of logarithms.

$$\begin{aligned}\log(800(1+x)^{20}) = \log 5000 &\rightarrow \log 800 + \log(1+x)^{20} = \log 5000\\ &\rightarrow 20\log(1+x) = \log 5000 - \log 800\\ &\rightarrow \log(1+x) = \frac{\log 5000 - \log 800}{20}\\ &\rightarrow x = 10\text{^}\left(\frac{\log 5000 - \log 800}{20}\right) - 1\end{aligned}$$

The calculation could then be completed in the calculator as shown in the strip "**Act-1C**".



Note that we have gotten only one solution here when earlier graphical exploration showed two possible solutions. **What went wrong?** If we examine closely on our algebraic approach above, we notice that

$$\log(1+x)^{20} \neq 20\log(1+x).$$

We leave this as an exercise to readers. [Hint: It is not hard to verify that the graphs for y= $\log(1+x)^{20}$ and y= $20\log(1+x)$ are not identical.] Furthermore, $\log(800(1+x)^{20}) = \log 5000$ does have two possible solutions, and the fact the expression $(1+x)^{20}$ is always > 0 except at x=-1, means that the two possible solutions, especially $x\approx$-2.096, are indeed feasible although we have $1+x = 1+(-2.096) = -876.8$.

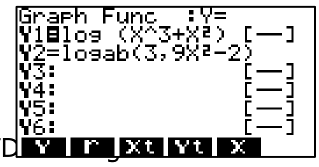
□

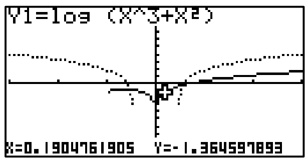
We saw in Activity 1 how useful it is to visualize the problem and the solutions before tackle the problem algebraically as logarithmic equation problems can be difficult to solve algebraically, like this next activity.

Activity 2: Solve the equation $\log(x^3 + x^2) = \log_3(9x^2 - 2)$ for x.

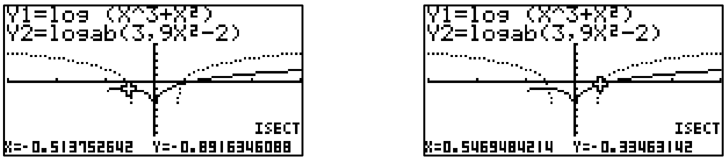
Solution:

(a) Open the Graph Editor strip "**Act-2A**" and enter the left hand side to Y1 and the right hand side to Y2. For entering the right hand side to Y2 we can access [logab] through the calculator catalog by tapping SHIFT 4 and scroll down the catalog to find [logab]. Graph both functions and study the intersections intently.

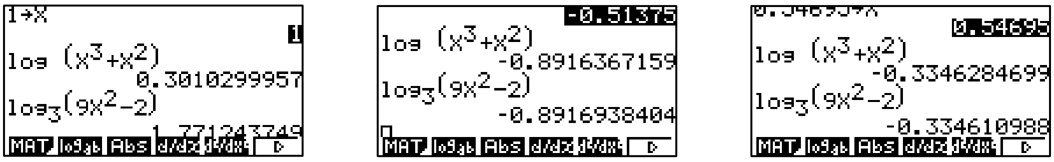




****Remind readers that log_3(9x^2-2)=log(9x^2-2)/log(3).**
Here we again find the problem has two possible solutions. Now solve for the intersection points with [G-Solve] where we should find the solutions are $x \approx -0.51375$ and 0.54695.



(b) Checking the solution with the solver is not feasible (the solver does not compute [logab].) However we can use a different method to do so. Open the <u>Run</u> strip "**Act-2B**" and assign the solutions obtained to X and compare the computations from $\log(x^3 + x^2)$ and $\log_3(9x^2 - 2)$.

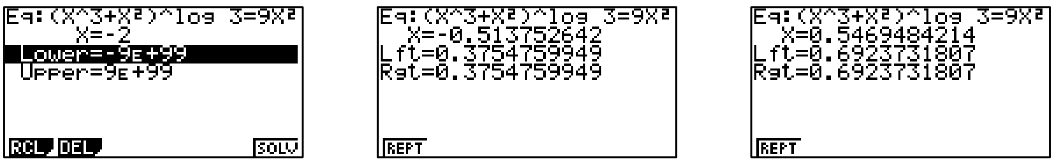


By assigning more accurate version of the solutions we will find the computations of both expressions converge to a similar value. This "substitution-like" investigation supports that both $x \approx -0.51375$ and 0.54695 are solutions to $\log(x^3 + x^2) = \log_3(9x^2 - 2)$.

<u>Discussion</u>
We begin our discussion with a probable algebraic solution which solves the equation using a few fundamental properties of logarithms.

$$
\begin{aligned}
\log(x^3 + x^2) = \log_3(9x^2 - 2) &\to \log(x^3 + x^2) = \frac{\log(9x^2 - 2)}{\log 3} \\
&\to \log 3 \times \log(x^3 + x^2) = \log(9x^2 - 2) \\
&\to \log(x^3 + x^2)^{\log 3} = \log(9x^2 - 2) \\
&\to (x^3 + x^2)^{\log 3} = 9x^2 - 2 \qquad \text{(A)}
\end{aligned}
$$

The last expression is clearly not in a readily solvable form. On the other hand this last expression is equivalent to the original problem, so we can use this new equation to check the solutions obtained in the solver.



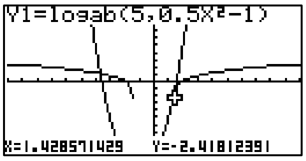
Open the Solver strip "**Act-2C**" and enter the equation (A). Use the initial values of -2 and 2 to check for the solutions respectively. □

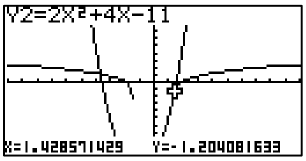
The previous activities show us key strategies which are effective in dealing with logarithmic equations problem with graphics calculator. Let's discuss them in length in Activity 3.

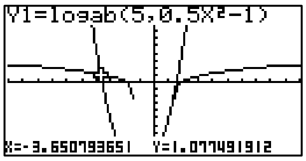
Activity 3: Solve $\log_5(\frac{1}{2}x^2 - 1) = 2x^2 + 4x - 11$ for x.

Solution:
This is a rather difficult equation which we seldom discuss in class without special tool. However with the aid of the graphics calculator we can try solving as follow. First we begin with a graphical exploration of the problem given.

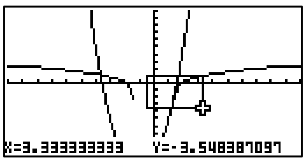
A. Graphical Exploration
Open the Graph Editor strip "**Act-3A**" and graph both Y1 and Y2. I have entered to Y1 and $2x^2 + 4x - 11$ to Y2.

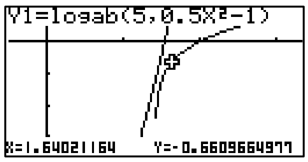




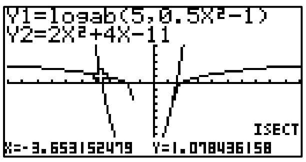


There are possibly 1, 2 or 3 intersection points from the graph. We can zoom in to the "unsure" area to explore further.





It seems from the exploration there is just 1 solution to the equation which is the intersection point occurs at $x \approx -3.6$. An accurate solution could be obtained with [G-Solve].



From the [G-Solve] the solution is $x \approx -3.653$.

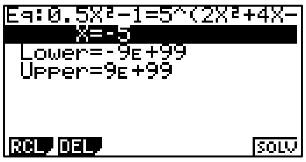
B. Check Solutions

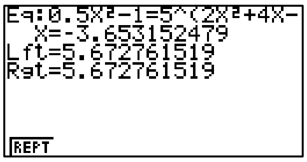
Using the original equation in the Solver will not work. But from Activity 2 we know we can find an equivalent equation which does not have not logarithmic terms.

$$\log_5(\tfrac{1}{2}x^2 - 1) = 2x^2 + 4x - 11 \rightarrow \tfrac{1}{2}x^2 - 1 = 5\text{^}(2x^2 + 4x - 11)$$

Now scroll down to open the <u>Solver</u> strip "**Act-3B**" and enter this new equation. We should test with a few initial values to be sure that there is no other solution present. For this purpose we use the initial values of -5, -1, 0 and 2.

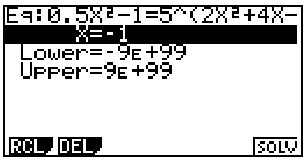
When initial value is -5:

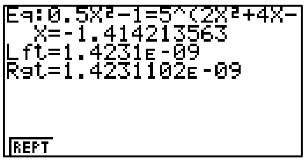




So the solution $x \approx -3.653$ is supported.

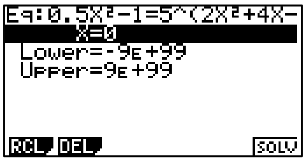
When initial value is -1:

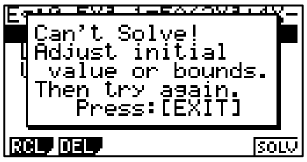




So there is one solution which we did not get at the graphical exploration.

When initial value is 0 or 2:



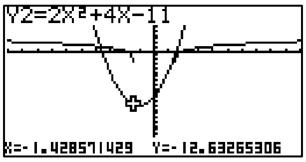


So there is no other solution for positive values of x.

<u>C. Analysis</u>

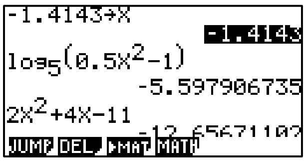
We should end the investigation by solving the given algebraic equation if possible, and such was the case for the first activity. The equation $\log_5(\tfrac{1}{2}x^2 - 1) = 2x^2 + 4x - 11$ is not easily solvable though, so we leave solving it.

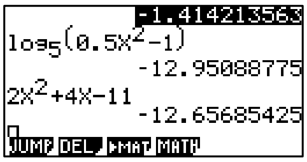
The graphical exploration did not show the solution $x \approx -1.4142$ but the solver did. Our concern is although we are confident of the solution $x \approx -3.653$ we have not check the second solution of $x \approx -1.4142$. If we study the graphs and this second solution does exist at $x \approx$-1.4142, then the intersection point is within the place where the cursor is in the displayed diagram below.



So if we can show that $\log_5(\frac{1}{2}x^2-1)$ is less than $2x^2+4x-11$ at this solution then we have shown that this solution indeed exists.

Open the <u>Run</u> strip "**Act-3C**". The solution correct up to 9 d.p.s is x=-1.414213563. Assign this value to X and compare the two computation outputs below it.





The computations show that $\log_5(\frac{1}{2}x^2-1)$ is less than $2x^2+4x-11$ at the point the solution occurs, which implies that both curves intersected at least once there, which is consistent with the claim that a second solution to $\log_5(\frac{1}{2}x^2-1)=2x^2+4x-11$ occurs at $x \approx$-1.4142. □

EXERCISES

Exercise 1

Solve $\log 800+20\log(1+x)=\log 5000$ for x.

Exercise 2

Solve $\ln|2-x|=\log(8x+3)-(x^3-5)$ for x.

SOLUTIONS to EXERCISES

Exercise 1

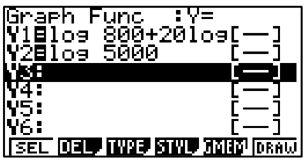
<u>Solution</u>:

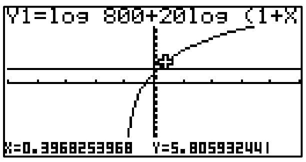
This is a variation from the equation in Activity. We saw from Activity 1 that

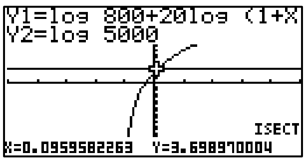
$$\log 800+20\log(1+x)=\log 5000 \quad \rightarrow 20\log(1+x)=\log 5000-\log 800$$

$$\rightarrow x=10\wedge\left(\frac{\log 5000-\log 800}{20}\right)-1$$

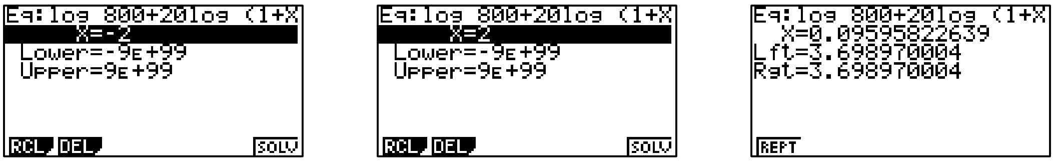
Open the <u>Graph Editor</u> strip "**Exp-1**" and explore the problem graphically.





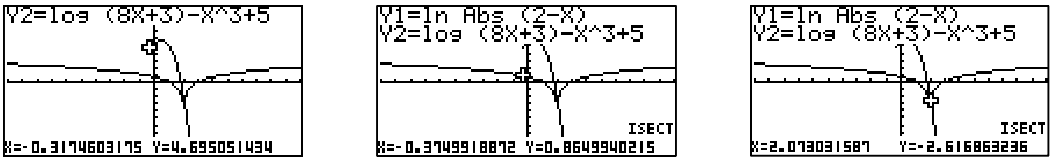


In this case we should expect to see only one intersection point which means just one solution to the problem. Then use Solver strip "**Exp-2**" to check the solution. Try using one negative initial value and one positive initial value to solve the equation.

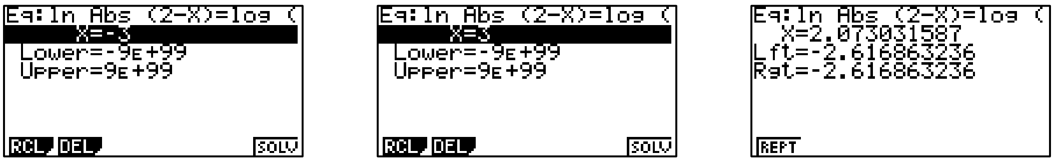


Hence the solution is indeed $x \approx 0.09596$. □

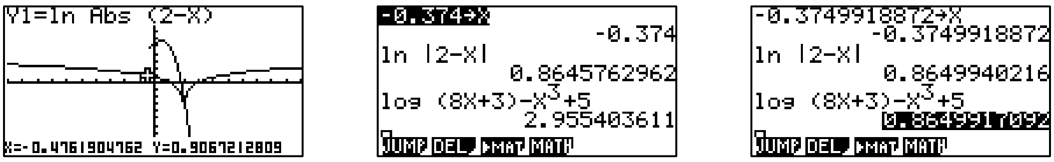
Exercise 2
Solution:
Try solving this equation following the approach studied in Activity 3. Open the Graph Editor strip "**Exp-3**" and graph both Y1 and Y2 then explore them.



There seems to be 2 possible intersection points from the graph. Using [G-Solve] we would have the solutions as $x \approx -0.375$ and 2.073. Now open the Solver strip "**Exp-4**" and enter the equation for checking. For this purpose we use the initial values of -3 and 3.



At this point we could find that we have a situation just the opposite of Activity 3: the solver could not check the solution $x \approx -0.375$ found at the graphs. Similarly if we can show that $\log(8x+3)-(x^3-5)$ is less than $\ln|2-x|$ at $x \approx -0.374991872$ then it is proved that this solution indeed valid.



Use the Run strip "**Exp-5**" and compare the two functions at $x \approx -0.374991872$. □

REFERENCE

[1] Mark Dugopolski, *College Algebra and Trigonometry*, Addison Wesley, 1996. ISBN: 0-201-88952-8.